

## SUPERCOMPACTNESS OF COMPACTIFICATIONS AND HYPERSPACES

BY

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**ABSTRACT.** We prove a theorem which implies that if  $\gamma\omega$  is a supercompact compactification of the countable discrete space  $\omega$ , then  $\gamma\omega - \omega$  is separable. This improves an earlier result of the author's that such a  $\gamma\omega$  must have  $\gamma\omega - \omega$  ccc.

We prove a theorem which implies that the hyperspace of closed subsets of  $2^{\omega_2}$  is not a continuous image of a supercompact space. This improves an earlier result of L. Šapiro that the hyperspace of closed subsets of  $2^{\omega_2}$  is not dyadic.

**1. Introduction.** A space  $X$  is supercompact, de Groot [9], if it possesses an open subbase  $\mathfrak{S}$  such that every open cover of  $X$  from  $\mathfrak{S}$  has a 2 subcover. The vague finite of compactness is replaced by the concrete two of supercompactness. Many compact spaces are supercompact, see van Mill [11], but not all.

We had earlier proved [2] that if  $\gamma X$  was a supercompactification of a locally compact space  $X$ , then any collection of disjoint open sets of  $\gamma X - X$  had size at most the weight of  $X$ . In §3, we prove a stronger theorem via a simpler proof. Namely, that if  $\gamma X$  is a supercompactification of a locally compact space  $X$ , then  $\gamma X - X$  has a dense subspace of size at most the weight of  $X$ .

The remaining sections are devoted to the proof of Theorem 6.1, which implies that the hyperspace of  $2^{\omega_2}$  is not the continuous image of a supercompact space. We mention that all powers of 2 are supercompact spaces. It is well known that  $\exp 2^\omega$  is homeomorphic to  $2^\omega$ . Sirola [15] proved that  $\exp 2^{\omega_1}$  is homeomorphic to  $2^{\omega_1}$ . Šapiro [14] proved that not only was  $\exp 2^{\omega_2}$  not homeomorphic to  $2^{\omega_2}$ , it was not even dyadic. We acknowledge a debt to the paper of Šapiro. It suggested to us the Subbase Lemma of §4, which is of independent interest, as well as the line of attack towards our generalization.

**2. Notation and definitions.** For a cardinal  $\kappa$ ,  $\kappa^+$  is the successor cardinal and  $2^\kappa = \{f : f \text{ is a function from } \kappa \text{ to } 2\}$ . If  $H \subseteq 2^\kappa$  and  $\alpha < \kappa$ , then  $H \upharpoonright \alpha$  denotes  $\{f \upharpoonright \alpha : f \in H\}$ . If  $\mathfrak{S}$  is a collection of sets and  $S$  is a set, then  $\mathfrak{S}^\cap$  denotes  $\{\bigcap \mathfrak{S}' : \mathfrak{S}' \subseteq \mathfrak{S}\}$  and  $\mathfrak{S} \upharpoonright S$  denotes  $\{T \cap S : T \in \mathfrak{S}\}$ .  $\mathfrak{S}$  is linked if all 2-fold intersections of members of  $\mathfrak{S}$  are nonempty.  $\mathfrak{S}$  is centered if all finite intersections of

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Received by the editors November 17, 1982.

1980 *Mathematics Subject Classification.* Primary 54D30; Secondary 54B20.

*Key words and phrases.* Supercompact, hyperspace, dyadic.

<sup>1</sup>This research was supported by Grant No. U0070 from the Natural Sciences and Engineering Research Council of Canada.

members of  $\mathfrak{S}$  are nonempty.  $\mathfrak{S}$  is  $\kappa$ -centered if  $\mathfrak{S}$  is the union of  $\kappa$  many centered subcollections.  $\mathfrak{S}$  is binary if all linked subcollections of  $\mathfrak{S}$  have a nonempty intersection.

We assume that all of our spaces are completely regular and Hausdorff. If  $Y$  is a continuous image of  $X$ , then we write  $X \twoheadrightarrow Y$ . A  $G_\kappa$  subset of a space  $X$  is a subset that is an intersection of at most  $\kappa$  open subsets of  $X$ . The collection of all nonempty open subsets of  $X$  is denoted by  $\iota(X)$ .  $X$  is said to be  $\kappa$ -centered if  $\iota(X)$  is  $\kappa$ -centered. We will use the following standard cardinal functions and refer the reader to Juhász [10]: Cardinality  $|X|$ , weight  $w(X)$ , net weight  $nw(X)$ , density  $d(X)$ , spread  $s(X)$  and cellularity  $c(X)$ . If  $c(X) \leq \omega$ , then we say that  $X$  is ccc.

