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THE $\bar{\partial}$-NEUMANN SOLUTION TO THE INHOMOGENEOUS CAUCHY-RIEMANN EQUATION IN THE BALL IN $\mathbf{C}^n$

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ABSTRACT. Let ϑ denote the formal adjoint of the Cauchy-Riemann operator $\bar{\partial}$ on $\mathbf{C}^n$, and let N denote the Kohn-Neumann operator on the unit ball in $\mathbf{C}^n$. The operator $\vartheta \circ N$ provides a natural fundamental solution for $\bar{\partial} f = g$ on the ball. It is our purpose to present the kernel P for this operator $\vartheta \circ N$ explicitly—the coefficients are exhibited as rational functions.

1. Introduction. The $\bar{\partial}$-Neumann problem was proposed by Spencer. The motivation for the problem comes from a desire to solve the inhomogeneous Cauchy-Riemann equations. This problem was successfully carried through by Kohn (see **[K]** and **[FK]**). However, none of the kernels shown to exist by Kohn are explictly known even in the simplest cases. It is our purpose to present one of these kernels explicitly (the coefficients of the kernel are rational functions).

Let $D \subset \mathbf{C}^n$ denote the unit ball. Let ϑ denote the formal adjoint of $\bar{\partial}$ with respect to the euclidean metric on D, and let N denote the Kohn-Neumann operator (see §2 for the defining properties of N). The operator $P = \vartheta \circ N$ is a modified fundamental solution for $\bar{\partial}$ on D; i.e., if K is the Bergman projection, and if ϕ is a smooth form on $\bar{D}$, then

$$\phi = \bar{\partial} P\phi + P\bar{\partial}\phi + K\phi.$$

In §2 we will present the kernel of P explicitly.

In §3(A) we will derive five properties of the kernel of the operator P. In §3(B) we will start with these five properties and show that they uniquely determine a kernel $P(\zeta, z)$ which must then be the kernel of $\vartheta \circ N$.

In §4, motivated by the kernel P on the ball, we describe a kernel for solving $\bar{\partial} f = g$ on a general strictly pseudoconvex domain. Then we put $P(\zeta, z)$ into an alternate form which makes certain properties of P more accessible. For example, applying the identity above with ϕ replaced by $\chi_D g$, where g is a smooth form on $\bar{D}$ and χ_D denotes the characteristic of D, yields

$$\chi_D g = \bar{\partial}\big(P(\chi_D g)\big) + P\big([bD]^{0,1} \wedge g\big)$$

if $\bar{\partial} g = 0$. Thus, in order for $f \equiv P(\chi_D g)$ to solve $\bar{\partial} f = g$ on D, one must show that $P([\partial D]^{0,1} \wedge g)$ vanishes. This is immediate from Theorem 4.14 which proves, in particular, that the part of P tangential in ζ vanishes on $|\zeta| = 1$.

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In related work Charpentier [C] has presented an explicit kernel $C(\zeta, z)$ which has the property that $C\phi = P\phi$ if ϕ is a $\bar\partial$ closed 0, 1 form. Harvey and Polking [**HP2**] have presented a representation for the $\bar\partial$-Neumann kernel on the ball in $\mathbf{C}^n$ while Stanton [**St**] has presented a representation for the $\bar\partial$-Neumann kernel on the unbounded ball in $\mathbf{C}^n$. Phoung [**Ph**] has given an asymptotic expansion for the $\bar\partial$-Neumann kernel in some generality.

2. Statement of the main result. Let $D = D_n$ denote the unit ball in $\mathbf{C}^n$, and let bD denote the boundary of D. We consider the euclidean metric on D; and on forms we adopt the convention that the inner product is generated by the relations

$$\langle dz_i, dz_j \rangle = \delta_{ij}.$$

Let $L_2^{p,q}$ denote the space of p, q forms on D with square integrable coefficients together with the inner product

$$(\phi, \psi) = \int_D \langle \phi, \psi \rangle \, d\lambda(z). \tag{2.1}$$

$\mathfrak{D}^{p,q} = \mathfrak{D}^{p,q}(D)$ and $\mathfrak{E}^{p,q} = \mathfrak{E}^{p,q}(\bar D)$ will denote the spaces of smooth forms of bidegree p, q which have compact support in D and which are smooth on $\bar D$, respectively. The corresponding spaces of all forms will be denoted by L_2^*, $\mathfrak{D}^*$, and $\mathfrak{E}^*$.

Let ϑ denote the formal adjoint of the operator $\bar\partial$ with respect to the inner product (2.1). We have

$$\vartheta\phi = -\sum_{k=1}^{n} \frac{\partial}{\partial z_k}\left(\frac{\partial}{\partial \bar z_k} \lrcorner \phi\right). \tag{2.2}$$

The $\bar\partial$-Laplacian is $\Box = \bar\partial\vartheta + \vartheta\bar\partial$. For $\phi = \sum \phi_{IJ}\, dz^I \wedge d\bar z^J \in \mathfrak{E}^*$, we have

$$\Box\phi = \sum (\Delta\phi_{IJ})\, dz^I \wedge d\bar z^J, \tag{2.3}$$

where

$$\Delta = -\sum_{k=1}^{n} \frac{\partial^2}{\partial z_k\, \partial \bar z_k}. \tag{2.4}$$

Let K denote the L_2^* projection onto the null space of $\Box$. The operator K is called the Bergman projection, and its kernel, the *Bergman kernel*, is given by

$$K(\zeta, z) = (2\pi i)^{-n}[d\zeta \cdot d(\zeta - z)]^n / (1 - \bar\zeta \cdot z)^{n+1}. \tag{2.5}$$

I.e.,

$$K\phi(z) = \int_D K(\zeta, z) \wedge \phi(\zeta), \qquad \phi \in L_2^*. \tag{2.6}$$

The conventions to be used in interpreting (2.6) and other such formulas used in this paper are explained in §2 of [**HP1**]. Briefly, all dz's are moved to the left in the product $K(\zeta, z) \wedge \phi(\zeta)$, and then only terms of bidegree n, n in ζ are integrated.

Let

$$\bar Z = \sum_{k=1}^{n} \bar z_k \frac{\partial}{\partial \bar z_k}. \tag{2.7}$$

A form $\phi \in \mathcal{E}^*$ satisfies the $\bar\partial$-*Neumann conditions* if

$$\bar Z \lrcorner \phi = 0 \quad \text{and} \quad \bar Z \lrcorner \bar\partial\phi = 0 \qquad \text{on } |z| = 1. \tag{2.8}$$

The $\bar\partial$-Neumann problem is to solve the equations

$$\begin{aligned} \Box u &= f && \text{for } |z| < 1, \\ \bar Z \lrcorner u = 0, \quad \bar Z \lrcorner \bar\partial u &= 0 && \text{for } |z| = 1. \end{aligned}$$

J. J. Kohn has shown that for each $f \in L_2^*$, there is a unique $u \in L_2^*$ for which

$$\begin{aligned} \Box u &= f - Kf && \text{for } |z| < 1, \\ \bar Z \lrcorner u = 0, \quad \bar Z \lrcorner \bar\partial u &= 0 && \text{for } |z| = 1, \\ Ku &= 0 && \text{for } |z| < 1. \end{aligned} \tag{2.9}$$

The operator N: $L_2^* \to L_2^*$ defined by $Nf = u$, where u satisfies (2.9), is called the *Neumann operator*. In addition to its defining property (2.9), the Neumann operator has the following properties:

$$K \circ N = N \circ K = 0. \tag{2.10}$$

$$\bar\partial N\phi = N\bar\partial\phi, \qquad \phi \in \mathcal{D}^*. \tag{2.11}$$

$$\vartheta N\phi = N\vartheta\phi, \qquad \phi \in \mathcal{D}^*. \tag{2.12}$$

$$N\phi \in \mathcal{E}^* \quad \text{if } \phi \in \mathcal{E}^*. \tag{2.13}$$

For a complete exposition of the $\bar\partial$-Neumann problem see **[FK]**.

It is our purpose here to present explicitly the kernel $P(\zeta, z)$ of the operator $\vartheta \circ N$. First we recall some classical kernels.

Let

$$b = (2\pi i)^{-1} |\zeta - z|^{-2} (\bar\zeta - \bar z) \cdot d(\zeta - z).$$

Then

$$B(\zeta, z) = b \wedge (\bar\partial b)^{n-1} \tag{2.14}$$

is the *Bochner-Martinelli kernel* in $\mathbf{C}^n$ and satisfies

$$\bar\partial B = [\Delta] \quad \text{on } \mathbf{C}^n \times \mathbf{C}^n, \tag{2.15}$$

where Δ denotes the diagonal in $\mathbf{C}^n \times \mathbf{C}^n$, and $[\Delta]$ is the current given by integration over Δ. (Note that the coefficients of B are locally integrable functions on $\mathbf{C}^n \times \mathbf{C}^n$.)

Let

$$\alpha = (2\pi i)^{-1} (1 - \bar\zeta \cdot z)^{-1} \bar\zeta \cdot d(\zeta - z). \tag{2.16}$$

Then

$$S(\zeta, z) = \alpha \wedge (\bar\partial \alpha)^{n-1} \tag{2.17}$$

is the *Cauchy-Szegö kernel* on D_n and satisfies

$$\bar\partial S = K \quad \text{on } D \times D. \tag{2.18}$$

Finally, in order to describe P, we introduce a new kernel $A(\zeta, z)$.

DEFINITION 2.19. For each $1 \leqslant k \leqslant n-1$, let

(2.20)
$$\omega_k = \bar{z} \cdot d(\zeta - z) \wedge \bar{\zeta} \cdot d(\zeta - z) \wedge [d\bar{z} \cdot d(\zeta - z)]^{k-1} \wedge [d\bar{\zeta} \cdot d(\zeta - z)]^{n-k-1}.$$

Also, let

$$\chi_k = (-1)^{k-1}(2\pi i)^{-n} \sum_{j=1}^{n-k} \binom{n-j-1}{k-1} \sigma^{-j} \tau^{j-n}, \tag{2.21}$$

where

$$\sigma = 1 - \bar{\zeta} \cdot z \quad \text{and} \quad \tau = 1 - z \cdot \bar{\zeta} - \bar{z} \cdot \zeta + |z|^2 |\zeta|^2. \tag{2.22}$$

Then the kernel A is defined by

$$A(\zeta, z) = \sum_{k=1}^{n-1} \chi_k(\zeta, z) \omega_k. \tag{2.23}$$

Now we can state the main result.

THEOREM 2.24. *The kernel $P(\zeta, z)$ corresponding to the operator $\vartheta \circ N$ on the unit ball in $\mathbf{C}^n$ is represented explicitly as*

$$P = B - S + \bar{\partial} A, \tag{2.25}$$

where B is the Bochner-Martinelli kernel, S is the Cauchy-Szegö kernel and A is a new kernel defined by (2.23).

The proof will be given in the next section.

Several equivalent descriptions of the new kernels A and P will be given in §4. There is one other description of A that we wish to mention immediately because of its elegance.

PROPOSITION 2.26. *The kernel A defined by* (2.23) *can be rewritten as*

$$A = \alpha \wedge \beta \wedge \sum_{k=1}^{n-1} (\bar{\partial}\alpha)^{k-1} \wedge (\bar{\partial}\beta)^{n-k-1}, \tag{2.27}$$

where

$$\alpha = \frac{1}{2\pi i} \frac{\bar{\zeta} \cdot d(\zeta - z)}{\sigma} \tag{2.28}$$

and

$$\beta = \frac{1}{2\pi i} \frac{(\bar{\zeta} - \bar{z}) \cdot d(\zeta - z)}{\tau}. \tag{2.29}$$

PROOF.

$$
\begin{aligned}
A &= \sum_{k=1}^{n-1} \chi_k \omega_k \\
&= (2\pi i)^{-n} \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} (-1)^{k-1} \binom{n-j-1}{k-1} \sigma^{-j} \tau^{j-n} \bar{z} \cdot d(\zeta - z) \\
&\quad \wedge \bar{\zeta} \cdot d(\zeta - z) \wedge [d\bar{z} \cdot d(\zeta - z)]^{k-1} \wedge [d\bar{\zeta} \cdot d(\zeta - z)]^{n-k-1} \\
&= (2\pi i)^{-n} \sum_{j=1}^{n-1} \sigma^{-j} \tau^{j-n} \bar{z} \cdot d(\zeta - z) \wedge \bar{\zeta} \cdot d(\zeta - z) \wedge [d\bar{\zeta} \cdot d(\zeta - z)]^{j-1} \\
&\quad \wedge \sum_{k=1}^{n-j} \binom{n-j-1}{k-1} (-1)^{k-1} [d\bar{z} \cdot d(\zeta - z)]^{k-1} \wedge [d\bar{\zeta} \cdot d(\zeta - z)]^{n-j-k} \\
&= (2\pi i)^{-n} \sum_{j=1}^{n-1} \sigma^{-j} \tau^{j-n} \bar{\zeta} \cdot d(\zeta - z) \wedge (\bar{\zeta} - \bar{z}) \cdot d(\zeta - z) \\
&\quad \wedge [d\bar{\zeta} \cdot d(\zeta - z)]^{j-1} \wedge [d(\bar{\zeta} - \bar{z}) \cdot d(\zeta - z)]^{n-j-1} \\
&= \alpha \wedge \beta \wedge \sum_{j=1}^{n-1} (\bar{\partial}\alpha)^{j-1} \wedge (\bar{\partial}\beta)^{n-j-1}.
\end{aligned}
$$

3. Construction of the kernel P for $\vartheta \circ N$. The purpose of this section is to give the proof of Theorem 2.24. In part (A) of this section we review five properties of $\vartheta \circ N$. In part (B) we will show that these five properties uniquely determine an explicit kernel $P(\zeta, z)$ which then must be the kernel of $\vartheta \circ N$.

