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GENERALIZED HUA-OPERATORS AND PARABOLIC SUBGROUPS. THE CASES OF SL(n, C) AND SL(n, R)[1]

BY

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ABSTRACT. Suppose $G = \mathrm{SL}(n, \mathbf{C})$ or $\mathrm{SL}(n, \mathbf{R})$ and K is a maximal compact subgroup of G. If P is any parabolic subgroup of G, we determine a system of differential equations on G/K with the property that any function on G/K satisfies these differential equations if and only if it is the Poisson integral of a hyperfunction on G/P.

1. Introduction. Let G be a connected noncompact semisimple Lie group with finite center. Fix $G = KAN$, an Iwasawa decomposition of G. That is, K is a maximal compact subgroup of G, A a maximal vector subgroup of G with Ad A consisting of semisimple elements normalizing N, a maximal simply connected unipotent subgroup of G. The space $\tilde{X} = G/K$ is a Riemannian symmetric space and the group AN acts transitively on $\tilde{X}$. Now if M is the centralizer of A in K, the group $B = MAN$ is a minimal parabolic subgroup of G, and the space $S_0 = G/B = K/M$ is called the maximal or Furstenberg boundary of $\tilde{X}$.

More generally, a boundary S of $\tilde{X}$ is a homogeneous space G/P, where P is a parabolic subgroup of G, i.e. P is conjugate to a closed subgroup of G which contains B. Henceforth, we shall consider only parabolic subgroups which contain B.

Let $D(\tilde{X})$ denote the space of all G-invariant differential operators on $\tilde{X}$ and let $D_+(\tilde{X}) = \{D \in D(\tilde{X}) \colon D1 = 0\}$. We call $F \in C^\infty(\tilde{X})$ harmonic if $DF = 0$ for every $D \in D_+(\tilde{X})$. Deep analysis of [8] shows that a C^∞-function F on $\tilde{X}$ is harmonic if and only if F is the Poisson integral of a hyperfunction T on S_0. In other words (abusing notation), we have

$$F(x) = \int_{S_0} P_0(x, b)\, dT(b),$$

where $P_0 \colon \tilde{X} \times S_0 \to \mathbf{R}^+$ is the Poisson kernel of $\tilde{X}$ with respect to S.

If P is a parabolic subgroup of G and $S = G/P = K/L$, where $L = K \cap P$, there is a Poisson kernel $P_s \colon \tilde{X} \times S \to \mathbf{R}^+$ of $\tilde{X}$ with respect to S with

$$P_s(x, u) = \int_L P_0(x, klM)\, dl,$$

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where $k \in u$ and dl is Haar measure on L with $|L| = 1$. Now suppose that $\tau(S)$ is the space of functions on $\tilde{X}$ which are the Poisson integrals of hyperfunctions on S. Clearly, $\tau(S) \subset \tau(S_0)$ where $\tau(S_0)$ is the space of harmonic functions on $\tilde{X}$. As is the case with $\tau(S_0)$ we wish to know if the space $\tau(S)$ can be characterized by means of differential equations. To be specific we state the problem as follows.

Problem. Determine a collection $\mathcal{C}(S)$ of partial differential operators on $\tilde{X}$ such that $F \in \tau(S)$ if and only if $DF = 0$ for every $D \in \mathcal{C}(S)$.

In this paper, we shall solve this problem for any boundary S of $\tilde{X}$ where G is either $\mathrm{SL}(n, \mathbf{C})$ or $\mathrm{SL}(n, \mathbf{R})$. The results for these groups may be stated in a clean and unencumbered fashion. In a subsequent paper, using various projections of the operators defined here, we shall examine the remaining simple noncompact groups.

In the case where $\tilde{X}$ is a bounded symmetric domain and S its Shilov boundary, this problem was first examined in [**10**], solved for tube type domains in [7] and finished completely in [**1**].

§2 deals with some basic algebra of semisimple Lie groups. In §3 differential operators are introduced, which play a crucial role in our results. §§4–7 are devoted to the proof of our main theorem, with §§5 and 6 used to recall some known results concerning asymptotic differential operators.

2. Some algebraic preliminaries. Let θ be the Cartan involution of G which fixes K and denote the Lie algebras of G, K, M, A, N, and θN by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$ and $\theta\mathfrak{n}$ respectively. For $\lambda \in \mathfrak{a}^*$ set

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g}: [H, X] = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}.$$

Recall that λ is called a restricted root if $\mathfrak{g}_\lambda \neq \{0\}$; we call λ a positive restricted root if, in addition, $\mathfrak{g}_\lambda \subset \mathfrak{n}$. Now, if Φ is the set of restricted roots and Φ^+ is the set of positive restricted roots, $\Phi = \Phi^+ \cup (-\Phi^+)$. We now wish to describe all parabolic subgroups of G up to conjugation.

Let $\mathfrak{a}^+ = \{H \in \mathfrak{a}: \lambda(H) > 0 \text{ for all } \lambda \in \Phi^+\}$. If $H \in \overline{\mathfrak{a}^+}$, let $\Phi^+(H) = \{\lambda \in \Phi^+ : \lambda(H) > 0\}$ and $\mathfrak{n}(H) = \Sigma\{\mathfrak{g}_\lambda: \lambda \in \Phi^+(H)\}$. Now $\mathfrak{n}(H)$ is a nilpotent Lie algebra and $N(H) = \exp \mathfrak{n}(H)$ is a unipotent Lie group. Now, if, $Z_G(H)$ is the centralizer of H in G, the group $P(H) = Z_G(H) \cdot N(H)$ is a parabolic subgroup of G, and up to conjugation all parabolic subgroups of G may be obtained in this way. Suppose $A(H) = Z(Z_G(H)) \cap A$ and $M(H) = [Z_G(H), Z_G(H)] \cdot (Z(Z_G(H)) \cap K)$; $A(H)$ is called the split component of $P(H)$, $N(H)$ is the nilradical and $P(H) = M(H)A(H)N(H)$ is called the Langlands decomposition of $P(H)$. Note that for H_1, $H_2 \in \overline{\mathfrak{a}^+}$, $P(H_1 + H_2) = P(H_1) \cap P(H_2)$ and $P(H_1) = P(H_2)$ if and only if $\Phi^+(H_1) = \Phi^+(H_2)$. Hence, we see that every parabolic subgroup is the intersection of maximal nontrivial parabolic subgroups.