As a space,  $2^\kappa$  is endowed with the product topology. The Alexandroff one-point compactification of the discrete space  $\kappa$  is denoted by  $\alpha\kappa$  and the Stone-Čech compactification of the discrete space  $\kappa$  is denoted by  $\beta\kappa$ .  $\exp X$  is the hyperspace of all closed subsets of  $X$  endowed with the topology which has an open base consisting of all sets of the form  $\langle O_1, \dots, O_n \rangle = \{F : F \text{ is a closed subset of } X, F \subseteq \bigcup_{i=1}^n O_i \text{ and for all } i \leq n, F \cap O_i \neq \emptyset\}$  where the  $O_i$ 's are open subsets of  $X$ . It is well known, Vietoris [16], that  $\exp X$  is compact iff  $X$  is compact.

A space is dyadic, Alexandroff [1], if it is a continuous image of a power of 2. A space is supercompact, de Groot [9], if it possesses a binary closed subbase. If  $\mathfrak{S}$  is a binary closed subbase for  $X$ , then  $\mathfrak{S}^\cap$  is also a binary closed subbase for  $X$ . Continuous images of supercompact spaces will be called superadic spaces. Indeed, there are superadic spaces that are not supercompact, van Mill and Mills [12]. Since a power of 2 is supercompact, superadic spaces are a generalization of dyadic spaces. In fact, they generalize the  $m$ -adic and  $\xi$ -adic spaces of Mrówka [13]. Another interesting generalization of dyadic spaces are the hyadic spaces of van Douwen [6], i.e., continuous images of hyperspaces of compact spaces. Both the superadic spaces, van Douwen and van Mill [7], and the hyadic spaces, van Douwen [6], have nontrivial convergent sequences. I do not know whether all supercompact (and therefore superadic) spaces are hyadic.

**3. Superadic compactifications.** In this section,  $\gamma X$  represents a compactification of  $X$ . A collection of sets  $\mathfrak{S}$  is said to be  $r$ -centered in a set  $S$  if  $\mathfrak{S} = \bigcup_{i < r} \mathfrak{S}_i$  where for each  $i < r$ ,  $\{S \cap T : T \in \mathfrak{S}_i\}$  is a centered collection.

3.1. THEOREM. *If  $\gamma X$  is superadic, then  $\gamma X - X$  is  $nw(X)$ -centered.*

PROOF. Let  $f$  map  $Y$  continuously onto  $\gamma X$  and let  $\mathfrak{S} = \mathfrak{S}^\cap$  be a binary closed subbase for  $Y$ . Let  $\mathfrak{U}$  be a network for  $X$  of size  $nw(X)$  closed under finite unions and such that  $\emptyset \notin \mathfrak{U}$ . Put  $\mathfrak{U} = \{U : U \text{ is open in } \gamma X \text{ and } U \not\subseteq X\}$ . Choose  $W \subseteq Y$  of size at most  $d(X)$  such that  $f(W)$  is a dense subspace of  $X$ .

For each  $U \in \mathfrak{U}$  and for each  $w \in W$ , set  $\mathfrak{S}(U, w) = \{S \in \mathfrak{S} : w \in S \text{ and } f(S) \not\subseteq U\}$ . For each  $w \in W$  and for each  $N \in \mathfrak{U}$  set  $\mathfrak{U}(w, N) = \{U \in \mathfrak{U} : \text{there exists } S \in \mathfrak{S} \text{ and there exists a finite subset } F \text{ of } Y - f^{-1}(f(S)) \text{ such that } w \in S \subseteq f^{-1}(U), f(S) \not\subseteq X, f(F) \cap X \subseteq N \subseteq X - f(S) \text{ and for all } T \in \mathfrak{S}(U, w), T \cap F \neq \emptyset\}$ .