(A) Properties of $\vartheta \circ N$. For the time being, let $P(\zeta, z)$ denote the kernel of the operator $\vartheta \circ N$. Then

$$
\vartheta N \phi(z) = \int_{\zeta} P(\zeta, z) \wedge \phi(\zeta), \qquad \phi \in \mathfrak{D}^*.
$$

Let Δ denote the diagonal in $D \times D$, and let $[\Delta]$ denote the n, n current in $D \times D$ consisting of integration over Δ. The current $[\Delta]$ is the kernel of the identity operator on $\mathfrak{D}^*$ (see §2 in **[HP1]**).

PROPOSITION 3.1. *The kernel $P(\zeta, z)$ of the operator $\vartheta \circ N$ has the following five properties*:

(a) $\bar{\partial} P(\zeta, z) = [\Delta] - K(\zeta, z)$ *in* $D \times D$.

(b) $\vartheta_z P(\zeta, z) = 0$ *in* $D \times D$.

(c) $\bar{Z} \lrcorner P(\zeta, z) = 0$ *for* $|z| = 1, |\zeta| < 1$.

(d) $P(\zeta, z) \wedge d(\zeta_j - z_j) = 0$, $1 \leqslant j \leqslant n$.

(e) *P is invariant under the diagonal action of the unitary group $U(n)$ on $D \times D$.*

REMARK 3.2. The list of properties of the kernel $P(\zeta, z)$ in Proposition 3.1 can easily be reformulated as an equivalent list of properties of the operator $P = \vartheta \circ N$

(see [**HP1**, Theorem 2.8] for part (a)):

(a) $\phi - K\phi = \bar\partial P\phi + P\bar\partial\phi, \qquad \phi \in \mathfrak{D}^*.$

(b) $\vartheta P\phi = 0, \qquad \phi \in \mathfrak{D}^*.$

(c) $\bar Z \lrcorner P\phi = 0 \quad \mathit{for}\,|z| = 1, \qquad \phi \in \mathfrak{D}^*.$

(d) $P(dz_j \wedge \phi) = -dz_j \wedge P\phi, \qquad \phi \in \mathfrak{D}^*,\ 1 \leqslant j \leqslant n.$

(e) $P(u^*\phi) = u^*P\phi, \qquad u \in U(n),\ \phi \in \mathfrak{D}^*.$

In proving Proposition 3.1 it will frequently be more convenient to prove the equivalent points of Remark 3.2.

PROOF. (a) By (2.9) we have $\square N\phi = \phi - K\phi$, and by (2.11), (2.12),

$$\square N\phi = \bar\partial\vartheta N\phi + \vartheta\bar\partial N\phi = \bar\partial P\phi + \vartheta N\bar\partial\phi = \bar\partial P\phi + P\bar\partial\phi.$$

(b) Since $\vartheta^2 = 0$, $\vartheta P = \vartheta^2 N = 0$.

(c) By (2.9) $\bar Z \lrcorner N(\zeta, z) = 0$ if $|z| = 1$. Thus there is a form $h(\zeta, z)$ such that

$$\bar Z \lrcorner N(\zeta, z) = (1 - |z|^2)h(\zeta, z).$$

Clearly, $\bar Z \lrcorner h(\zeta, z) \equiv 0$. But then

$$\begin{aligned} \bar Z \lrcorner P(\zeta, z) &= \bar Z \lrcorner(\vartheta_z N(\zeta, z)) = -\vartheta_z(\bar Z \lrcorner N(\zeta, z)) \\ &= -\vartheta_z\big[(1 - |z|^2)h(\zeta, z)\big] = -(1 - |z|^2)\vartheta_z h(\zeta, z) - \bar Z \lrcorner h(\zeta, z) \end{aligned}$$

vanishes if $|z| = 1$.

(d) Notice that

$$\square(dz_j \wedge \phi) = dz_j \wedge \square\phi, \qquad \bar Z \lrcorner(dz_j \wedge \phi) = -dz_j \wedge (\bar Z \lrcorner \phi),$$

and

$$K(dz_j \wedge \phi) = dz_j \wedge K\phi.$$

It follows from (2.9) that $N(dz_j \wedge \phi) = dz_j \wedge N\phi$, and

$$P(dz_j \wedge \phi) = \vartheta N(dz_j \wedge \phi) = \vartheta(dz_j \wedge N\phi) = -dz_j \wedge P\phi.$$

(e) Let $u \in U(n)$. Then

$$\bar\partial(u^*\phi) = u^*\bar\partial\phi, \qquad \vartheta(u^*\phi) = u^*\vartheta\phi, \qquad \square(u^*\phi) = u^*\square\phi,$$
$$\bar Z \lrcorner(u^*\phi) = u^*(\bar Z \lrcorner \phi), \qquad K(u^*\phi) = u^*K\phi,$$

and as before it follows that

$$N(u^*\phi) = u^*N\phi, \qquad P(u^*\phi) = u^*P\phi.$$

(B) The kernel $P(\zeta, z)$. Since $\bar\partial P = [\Delta] - K$ and $\bar\partial(B - S) = [\Delta] - K$, there exists an $n, n-2$ current A on $D \times D$ such that

(3.3) $$P = B - S + \bar\partial A.$$

The kernels P, B, and S all satisfy Proposition 3.1(d), (e). Thus $\bar\partial A$ also satisfies these conditions and, without losing anything, we may assume that A does too. We will state the effect of these properties as a lemma.

LEMMA 3.4. *Suppose A is an n, $n-2$ form on $D \times D$ which is invariant under the diagonal action of the unitary group $U(n)$ on $D \times D$ and satisfies $A \wedge d(\zeta_j - z_j) = 0$, $1 \leqslant j \leqslant n$. Then*

$$A = \sum_{k=1}^{n-1} \chi_k(\zeta, z)\omega_k, \tag{3.5}$$

where the coefficients χ_k are invariant under the diagonal action of $U(n)$, and

$$\omega_k = \bar{z} \cdot d(\zeta - z) \wedge \bar{\zeta} \cdot d(\zeta - z) \wedge [d\bar{z} \cdot d(\zeta - z)]^{k-1} \wedge [d\bar{\zeta} \cdot d(\zeta - z)]^{n-k-1},$$

as in Definition 2.19.

Lemma 3.4 is an essential part of the analysis. It says in effect that up to multiplication by an invariant function there is only one invariant n, $n-2$ form (with $k-1$ $d\bar{z}$'s and $n-k-1$ $d\bar{\zeta}$'s). Lemma 3.4 reduces the problem to finding the coefficients χ_k. The proof of Lemma 3.4 is tedious and uninspiring so we will relegate the proof to an appendix and proceed.

Since the functions $\chi_k(\zeta, z)$ are invariant we can write

$$\chi_k(\zeta, z) = \chi_k(r, \rho, w, \bar{w}), \tag{3.6}$$

where

$$r = |z|^2, \quad \rho = |\zeta|^2, \quad w = \bar{\zeta} \cdot z, \quad \bar{w} = \zeta \cdot \bar{z}. \tag{3.7}$$

The problem now is to find the coefficient functions χ_k.

An easy calculation shows that $\vartheta_z B = \vartheta_z S \equiv 0$. Thus using Proposition 3.1(b) and (3.3) we have

$$\vartheta_z \bar{\partial} A \equiv 0. \tag{3.8}$$

Notice that if the coefficients χ_k are invariant, then $\vartheta_z A = 0$ is automatic. Thus

$$\vartheta_z \bar{\partial}_\zeta A = -\bar{\partial}_\zeta \vartheta_z A = 0,$$

and (3.8) becomes

$$\Box_z A = [\vartheta_z \bar{\partial}_z + \bar{\partial}_z \vartheta_z] A = 0. \tag{3.9}$$

Using (2.3) and (3.6) we see that

$$\Box_z(\chi_k \omega_k) = [\Delta \chi_k - \partial \chi_k / \partial r]\omega_k.$$

Thus the condition that $\vartheta_z P = 0$ has been uncoupled into a set of scalar conditions on the functions χ_k; namely

$$\Delta \chi_k - \partial \chi_k / \partial r \equiv 0, \qquad 1 \leqslant k \leqslant n-1. \tag{3.10}$$

Notice that if

$$\Delta_n = -\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$$

and if $\phi = \phi(r, \rho, w, \bar{w})$ is an invariant function, then

$$\Delta_n \phi = -\left[r\frac{\partial^2 \phi}{\partial r^2} + w\frac{\partial^2 \phi}{\partial r \partial w} + \bar{w}\frac{\partial \phi}{\partial r \partial \bar{w}} + \rho\frac{\partial \phi}{\partial w \partial \bar{w}} + n\frac{\partial \phi}{\partial r} \right].$$

Consequently (3.10) can be rewritten

$$\Delta_{n+1}\chi_k \equiv 0, \qquad 1 \leqslant k \leqslant n-1. \tag{3.11}$$

Thus the invariant functions χ_k must be harmonic in D_{n+1}.

Next we interpret the boundary condition (property (c) in Proposition 3.1) on P. By (3.3) on $|z| = 1$ we must have

$$\begin{aligned} \bar{Z} \lrcorner \bar{\partial} A &= -\bar{Z} \lrcorner B \\ &= (2\pi i)^{-n}(n-1)|\zeta - z|^{-2n}\bar{z} \cdot d(\zeta - z) \wedge \bar{\zeta} \cdot d(\zeta - z) \\ &\qquad \wedge \left[d(\bar{\zeta} - \bar{z}) \cdot d(\zeta - z)\right]^{n-2} \\ &= (2\pi i)^{-n}(n-1)|\zeta - z|^{-2n} \sum_{k=1}^{n-1} (-1)^{k-1}\binom{n-2}{k-1}\omega_k. \end{aligned}$$

Using (3.5) this becomes, with $\bar{Z}_k = \bar{Z} + k$,

$$\bar{Z}_k\chi_k = (-1)^{k-1}(2\pi i)^{-n}(n-1)\binom{n-2}{k-1}|\zeta - z|^{-2n} \quad \text{on } |z| = 1. \tag{3.12}$$

Thus we have also succeeded in uncoupling the boundary condition on P into a set of scalar boundary conditions.

Since χ_k is invariant, we may interpret (3.12) as an equation in D_{n+1}. Thus to find χ_k we must solve the boundary value problems (3.11) and (3.12) in D_{n+1}.

We will solve the boundary value problem in two steps. First let

$$\phi_k = \bar{Z}_k\chi_k. \tag{3.13}$$

Since $\Delta_{n+1}\bar{Z}_k - \bar{Z}_k\Delta_{n+1} = \Delta_{n+1}$, ϕ_k must solve

$$\Delta_{n+1}\phi_k(\zeta, z) \equiv 0, \qquad z \in D_{n+1},$$

$$\phi_k(\zeta, z) = (-1)^{k-1}(2\pi i)^{-n}(n-1)\binom{n-2}{k-1}|\zeta - z|^{-2n}, \qquad z \in bD_{n+1}.$$

This is the Dirichlet problem in D_{n+1}, and with the given boundary data we can solve for ϕ_k explicitly using the Kelvin transform (inversion in bD_{n+1}). The unique solution is

$$\phi_k(\zeta, z) = (-1)^{k-1}(2\pi i)^{-n}(n-1)\binom{n-2}{k-1}\tau^{-n}, \tag{3.14}$$

where

$$\tau(\zeta, z) = 1 - z \cdot \bar{\zeta} - \bar{z} \cdot \zeta + |z|^2|\zeta|^2.$$

It remains to find a harmonic solution to (3.13).