Now suppose $G = \mathrm{SL}(n, \mathbf{C})$ (resp. $\mathrm{SL}(n, \mathbf{R})$). Choose $K = \mathrm{SU}(n)$ (resp. $\mathrm{SO}(n)$), A the group of diagonal elements of G with positive entries down the diagonal, and take N to be the group of upper triangular matrices in G with l's down the diagonal. In both cases the Cartan involution $\theta: G \to G$ is the map $\theta g = (\bar{g}^t)^{-1}$. Now $\mathfrak{a}$ is the space of real diagonal $n \times n$ matrices of trace 0, and $\mathfrak{a}^+$ is the set of matrices $H = (h_{ij})$ in $\mathfrak{a}$ where $h_{ii} > h_{i+1\, i+1}$. If, for any integer $l \geq 0$, I_l is the $l \times l$-identity matrix and k is a positive integer $\leqslant n-1$, let

$$\hat{H}_k = \begin{pmatrix} I_k & 0 \\ 0 & -c_k I_{n-k} \end{pmatrix}$$

with $(n-k)c_k = k$. Then $\hat{H}_k \in \overline{\mathfrak{a}^+}$ and $P(\hat{H}_k) = P(k)$ is a maximal parabolic subgroup, and any parabolic subgroup is conjugate to one of the form

$$P(k_1, k_2, \ldots, k_i) = P(k_1) \cap \cdots \cap P(k_i),$$

with $k_1 < k_2 < \cdots < k_i \leqslant n-1$, and any minimal one to $B = P(1, 2, \ldots, n-1)$. Finally, for $1 \leqslant k_1 < \cdots < k_i \leqslant n-1$, let

$$S(k_1, \ldots, k_i) = G/P(k_1, \ldots, k_i).$$

REMARK. We use the notation "$S(k_1, \ldots, k_i)$, $P(k_1, \ldots, k_i)$" in the real and complex cases. It will be clear with which case we are dealing.

3. Construction of our operators. For $G = \mathrm{SL}(n, \mathbf{C})$ (resp. $\mathrm{SL}(n, \mathbf{R})$), the map $g \to gg^*$ allows us to identify $\tilde{X}$ with the space of positive definite hermitian (resp. real) $n \times n$-matrices of determinant one. Hence, $\tilde{X} \subset H_n$ or P_n, where H_n is the space of positive definite hermitian $n \times n$-matrices, and P_n is the space of positive definite real $n \times n$-matrices, or in other words, the real points of H_n. Thus, we see if $\tilde{X} \subset H_n$ or in P_n the C^∞-functions on $\tilde{X}$ may be identified with C^∞-functions on H_n or P_n which are homogeneous of degree 0.

For an arbitrary $Z \in H_n$, let dZ be the $n \times n$-matrix of differentials whose i, jth entry is dz_{ij}, let ∂ be the $n \times n$-matrix whose i, jth entry is $\partial/\partial z_{ij}$, and let ∂' be the transpose of ∂. Now $\mathrm{GL}(n, \mathbf{C})$ acts transitively on H_n by setting $g(Z) = gZg^*$, where $g \in \mathrm{GL}(n, \mathbf{C})$ and $Z \in H_n$. As $\operatorname{tr} dZ(\partial') = n^2$ identically on H_n and $dg(Z) = gdZg^*$, we have

$$(3.1) \qquad (\partial' f)(g(Z)) = g^{*-1}\left[\partial'(f \circ g)(Z)\right] g^{-1},$$

where $(f \circ g)(Z) = f(g(Z))$ for $f \in C^\infty(H_n)$.

Set $\mathfrak{D} = Z\partial'$ and consider for any integer $k > 0$ the operator $\mathfrak{D}^k$. If $Z \in H_n$, let $(\mathfrak{D}^k)_Z$ be the local expression of $\mathfrak{D}^k$ at Z. Then, for $g \in \mathrm{GL}(n, \mathbf{C})$ and $f \in C^\infty(H_n)$, we have

$$(3.2) \qquad (\mathfrak{D}^k)_{g(Z)} f(g(Z)) = g\left[(\mathfrak{D}^k)_Z (f \circ g)(Z)\right] g^{-1}.$$

From **[14]** we know that the ring of $\mathrm{GL}(n, \mathbf{C})$-invariant differential operators on H_n is generated by the constants and the operators $D_k = \operatorname{tr}(\mathfrak{D}^k)$ for $k \leqslant n$. Then we have that the harmonic functions on $\tilde{X}$ are identified with the functions on H_n which are annihilated by the D_k, and we call such functions harmonic on H_n.

If P is a parabolic subgroup of $\mathrm{SL}(n, \mathbf{C})$, then $\tilde{P} = \{\mu g \colon \mu \in \mathbf{C}^*, \ g \in P\}$ is a parabolic subgroup of $\mathrm{GL}(n, \mathbf{C})$ and

$$S = \mathrm{SL}(n, \mathbf{C})/P = \mathrm{GL}(n, \mathbf{C})/\tilde{P}.$$

Moreover, the Poisson kernel $P_s \colon H_n \times S \to \mathbf{R}^+$ has the property that $P_s(\mu x, b) = P_s(x, b)$ for $\mu \in \mathbf{C}^*$, $x \in \tilde{X}$ and $b \in S$, where $P_s(x, b)$ is the Poisson kernel of $\tilde{X}$ with respect to S.