*Claim 1.* For each  $w \in W$  and for each  $N \in \mathfrak{U}$ , there exists an  $r < \omega$  such that  $\mathfrak{U}(w, N)$  is  $r$ -centered in  $\gamma X - X$ .

**PROOF OF CLAIM.** If  $\mathcal{Q}(w, N)$  is 1-centered in  $\gamma X - X$ , then let  $r = 1$ . If not, then there exists a finite  $\mathcal{F} \subseteq \mathcal{Q}(w, N)$  such that  $\bigcap \mathcal{F} \subseteq X$ . For each  $U \in \mathcal{F}$ , let  $S_u$  and  $F_u$  witness the fact that  $U \in \mathcal{Q}(w, N)$ . Then, every  $V \in \mathcal{Q}(w, N)$  contains an element of  $\bigcup_{U \in \mathcal{F}} f(F_u) \cap (\gamma X - X)$ , so we can let  $r = |\bigcup_{U \in \mathcal{F}} f(F_u) \cap (\gamma X - X)|$ . For, if  $V \in \mathcal{Q}(w, N)$ , then let  $S_v$  and  $F_v$  witness this fact. Since  $f(S_v) \not\subseteq X$ , we have  $f(S_v) \not\subseteq \bigcap \mathcal{F}$ , therefore, there exists  $U \in \mathcal{F}$  such that  $f(S_v) \not\subseteq U$ . Hence,  $S_v \in \mathcal{S}(U, w)$  and because  $U \in \mathcal{Q}(w, N)$  we have  $S_v \cap F_u \neq \emptyset$ . So,  $f(S_v \cap F_u) \neq \emptyset$ . If  $f(S_v \cap F_u) \subseteq X$ , then  $f(S_v \cap F_u) \subseteq f(F_u) \cap X \subseteq N \subseteq X - f(S_v)$ , contradiction. Therefore,  $f(S_v \cap F_u) \cap (\gamma X - X) \neq \emptyset$  and so  $f(S_v) \cap f(F_u) \cap (\gamma X - X) \neq \emptyset$ . Since  $f(S_v) \subseteq V$ , we have  $V \cap f(F_u) \cap (\gamma X - X) \neq \emptyset$ .

*Claim 2.*  $\mathcal{Q} = \bigcup \{ \mathcal{Q}(w, N) : w \in W \text{ and } N \in \mathcal{N} \}$ .

**PROOF OF CLAIM.** Let  $U \in \mathcal{Q}$ . Pick  $x \in U - X$ . By regularity, there exists an open subset  $O$  of  $\gamma X$  such that  $x \in O \subseteq \bar{O} \subseteq U$ . Therefore,  $f^{-1}(x) \subseteq f^{-1}(O) \subseteq f^{-1}(\bar{O}) \subseteq f^{-1}(U)$ . There exists a finite  $\mathcal{S}' \subseteq \mathcal{S}$  such that  $f^{-1}(\bar{O}) \subseteq \bigcup \mathcal{S}' \subseteq f^{-1}(U)$ . Since  $f$  is a closed map and  $f(W)$  is dense in  $X$ , there must exist an  $S_u \in \mathcal{S}'$  such that  $S_u \cap f^{-1}(x) \neq \emptyset$  and  $S_u \cap W \neq \emptyset$ . Pick  $w \in S_u \cap W$ . Note that since  $x \in f(S_u)$ ,  $f(S_u) \not\subseteq X$ .

Since  $Y - f^{-1}(U) \subseteq Y - f^{-1}(f(S_u))$ , there exists  $\{S_i : i < m\} \subseteq \mathcal{S}$  such that  $Y - f^{-1}(U) \subseteq \bigcup_{i < m} S_i \subseteq Y - f^{-1}(f(S_u))$ . For each  $i < m$ , set  $\mathcal{S}_i = \{S \in \mathcal{S}(U, w) : S \cap S_i \neq \emptyset\}$ . Then,  $\mathcal{S}(U, w) = \bigcup_{i < m} \mathcal{S}_i$ . For each  $i < m$ ,  $\mathcal{S}_i \cup \{S_i\}$  is linked. Hence, there exists  $x_i \in \bigcap \mathcal{S}_i \cap S_i \subseteq Y - f^{-1}(f(S_u))$ . Set  $F_u = \{x_i : i < m\}$ . Since  $f(F_u) \cap X \subseteq X - f(S_u)$ , there exists  $N \in \mathcal{N}$  such that  $f(F_u) \cap X \subseteq N \subseteq X - f(S_u)$ . Finally,  $S_u$  and  $F_u$  witness the fact that  $U \in \mathcal{Q}(w, N)$ .