LEMMA 3.15. *Suppose ϕ is real analytic near the origin in $\mathbf{C}^{n+1}$. Then for $k \geqslant 1$ the unique real analytic function χ, such that $\bar{Z}_k\chi = \phi$, is given by*

$$\chi(z, \bar{z}) = \int_0^1 \phi(z, t\bar{z})t^{k-1}\,dt. \tag{3.16}$$

PROOF. Let

$$\phi(z) = \sum a_{\alpha\beta}z^\alpha\bar{z}^\beta.$$

Then if $\chi(z) = \sum b_{\alpha\beta} z^\alpha \bar z^\beta$, we have

$$\bar Z_k \chi = \sum (k + |\beta|) b_{\alpha\beta} z^\alpha \bar z^\beta.$$

Thus $\bar Z_k \chi = \phi$ if and only if $b_{\alpha\beta} = a_{\alpha\beta}(k + |\beta|)^{-1}$. The power series for χ clearly converges and we have

$$\chi(z, \bar z) = \sum \frac{a_{\alpha\beta}}{k + |\beta|} z^\alpha \bar z^\beta = \sum a_{\alpha\beta} z^\alpha \int_0^1 t^{k-1} (t\bar z)^\beta \, dt$$
$$= \int_0^1 \phi(z, t\bar z) t^{k-1} \, dt.$$

In the case at hand we have, therefore,

$$\text{(3.17)} \quad \chi_k = (-1)^{k-1} (2\pi i)^{-n} (n-1) \binom{n-2}{k-1} \int_0^1 \frac{t^{k-1}\, dt}{\left(1 - z \cdot \bar\zeta - t\bar z \cdot \zeta + t|z|^2|\zeta|^2\right)^n}.$$

Notice that χ_k is harmonic in D_{n+1} and satisfies (3.13). Using the notation $\sigma = 1 - \bar\zeta \cdot z$ we have

$$\chi_k = (-1)^{k-1} (2\pi i)^{-n} \frac{\Gamma(n)}{\Gamma(k)\Gamma(n-k)} \int_0^1 t^{k-1} [\tau t + \sigma(1-t)]^{-n} \, dt.$$

The integral can be evaluated explicitly (substitute $t = \sigma s/(\sigma s + \tau(1-s))$) and we get

$$\text{(3.18)} \qquad \chi_k = (-1)^{k-1} (2\pi i)^{-n} \sum_{j=1}^{n-k} \binom{n-j-1}{k-1} \sigma^{-j} \tau^{j-n},$$

completing the proof of Theorem 2.24.

REMARK 3.19. The functions χ_k, $1 \leqslant k \leqslant n-1$, depend on n. For fixed k and n, let $p = k$ and $q = n - k$, and let $\chi_{p,q}(r, \rho, w, \bar w)$ denote $\chi_k(\zeta, z)$ considered as a function of $r, \rho \in \mathbf{R}^+$ and $w \in \mathbf{C}$. Let

$$\mathcal{L}_{p,q} \equiv r \frac{\partial^2}{\partial r^2} + w \frac{\partial^2}{\partial r\, \partial w} + \bar w \frac{\partial^2}{\partial r \partial \bar w} + \rho \frac{\partial^2}{\partial w \partial \bar w} + (p+q) \frac{\partial}{\partial r},$$

and let

$$\bar Z_p = r \frac{\partial}{\partial r} + \bar w \frac{\partial}{\partial \bar w} + p.$$

Finally, let

$$\eta_{p,q} = (-1)^{p-1} (2\pi i)^{-p-q} (p+q-1) \binom{p+q-2}{p-1} (r - w - \bar w + \rho)^{-n}.$$

Then the boundary value problem (3.11), (3.12) on D_{n+1}, in the r, ρ, w coordinates, becomes

$$\text{(3.20)} \qquad \mathcal{L}_{p,q} \chi_{p,q} = 0 \quad \text{and} \quad \bar Z_p \chi_{p,q} = \eta_{p,q} \qquad \text{on } r = 1.$$

Assuming one can guess the solution

$$\text{(3.21)} \qquad \chi_{1,1} = 1/\sigma\tau$$

in the special case $p = q = 1$, the solutions $\chi_{p,q}$ can be found in general using the method of Dadok and Harvey [**DH**]. Namely, define $\chi_{p,q}$ recursively by

$$q\chi_{p,q+1} = \partial\chi_{p,q}/\partial w, \tag{3.22}$$

$$p\chi_{p+1,q} = -\partial\chi_{p,q}/\partial\bar{w}, \tag{3.23}$$

and then verify that $\chi_{p,q}$ satisfies (3.20) by induction. Perhaps it is also worth noting that

$$\partial\chi_{p,q}/\partial r = p\rho\chi_{p+1,q} \quad \text{and} \quad \partial\chi_{p,q}/\partial\rho = pr\chi_{p+1,q}. \tag{3.24}$$

4. Explicit kernels for other domains. Motivated by the expression for the kernel of $\vartheta \circ N$ on the ball derived in the preceding sections, we can give an explicit kernel for a general strictly pseudoconvex domain $\Omega \subset \mathbf{C}^n$ that retains many of the properties of $\vartheta \circ N$. We will again denote this kernel by $P(\zeta, z)$ and it is given by the same formula as (2.25).

$$P = B - S + \bar{\partial}A. \tag{4.1}$$

Here $B(\zeta, z)$ is again the Bochner-Martinelli kernel (see (2.16)), $S(\zeta, z)$ is a "Szegö-like" kernel given by

$$S(\zeta, z) = \alpha \wedge (\bar{\partial}\alpha)^{n-1}, \tag{4.2}$$

and (see (2.27))

$$A(\zeta, z) = \alpha \wedge \beta \wedge \sum_{k=1}^{n-1} (\bar{\partial}\alpha)^{k-1} \wedge (\bar{\partial}\beta)^{n-k-1}. \tag{4.3}$$

We need only define the basic one-forms α and β for the domain Ω.

Let r denote a defining function for Ω (i.e. $\Omega = \{z \mid r(z) < 0\}$, $\nabla r \neq 0$ on $b\Omega$) assumed to be infinitely differentiable and strictly plurisubharmonic near $b\Omega$. Set

$$\tau(\zeta, z) = |\zeta - z|^2 + r(\zeta)r(z) \tag{4.4}$$

and

$$\beta = (2\pi i)^{-1}\tau^{-1}(\bar{\zeta} - \bar{z})\cdot d(\zeta - z) \tag{4.5}$$

(compare (2.29); notice that if $\Omega = D_n$, and $r(z) = |z|^2 - 1$, then $\tau(\zeta, z) = 1 - \bar{\zeta}\cdot z - \zeta\cdot\bar{z} + |\zeta|^2|z|^2$).

To construct α we must work harder. It is instructive to first consider the case when r is strictly convex. In this case set

$$u_j(\zeta) = \partial r(\zeta)/\partial\zeta_j, \qquad 1 \leqslant j \leqslant n,$$

$$\sigma(\zeta, z) = u(\zeta)\cdot(\zeta - z) - r(\zeta), \quad \alpha = (2\pi i)^{-1}\sigma^{-1}u\cdot d(\zeta - z). \tag{4.6}$$

Since ρ is strictly convex, we have, by Taylor's formula,

$$\begin{aligned} 2\operatorname{Re}\sigma(\zeta, z) &= 2\operatorname{Re} u(\zeta)\cdot(\zeta - z) - 2r(\zeta) \\ &= -r(\zeta) - \left[r(\zeta) + 2\operatorname{Re}\sum \frac{\partial r}{\partial\zeta_j}(\zeta)(z_j - \zeta_j)\right] = -r(\zeta) - r(z) + R(\zeta, z), \end{aligned}$$

where the remainder term satisfies $R(\zeta, z) \geq C|\zeta - z|^2$. Thus σ is well defined and not zero in $\bar\Omega \times \bar\Omega - \Delta_{b\Omega}$. Thus α, A, and S are well defined. In addition we have

$$2\,\mathrm{Re}\,\sigma(\zeta, z) \geq c\tau(\zeta, z). \tag{4.7}$$

In the nonconvex case we have to imitate the Henkin construction (see **[H2]** or **[O]**). Let

$$\sigma_0(\zeta, z) = -r(\zeta) - \sum_{j=1}^{n} \frac{\partial r}{\partial \zeta_j}(\zeta)(z_j - \zeta_j) - \frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial\zeta_j\partial\zeta_k}(\zeta)(z_j - \zeta_j)(z_k - \zeta_k).$$

Then σ_0 is holomorphic in z and, by Taylor's theorem,

$$2\,\mathrm{Re}\,\sigma_0(\zeta, z) = -r(\zeta) - r(z) + R(\zeta, z)$$

where $|R(\zeta, z)| \leq C_1|\zeta - z|^2$. Since r is strictly plurisubharmonic we also have

$$R(\zeta, z) \geq C_2|\zeta - z|^2 \quad \text{if } |\zeta - z| \leq \delta.$$

Let

$$\omega = \log\sigma_0(\zeta, z)\bar\partial_z[\chi(|z - \zeta|)], .$$

where $\chi \in C_0^\infty(\mathbf{R})$ satisfies $\chi(t) = 1$ if $t \leq \delta/2$ and $\chi(t) = 0$ if $t \geq \delta$. For ε sufficiently small the form ω is smooth and $\bar\partial$ closed in $\Omega_\varepsilon = \{z \,|\, r(z) < \varepsilon\}$ for all ζ with $|r(\zeta)| < \varepsilon$. By standard arguments there is a smooth function $\gamma(\zeta, z)$ defined for $|\rho(\zeta)| < \varepsilon$ and $z \in \Omega_\varepsilon$ such that $\bar\partial_z\gamma(\zeta, z) = \omega$, and $\gamma(\zeta, \zeta) \equiv 0$. Set

$$\sigma(\zeta, z) = \begin{cases} \sigma_0(\zeta, z)e^{-\gamma(\zeta, z)} & \text{if } |\zeta - z| < \delta/2, \\ e^{\chi\log\sigma_0 - \gamma} & \text{if } |\zeta - z| > \delta/2. \end{cases} \tag{4.8}$$

Then

(4.9) $\sigma(\zeta, z)$ is holomorphic for $z \in \Omega_\varepsilon$ for fixed ζ with $|r(\zeta)| < \varepsilon$.

$$\sigma(\zeta, \zeta) = \sigma_0(\zeta, \zeta),$$

$$\mathrm{Re}\,\sigma(\zeta, z) \geq C[-r(\zeta) - r(z) + |\zeta - z|^2] \quad \text{if } |r(\zeta)|, r(z) < \varepsilon. \tag{4.10}$$

If we set $G(\zeta, w, z) = \sigma(\zeta, z) - \sigma(\zeta, w)$ then $G(\zeta, z, z) \equiv 0$. By standard solutions of the division problem there exist smooth functions $g_j(\zeta, w, z)$ holomorphic in (w, z) such that

$$G(\zeta, w, z) = \sum_{j=1}^{n} g_j(\zeta, w, z)(w_j - z_j).$$

Since $\sigma(\zeta, \zeta) = \sigma_0(\zeta, \zeta) = -r(\zeta)$, we have

$$\sigma(\zeta, z) = \sum_{j=1}^{n} g_j(\zeta, \zeta, z)(\zeta_j - z_j) - r(\zeta).$$

Thus if we set $u_j(\zeta, z) = g_j(\zeta, \zeta, z)$, we have

$$\sigma(\zeta, z) = u(\zeta, z)\cdot(\zeta - z) - r(\zeta).$$

Now to define the 1-form α we choose $\phi \in C^\infty(\mathbf{C}^n)$ with

$$\phi(\zeta) = \begin{cases} 0 & \text{if } r(\zeta) < -2\varepsilon/3, \\ 1 & \text{if } r(\zeta) > -\varepsilon/3. \end{cases}$$

Then we define

$$\alpha = (2\pi i)^{-1}\phi(\zeta)\sigma(\zeta, z)^{-1}u(\zeta, z)\cdot d(\zeta - z). \tag{4.11}$$

By (4.10) $\sigma(\zeta, z) \neq 0$ for $(\zeta, z) \in \bar{\Omega} \times \bar{\Omega}$ unless $\zeta = z \in b\Omega$. Thus the kernels S and A defined by (4.2) and (4.3) are well defined on $\Omega \times \Omega$. Furthermore $S(\zeta, z)$ is an $n, n-1$ form with no $d\bar{z}$'s because of (4.9). Thus the kernel

$$K(\zeta, z) = \bar{\partial}S(\zeta, z) = (\bar{\partial}\alpha)^n \tag{4.12}$$

is an n, n form with no $d\bar{z}$'s. From (4.1) and (4.12) we have

$$\bar{\partial}P(\zeta, z) = [\Delta] - K(\zeta, z). \tag{4.13}$$

We will show later that K is a projection onto holomorphic $(p, 0)$ forms.