Suppose $\vec{a} = (a_1, \ldots, a_m) \in \mathbf{C}^m$, and set

$$\mathcal{D}(\vec{a}) = \sum_{i=1}^{m} a_i \mathcal{D}^i. \tag{3.3}$$

From (3.2) we have

PROPOSITION 3.1. *For $F \in C^\infty(H_n)$ and $\vec{a} \in \mathbf{C}^m$, $\mathcal{D}(\vec{a})F = 0$ if and only if $\mathcal{D}(\vec{a})_I(f \circ g)(I) = 0$ for every $g \in \mathrm{GL}(n, \mathbf{C})$, where I is the identity.*

To study harmonic functions on $\tilde{X} = \mathrm{SL}(n, \mathbf{R})/\mathrm{SO}(n)$, we consider $\tilde{X} \subset P_n = \mathrm{GL}^+(n, \mathbf{R})/\mathrm{SO}(n)$. If X is an arbitrary element of P_n, let ∂_0' be the matrix whose i, jth coordinate is $(1 + \delta_{ij})\ (\partial/\partial x_{ij})$ and set $\mathcal{D}_0 = X\partial_0'$. Now $\mathcal{D}_0$ is merely the restriction of $2\mathcal{D}$ to P_n. If again $\vec{a}(a_1, \ldots, a_m) \in \mathbf{C}^m$, define

$$\mathcal{D}_0(\vec{a}) = \sum_{i=1}^{m} a_i (\mathcal{D}_0)^i.$$

The corresponding statements concerning harmonic functions on $\mathrm{SL}(n, \mathbf{R})/\mathrm{SO}(n)$ are now easily seen to be valid in this case also.

4. The Poisson kernel. Let $\tilde{X}$ be as in §3 and suppose S is a boundary of $\tilde{X}$ with T a hyperfunction on S. Let $\mathcal{P}_S(T)$ be the Poisson integral of T. It is well known that a differential operator on $\tilde{X}$ annihilates the Poisson kernel if and only if it annihilates all Poisson integrals $\mathcal{P}_S(T)$. In fact, if the operator is $\mathcal{D}(\vec{a})$ or $\mathcal{D}_0(\vec{a})$, it suffices to show that the operator annihilates all $\mathcal{P}_S(T)$ where T is a K-finite function on S.

For T a function on S and $g \in G$, set $L_g T(b) = T(g^{-1}b)$. From the standard properties of the Poisson integral we have

$$L_{g^{-1}} \mathcal{P}_S(T) = \mathcal{P}_S(L_{g^{-1}} T), \tag{4.1}$$

and hence if $\tilde{X} \subset H_n$ and $\vec{a} \in \mathbf{C}^m$, (3.2) yields

$$\mathcal{D}(\vec{a}) \mathcal{P}_S(T)(g(Z)) = g\left[\mathcal{D}(\vec{a}) \mathcal{P}_S(L_{g^{-1}} T)(Z)\right] g^{-1}. \tag{4.2}$$

The corresponding equation holds when $\tilde{X} \subset P_n$ and $\mathcal{D}(\vec{a})$ is replaced by $\mathcal{D}_0(\vec{a})$. Thus we obtain

PROPOSITION 4.1. *The operator $\mathcal{D}(\vec{a})$ (resp. $\mathcal{D}_0(\vec{a})$) annihilates the Poisson kernel of $\tilde{X}$ with respect to S if and only if $\mathcal{D}(\vec{a})_I \mathcal{P}_S(T)(I) = 0$ (resp. $\mathcal{D}_0(\vec{a})_I \mathcal{P}_S(T)(I) = 0$) for all K-finite functions T on S.*

Suppose $\tilde{X} \subset H_n$ and $U \in \mathrm{End}(\mathbf{C}^n)$, and consider the pairing with K-finite functions on S given by

$$\langle U, T \rangle = [\mathrm{tr}\, \mathcal{D}(\vec{a}) U] \mathcal{P}_S(T)(I).$$

From (4.2) we have

$$\langle \mathrm{Ad}\, kU, L_k T \rangle = \langle U, T \rangle \qquad (k \in K), \tag{4.3}$$

with the corresponding equation holding for $\tilde{X} \subset P_n$.

Let G be $\mathrm{SL}(n, \mathbf{C})$ or $\mathrm{SL}(n, \mathbf{R})$ and let $\mathfrak{g}$ be the Lie algebra of G. Suppose $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$. Identify $\mathfrak{g}$ with its image in $\mathrm{End}\, \mathbf{C}^n$ or $\mathrm{End}\, \mathbf{R}^n$.

PROPOSITION 4.2. *The operator $\mathfrak{D}(\vec{a})$ (resp. $\mathfrak{D}_0(\vec{a})$) annihilates the Poisson kernel of $\tilde{X}$ with respect to S if and only if for all $U \in \mathfrak{p}$ and all K-finite functions T on S, $\langle U, T \rangle = 0$.*

PROOF. Clearly, if $\mathfrak{D}(\vec{a})$ (resp. $\mathfrak{D}_0(\vec{a})$) annihilates the Poisson kernel, $\langle U, T \rangle = 0$ for all such U and T.

If $G = \mathrm{SL}(n, \mathbf{C})$, $\mathfrak{p}_{\mathbf{C}} = \mathrm{sl}(n, \mathbf{C})$, and as $\operatorname{tr} \mathfrak{D}(\vec{a})$ is invariant, we are done in this case.

If $G = \mathrm{SL}(n, \mathbf{R})$, the Furstenberg boundary of $\tilde{X}$ is K/M where $K = \mathrm{SO}(n)$ and M is the group of diagonal matrices in K. As there are no M-fixed vectors in $\mathfrak{so}(n)$, we have automatically that $\langle U, T \rangle = 0$ for all $U \in \mathfrak{so}(n)$. Since $\operatorname{tr} \mathfrak{D}_0(\vec{a})$ is invariant, we are done.