From Claims 1 and 2, it follows that

$$t(\gamma X - X) = \bigcup \{ \{ U \cap (\gamma X - X) : U \in \mathcal{Q}(w, N) \} : w \in W \text{ and } N \in \mathcal{N} \}$$

where each of these collections is  $r$ -centered for some  $r < \omega$ . Since  $|W| = d(X) \leq nw(X)$ , we get that  $\gamma X - X$  is  $nw(X)$ -centered.

**REMARK 1.** In the theorem, if  $X$  is nowhere locally compact, then the conclusion is true regardless of whether  $\gamma X$  is superadic or not. The theorem has nontrivial content for somewhere locally compact  $X$ 's.

**3.2. COROLLARY.** *If  $X$  is locally compact and  $\gamma X$  is superadic, then  $d(\gamma X - X) \leq nw(X)$ . In particular, if  $\gamma\omega$  is superadic, then  $\gamma\omega - \omega$  is separable.*

**PROOF.** From Theorem 3.1 we conclude that  $\gamma X - X$  is  $nw(X)$ -centered. Since  $X$  is locally compact,  $\gamma X - X$  is compact and hence  $d(\gamma X - X) \leq nw(X)$ . Note that  $nw(X) = w(X)$  for locally compact spaces  $X$ .

**REMARK 2.** This corollary improves an earlier result of the author [2], that if  $\gamma\omega$  is superadic, then  $\gamma\omega - \omega$  is ccc. The earlier proof had an unnecessary use of infinitary combinatorics. The present proof is elementary. So elementary, in fact, that if one follows through the proof of Theorem 3.1, with  $X = \omega$ ,  $Y = \gamma\omega$ ,  $f = \text{identity}$ ,  $\mathcal{N} = \{F : F \text{ is a nonempty finite subset of } \omega\}$  and  $W = \omega$ , then one gets a proof without the aid of the Axiom of Choice of the following: (ZF) If  $\gamma\omega$  is supercompact, then  $t(\gamma\omega - \omega)$  is the union of countably many subcollections each of which is

$r$ -centered for some  $r < \omega$ . Since there is in ZF a compactification  $\gamma\omega$  for which  $\gamma\omega - \omega$  has  $c$  isolated points, for example, Example 1.4 in [7], and  $c$  is not the union of countably many finite sets, we get that ZF alone implies that not all compact Hausdorff spaces are supercompact.

REMARK 3. In [3], compactifications  $\gamma\omega$  are constructed for which  $\gamma\omega - \omega$  is ccc but not separable. By the theorem, none of these compactifications are superadic. However, they can be constructed to have arbitrary compactness number  $> 2$ , see [4]. The compactness number of a compact space  $X$ ,  $\text{cmpn}(X)$ , is the least  $n < \omega$ , if one exists, such that  $X$  possesses an  $n$ -ary closed subbase, Bell and van Mill [5]. If no such  $n < \omega$  exists, then one says  $\text{cmpn}(X) = \infty$ .

No results similar to the theorem or corollary exist for compactness number  $> 2$ . There is a compactification  $\gamma\omega$  such that  $\text{cmpn } \gamma\omega = 3$  and  $\gamma\omega - \omega$  is not ccc, cf. van Douwen and van Mill [7].

3.3. COROLLARY. *If  $\text{exp}(\gamma X)$  is superadic, then  $\gamma X - X$  is  $nw(X)$ -centered.*

PROOF. Set  $X^* = \{F : F \text{ is a finite subset of } X\}$ . Then  $X^*$  is a dense subspace of  $\text{exp}(\gamma X)$ . Furthermore,  $nw(X^*) = nw(X)$ . By our theorem,  $\text{exp}(\gamma X) - X^*$  is  $nw(X)$ -centered. In particular,

$$\{\langle O \rangle - X^* : O \in t(\gamma X)\} - \{\phi\} = \cup \{\mathcal{C}'_\alpha : \alpha < nw(X)\}$$

where each  $\mathcal{C}'_\alpha$  is centered. For each  $\alpha < nw(X)$ , set  $\mathcal{C}'_\alpha = \{O - X : O \in t(X) \text{ and } \langle O \rangle - X^* \in \mathcal{C}'_\alpha\}$ . Thus,  $t(\gamma X - X) = \cup \{\mathcal{C}'_\alpha : \alpha < nw(X)\}$  and each  $\mathcal{C}'_\alpha$  is centered.  $\square$