For estimating the kernel P it is convenient to separate it into its tangential and normal parts in ζ, the variable of integration. This is achieved by calculating $\bar{\partial}A$.

Theorem 4.14. *For ζ near $b\Omega$, we have*

$$P(\zeta, z) = b \wedge (\bar{\partial}b)^{n-1} - \beta \wedge (\bar{\partial}\beta)^{n-1} + r(\zeta)E_1 + \bar{\partial}r(\zeta) \wedge E_2 + r(\zeta)r(z)E_3 + \bar{\partial}[r(\zeta)r(z)] \wedge E_4,$$

where

$$E_1(\zeta, z) = -(2\pi i)^{-n}\sum_{j=0}^{n-1}\sigma^{-j-1}\tau^{j-n}(\bar{\zeta} - \bar{z})\cdot d(\zeta - z) \wedge [\bar{\partial}u \cdot d(\zeta - z)]^j \wedge [d(\bar{\zeta} - \bar{z})\cdot d(\zeta - z)]^{n-j-1},$$

$$E_2(\zeta, z) = (2\pi i)^{-n}\sum_{j=0}^{n-1} j\sigma^{-j-1}\tau^{j-n}u \cdot d(\zeta - z) \wedge (\bar{\zeta} - \bar{z})\cdot d(\zeta - z) \wedge [\bar{\partial}u \cdot d(\zeta - z)]^{j-1} \wedge [d(\bar{\zeta} - \bar{z})\cdot d(\zeta - z)]^{n-j-1},$$

$$E_3(\zeta, z) = -(2\pi i)^{-n}\sum_{j=0}^{n-1}\sigma^{-j-1}\tau^{j-n}u \cdot d(\zeta - z) \wedge [\bar{\partial}u \cdot d(\zeta - z)]^j \wedge [d(\bar{\zeta} - \bar{z})\cdot d(\zeta - z)]^{n-j-1},$$

$$E_4(\zeta, z) = (2\pi i)^{-n}\sum_{j=0}^{n-2}(n - j - 1)\sigma^{-j-1}\tau^{j-n}u \cdot d(\zeta - z) \wedge (\bar{\zeta} - \bar{z})\cdot d(\zeta - z) \wedge [\bar{\partial}u \cdot d(\zeta - z)]^j \wedge [d(\bar{\zeta} - \bar{z})\cdot d(\zeta - z)]^{n-j-2}.$$

Proof. We take ζ close enough to $b\Omega$ so that $\phi(\zeta) \equiv 1$. This is then a special case of Theorem B.6.

It will be useful to rewrite the result of Theorem 4.14. First notice that

$$|\zeta - z|^{-2n} - \tau^{-n} = r(z)r(\zeta)\tau^{-1}\sum_{k=0}^{n-1}|\zeta - z|^{-2(n-k)}\tau^{-k}.$$

Hence

$$b \wedge (\bar\partial b)^{n-1} - \beta \wedge (\bar\partial \beta)^{n-1} = r(z)r(\zeta)\tau^{-1}K_1(\zeta, z),$$

where

$$K_1(\zeta, z) = (2\pi i)^{-n}\sum_{k=0}^{n-1}|\zeta - z|^{-2(n-k)}\tau^{-k}(\bar\zeta - \bar z)\cdot d(\zeta - z) \wedge\left[d(\bar\zeta - \bar z)\cdot d(\zeta - z)\right]^{n-1}. \tag{4.15}$$

Set

$$\begin{aligned} K_2(\zeta, z) &= E_1 + r(z)E_3 + \bar\partial r(z) \wedge E_4, \\ K_3(\zeta, z) &= E_2 + r(z)E_4. \end{aligned} \tag{4.16}$$

Then by Proposition 4.4,

$$P(\zeta, z) = r(z)r(\zeta)\tau^{-1}K_1 + r(\zeta)K_2 + \bar\partial r(\zeta) \wedge K_3.$$

To facilitate the estimation of these kernels we note the following inequalities.

$$\tau = |\zeta - z|^2 + r(z)r(\zeta) \leqslant |\zeta - z|^2 + r(z)^2 + r(\zeta)^2 \leqslant C\tau. \tag{4.17}$$

$$C_1|\sigma|^2 \leqslant \tau \leqslant C_2|\sigma|. \tag{4.18}$$

The first inequality in (4.17) is trivial. The second follows by showing that τ majorizes each term. To show that τ majorizes $r(z)^2$, for example, we look for the minimum point of $\tau(\zeta, z)$ for $\zeta \in \bar\Omega$, with z fixed. If the minimum occurs with $\zeta \in b\bar\Omega$, then $\tau(\zeta, z) = d(z, b\Omega)^2 \geqslant Cr(z)^2$. If the minimum occurs with $\zeta \in \Omega$, then $\nabla_\zeta\tau(\zeta, z) = 0$, so

$$|\zeta - z| = \tfrac{1}{2}|r(z)||\nabla_\zeta r(\zeta)|$$

and, again, $\tau(\zeta, z) \geqslant |\zeta - z|^2 \geqslant C|r(z)|^2$.

This first inequality in (4.18) follows from (4.17) and the elementary inequality

$$|\sigma(\zeta, z)| \leqslant C(|\zeta - z| + |r(\zeta)|).$$

The second inequality is a consequence of (4.10).

The next proposition will summarize the results of the last few paragraphs.

PROPOSITION 4.19. *We have*

$$P(\zeta, z) = P_1(\zeta, z) + P_2(\zeta, z) + P_3(\zeta, z)$$

where

$$P_1(\zeta, z) = r(z)r(\zeta)\tau^{-1}K_1(\zeta, z),$$

$$P_2(\zeta, z) = r(\zeta)K_2(\zeta, z) \quad \textit{and} \quad P_3(\zeta, z) = \bar\partial r(\zeta) \wedge K_3(\zeta, z).$$

The kernels satisfy the following estimates:

$$|K_1(\zeta, z)| \leq C|\zeta - z|^{1-2n}, \quad |K_2(\zeta, z)| \leq C|\sigma|^{-1}\tau^{1/2-n},$$
$$|K_3(\zeta, z)| \leq C|\sigma|^{-2}\tau^{3/2-n}.$$

Proposition 4.19 shows the character of the kernel P as exhibited by the three summands. The term P_1 contains the singularity along the diagonal of Ω, but it has the nice feature that it vanishes if either variable is on the boundary. The term P_2 is smooth in $\bar{\Omega} \times \bar{\Omega}$ except on the diagonal of the boundary of Ω. It vanishes for $\zeta \in b\Omega$ and satisfies an estimate that is elliptic in character. The last term, P_3, is also smooth on $\bar{\Omega} \times \bar{\Omega}$ except on the diagonal of $b\Omega$. It satisfies the worst estimate, but acts only on the tangential components of forms.

It is interesting to notice that the distribution transpose (*not* the L^2 adjoint) of the kernel N for the $\bar{\partial}$-Neumann problem is the solution kernel for the $\bar{\partial}$-Dirichlet problem. Consequently, it is true on any strictly pseudoconvex domain that the kernel of $\vartheta \circ N$ has vanishing tangential part in ζ (the variable of integration). Proposition 4.19 shows that it is this boundary property that the kernel P has in common with the kernel of $\vartheta \circ N$ on general domains. This fact is of some importance, since it is this property of P that enables us to use P to solve the inhomogeneous Cauchy-Riemann equations on Ω. This fact is made explicit in (c) of the next result and the following remark.

DEFINITION 4.20. Let $\mathfrak{M}^{p,q}$ denote the space of p, q forms on Ω having finite measures for coefficients.

THEOREM 4.21. *Suppose rf, $\bar{\partial}r \wedge f \in \mathfrak{M}^*$.*

(a) *The integral*

$$Pf(z) = \int_{\zeta \in \Omega} P(\zeta, z) \wedge f(\zeta)$$

exists a.e. and $Pf \in L^1_{\mathrm{loc}}(\Omega)$.

(b) *Suppose $\{\phi_k\}$ is a sequence of bounded functions with compact support in Ω, which converges boundedly to the characteristic function of Ω. Then*

$$Pf = \lim_{k \to \infty} P(\phi_k f),$$

where the limit is in $L^1_{\mathrm{loc}}(\Omega)$.

(c) *Suppose, in addition, that $r\bar{\partial}f$, $\bar{\partial}r \wedge \bar{\partial}f \in \mathfrak{M}^*$. Then $P\bar{\partial}f \in L^1_{\mathrm{loc}}(\Omega)$, $Kf \in L^1_{\mathrm{loc}}(\Omega)$ and*

$$f - Kf = \bar{\partial}Pf + P\bar{\partial}f.$$

PROOF. (a) and (b) are immediate from Proposition 4.19.

To prove (c) choose $\psi \in C^1(\mathbf{R})$ such that

$$\psi(t) = \begin{cases} 1, & t \leq -1, \\ 0, & t \geq -\frac{1}{2}, \end{cases}$$

and set $\phi_k(z) = \psi(kr(z))$. Then since $\phi_k f$ has compact support in Ω we have

(4.22)

$$\phi_k f - K(\phi_k f) = \bar{\partial}P(\phi_k f) + P\bar{\partial}(\phi_k f) = \bar{\partial}P(\phi_k f) + P(\phi_k \bar{\partial}f) + P(\bar{\partial}\phi_k \wedge f).$$

Notice that if f is a (p, o) form, then $f \in \mathfrak{M}^{p,o}$, so $K(\phi_k f) \to Kf$. All other terms in (4.22) except $P(\bar\partial\phi_k \wedge f)$ converge using (b) and we get

$$f - Kf = \bar\partial Pf + P\bar\partial f + \lim_{k\to\infty} P(\bar\partial\phi_k \wedge f).$$

Since $\bar\partial\phi_k(\zeta) = k\psi'(kr(\zeta))\bar\partial r(\zeta)$,

$$P(\bar\partial\phi_k \wedge f) = \int P(\zeta, z) \wedge k\psi'(kr(\zeta))\bar\partial r(\zeta) \wedge f(\zeta)$$

$$= \int kr(\zeta)\psi'(kr(\zeta))\left[r(z)\tau^{-1}K_1(\zeta, z) + K_2(\zeta, z)\right] \wedge \bar\partial r(\zeta) \wedge f(\zeta).$$

Now $\bar\partial r \wedge f$ has measure coefficients. The remainder of the integrand is uniformly bounded for z in compact subsets of Ω and tends to zero pointwise as $k \to \infty$. Therefore the integral converges to zero.

REMARK. Suppose rf, $\bar\partial r \wedge f \in \mathfrak{M}^*$ and $\bar\partial f \equiv 0$. Then by (c) of Theorem 4.21, $f - Kf = \bar\partial Pf$.

Suppose $f \in \mathfrak{M}^{p,o}$. Then we have $Pf = 0$, since $P(\zeta, z) \wedge f(\zeta)$ is of degree at most $n - 1$ in $d\bar\zeta$. Hence,

$$Kf = f \quad \text{if } f \in \mathfrak{M}^{p,o} \text{ and } \bar\partial f \equiv 0,$$

so K is a projection onto holomorphic p, o forms as we have advertised. On the other hand if $rf \in \mathfrak{M}^{p,q}$, $\bar\partial r \wedge f \in \mathfrak{M}^{p,q+1}$, and $\bar\partial f \equiv 0$, then $Kf \equiv 0$, so

$$f = \bar\partial Pf \quad \text{if } rf \in \mathfrak{M}^{p,q},\ \bar\partial r \wedge f \in \mathfrak{M}^{p,q+1}, \text{ and } \bar\partial f = 0.$$

Thus P is a solution operator for $\bar\partial$ in Ω.