For k a positive integer $\leqslant n - 1$, let $P(k)$ denote the parabolic subgroup defined in §2, and, similarly, for $1 \leqslant k_1 < k_2 < \cdots < k_r \leqslant n - 1$, define $P(k_1, \ldots, k_r)$ as in §2. Now $A(k_1, \ldots, k_r) = Z(M(k_1, \ldots, k_r)) \cap A$ is the split component of $P(k_1, \ldots, k_r)$, and $r = \dim A(k_1, \ldots, k_r)$ is the split rank of $P(k_1, \ldots, k_r)$. Let $L(k_1, \ldots, k_r) = K \cap P(k_1, \ldots, k_r)$ and let V be the space of all K-finite functions on $K/L(k_1, \ldots, k_r)$.

LEMMA 4.3. *The representation of K on $\mathfrak{p}$ occurs in V with multiplicity r.*

PROOF. The space of $L(k_1, \ldots, k_r)$ fixed vectors in $\mathfrak{p}$ is precisely the Lie algebra of $A(k_1, \ldots, k_r)$. Hence we are done by Frobenius reciprocity.

PROPOSITION 4.4. *Let $1 \leqslant k_1 < k_2 < \cdots < k_r \leqslant n - 1$ be integers. If $\tilde{X} = \mathrm{SL}(n, \mathbf{C})/\mathrm{SU}(n)$, let $H^m(k_1, \ldots, k_r) = \{\vec{a} \in \mathbf{C}^m$: $\mathfrak{D}(\vec{a})$ annihilate the Poisson kernel of $\tilde{X}$ with respect to $S(k_1, \ldots, k_r)\}$, and if $\tilde{X} = \mathrm{SL}(n, \mathbf{R})/\mathrm{SO}(n)$, define $H_0^m(k_1, \ldots, k_r)$ similarly with $\mathfrak{D}_0(\vec{a})$ substituted for $\mathfrak{D}(\vec{a})$ above. Then we have:*

(i) $\dim H^m(k_1, \ldots, k_r) \geqslant m - r$;

(i′) $\dim H_0^m(k_1, \ldots, k_r) \geqslant m - r$;

(ii) $H^m(k_1, \ldots, k_r) \subseteq H^m(k_1) \cap \cdots \cap H^m(k_r)$;

(ii′) $H_0^m(k_1, \ldots, k_r) \subseteq H_0^m(k_1) \cap \cdots \cap H_0^m(k_r)$; *and, in fact, for any integer k with $1 \leqslant k \leqslant n - 1$,*

(iii) $\dim H^m(k) = m - 1$; *and*

(iii′) $\dim H_0^m(k) = m - 1$.

PROOF. (i) and (i′) follow from Lemma 4.3. (ii) and (ii′) are obvious. Since $\mathfrak{D}(\vec{e}_1)$ and $\mathfrak{D}_0(\vec{e}_1)$ annihilate only constants for $\vec{e}_1 = (1, 0, \ldots, 0)$, (iii) and (iii′) follows. This completes our proof.

In §7 we show that

$$H^m(k_1, \ldots, k_r) = H^m(k_1) \cap \cdots \cap H^m(k_r),$$

and

$$H_0^m(k_1, \ldots, k_r) = H_0^m(k_1) \cap \cdots \cap H_0^m(k_r),$$

by showing that the hyperplanes $H^m(k)$ and $H_0^m(k)$ $(k \leqslant n - 1)$ are in general position.

5. Asymptotic differential operators. We now recall some results from **[6]** which will be useful in the proof of our main theorem. Since these results are stated there for semisimple groups with finite center, we shall recall them in this framework. The extension of these results to reductive groups will be clear.

Fix G, K, M, A and N as in §2. Let θ be the Cartan involution of G which is the identity on K and set $\bar{N} = \theta N$. Let $\mathfrak{g}$ (resp. $\mathfrak{k}$, $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{n}$ and $\bar{\mathfrak{n}}$) denote the Lie algebra of G (resp., K, M, A, N and $\bar{N}$).

If $g \in G$, g may be written uniquely as

$$g = k(g)\exp H(g)n(g),$$

where $k(g) \in K$, $H(g) \in \mathfrak{a}$ and $n(g) \in N$. Defining $\rho \in \mathfrak{a}^*$ so that $2\rho(H) = \operatorname{tr}(\operatorname{ad} H|_{\mathfrak{n}})$ for $H \in \mathfrak{a}$, the Poisson kernel P_0: $G/K \times K/M \to \mathbf{R}^+$ is defined by setting

$$P_0(gK, kM) = \exp(-2\rho H(g^{-1}k)).$$

With $\tilde{X} = G/K$, $S_0 = K/M$ we say $F \in C^\infty(\tilde{X})$ is the Poisson integral of $f \in C^\infty(S_0)$ if

$$F(x) = \int_{S_0} P_0(x, b)f(b)\, db,$$

where db is the unique K-invariant measure on S_0 normalized with $|S_0| = 1$.

If $F \in C^\infty(\tilde{X}) \subset C^\infty(G)$ and $H \in \overline{\mathfrak{a}^+}$ (see §2) define

$$F_H(g) = \lim_{t\to\infty} F(g\exp tH) = \lim_{t\to\infty} F(g\exp tHK).$$

Let $M(H)$, $A(H)$ and $N(H)$ be as in §2 and set $K(H) = M(H) \cap K$. Then, if F is the Poisson integral of $f \in C^\infty(S_0)$ and $g = umn$ with $u \in K$, $m \in M(H)$ and $n \in N(H)$,

$$F_H(g) = c\int_{K(H)/M} \left(\exp(-2\rho_0 H(m^{-1}k))\right)f(ukM)\, dkM,$$

where $2\rho_0(\bar{H}) = \operatorname{tr}\operatorname{ad}\bar{H}|_{\mathfrak{n}(H)}$ with $\mathfrak{n}(H)$ the Lie algebra of $N(H)$, and

$$0 \neq c = \int_{M(H)\cap \bar{N}} \exp(-2\rho H(\bar{n}))\, d\bar{n}.$$

Now $F_H \in C^\infty(G/Q(H))$ where $Q(H) = K(H)A(H)N(H)$, so F_H is uniquely determined by its restriction to $\bar{N}(H)M(H)$. Moreover, for any $\bar{n} \in \bar{N}(H)$ the function $g \to F_H(\bar{n}g)$ is a harmonic function on $M(H)/K(H)$ provided $g \in M(H)$.