3.4. EXAMPLE. A 0-dimensional space  $X$  with  $w(X) = \omega_1$  and  $\text{exp } X$  not superadic: Let  $X$  be any compactification  $\gamma\omega$  of  $\omega$  of weight  $\omega_1$  and having  $\gamma\omega - \omega$  homeomorphic to  $\alpha\omega_1$ .

3.5. EXAMPLE. A 0-dimensional space  $X$  with  $\text{exp } X$  superadic but not dyadic: Let  $\kappa$  be an uncountable cardinal and set  $X = \alpha\kappa$ .  $\text{exp } X$  is not dyadic because  $\text{exp } X$  is not ccc. We will show that  $\text{exp } X$  is supercompact by producing a binary closed subbase for  $\text{exp } X$ . Set  $\mathcal{Q} = \{\langle \{\gamma\} \rangle : \gamma < \kappa\} \cup \{\langle X - \{\gamma\} \rangle : \gamma < \kappa\}$ . It is easily checked that  $\mathcal{Q} \cup \{\text{exp } X - A : A \in \mathcal{Q}\} \cup \{\{F\} : F \text{ is a finite subset of } \kappa\}$  is a binary closed subbase for  $\text{exp } X$ . Note that superadicity of  $\text{exp } X$  places no weight restrictions on  $X$ . This contrasts with Šapiro's result [13] that dyadicity of  $\text{exp } X$  implies  $w(X) \leq c$ .

**4. The Subbase Lemma.** The following general result on subbases plays a key role in our main theorem of §6.

4.1. THE SUBBASE LEMMA. *Let  $\mathfrak{S} = \mathfrak{S}^\cap$  be a closed subbase for a compact space  $X$ . Let  $f$  be a continuous map from  $X$  onto a space  $Y$ . If  $F$  is a nonempty closed  $G_\kappa$  subset of  $Y$ , then there exists a nonempty  $S \in \mathfrak{S}$  such that  $f(S) \subseteq F$  and  $f(S)$  is a  $G_\kappa$  subset of  $Y$ .*

PROOF. Let  $F = \bigcap_{\alpha < \kappa} O_\alpha$  where each  $O_\alpha$  is open in  $Y$ . For each  $\alpha < \kappa$ , choose a finite set  $\{S_i^\alpha : i < n_\alpha\} \subseteq \mathfrak{S}$  such that  $f^{-1}(F) \subseteq \bigcup_{i < n_\alpha} S_i^\alpha \subseteq f^{-1}(O_\alpha)$ . Then,

$$f^{-1}(F) = \bigcap_{\alpha < \kappa} \left( \bigcup_{i < n_\alpha} S_i^\alpha \right) = \bigcup \left\{ \bigcap_{\alpha < \kappa} S_{\varphi(\alpha)}^\alpha : \varphi \in \prod_{\alpha < \kappa} n_\alpha \right\}.$$

Since  $\mathfrak{S} = \mathfrak{S}^\cap$ , each  $\bigcap_{\alpha < \kappa} S_{\varphi(\alpha)}^\alpha \in \mathfrak{S}$ . Give each  $\{S_i^\alpha : i < n_\alpha\}$  the discrete topology and give  $P = \prod_{\alpha < \kappa} \{S_i^\alpha : i < n_\alpha\}$  the product topology. Note that  $w(P) \leq \kappa$ .

*Claim.* There exists  $S_0 \in \mathfrak{S}$  and a nonempty closed  $C_0 \subseteq S_0$  such that  $f(S_0) \subseteq F$  and  $f(C_0)$  is a  $G_\kappa$  subset of  $Y$ .