The different characters of the pieces of the kernel P are reflected in the estimates that each satisfies. We will give two examples. The first result (Theorem 4.23) provides a more direct proof of the Henkin-Skoda Theorem **[H]**, **[S]**. The second result (Theorem 4.26) is an accumulation of estimates for the kernel P which are standard for other similar kernels.

THEOREM 4.23. *Suppose f and $|r|^{-1/2}\bar\partial r \wedge f \in \mathfrak{M}^*$. Then Pf has boundary values in $L^1(b\Omega)$.*

PROOF. We will show that

$$\int_{b\Omega} |Pf(z)|\, dS(z) \leq C\int_\Omega \left[|f(\zeta)| + |r(\zeta)|^{-1/2}|\bar\partial r(\zeta) \wedge f(\zeta)|\right] d\lambda(\zeta) \tag{4.24}$$

for f having compact support in Ω. The result then follows using Theorem 4.21(b). We have

$$P_1 f(z) = 0, \qquad z \in b\Omega,$$

$$P_2 f(z) = \int_\Omega r(\zeta)K_2(\zeta, z) \wedge f(\zeta),$$

$$P_3 f(z) = \int_\Omega K_3(\zeta, z) \wedge \bar\partial r(\zeta) \wedge f(\zeta).$$

Thus to prove (4.24) we must prove

$$\int_{z\in b\Omega}|K_2(\zeta,z)|\,dS(z)\leq M|r(\zeta)|^{-1},$$

$$\int_{z\in b\Omega}|K_3(\zeta,z)|\,dS(z)\leq M|r(\zeta)|^{-1/2}$$

with M a constant independent of ζ.

For ζ close to $b\Omega$ fixed, we need only estimate the integral over a neighborhood of $z_0\in b\Omega$, where z_0 is the closest point in $b\Omega$ to ζ which also satisfies $\operatorname{Im}\sigma_0(\zeta,z_0)=0$. We choose coordinates $(y,w)\in\mathbf{R}\times\mathbf{C}^{n-1}$ for $b\Omega$ with $y=\operatorname{Im}\sigma_0(\zeta,z)$ and $w(z_0)=0$. Then

$$|\sigma(\zeta,z)|\geq C|\sigma_0(\zeta,z)|\geq C\big[|r(\zeta)|+|y|+|w|^2\big],$$

$$\tau(\zeta,z)=|\zeta-z|^2\geq C\big(|r(\zeta)|+|y|+|w|\big)^2.$$

In the first case $|K_2(\zeta,z)|\leq C|\sigma|^{-1}\tau^{1/2-n}$, so it suffices to estimate (with $s=|w|$)

$$\int_0^\infty\int_0^\infty\frac{s^{2n-3}\,ds\,dy}{\big(|r(\zeta)|+y+s^2\big)\big(|r(\zeta)|+y+s\big)^{2n-1}}.$$

Set $y=|r(\zeta)|y'$, $s=|r(\zeta)|s'$ and this becomes

$$\leq|r(\zeta)|^{-1}\int_0^\infty\int_0^\infty\frac{s^{2n-3}\,ds\,dy}{(1+y)(1+y+s)^{2n-1}}=M|r(\zeta)|^{-1}.$$

In the second case we use $|K_3(\zeta,z)|\leq C|\sigma|^{-2}\tau^{3/2-n}$, so it suffices to estimate

$$\int_0^\infty\int_0^\infty\frac{s^{2n-3}\,ds\,dy}{\big(|r(\zeta)|+y+s^2\big)^2\big(|r(\zeta)|+y+s\big)^{2n-3}}$$

$$\leq\int_0^\infty\int_0^\infty\frac{dy\,ds}{\big(|r(\zeta)|+y+s^2\big)^2}=\int_0^\infty\frac{ds}{|r(\zeta)|+s^2}=|r(\zeta)|^{-1/2}\int_0^\infty\frac{ds}{1+s^2}.$$

Perhaps the most important application of the kernel approach to complex analysis is the Henkin-Skoda Theorem **[H]**, **[S]**.

THEOREM 4.25. *If T is a Blaschke divisor on Ω (i.e. $rT\in\mathfrak{M}^{1,1}$) which vanishes in $H^2(\Omega,Z)$, then there exists a meromorphic function F on Ω in the Nevanlinna class with the prescribed divisor T.*

This result is a consequence of being able to solve $\bar\partial\phi=f$ with the estimates of Theorem 4.23.

THEOREM 4.26. (a) *For $j=1,2$ we have*

$$P_j \textit{ is of weak type } (1,2n/(2n-1)),$$

$$\|P_jf\|_q\leq C\|f\|_p,\quad 1/q=1/p-1/2n,\quad 1<p<2n,$$

$$\|P_jf\|_{\Lambda_\alpha}\leq C\|f\|_p,\quad \alpha=1-2n/p,\quad 2n<p\leq\infty.$$

(b) *For* $j = 3$ *we have*

$$P_3 \text{ is of weak type } (1, (2n+2)/(2n+1)),$$
$$\|P_3 f\|_q \leqslant C\|f\|_p, \quad 1/q = 1/p - 1/(2n+2), \quad 1 < p < 2n+2,$$
$$\|P_3 f\|_{\Lambda_\alpha} \leqslant C\|f\|_p, \quad \alpha = 1/2 - (n+1)/p, \quad 2n+2 < p \leqslant \infty.$$

PROOF. Estimates of this type are fairly standard for explicit kernels of the type of P_j. See for instance **[HR]** or **[Kr]**. We will give an outline only. See **[GS]** for a complete discussion of estimates for the $\bar\partial$-Neumann problem.

From Proposition 4.19 and inequalities (4.10), (4.17), and (4.18) we see that

$$|P_j(\zeta, z)| \leqslant C|\zeta - z|^{1-2n}, \qquad j = 1, 2.$$

Consequently, the first two parts of (a) follow from standard fractional integral inequalities. $P_1 = B - B_1$, where $B = b \wedge (\bar\partial b)^{n-1}$ is the Bochner-Martinelli kernel and $B_1 = \beta \wedge (\bar\partial \beta)^{n-1}$. The third part of (a) is standard for B. For B_1 we notice that

$$|\nabla_z B_1(\zeta - z)| \leqslant C\tau^{-n} \leqslant C|\sigma|^{-1}\tau^{1/2-n}.$$

Similarly we notice that

$$|\nabla_z P_2(\zeta, z)| \leqslant C|\sigma|^{-1}\tau^{1/2-n}.$$

We have, for example, with $1/p + 1/q = 1$,

$$|\nabla P_2 f(z)| \leqslant \left(\int |\nabla_2 P_2(\zeta, z)|^q \, d\lambda(\zeta)\right)^{1/q} \|f\|_p.$$

If we show that

$$\int_\Omega \left(|\sigma|^{-1}\tau^{1/2-n}\right)^q d\lambda(\zeta) \leqslant C|r(z)|^{2n(1-q)}, \tag{4.27}$$

then it follows that

$$|\nabla P_2 f(z)| \leqslant C|r(z)|^{-2n/p}\|f\|_p,$$

and the inequality

$$\|P_2 f\|_{\Lambda\alpha} \leqslant C\|f\|_p, \qquad \alpha = 1 - 2n/p,$$

follows by a standard argument.

It remains to prove (4.27). For z fixed but near $b\Omega$, we choose coordinates $(x, y, w) \in \mathbf{R}^{2n}$ with $x = -r(\zeta)$, $y = \operatorname{Im} \sigma_0(\zeta, z)$. Then

$$|\sigma| \geqslant C(|r(z)| + x + |y|), \quad \tau \geqslant C(|r(z)| + x + |y| + |w|)^2.$$

Thus it suffices to estimate

$$\iiint_{\mathbf{R}^{2n}} \frac{dx\, dy\, dw}{(|r(z)| + |x| + |y|)^q (|r(z)| + |x| + |y| + |w|)^{q(2n-1)}}.$$

Setting $w = (|r(z)| + |x| + |y|)w'$ we get

$$\int_{\mathbf{R}^{2n-2}} \frac{dw}{(1 + |w|)^{q(2n-1)}} \iint_{\mathbf{R}^2} (|r(z)| + |x| + |y|)^{2n(1-q)-2} \, dx\, dy$$
$$= |r(z)|^{2n(1-q)} \int \frac{dw}{(1 + |w|)^{q(2n-1)}} \iint \frac{dx\, dy}{(1 + |x| + |y|)^{2n(q-1)+2}}.$$

We now turn to the proof of (b). For the first two inequalities, we reduce the proof to a theorem of Folland and Stein [**FS**] by showing

$$\lambda\{\zeta \in \Omega \,|\, |P_3(\zeta, z)| > t\} \leq Ct^{-(2n+2)/(2n+1)},$$
$$\lambda\{z \in \Omega \,|\, |P_3(\zeta, z)| > t\} \leq Ct^{-(2n+2)/(2n+1)},$$

where C is a constant independent of z or ζ.

If t is large and $|P_3(\zeta, z)| > t$, then z and ζ must be close and near the boundary of Ω. Thus with z fixed, for example, the set of ζ where $|P_3(\zeta, z)| > t$ is contained in a coordinate patch with coordinates $(x, y, w) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^{2n-2}$, where $x = -\rho(\zeta)$ and $y = \operatorname{Im} \sigma_0(\zeta, z)$. Then since

$$|P_3(\zeta, z)| \geq C|\sigma|^{-2}\tau^{3/2-n},$$

and

$$|\sigma| \geq C(|r(z)| + x + |y| + |w|^2) \geq C(x + |y| + |w|^2),$$
$$\tau \geq C(|r(z)| + x + |y| + |w|)^2 \geq C|w|^2,$$

it suffices to show that

$$\lambda(U_t) \leq Ct^{-(2n+2)/(2n+1)}, \tag{4.28}$$

where

$$U_t = \left\{(x, y, w) \,|\, |w|^{2n-3}\left(|x| + |y| + |w|^2\right)^2 < 1/t\right\}.$$

Under the diffeomorphism

$$x = t^{-2/(2n+1)}x', \quad y = t^{-2/(2n+1)}y', \quad w = t^{-1/(2n+1)}w',$$

U_t is the image of U_1. Thus

$$\lambda(U_t) = t^{-(2n+2)/(2n+1)}\lambda(U_1).$$

Since $\lambda(U_1) < \infty$, (4.28) is verified. The result with ζ fixed is proved in exactly the same way.

To prove the third inequality in (b) we proceed as we did in (a). We notice that

$$|\nabla_z P_3(\zeta, z)| \leq C|\sigma|^{-3}\tau^{3/2-n}.$$

We will show that

$$\int_\Omega \left(|\sigma|^{-3}\tau^{3/2-n}\right)^q d\lambda(\zeta) \leq C|r(z)|^{-q(1/2+(n+1)/p)}$$

if $p > n - \frac{1}{2}$. The result then follows as before. Using the same coordinate system as before, with the inequalities

$$|\sigma| \geq C(|r(z)| + |x| + |y| + |w|^2), \quad \tau \geq C|w|^2,$$

we must estimate

$$\iiint \left(|r(z)| + |x| + |y| + |w|^2\right)^{-3q} |w|^{-(2n-3)q}\, dx\, dy\, dw.$$

Setting $w = (|r(z)| + |x| + |y|)^{1/2} w'$ we get

$$\int_{\mathbf{R}^{2n-2}} \left(1 + |w|^2\right)^{-3q} |w|^{-(2n-3)q}\, dw \cdot \iint \left(|r(z)| + |x| + |y|\right)^{n-1-(n+3/2)q}\, dx\, dy$$
$$= C|r(z)|^{-q(1/2+(n+1)/p)}$$

provided $q < (2n-2)/(2n-3)$ (or $p > 2n-2$).

Appendix A: Invariant forms. One of the key elements in the identification of the kernel of $\vartheta \circ N$ is Lemma 3.4, which characterizes the $n, n-2$ forms ω on $D \times D$ which are invariant under the diagonal action of the unitary group $U(n)$ and satisfy

$$\text{(A.1)} \qquad \omega \wedge d(\zeta_j - z_j) = 0, \qquad j = 1, 2, \ldots, n.$$

In this appendix we will prove Lemma 3.4 together with some other results about invariant forms and functions.

Suppose $u \in U(n)$. We will indicate the diagonal action of u on $D \times D$ by $u(\zeta, z) = (u\zeta, uz)$. If ω is a differential form on $D \times D$, then $u^*\omega$ is the pull-back of ω under this action. The form ω is *invariant* if $u^*\omega = \omega$ for every $u \in U(n)$.