For $u \in C^\infty(G)$ and $X \in \mathfrak{g}$ set

$$Xu(g) = \frac{d}{dt}u((\exp - tX)g)\,|_{t=0}.$$

If F is the Poisson integral of $f \in C^\infty(S_0)$, $X_1, \ldots, X_p \in \bar{\mathfrak{n}}(H) = \theta(\mathfrak{n}(H))$, and $Y_1, \ldots, Y_q \in \mathfrak{m}(H) + \mathfrak{n}(H)$ with $Y_i = Y_i^0 + Z_i$ ($Y_i^0 \in \mathfrak{m}(H)$, $Z_i \in \mathfrak{n}(H)$), Lemma 2.1 of **[6]** states that for $\bar{n} \in \bar{N}(H)$ and $m \in M(H)$,

$$(5.1) \qquad \left(Y_1 \cdots Y_q X_1 \cdots X_p L_{\bar{n}}^{-1} F\right)_H(m) = Y_1^0 \cdots Y_p^0 \tilde{F}_H(m),$$

where

$$\tilde{F}(m)=\int_{S_0}P_0(mK,b)X_1\cdots X_pL_{\bar{n}}^{-1}f(b)\,db.$$

Fixing F, we now proceed to rewrite (5.1).

As F is a function on G/K, we may consider F as a function on $\bar{N}\times G/K$ by setting

$$F(\bar{n},g)=L_{\bar{n}}^{-1}F(g)=F(\bar{n}g).$$

If $X_1,\ldots,X_p$ and $Y_1\cdots Y_q$ are as above, then

$$Y_1\cdots Y_qX_1\cdots X_pL_{\bar{n}}^{-1}F(g)=(-1)^pY_1\cdots Y_qR(X_p)\cdots R(X_1)F(\bar{n},g),$$

where for $X\in\bar{\mathfrak{n}}$,

$$R(X)F(\bar{n},g)=\frac{d}{dt}F(\bar{n}\exp tX,g)\,|_{t=0},$$

and for $Y\in g$,

$$YF(\bar{n},g)=\frac{d}{dt}F(\bar{n}\exp -tYg)\,|_{t=0}.$$

Setting

$$F_H(\bar{n},g)=\lim_{t\to\infty}F(\bar{n},g\exp tH),$$

we have $F_H(n,g)=F_H(ng)=(L_{\bar{n}}^{-1}F)_H(g)$. (5.1) now becomes

$$\begin{aligned}(5.2)\qquad &(-1)^p\big(Y_1\cdots Y_qR(X_p)\cdots R(X_1)F\big)_H(\bar{n},m)\\ &\quad=(-1)^p\big(Y_1^0\cdots Y_q^0R(X_p)\cdots R(X_1)F_H\big)(\bar{n},m).\end{aligned}$$

Observe that each Y_i commutes with each $R(X_j)$.

If $D=Y_1\cdots Y_qR(X_p)\cdots R(X_1)$, set $D_H=Y_1^0\cdots Y_q^0R(X_p)\cdots R(X_1)$. Then $(DF)_H=D_HF_H$. Suppose $D_1,\ldots,D_k$ are operators of the same type as D and $\alpha_i\in C^\infty(\bar{N}\times G/K)$ for $i\leqslant k$. If $\alpha_{i,H}$ exists for each i, we have

$$(5.3)\qquad \left[\left(\sum_{i=1}^k\alpha_iD_i\right)F\right]_H\cdot(\bar{n},m)=\sum_{i=1}^k\alpha_{i,H}(\bar{n},m)D_{i,H}F_H(\bar{n},m).$$

If $D\in\mathfrak{U}(\mathfrak{g})$, the enveloping algebra of $\mathfrak{g}$, $DF(\bar{n},m)=(\mathrm{Ad}\,\bar{n}^{-1}D)L_{\bar{n}}^{-1}F(m)$. Now

$$\mathrm{Ad}\,\bar{n}^{-1}D=\sum_{i=1}^k\alpha_i(\bar{n})D_i,$$

where each $D_i\in\mathfrak{U}(\mathfrak{g})$ and each $\alpha_i\in C^\infty(\bar{N}(H))$. From the Poincaré-Birkhoff-Witt Theorem each D_i may be written as $Y_1\cdots Y_qR(X_p)\cdots R(X_1)$. Thus, $(\mathrm{Ad}\,\bar{n}^{-1}D)_H$ exists and $(DF)_H(\bar{n},m)=(\mathrm{Ad}\,\bar{n}^{-1}D)_HF_H(\bar{n},m)$.

6. The operators $(\mathfrak{D}^k)_H$ and $(\mathfrak{D}_0^k)_H$. We now apply the results of the previous section to our operators $\mathfrak{D}^k$ and $\mathfrak{D}_0^k$, and calculate explicitly $(\mathfrak{D}^k)_H$ and $(\mathfrak{D}_0^k)_H$ for $H\in\overline{\mathfrak{a}^+}$. We first consider the case of H_n and proceed to rewrite the entries of $\mathfrak{D}$ in terms of the Lie algebra $\mathfrak{gl}(n,\mathbf{C})$.

As we shall find it necessary to distinguish the abstract Lie algebra $\mathfrak{gl}(n, \mathbf{C})$ from its standard realization, let ν be the standard representation of $\mathrm{GL}(n, \mathbf{C})$ and $\mathfrak{gl}(n, \mathbf{C})$ on $\mathbf{C}^n$. Fix $i\sqrt{-1}$ and suppose J is the linear operator on $\mathfrak{gl}(n, \mathbf{C})$ for which $\nu(JX) = i\nu(X)$ for all $X \in \mathfrak{gl}(n, \mathbf{C})$. Where there is no possibility of confusion we shall write X for $\nu(X)$.