**PROOF OF CLAIM.** Assume not. We will get a contradiction by constructing a  $\kappa^+$  strictly decreasing sequence of closed sets of  $P$ . Define the (not necessarily continuous) function  $\pi : \exp X \rightarrow \exp P$  by

$$\pi(C) = \left\{ (S_{\varphi(\alpha)}^\alpha) : C \cap \bigcap_{\alpha < \kappa} S_{\varphi(\alpha)}^\alpha \neq \emptyset \right\}.$$

Set  $F_0 = X$ . Assume we have constructed nonempty closed subsets  $F_\beta$  of  $X$  for all  $\beta < \gamma < \kappa^+$  such that

(1)  $\{F_\beta : \beta < \gamma\}$  is a decreasing  $\gamma$ -sequence and  $\{\pi(F_\beta) : \beta < \gamma\}$  is a strictly decreasing  $\gamma$ -sequence,

(2)  $f(F_\beta)$  is a  $G_\kappa$  subset of  $Y$ .

If  $\gamma$  is a limit ordinal, set  $F_\gamma = \bigcap_{\beta < \gamma} F_\beta$ . Since  $f(F_\gamma) = f(\bigcap_{\beta < \gamma} F_\beta) = \bigcap_{\beta < \gamma} f(F_\beta)$  and each  $f(F_\beta)$  is a  $G_\kappa$  subset of  $Y$ , we have that  $f(F_\gamma)$  is a  $G_\kappa$  subset of  $Y$ . Also,

$$\emptyset \neq \pi(F_\gamma) = \left\{ (S_{\varphi(\alpha)}^\alpha) : \bigcap_{\beta < \gamma} F_\beta \cap \bigcap_{\alpha < \kappa} S_{\varphi(\alpha)}^\alpha \neq \emptyset \right\} = \bigcap_{\beta < \gamma} \pi(F_\beta) \subseteq \pi(F_\beta)$$

for all  $\beta < \gamma$ .

If  $\gamma = \beta + 1$ , then choose a  $(S_{\varphi(\alpha)}^\alpha) \in \pi(F_\beta)$ . Therefore,  $F_\beta \cap \bigcap_{\alpha < \kappa} S_{\varphi(\alpha)}^\alpha \neq \emptyset$ . If  $f(F_\beta \cap \bigcap_{\alpha < \kappa} S_{\varphi(\alpha)}^\alpha) = f(F_\beta)$ , then the claim is true, for we can set  $C_0 = F_\beta \cap \bigcap_{\alpha < \kappa} S_{\varphi(\alpha)}^\alpha$  and  $S_0 = \bigcap_{\alpha < \kappa} S_{\varphi(\alpha)}^\alpha$ . Since we have assumed the claim is false, we can choose a nonempty zero set  $Z$  of  $f(F_\beta)$  such that  $Z \subseteq f(F_\beta) - f(F_\beta \cap \bigcap_{\alpha < \kappa} S_{\varphi(\alpha)}^\alpha)$ . Now set  $F_\gamma = F_\beta \cap f^{-1}(Z)$ . Since  $f(F_\gamma) = Z$  and  $Z$  is a  $G_\omega$  subset of  $f(F_\beta)$  we have that  $f(F_\gamma)$  is a  $G_\kappa$  subset of  $Y$ . Since  $(S_{\varphi(\alpha)}^\alpha) \in \pi(F_\beta) - \pi(F_\gamma)$  we have that  $\pi(F_\gamma) \subseteq \pi(F_\beta)$ .

Now, the  $\{\pi(F_\gamma) : \gamma < \kappa^+\}$  forms a  $\kappa^+$  strictly decreasing sequence of closed sets of  $P$ . Our claim is proven.

To prove the lemma, we repeatedly apply the claim. Given an  $S_0$  and  $C_0$  as in the claim, we consider  $f \upharpoonright S_0 : S_0 \rightarrow f(S_0)$ .  $\mathfrak{S} \upharpoonright S_0$  is a closed subbase for  $S_0$  and  $\mathfrak{S} \upharpoonright S_0 = (\mathfrak{S} \upharpoonright S_0)^\cap$ . Also  $f(C_0)$  is a nonempty  $G_\kappa$  subset of  $Y$ , hence also of  $f(S_0)$ . So, we have a new set-up for which we can apply the claim again. In this way, we get  $S_i$ 's  $\in \mathfrak{S}$  and closed sets  $C_i \subseteq S_i$  such that  $S_0 \supseteq S_1 \supseteq S_2 \supseteq \dots$  and  $f(S_{i+1}) \subseteq f(C_i) \subseteq f(S_i)$  with the  $f(C_i)$ 's being  $G_\kappa$  subsets of  $Y$  and  $f(S_0) \subseteq F$ . Now,  $f(\bigcap_{i < \omega} S_i) = \bigcap_{i < \omega} f(S_i) = \bigcap_{i < \omega} f(C_i)$  which is a  $G_\kappa$  subset of  $Y$  contained in  $F$ . Setting  $S = \bigcap_{i < \omega} S_i$  completes the proof of the lemma.