We will start by characterizing the invariant functions. Obviously,

$$r = |z|^2, \quad \rho = |\zeta|^2, \quad w = \bar{\zeta} \cdot z, \quad \text{and} \quad \bar{w} = \zeta \cdot \bar{z}$$

are invariant. Let $S = \{(r, \rho, w) \in \mathbf{R} \times \mathbf{R} \times \mathbf{C}\colon 0 \leqslant r < 1,\ 0 \leqslant \rho < 1,\ |w|^2 \leqslant r\rho\}$.

PROPOSITION A.2. *A real analytic function $f\colon D \times D \to \mathbf{C}$ is $U(n)$ invariant if and only if there exists a real analytic function $g\colon S \to \mathbf{C}$ such that*

$$f(\zeta, z) = g\left(|z|^2, |\zeta|^2, z \cdot \bar{\zeta}\right).$$

Moreover, f uniquely determines g if $n \geqslant 2$, while for $n = 1$, g is unique modulo the ideal generated by $r\rho - |w|^2$.

PROOF. We will give a proof for $n \geqslant 2$. For $n = 1$ the proof is similar.

Let $\psi\colon D \times D \to S$ denote the map $\psi(\zeta, z) = (|z|^2, |\zeta|^2, z \cdot \bar{\zeta})$. Then

$$\psi^*\colon \mathcal{A}(S) \to \mathcal{A}_{\text{inv.}}(D \times D),$$

where $\mathcal{A}(S)$ denotes the space of real analytic functions on S and $\mathcal{A}_{\text{inv.}}(D \times D)$ denotes the real analytic functions on $D \times D$ which are $U(n)$ invariant. We must show that ψ^* is an isomorphism. Obviously ψ^* is one-to-one since ψ is onto for $n \geqslant 2$. It suffices to show that ψ^* is onto near each point $(\zeta_0, z_0) \in D \times D$. We will give the proof when $(\zeta_0, z_0) = (0, 0)$, which is the most difficult case.

Since it suffices to prove that $\psi^*(g_k) = f_k$ for each homogeneous polynomial f_k in the Taylor expansion of f, we may assume that $f \in \mathcal{A}_{\text{inv.}}(D \times D)$ is a homogeneous polynomial. Let $X = \{(\lambda, s, t) \in \mathbf{C} \times \mathbf{R} \times \mathbf{R}\}$ and define $\phi\colon X \to \mathbf{C}^n \times \mathbf{C}^n$ by $\phi(\lambda, s, t) = (\lambda, s, 0, \ldots, 0; t, 0, \ldots, 0)$. Then

$$\phi^*(f)(\lambda, s, t) = f(\lambda, s, 0, \ldots, 0; t, 0, \ldots, 0).$$

The invariance of f implies that

$$\phi^*(f)(\lambda, s, t) = \phi^*(f)(-\lambda, -s, -t) \quad \text{and} \quad \phi^*(f)(\lambda, s, t) = \phi^*(f)(\lambda, -s, t).$$

Thus $\phi^*(f)$ is an even polynomial of degree $2m$ which is even in s. Therefore,

$$\phi^*(f) = \sum_{k+2j \leq 2m} a_{2(m-j)-k}(\lambda, \bar{\lambda}) t^k (s^2 + |\lambda|^2)^j,$$

where each $a_p(\lambda, \bar{\lambda})$ is a homogeneous polynomial of degree p. Hence

$$\begin{aligned}\phi^*(|z|^{2m} f) &= \sum_{k+2j \leq m} a_{2(m-j)-k}(t\lambda, t\bar{\lambda}) t^{2(k+j)} (s^2 + |\lambda|^2)^j \\ &= \phi^*\left(\sum a_{2(m-j)-k}(w, \bar{w}) r^{k+j} \rho^j\right).\end{aligned}$$

For $z, \zeta \in \mathbf{C}^n$ given, there exists $u \in U(n)$ with

$$uz = (|z|, 0, \ldots, 0) = (\sqrt{r}, 0, \ldots, 0)$$

and

$$u\zeta = \left(\frac{z \cdot \bar{\zeta}}{|z|}, \frac{|z \wedge \zeta|}{|z|}, 0, \ldots, 0\right) = \left(\frac{w}{\sqrt{r}}, \sqrt{\frac{r\rho - |w|^2}{r}}, 0, \ldots, 0\right).$$

Therefore ϕ^* is 1-1. In particular, $r^m f = \sum a_{2(m-j)-k}(w, \bar{w}) r^{k+j} \rho^j$. By rearranging terms and cancelling powers of r we see that there is a nonnegative integer k_0 and polynomials $b_k(\rho, w, \bar{w})$ such that

$$\text{(A.3)} \qquad r^{k_0} f = \sum_{k=0}^{N} r^k b_k(\rho, w, \bar{w})$$

and b_0 is not identically zero. It remains to show that $k_0 = 0$, so suppose $k_0 > 0$. Since b_0 is not identically zero, there is a $\rho_0 \neq 0$ and $w_0 \neq 0$ such that $b_0(\rho_0, w_0, \bar{w}_0) \neq 0$. If we evaluate (A.3) at $z = \rho_0^{-1/2}(w_0, t, 0, \ldots, 0)$, $s = (\rho_0^{-1/2}, 0, \ldots, 0)$, we get

$$\text{(A.4)} \qquad \rho_0^{-k_0}(|w_0|^2 + t^2)^{k_0} F(t) = \sum_{k=0}^{N} (|w_0|^2 + t^2)^k b_k(\rho_0, w_0, \bar{w}_0)$$

where F is a polynomial in t. The identity between polynomials in the real variable t extends to the complex plane, and if we set $t = i|w_0|$, we conclude that $b_0(\rho_0, w_0, \bar{w}_0) = 0$, which is a contradiction. Thus $k_0 = 0$ and (A.3) expresses $f \in \mathcal{Q}_{\text{inv.}}(D \times D)$ as a polynomial in r, ρ, w and $\bar{w}$. This proves $f = \psi^*(g)$.

Let $\mathcal{Q}^{n,p}_{\text{inv.}}(D \times D)$ denote the space of real analytic forms ω on $D_n \times D_n$ which are $U(n)$ invariant *and* satisfy (A.1), or, equivalently,

$$\omega = \alpha \wedge d(\zeta_1 - z_1) \wedge \cdots \wedge d(\zeta_n - z_n),$$

where α is a $0, p$ form which satisfies

$$\text{(A.5)} \qquad u^*\alpha = (\det u)^{-1}\alpha \quad \text{for all } u \in U(n).$$

Let $\mathcal{K}$ denote the space of all functions α $(= f)$ satisfying (A.5) (the special case $p = 0$).

THEOREM A.6. (a) *For $n = 1$, $\mathcal{K}$ is the module over $\mathcal{Q}_{\text{inv.}}(D \times D)$ generated by $\bar{z}$ and $\bar{\zeta}$ with relations $r\bar{\zeta} - w\bar{z} = \bar{w}\bar{\zeta} - \rho\bar{z} = 0$.*

(b) *For $n = 2$, $\mathcal{K}$ is the free module over $\mathcal{Q}_{\text{inv.}}(D \times D)$ generated by the function $\alpha \equiv \bar{z}_1\bar{\zeta}_2 - \bar{z}_2\bar{\zeta}_1$.*

(c) *For $n \geq 3$, $\mathcal{K} = \{0\}$.*

PROOF. As in the proof of Theorem A.2 we may assume that f is a homogeneous polynomial. Consider $u(z_1,\dots,z_n) = (z_1,\dots,z_{n-1},-z_n)$. Then

$$\phi^*(f) = -\phi^*(u^*f). \tag{A.7}$$

If $n \geqslant 3$ then $\phi^*(u^*f) = \phi^*(f)$ and, hence, $\phi^*f = 0$, so $f = 0$, completing the proof of (c).

For $n = 2$, the proof is very similar to that of Theorem A.2. The invariance property (A.7) implies that $\phi^*f(\lambda,-s,t) = -\phi^*f(\lambda,s,t)$. There are polynomials $a_{2(m-j)-k-1}$ such that

$$\phi^*(f)(\lambda,s,t) = s\sum a_{2(m-j)-k-1}(\lambda,\bar\lambda)t^k\left(s^2+|\lambda|^2\right)^j.$$

Hence

$$\begin{aligned}\phi^*(r^{m+1}f) &= st\sum a_{2(m-j)-k-1}(t\lambda,t\bar\lambda)t^{2(k+j)}\left(s^2+|\lambda|^2\right)^j\\ &= \phi^*\left(\alpha\sum a_{2(m-j)-k-1}(\bar w,w)r^{k+j}\rho^j\right).\end{aligned}$$

From this point the proof proceeds as before.

The proof for $n = 1$ is left to the reader.

The next result includes Lemma 3.4 as part (b). Let $\mathcal{A}^{n,n-2}_{\text{inv.}}(k-1,n-k-1)$ denote the space of real analytic forms on $D_n \times D_n$ of bidegree n, $n-2$ which contain exactly $k-1$ $d\bar z$'s and $n-k-1$ $d\bar\zeta$'s. Here $1 \leqslant k \leqslant n-1$.

THEOREM A.8. (a) $\mathcal{A}^{n,p}_{\text{inv.}}(D\times D) = 0$ *for* $p \leqslant n-3$.
(b)

$$\mathcal{A}^{n,n-2}_{\text{inv.}}(D\times D) = \sum_{k=1}^{n-1}\oplus\,\mathcal{A}^{n,n-2}_{\text{inv.}}(k-1,n-k-1)$$

and each $\mathcal{A}^{n,n-2}_{\text{inv.}}(k-1,n-k-1)$ *is the free module over* $\mathcal{A}_{\text{inv.}}(D\times D)$ *generated by*

(A.9)
$$\omega_k \equiv \bar z\cdot d(\zeta-z)\wedge\bar\zeta\cdot d(\zeta-z)\wedge\left[d\bar z\cdot d(\zeta-z)\right]^{k-1}\wedge\left[d\bar\zeta\cdot d(\zeta-z)\right]^{n-k-1}.$$

PROOF. First consider the case $n = 2$. We must have $p = n-2 = 0$ and $k = 1$.
(a) is vacuous and (b) is just Theorem A.6(b) since

$$\left(\bar\zeta_1\bar z_2 - \bar\zeta_2\bar z_1\right)d(\zeta_1-z_1)\wedge d(\zeta_2-z_2) = \bar\zeta\cdot d(\zeta-z)\bar z\cdot d(\zeta-z) = -\omega_1.$$

The result for general n will be proved by reduction to the case $n = 2$ using the next lemma.