Let θ be the Cartan involution of $\mathfrak{gl}(n, \mathbf{C})$ which is the identity on $\mathfrak{u}(n)$. For $Z, W \in \mathfrak{gl}(n, \mathbf{C})$ set

$$(Z + iW)^* = Z^* + iW^* = -(\theta Z + i\theta W)$$

and

$$\overline{Z + iW} = Z - iW.$$

On $\mathrm{GL}(n, \mathbf{C})$ we define $g \to g^*$ accordingly.

If $F \in C^\infty(H_n)$, F lifts to a function on $\mathrm{GL}(n, \mathbf{C})$ and we set $L_g^{-1}F(Z) = F(gZg^*)$. For $X \in \mathfrak{gl}(n, \mathbf{C})$,

$$\begin{aligned} XF(Z) &= \frac{d}{dt} F((\exp - tX)Z(\exp - tX^*))\big|_{t=0} \\ &= \frac{d}{dt} F(Z - t(XZ + ZX^*) + O(t^2))\big|_{t=0} \\ &= -\mathrm{Tr}(\partial' F(Z))(XZ + ZX^*) = -\mathrm{Tr}\, \nu(X)\mathfrak{D}F(Z) - \mathrm{Tr}\, \nu(X^*)\mathfrak{D}^*F(Z). \end{aligned}$$

Hence,

$$\tfrac{1}{2}(X - iJX)F(Z) = -\mathrm{Tr}\, \nu(X)\mathfrak{D}F(Z),$$

and

$$\tfrac{1}{2}(X + iJX)F(Z) = -\mathrm{Tr}\, \nu(X^*)\mathfrak{D}^*F(Z).$$

Thus, if for $j, k \leqslant n$, $E_{jk} \in \mathfrak{gl}(n, \mathbf{C})$ with $\nu(E_{jk})$, the matrix with 1 in the (j, k) entry and zeroes elsewhere,

$$\mathfrak{D} = \sum \nu(E_{jk})Z_{kj},$$

where $Z_{kj} = \frac{1}{2}(E_{kj} - iJE_{kj})$ and the sum ranges over $1 \leqslant j, k \leqslant n$.

Fix $l \leqslant n$ and set $H = \hat{H}_l$, where

$$\hat{H}_l = \begin{pmatrix} I_l & 0 \\ 0 & -c_l I_{n-l} \end{pmatrix}$$

with $(n - l)c_l = l$. Define $M(H)$, $K(H)$, $N(H)$ and $\overline{N}(H)$ as in §5. For $\bar{n} \in \overline{N}(H)$ and $g \in \mathrm{GL}(n, \mathbf{C})$,

$$F(\bar{n}, g) = F(\bar{n}gg^*\bar{n}^*).$$

We shall now find it convenient to introduce some additional notation.

Suppose $\mathfrak{S}$ is an $n \times n$-matrix of differential operators. The notation $g \to \mathfrak{S}\nu(g)$ (or $g \to \mathfrak{S}\nu(g)^{-1}$) indicates that $\mathfrak{S}$ does not differentiate the entries of $\nu(g)$, while the notation $\mathfrak{S}\,\circ$ will indicate that the entries of $\mathfrak{S}$ differentiate everything to the right of $\mathfrak{S}$. As usual, $\mathfrak{S}^r = \mathfrak{S} \circ \mathfrak{S}^{r-1}$.

Keeping the notation of §5, set

$$DF(\bar{n}, g) = \sum_{j \leq l < k} \nu(E_{jk}) R(Z_{kj}) F(\bar{n}, g),$$

$$\hat{D}F(\bar{n}, g) = \sum_{k \leq l < j} \nu(E_{jk}) R(\bar{Z}_{jk}) F(\bar{n}, g),$$

$$-D_0 F(\bar{n}, g) = \left(\sum_{j, k \leq l} \nu(E_{jk}) Z_{kj} + \sum_{j, k > l} \nu(E_{jk}) Z_{kj} \right) F(\bar{n}, g),$$

where $2\bar{Z}_{jk} = E_{jk} + iJE_{jk}$.

PROPOSITION 6.1. *For $F \in C^\infty(H_n, \operatorname{End} \mathbf{C}^n)$, $\bar{n} \in \bar{N}(H)$ and $m \in M(H)$,*

$$\mathfrak{D}F(\bar{n}mm^*\bar{n}^*) = \nu(\bar{n})\left(D + D_0 + \nu(mm^*)\hat{D}\nu(mm^*)^{-1}\right)\nu(\bar{n})^{-1} F(\bar{n}, m).$$

PROOF. As $\mathfrak{D}F(\bar{n}mm^*\bar{n}^*) = \nu(\bar{n})\mathfrak{D}\nu(\bar{n})^{-1}F(\bar{n}, m)$, we obtain

$$\mathfrak{D}F(\bar{n}mm^*\bar{n}^*) = \nu(\bar{n})(D + D_0 + D_1)\nu(\bar{n})^{-1} F(\bar{n}, m)$$

where

$$D_1 = -\sum_{k \leq l < j} \nu(E_{jk}) Z_{kj}.$$

We now need only show that

$$D_1 = \nu(mm^*)\hat{D}\nu(mm^*)^{-1}.$$

Setting $G(m) = \nu(\bar{n})^{-1}F(\bar{n}, m)$, observe that

$$\begin{aligned} D_1 G(m) &= -\sum_{k \leq l < j} \nu(E_{jk}) \operatorname{Ad} m^{-1}(Z_{kj})(L_{m^{-1}}G)(1) \\ &= -\sum_{k \leq l < j} \nu(E_{jk}) \left[\operatorname{Ad} m^{-1}(Z_{kj})\right]^* (L_{m^{-1}}G)(1) \\ &= -\sum_{k \leq l < j} \nu(E_{jk}) \left[\operatorname{Ad} mm^*(\bar{Z}_{jk})\right] G(m). \end{aligned}$$