**5. The hyperspace of  $2^{\kappa^+}$ .** If  $\alpha < \kappa^+$  and  $F$  is a closed subset of  $2^\alpha$ , then we set  $\hat{F} = \{H : H \text{ is a closed subset of } 2^{\kappa^+} \text{ and } H \upharpoonright \alpha = F\}$ . Note that  $\hat{F} \subseteq \exp 2^{\kappa^+}$  and that if  $\pi_\alpha : 2^{\kappa^+} \rightarrow 2^\alpha$  is the projection map, then  $\hat{F} = (\exp \pi_\alpha)^{-1}(F)$ . We collect the following simple facts into a lemma.

5.1. LEMMA. Let  $\alpha < \kappa^+$ ,  $F$  a closed subset of  $2^\alpha$  and  $\mathfrak{F}$  a closed  $G_\kappa$  subset of  $\exp 2^{\kappa^+}$ . Then,

- (a)  $\hat{F}$  is a closed  $G_\kappa$  subset of  $\exp 2^{\kappa^+}$ ,
- (b)  $\hat{F} \rightarrow \exp F$ ,
- (c) there exists  $\beta < \kappa^+$  such that  $\mathfrak{F} = \bigcup_{F \in \mathfrak{F}} (F \uparrow \beta)^\wedge$ .

PROOF. (a) and (c) are well-known dyadic facts. For (b), define  $\varphi: \hat{F} \rightarrow \exp F$  by  $\varphi(H) = \{s \in F: s \uparrow \alpha + 1\}$ .

6.  $\exp 2^{\omega_2}$  is not superadic. For  $\kappa$  a cardinal, set  $P_\kappa = \bigcup_{\alpha < \kappa} \exp 2^\alpha$ . If  $F \in \exp 2^\alpha$  and  $G \in \exp 2^\beta$ , define  $F \leq G$  if  $\beta \leq \alpha$  and  $F \uparrow \beta = G$ . Then,  $(P_\kappa, \leq)$  is a poset in which every decreasing  $\kappa$ -sequence has an infimum. We note that if  $F \leq G$ , then  $F \rightarrow G$  by the projection map.

We are now ready to prove our main theorem.

6.1. THEOREM. Let  $\mathfrak{S} = \mathfrak{S}^\cap$  be a closed subbase for a compact space  $X$  and let  $X \rightarrow \exp 2^{\kappa^+}$ . If  $Y$  is a compact 0-dimensional space of weight at most  $\kappa$ , then there exists an  $S \in \mathfrak{S}$  such that  $S \rightarrow \exp Y$ .

PROOF. Assume that  $f: X \rightarrow \exp 2^{\kappa^+}$  and that  $Y$  is a closed subspace of  $2^\kappa$ . We will find an  $F \leq Y$  and an  $S \in \mathfrak{S}$  such that  $S \rightarrow \exp F$ . Since  $F \rightarrow Y$ , we have  $\exp F \rightarrow \exp Y$  and thus  $S \rightarrow \exp Y$ .

Put  $S_1 = X$ ,  $\alpha_1 = \kappa$  and  $F_{\alpha_1} = Y$ . Assume we have found  $S_n \subseteq S_{n-1} \subseteq \dots \subseteq S_1$  with all  $S_i \in \mathfrak{S}$  and  $\alpha_1 < \alpha_2 < \dots < \alpha_n < \kappa^+$  with  $F_{\alpha_i} \in \exp 2^{\alpha_i}$  such that

- (1)  $\hat{F}_{\alpha_i} \subseteq f(S_i)$ ,
- (2)  $f(S_i)$  is a  $G_\kappa$  subset of  $\exp 2^{\kappa^+}$ ,
- (3)  $S_i \subseteq S_{i-1} \cap f^{-1}(\hat{F}_{\alpha_{i-1}})$ , if  $i \geq 2$ .