LEMMA A.10. *Suppose* $\omega \in \mathcal{A}^{n,p}_{\text{inv.}}(k-n+p,n-k)$ *has constant coefficients. If* $p \neq n$, $\omega = 0$. *If* $p = n$, *then* ω *is a constant multiple of*

$$\left[d\bar z\cdot d(\zeta-z)\right]^k\wedge\left[d\bar\zeta\cdot d(\zeta-z)\right]^{n-k}.$$

PROOF. We will proceed by induction on n. For $n = 1$ the result is trivial. Suppose the result is true for $n-1$. If $\omega \in \mathcal{A}^{n,p}_{\text{inv.}}(k-n+p,n-k)$ then there exist forms $\eta_1,\eta_2,\eta_3,\eta_4 \in \mathcal{A}^{n-1,*}_{\text{inv.}}(D_{n-1}\times D_{n-1})$ such that

$$\omega = \left(\eta_1+\eta_2\wedge d\bar z_n+\eta_3\wedge d\bar\zeta_n+\eta_4\wedge d\bar z_n\wedge d\bar\zeta_n\right)\wedge d(\zeta_n-z_n).$$

Let $uz = (z_1, \dots, z_{n-1}, e^{i\theta} z_n)$. Then

$$\omega = u^*\omega = \left(e^{i\theta}\eta_1 + \eta_2 \wedge d\bar{z}_n + \eta_3 \wedge d\bar{\zeta}_n + e^{-i\theta}\eta_4 \wedge d\bar{z}_n \wedge d\bar{\zeta}_n\right) \wedge d(\zeta_n - z_n).$$

Thus $\eta_1 = \eta_4 = 0$. By the induction hypothesis

$$\eta_2 = \alpha\left[d\bar{z}' \cdot d(\zeta - z)\right]^{k-1} \wedge \left[d\bar{\zeta}' d(\zeta - z)\right]^{n-k},$$

$$\eta_3 = \beta\left[d\bar{z}' \cdot d(\zeta - z)\right]^{k} \wedge \left[d\bar{\zeta}' \cdot d(\zeta - z)\right]^{n-k},$$

where $\alpha, \beta \in \mathbf{C}$ and $z' = (z_1, \dots, z_{n-1}, 0)$. Hence

$$\begin{aligned}
\omega &= \alpha\left[d\bar{z} \cdot d(\zeta - z)\right]^{k-1} \wedge \left[d\bar{\zeta} \cdot d(\zeta - z)\right]^{n-k} \wedge d\bar{z}_n \wedge d(\zeta_n - z_n) \\
&\qquad + \beta\left[d\bar{z} \cdot d(\zeta - z)\right]^{k} \wedge \left[d\bar{\zeta} \cdot d(\zeta - z)\right]^{n-k-1} \wedge d\bar{\zeta}_n \wedge d(\zeta_n - z_n) \\
&= \alpha(k-1)\left[d\bar{z} \cdot d(\zeta - z)\right]^{k-2} \wedge \left[d\bar{\zeta} \cdot d(\zeta - z)\right]^{n-k} \\
&\qquad \wedge d\bar{z}_{n-1} \wedge d(\zeta_{n-1} - z_{n-1}) \wedge d\bar{z}_n \wedge d(\zeta_n - z_n) \\
&\qquad + \alpha(n-k)\left[d\bar{z} \cdot d(\zeta - z)\right]^{k-1}\left[d\bar{\zeta} \cdot d(\zeta - z)\right]^{n-k-1} \\
&\qquad \wedge d\bar{\zeta}_{n-1} \wedge d(\zeta_{n-1} - z_{n-1}) \wedge d\bar{z}_n \wedge d(\zeta_n - z_n) \\
&\qquad + \beta_k\left[d\bar{z} \cdot d(\zeta - z)\right]^{k-1} \wedge \left[d\bar{\zeta} \cdot d(\zeta - z)\right]^{n-k-1} \\
&\qquad \wedge d\bar{z}_{n-1} \wedge d(\zeta_{n-1} - z_{n-1}) \wedge d\bar{\zeta}_n \wedge d(\zeta_n - z_n) \\
&\qquad + \beta(n-k-1)\left[d\bar{z} \cdot d(\zeta - z)\right]^{k} \wedge \left[d\bar{\zeta} \cdot d(\zeta - z)\right]^{n-k-2} \\
&\qquad \wedge d\bar{\zeta}_{n-1} \wedge d(\zeta_{n-1} - z_{n-1}) \wedge d\bar{\zeta}_n \wedge d(\zeta_n - z_n).
\end{aligned}$$

Let $uz = (z_1, \dots, z_{n-2}, z_n, z_{n-1})$. Then $u^*\omega = \omega$, so $\beta k = \alpha(n-k)$. Consequently, there is a constant γ such that $\alpha = k\gamma$ and $\beta = (n-k)\gamma$. From (A.11) it follows that

$$\omega = \gamma\left[d\bar{z} \cdot d(\zeta - z)\right]^{k} \wedge \left[d\bar{\zeta} \cdot d(\zeta - z)\right]^{n-k}$$

and the lemma is proved.

For $\omega \in \mathcal{Q}^{n,p}_{\text{inv.}}(D \times D) = \mathcal{Q}^{n,p}_{\text{inv.}}$ define $\psi(\omega)$: $D^2 \times D^2 \to \Lambda^{n,p}(\mathbf{C}^n \times \mathbf{C}^n)$ by

$$\psi(\omega)(\zeta, z) = \omega(\zeta_1, \zeta_2, 0, \dots, 0; z_1, z_2, 0, \dots, 0).$$

Clearly

$$\begin{aligned}
\psi(\omega) \in \mathcal{Q}^{2,0} &\otimes \Lambda^{n-2,p}\left(\mathbf{C}^{n-2} \times \mathbf{C}^{n-2}\right) \oplus \mathcal{Q}^{2,1} \otimes \Lambda^{n-2,p-1}\left(\mathbf{C}^{n-2} \times \mathbf{C}^{n-2}\right) \\
&\oplus \mathcal{Q}^{2,2} \otimes \Lambda^{n-2,p-2}\left(\mathbf{C}^{n-2} \times \mathbf{C}^{n-2}\right).
\end{aligned}$$

The invariance of ω implies that $\psi(\omega)$ is invariant under the diagonal action of $U(n-2)$ on $\mathbf{C}^{n-2} \times \mathbf{C}^{n-2}$. But then Lemma A.10 implies that $\omega = 0$ if $p \leqslant n-3$ and $\psi(\omega) \in \mathcal{Q}^{2,0} \otimes \Lambda^{n-2,n-2}$ if $p = n-2$. Suppose $\omega \in \mathcal{Q}^{n,n-2}_{\text{inv.}}(k-2, n-k)$. Then $\psi(\omega)$ is invariant under $U(2)$ and $U(n-2)$. Hence the case $n = 2$ and Lemma A.10 imply that there is a function $f(r, w, \bar{w}, \rho)$ such that, with $z' = (z_1, z_2, 0, \dots, 0)$,

$$\begin{aligned}
\psi(\omega) &= f(r, w, \bar{w}, \rho)\bar{z}' \cdot d(\zeta - z) \wedge \bar{\zeta}' \cdot d(\zeta - z) \\
&\qquad \wedge \left[d\bar{z} \cdot d(\zeta - z)\right]^{k-1} \wedge \left[d\bar{\zeta} \cdot d(\zeta - z)\right]^{n-k-1} \\
&= \psi\left(f\bar{z} \cdot d(\zeta - z) \wedge \bar{\zeta} \cdot d(\zeta - z) \wedge \left[d\bar{z} \cdot d(\zeta - z)\right]^{k-1} \wedge \left[d\bar{\zeta} \cdot d(\zeta - z)\right]^{n-k-1}\right).
\end{aligned}$$

Since ψ is 1-1 it follows that $\omega = f\omega_k$ and Theorem A.8(b) is proved.

Appendix B: A formal identity. Let u and v denote $\mathbf{C}^n$ valued functions of $z, \zeta \in \mathbf{C}^n$, and consider the important $1,0$ forms:

$$a \equiv \frac{u \cdot d(\zeta - z)}{u \cdot (\zeta - z)}, \qquad b \equiv \frac{v \cdot d(\zeta - z)}{v \cdot (\zeta - z)},$$

where the dot denotes summation from 1 to n.

Let $\bar\partial$ denote the total $\bar\partial$ on $\mathbf{C}^n \times \mathbf{C}^n$.

THEOREM B.1.

$$a \wedge (\bar\partial a)^{n-1} - b \wedge (\bar\partial b)^{n-1} = \bar\partial \left[a \wedge b \wedge \sum_{j=1}^{n-1} (\bar\partial a)^{j-1} \wedge (\bar\partial b)^{n-j-1} \right]$$

on the complement of the set where either $u \cdot (\zeta - z)$ or $v \cdot (\zeta - z)$ vanish.

PROOF.

$$\begin{aligned}
\bar\partial &\left[a \wedge b \wedge \sum_{j=1}^{n-1} (\bar\partial a)^{j-1} \wedge (\bar\partial b)^{n-j-1} \right] \\
&= b \wedge \sum_{j=1}^{n-1} (\bar\partial a)^{j} \wedge (\bar\partial b)^{n-j-1} - a \wedge \sum_{j=1}^{n-1} (\bar\partial a)^{j-1} \wedge (\bar\partial b)^{n-j} \\
&= a \wedge (\bar\partial a)^{n-1} - b \wedge (\bar\partial b)^{n-1} \\
&\quad + b \wedge \sum_{j=0}^{n-1} (\bar\partial a)^{j} \wedge (\bar\partial b)^{n-j-1} - a \wedge \sum_{j=1}^{n} (\bar\partial a)^{j-1} \wedge (\bar\partial b)^{n-j}.
\end{aligned}$$

That is, for any $1,0$ forms a and b, we have the useful identity

$$\text{(B.2)} \quad \bar\partial \left[a \wedge b \wedge \sum_{j=1}^{n-1} (\bar\partial a)^{j-1} \wedge (\bar\partial b)^{n-j-1} \right]$$
$$= a \wedge (\bar\partial a)^{n-1} - b \wedge (\bar\partial b)^{n-1} + (b - a) \wedge \sum_{j=0}^{n-1} (\bar\partial a)^{j} \wedge (\bar\partial b)^{n-j-1}.$$

Let $X \equiv (\zeta - z) \cdot \partial/\partial\zeta$ denote the Euler vector field. Note that

$$\text{(B.3)} \qquad X \lrcorner a = 1 \quad \text{and} \quad X \lrcorner b = 1.$$

Since $X \lrcorner \bar\partial a = -\bar\partial (X \lrcorner a)$,

$$\text{(B.4)} \qquad X \lrcorner \bar\partial a = 0 \quad \text{and} \quad X \lrcorner \bar\partial b = 0.$$

Therefore $X \lrcorner (a \wedge (\bar\partial a)^k) = (\bar\partial a)^k$ and similarly with a replaced by b.

The form $a \wedge b \wedge (\bar\partial a)^j \wedge (\bar\partial b)^{n-j-1}$, $j = 0, \ldots, n-1$, vanishes identically since it is of degree $n+1$ in $d(\zeta - z)$. Therefore

$$0 = X \lrcorner \left(a \wedge b \wedge (\bar\partial a)^j \wedge (\bar\partial b)^{n-j-1} \right) = (b - a) \wedge (\bar\partial a)^j \wedge (\bar\partial b)^{n-j-1},$$

completing the proof.

REMARK. The identity B.1 is a very important special case of Theorem 4.10 in Harvey and Polking **[HP1]**. We include the proof (of the special case) here so that it can be modified in the following.

Consider the 1, 0 forms

$$\text{(B.5)}\qquad \alpha \equiv \frac{u \cdot d(\zeta - z)}{\lambda}, \qquad \beta \equiv \frac{v \cdot d(\zeta - z)}{\mu}$$

where the denominators in a and b have been replaced by arbitrary scalar functions λ and μ of z and ζ.

THEOREM B.6.

$$\begin{aligned}
\bar\partial\left[\alpha \wedge \beta \wedge \sum_{j=1}^{n-1} (\bar\partial\alpha)^{j-1} \wedge (\bar\partial\beta)^{n-j-1}\right] &= \alpha \wedge (\bar\partial\alpha)^{n-1} - \beta \wedge (\bar\partial\beta)^{n-1} \\
&+ (1 - (X \lrcorner \alpha)) \sum_{j=0}^{n-1} \left(\frac{\bar\partial u \cdot d(\zeta - z)}{\lambda}\right)^{j} \wedge \beta \wedge (\bar\partial\beta)^{n-j-1} \\
&- \frac{\bar\partial[\lambda(1 - (X \lrcorner \alpha))]}{\lambda} \alpha \wedge \beta \wedge \sum_{j=0}^{n-1} j(\bar\partial\alpha)^{j-1} \wedge (\bar\partial\beta)^{n-j-1} \\
&- (1 - (X \lrcorner \beta)) \sum_{j=0}^{n-1} \alpha \wedge (\bar\partial\alpha)^{j} \wedge \left(\frac{\bar\partial v \cdot d(\zeta - z)}{\mu}\right)^{n-j-1} \\
&+ \frac{\bar\partial[\mu(1 - (X \lrcorner \beta))]}{\mu} \alpha \wedge \beta \wedge \sum_{j=0}^{n-1} (n - j - 1)(\bar\partial\alpha)^{j} \wedge (\bar\partial\beta)^{n-j-2}.
\end{aligned}$$

PROOF. The identity (B.2) remains valid with a replaced by α and b replaced by β. Therefore, it remains to compute

$$(\beta - \alpha) \wedge \sum_{j=0}^{n-1} (\bar\partial\alpha)^{j} \wedge (\bar\partial\beta)^{n-j-1}.$$

Again the forms $\alpha \wedge \beta \wedge (\bar\partial\alpha)^{j} \wedge (\bar\partial\beta)^{n-j-1}$ vanish for $j = 0, \ldots, n - 1$ and, hence, (B.7)

$$0 = \left(X \lrcorner\left(\alpha \wedge (\bar\partial\alpha)^{j}\right)\right) \wedge \beta \wedge (\bar\partial\beta)^{n-j-1} - \left(X \lrcorner\left(\beta \wedge (\bar\partial\beta)^{n-j-1}\right)\right) \wedge \alpha \wedge (\bar\partial\alpha)^{j}.$$