The proof now follows since $\nu(mm^*)$ is a selfadjoint matrix and, for $a, b \in \mathbf{C}$ and $X \in \mathfrak{gl}(n, \mathbf{C})$,

$$(a + bJ)X + iJ(a + bJ)X = (a - bi)(X + iJX).$$

Setting $\mathfrak{S} = \nu(\bar{n})\mathfrak{S}_1\nu(\bar{n})^{-1}$ with

$$\mathfrak{S}_1 = D + D_0 + \nu(mm^*)\hat{D}\nu(mm^*)^{-1},$$

we obtain easily that

$$\mathfrak{D}^p F(\bar{n}mm^*\bar{n}^*) = \mathfrak{S}^p F(\bar{n}, m) = \nu(\bar{n})(\mathfrak{S}_1 + C)^{p-1} \circ \mathfrak{S}_1 \nu(\bar{n})^{-1} F(\bar{n}, m),$$

where

$$C = \sum_{j \leq l < k} \nu(E_{jk})\nu(E_{kj}) = \begin{pmatrix} (n - l)I_l & 0 \\ 0 & 0 \end{pmatrix}.$$

PROPOSITION 6.2. *Suppose F is the Poisson integral of a C^∞-function and $\bar{n} \in \bar{N}(H)$, $m \in M(H)$ with $H = \hat{H}_l$. Then*

$$\lim_{t\to\infty} \mathfrak{D}^p F(\bar{n} m \exp 2tHm^*\bar{n}^*) = \nu(\bar{n})(D + C + D_0)^{p-1} \circ (D + D_0)\nu(\bar{n})^{-1} F_H(\bar{n}, m).$$

PROOF. As $\nu(m \exp 2tHm^*)\hat{D}\nu(m \exp 2tHm^*)^{-1}$ decays exponentially regardless of the number of entries of D_0 which differentiate it, the proof is an immediate consequence of (5.2) and (5.3).

In the case of P_n, again take ν to be the standard representation of $\mathrm{GL}(n, \mathbf{R})$ and $\mathfrak{gl}(n, \mathbf{R})$. Also, if $X \in \mathfrak{gl}(n, \mathbf{R})$, let $X' = -\theta X$, where θ is the Cartan involution of $\mathfrak{gl}(n, \mathbf{R})$ which is the identity on $\mathfrak{so}(n)$. Define $g \to g'$ accordingly for $g \in \mathrm{GL}(n, \mathbf{R})$. Then

$$\mathfrak{D}_0 F = -\sum \nu(E_{jk}) E_{kj} F,$$

where the sum is taken over all $j, k \leqslant n$. If $H = \hat{H}_l$, $m \in M(H)$ and $\bar{n} \in \bar{N}(H)$,

$$\mathfrak{D}_0 F(\bar{n} m m' \bar{n}') = \nu(\bar{n})\left(D + D_0 + \nu(mm')\hat{D}(mm')^{-1}\right)\nu(\bar{n})^{-1} F(\bar{n}, m),$$

where D, D_0 and $\hat{D}$ are defined as in the complex case except that Z_{pq} is replaced by E_{pq}. Setting

$$\mathfrak{S}_1 = D + D_0 + \nu(mm')\hat{D}\nu(mm')^{-1} \quad \text{and} \quad \mathfrak{S} = \nu(\bar{n})\mathfrak{S}_1\nu(\bar{n})^{-1},$$

$$\mathfrak{D}_0^p F(\bar{n} m m' \bar{n}') = \mathfrak{S}^p F(\bar{n}, m).$$

Unlike the complex case the entries of $\hat{D}$ do not act trivially on $\nu(\bar{n})$. However, we still obtain, using the same proof as in Proposition 6.2,

$$(\mathfrak{S}^p F)_H(\bar{n}, m) = \nu(\bar{n})(D + C + D_0)^{p-1} \circ (D + D_0)\nu(\bar{n})^{-1} F_H(\bar{n}, m).$$

7. The main theorem. In this section we give criteria for the operators $\mathfrak{D}(\vec{a})$ and $\mathfrak{D}_0(\vec{a})$ to characterize harmonic functions on $\tilde{X}$ which are the Poisson integrals of hyperfunctions on a boundary S of $\tilde{X}$.

DEFINITION 7.1. For $n \geqslant 2$ and l a positive integer $\leqslant n - 1$, let

$$U^m(l) = \left\{\vec{a} \in \mathbf{C}^m \colon \sum_{j=1}^m a_j (n-l)^j = 0\right\},$$

and for $1 \leqslant k_1 < \cdots < k_r \leqslant n - 1$, let

$$U^m(k_1, \ldots, k_r) = U^m(k_1) \cap \cdots \cap U^m(k_r).$$

PROPOSITION 7.1. *For any integers $1 \leqslant k_1 < \cdots < k_r \leqslant n - 1$, the hyperplanes $U^m(k_1), \ldots, U^m(k_r)$ are in general position.*

PROOF. This is immediate from the theory of Vandermonde determinants.

Suppose $U^m = \bigcup_{k=1}^{n-1} U(k)$.

THEOREM 7.2. *Suppose F is harmonic on H_n (resp. P_n) $(n \geqslant 2)$ and $\mathfrak{D}(\vec{a})F = 0$ (resp. $\mathfrak{D}_0(\vec{a})F = 0$) for some $\vec{a} \in \mathbf{C}^m$. Then, if $a \notin U^m$, F is constant.*

PROOF. We shall prove this result only in the case that F is harmonic on H_n, as this case contains one complication which does not occur in the real case.

From [3] we know that F is the convergent sum of K-finite harmonic functions which are annihilated by $\mathfrak{D}(\vec{a})$. Hence it suffices to prove the theorem for F a K-finite function.

As in §§5 and 6, let $l \leqslant n-1$ and set $H = \hat{H}_l$. From (5.2), (5.3) and Proposition 6.2, we obtain, for $\bar{n} \in \bar{N}(H)$ and $m_0 \in M(H)$,

$$\sum_{p=1}^{m} a_p \nu(\bar{n})(D + C + D_0)^{p-1} \circ (D + D_0)\nu(\bar{n})^{-1} F_H(\bar{n}, m_0) = 0.$$

If F_H is independent of m_0, F_H is a function on $S(l)$, the space of l-planes in $\mathbf{C}^m$, and we write $F_H(\bar{n}) = F_H(\bar{n}, -)$. As $D^2 = 0$ the above equation yields

$$\sum_{p=1}^{m} a_p C^{p-1} D F_H(\bar{n}) = \sum_{p=1}^{m} a_p (n-l)^{p-1} D F_H(\bar{n}) = 0.$$

As $R(X - iJX)F_H(\bar{n}) = 0$ for all $X \in \bar{\mathfrak{n}}(H)$, F_H is (anti) holomorphic on a dense open subset of $S(l)$ which is an orbit of $\bar{N}(H)$ is $S(l)$. As F_H is K-finite, F_H is (anti) holomorphic on $S(l)$. Since $S(l)$ is compact, F_H, and hence F, is constant. (In the real case we obtain $R(X)F_H = 0$ for all $X \in \bar{\mathfrak{n}}(H)$ which implies F_H is constant.) We now proceed by induction on n to show that $F_H(\bar{n}, m)$ is independent of m.

If $n = 2$, $M(H)/K(H) = H_1 \times H_1$, and since any harmonic function on H_1 is constant, we are done in this case. Now assume our result holds for $n < r$ and suppose $n = r$.

Fixing $\bar{n}$, $m_0 \to F_H(\bar{n}, m_0)$ is harmonic $M(H)/K(H) = H_l \times H_{r-l}$, and $D_0 = \left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)$, where α is the operator $\mathfrak{D}$ for H_l and β is the operator $\mathfrak{D}$ for H_{r-l}. As no entry of D_0 differentiates $\nu(\bar{n})^{-1}$, we obtain

$$\sum_{p=1}^{m} a_p (D_0 + C)^{p-1} D_0 F_H(\bar{n}, m_0) = 0.$$

Hence we have

$$\sum_{p=1}^{m} a_p (\alpha + n - l)^{p-1} \alpha F_H(\bar{n}, m_0) = 0$$

and

$$\sum_{p=1}^{m} a_p \beta^p F_H(\bar{n}, m_0) = 0.$$

As

$$\sum_{p=1}^{m} a_p (r - l - k)^p \neq 0$$

for any $1 \leqslant k \leqslant r - l$, we have by induction that F_H is constant on H_{r-l}. As

$$\sum_{p=1}^{m} a_p (r - l + l - k)^{p-1}(l-k) = \sum_{p=1}^{m} a_p (r-k)^{p-1}(l-k) \neq 0$$

for any $1 \leqslant k < l$, F_H is constant on H_l also. Hence F_H is constant on $M(H)/K(H)$. From our previous remarks this proves our theorem.

PROPOSITION 7.3. *Suppose F is the Poisson integral of a hyperfunction on $S(l)$ and $\mathfrak{D}(\vec{a})F = 0$ (resp. $\mathfrak{D}_0(\vec{a})F = 0$) for $\vec{a} \in \mathbf{C}^m$. If $\vec{a} \notin U^m(l)$, F is constant.*

PROOF. Again considering only the complex case, we assume F is K-finite and select $H = \hat{H}_l$. Now there is an $\alpha \in \mathbf{C}^*$ and $\vec{b} \in U^m(l)$ such that $\mathfrak{D}(\vec{a}) = \alpha\mathfrak{D} + \mathfrak{D}(\vec{b})$. As $\alpha \neq 0$, $(\mathfrak{D}F)_H = 0$ and hence F is constant.

Fix $1 \leqslant k_1 < k_2 < \cdots < k_r < n$ and set $I = \{k_1,\ldots,k_r\}$.

COROLLARY. *For any l, $U^m(l) = H^m(l) = H_0^m(l)$, and in general*

$$H^m(k_1,\ldots,k_r) = U^m(k_1,\ldots,k_r) = H_0^m(k_1,\ldots,k_r).$$

PROOF. From the above proposition, $U^m(l) \subset H^m(l)$ and $U^m(l) \subset H_0^m(l)$. As $H^m(l)$ and $H_0^m(l)$ are of codimension one in $\mathbf{C}^m$, we have equality. Our result now follows (i), (i′), (ii), and (ii′) of §4 and Proposition 7.1.

We now prove our main result.

THEOREM 7.4. *Suppose*

$$\vec{a} \in \bigcap_{l\in I} U^m(l) \sim \left[\bigcup_{j\notin I} U^m(j)\right]$$

and F is a harmonic function on H_n (resp. P_n). Then F is the Poisson integral of a hyperfunction on $S(k_1,\ldots,k_r)$ if and only if $\mathfrak{D}(\vec{a})F = 0$ (resp. $\mathfrak{D}_0(\vec{a})F = 0$).

PROOF. Necessity is immediate from the above corollary. Assuming F is K-finite set, for $i \leqslant r$,

$$F(1,\ldots,i) = F(1,\ldots,i-1)_{H_{k_i}}$$

and recall from §5 that $F_{H_1+H_2} = (F_{H_1})_{H_2}$. It suffices to show that $F(1,\ldots,r)$ is defined on $S(k_1,\ldots,k_r)$. As before we prove our result for F harmonic on H_n. Let $H = \hat{H}_{k_1}$.

Now $M(H)/K(H) = H_{k_1} \times H_{n-k_1}$ and, fixing $\bar{n} \in \overline{N}(H)$, we see from the proof of Theorem 7.2 that $F(1)(\bar{n},)$ is constant on H_{k_1}. If $H = \hat{H}_{k_1} + \hat{H}_{k_2}$,

$$M(H)/K(H) = H_{k_1} \times H_{k_2-k_1} \times H_{n-k_2},$$

and $F(1,2)$ is constant on $H_{k_1} \times H_{k_2-k_1}$. Continuing in this way we obtain that $F(1,\ldots,r)$ is defined on $S(k_1,\ldots,k_r)$ (i.e., constant on $H_{k_1} \times H_{k_2-k_1} \times \cdots \times H_{n-k_r}$). This completes the proof.

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