Since $\bar\partial\alpha = \bar\partial u \cdot d(\zeta - z)/\lambda - (\bar\partial\lambda/\lambda)\alpha$, the binomial theorem implies that

$$\begin{aligned}
\text{(B.8)}\qquad (\bar\partial\alpha)^{j} &= \left(\frac{\bar\partial u \cdot d(\zeta - z)}{\lambda}\right)^{j} - j\frac{\bar\partial\lambda}{\lambda} \wedge \alpha \wedge \left(\frac{\bar\partial u \cdot d(\zeta - z)}{\lambda}\right)^{j-1} \\
&= \left(\frac{\bar\partial u \cdot d(\zeta - z)}{\lambda}\right)^{j} - j\frac{\bar\partial\lambda}{\lambda} \wedge \alpha \wedge (\bar\partial\alpha)^{j-1},
\end{aligned}$$

$$\begin{aligned}
X \lrcorner\left(\alpha \wedge (\bar\partial\alpha)^{j}\right) &= (X \lrcorner \alpha)(\bar\partial\alpha)^{j} + j(X \lrcorner \bar\partial\alpha) \wedge \alpha \wedge (\bar\partial\alpha)^{j-1} \\
&= (X \lrcorner \alpha)(\bar\partial\alpha)^{j} - j\bar\partial(X \lrcorner \alpha) \wedge \alpha \wedge (\bar\partial\alpha)^{j-1} \\
&= (\bar\partial\alpha)^{j} - (1 - (X \lrcorner \alpha))(\bar\partial\alpha)^{j} + j\bar\partial(1 - (X \lrcorner \alpha)) \wedge \alpha \wedge (\bar\partial\alpha)^{j-1},
\end{aligned}$$

and using the expansion (B.8) for $(\bar\partial\alpha)^j$ this equals

$$
\begin{aligned}
&(\bar\partial\alpha)^j - (1-(X\lrcorner\alpha))(\bar\partial u\cdot d(\lambda - z)/\lambda)^j \\
&\qquad + j[\bar\partial(1-(X\lrcorner\alpha)) + (1-(X\lrcorner\alpha))(\bar\partial\lambda/\lambda)] \wedge \alpha \wedge (\bar\partial\alpha)^{j-1} \\
&= (\bar\partial\alpha)^j - (1-(X\lrcorner\alpha))\left(\frac{\bar\partial u\cdot d(\zeta - z)}{\lambda}\right)^j + j\frac{\bar\partial[\lambda(1-(X\lrcorner\alpha))]}{\lambda}\alpha\wedge(\bar\partial\alpha)^{j-1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
X\lrcorner(\alpha\wedge(\bar\partial\alpha)^j) &= (\bar\partial\alpha)^j - (1-(X\lrcorner\alpha))(\bar\partial u\cdot d(\zeta - z)/\lambda)^j \\
&\quad + j(\bar\partial[\lambda(1-(X\lrcorner\alpha))]/\lambda)\wedge\alpha\wedge(\bar\partial\alpha)^{j-1}.
\end{aligned}
\tag{B.9}
$$

Now, substitution of (B.9) and a similar expression for $X\lrcorner(\beta\wedge(\bar\partial\beta)^{n-j-1})$ into the expression (B.7) proves the theorem.

Appendix C: The new kernel A as a Cauchy principal value current on $\mathbf{C}^n\times\mathbf{C}^n$. The kernel

$$
\begin{aligned}
A &= (2\pi i)^{-n}\alpha\wedge\beta\wedge\sum_{p=1}^{n-1}(\bar\partial\alpha)^{p-1}\wedge(\bar\partial\beta)^{n-p-1} \\
&= -(2\pi i)^{-n}\sum_{p=1}^{n-1}\frac{\bar\zeta\cdot d(\zeta - z)\wedge\bar z\cdot d(\zeta - z)}{(1-\bar\zeta\cdot z)^p(|\zeta|^2|z|^2 - \zeta\cdot z - \bar z\cdot\zeta + 1)^{n-p}} \\
&\qquad \wedge[d\bar\zeta\cdot d(\zeta - z)]^{p-1}\wedge[d(\bar\zeta - \bar z)\cdot d(\zeta - z)]^{n-p-1}
\end{aligned}
$$

is singular on the set $1-\bar\zeta\cdot z = 0$ since

$$
|\zeta|^2|z|^2 - \bar\zeta\cdot z - \zeta\cdot\bar z + 1 = |1-\bar\zeta\cdot z|^2 + |z|^2|\zeta|^2 - |\bar\zeta\cdot z|^2 \geqslant |1-\bar\zeta\cdot z|^2.
$$

However, the kernel A can be defined as a Cauchy principal value current on all of $\mathbf{C}^n\times\mathbf{C}^n$.

THEOREM C.1. *The form A, which is smooth on $\mathbf{C}^n\times\mathbf{C}^n$ minus the zero locus of $1-\bar\zeta\cdot z$, defines a Cauchy principal value current on $\mathbf{C}^n\times\mathbf{C}^n$ by the formula*

$$
A = \lim_{\varepsilon\to 0^+}\chi_\varepsilon A,
\tag{C.2}
$$

where χ_ε denotes the characteristic function of $\{(\zeta, z): |1-\bar\zeta\cdot z| \geqslant \varepsilon|\zeta|\}$. Moroever, the kernel A on $\mathbf{C}^n\times\mathbf{C}^n$ is regular (i.e. A: $\mathfrak{D}(\mathbf{C}^n)\to\mathcal{E}(\mathbf{C}^n)$) and extendable (i.e. A: $\mathcal{E}'(\mathbf{C}^n)\to\mathfrak{D}'(\mathbf{C}^n)$). In particular, for all $\phi\in\mathfrak{D}^{p,q}(\mathbf{C}^n)$, the principal value integral

$$
(A\phi)(z) = \lim_{\varepsilon\to 0^+}\int_{|1-\bar\zeta\cdot z|\geqslant\varepsilon|\zeta|} A(\zeta, z)\wedge\phi(\zeta)
\tag{C.3}
$$

exists with $A\phi\in\mathcal{E}^{p,q-2}(\mathbf{C}^n)$.

PROOF. It suffices to show that for each point $(\zeta_0, z_0)\in\mathbf{C}^n\times\mathbf{C}^n$ there exists a product neighborhood $U_1\times U_2$ such that (C.3) is valid for all $\phi\in\mathfrak{D}^{p,q}(U_1)$. If $1-\zeta_0 z_0\neq 0$ this is automatic since A is smooth in a product neighborhood. If $1-\zeta_0 z_0 = 0$ then the change of variables $(\zeta, z)\mapsto(\zeta|\zeta|^{-2}, z)$ converts $1-\bar\zeta\cdot z$ to

$\bar{\zeta} \cdot (\zeta - z)/|\zeta|^2$, and $|\zeta|^2|z|^2 - \bar{\zeta} \cdot z - \zeta \cdot \bar{z} + 1$ to $|\zeta - z|^2/|\zeta|^2$. Now one can easily check that this change of variables converts the integral (C.3) to an integral covered by Proposition 5.16 of Harvey and Polking **[HP1]** (with $u(\zeta, z) \equiv \bar{\zeta}$). Thus Proposition 5.16 of **[HP1]** implies that the limit in (C.3) exists and $A\phi \in \mathcal{D}^{p,q-2}(\mathbf{C}^n)$ is smooth. The kernel A is extendable to $\mathcal{E}'(\mathbf{C}^n)$ if and only if the transpose kernel ${}^tA$ is regular. The proof that ${}^tA$ is regular is similar to the above proof that A is regular and is omitted.

REMARK. Since Proposition 5.16 of **[HP1]** with $u(\zeta, z) = \bar{\zeta}$ has been applied to the proof of Theorem C.1, perhaps it is worth noting that in this case ($u = \bar{\zeta}$) the coordinate change $w = \Phi_z(\zeta)$, necessary for the proof of Proposition 5.16 of **[HP1]**, can be alternately described as follows. Choose an orthonormal frame $e_1(\zeta), \ldots, e_n(\zeta)$ near $\zeta_0 \neq 0$ with $e_n(\zeta) \equiv \bar{\zeta}$. Let $w_j \equiv \langle e_j(\zeta), \zeta - z \rangle$, $j = 1, \ldots, n$. Then $|\zeta - z|^2 = |w|^2$ and $w_n = \bar{\zeta} \cdot (\zeta - z)$ are automatic.

Similar results hold for the Szegö kernel S and the Bergman kernel K.

THEOREM C.4. *The forms S and K, which are smooth on $\mathbf{C}^n \times \mathbf{C}^n$ minus the zero locus of $1 - \bar{\zeta} \cdot z$, define Cauchy principal value currents on $\mathbf{C}^n \times \mathbf{C}^n$ by the formulas*

$$S = \lim_{\varepsilon \to 0^+} \chi_\varepsilon S, \tag{C.5}$$

$$K = \lim_{\varepsilon \to 0^+} \chi_\varepsilon K. \tag{C.6}$$

Moreover, the kernels S and K on $\mathbf{C}^n \times \mathbf{C}^n$ are regular and extendable.

The proof is similar to that of Theorem C.1 and is omitted.

BIBLIOGRAPHY

[C] P. Charpentier, *Formules explicites pour les solutions minimales de l'equation $\bar{\partial}u = f$ dans le boule et dans le polydisque de $\mathbf{C}^n$*, Ann. Inst. Fourier **30** (1980), 121–154.

[DH] J. Dadok and R. Harvey, *The fundamental solution for the Kohn-Laplacian $\square_b$ on the sphere in $\mathbf{C}^n$*, Math. Ann. **244** (1979), 89–104.

[FK] G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Ann. of Math. Studies, no. 75, Princeton Univ. Press, Princeton, N.J., 1972.

[FS] G. B. Folland and E. M. Stein, *Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group*, Comm. Pure Appl. Math. **27** (1974), 429–522.

[GS] P. Greiner and E. M. Stein, *Estimates for the $\bar{\partial}$-Neumann problem*, Princeton Univ. Press, Princeton, N.J., 1977.

[H1] G. M. Henkin, *Solutions with estimates of the H. Lewy and Poincaré-Lelong equations. The construction of functions of a Nevanlinna class with given zeros in a strictly pseudoconvex domain*, Dokl. Akad. Nauk SSSR **224** (1975), 771–774; English transl., Soviet Math. Dokl. **16** (1975), 1310–1314.

[H2] ______, *Integral representations of functions holomorphic in strictly pseudoconvex domains, and some applications*, Mat. Sb. (N.S.) **78** (**120**) (1969), 611–632; English transl., Math. USSR-Sb. **7** (1969), 597–616.

[HP1] R. Harvey and J. Polking, *Fundamental solutions in complex analysis*. I, II, Duke Math. J. **46** (1979), 253–300, 301–304.

[HP2] R. Harvey and J. Polking, *The $\bar{\partial}$-Neumann kernel in the ball in $\mathbf{C}^n$*, Proceedings of AMS Symposium in Several Complex Variables, Madison, 1982 (to appear).

[HR] G. M. Henkin and A. V. Romenov, *Exact Hölder estimates for the solutions of the $\bar{\partial}$-equation*, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 1171–1185; English transl., Math. USSR-Izv. **5** (1971), 1180–1191.

[K] J. J. Kohn, *Harmonic integrals on strongly pseudo-convex manifolds*. I, II, Ann. of Math. (2) **78** (1963), 112–148, **79** (1964), 450–472.

[Kr] S. G. Krantz, *Estimates for integral kernels of mixed type, fractional integration operators, and optimal estimates for the $\bar{\partial}$ operator*, Manuscripta Math. **30** (1979), 21–52.

[O] N. Ovrelid, *Integral representation formulas and L^p estimates for the $\bar{\partial}$-equation*, Math. Scand. **29** (1971), 137–160.

[Ph] D. H. Phoung, *On integral representations for the Neumann operator*, Proc. Nat. Acad. Sci. U.S.A. **76** (1979), 1554–1558.

[S] H. Skoda, *Valeurs au bord pour les solutions de l'opérateur d, et caractérisation des zéors des functions de la classe de Nevanlinne*, Bull. Soc. Math. France **104** (1976), 225–299.

[St] N. Stanton, *The solution of the $\bar{\partial}$-Neumann problem in a strictly pseudoconvex Siegal domain*, Invent. Math. **65** (1981), 137–174.

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