At stage  $n + 1$ , consider  $f \upharpoonright S_n: S_n \rightarrow f(S_n)$ . Then,  $\mathfrak{S} \upharpoonright S_n = (\mathfrak{S} \upharpoonright S_n)^\cap$  is a closed subbase for  $S_n$ . By Lemma 5.1(a),  $\hat{F}_{\alpha_n}$  is a closed  $G_\kappa$  subset of  $\exp 2^{\kappa^+}$  and thus  $\hat{F}_{\alpha_n}$  is a closed  $G_\kappa$  subset of  $f(S_n)$ . The Subbase Lemma now implies that there is a nonempty  $S_{n+1} \in \mathfrak{S} \upharpoonright S_n$ ,  $S_{n+1} \subseteq S_n$  such that  $f(S_{n+1}) \subseteq \hat{F}_{\alpha_n}$  and  $f(S_{n+1})$  is a  $G_\kappa$  subset of  $f(S_n)$ . By inductive assumption (2), we conclude that  $f(S_{n+1})$  is a  $G_\kappa$  subset of  $\exp 2^{\kappa^+}$ . This verifies inductive assumptions (2) and (3) for  $S_{n+1}$ .

By Lemma 5.1(c), there exists  $\alpha_{n+1} > \alpha_n$  such that

$$f(S_{n+1}) = \bigcup_{F \in f(S_{n+1})} (F \uparrow \alpha_{n+1})^\wedge.$$

Set  $F_{\alpha_{n+1}} = H \uparrow \alpha_{n+1}$  for some  $H \in f(S_{n+1})$ . Then,  $\hat{F}_{\alpha_{n+1}} \subseteq f(S_{n+1})$ , verifying inductive assumption (1) for  $F_{\alpha_{n+1}}$  and  $S_{n+1}$ .

Having completed the induction, we see that the  $\hat{F}_{\alpha_i}$ 's are interlaced with the  $f(S_i)$ 's, hence  $\bigcap_{i < \omega} \hat{F}_{\alpha_i} = \bigcap_{i < \omega} f(S_i)$ . If we set  $F = \inf\{F_{\alpha_i}: i < \omega\}$ , we have that  $\hat{F} = \bigcap_{i < \omega} \hat{F}_{\alpha_i}$ . Since  $\bigcap_{i < \omega} f(S_i) = f(\bigcap_{i < \omega} S_i)$ , by setting  $S = \bigcap_{i < \omega} S_i$ , we have that  $S \in \mathfrak{S}$ ,  $F \leq F_{\alpha_1} = Y$ , and  $S \rightarrow \hat{F}$ . By Lemma 5.1(b),  $\hat{F} \rightarrow \exp F$  and thus  $S \rightarrow \exp F$ , completing the proof.

6.2. COROLLARY.  $\exp 2^{\omega_2}$  is not superadic.

PROOF. Assume there exists a space  $X$  with a binary closed subbase  $\mathfrak{S}$  (w.l.o.g.  $\mathfrak{S} = \mathfrak{S}^\cap$ ) such that  $X \rightarrow \exp 2^{\omega_2}$ . Observe that if  $S \in \mathfrak{S}$ , then  $\mathfrak{S} \upharpoonright S$  is a binary closed subbase for  $S$ , hence  $S$  is supercompact. Theorem 6.1 implies that if  $Y$  is a compact 0-dimensional space of weight  $\omega_1$ , then  $\exp Y$  is superadic. This is not true, see Example 3.4.

This corollary generalizes Šapiro's result [14] that  $\exp 2^{\omega_2}$  is not dyadic. It also shows that one of the simplest types of supercompact space need not have a supercompact hyperspace. Whether  $\exp X$  supercompact implies  $X$  supercompact is not known. Since Sirota [15] has shown that  $\exp 2^{\omega_1}$  is homeomorphic to  $2^{\omega_1}$ , we get that the continuum hypothesis is equivalent to the supercompactness of  $\exp 2^c$ .

Similarly, it can be shown that  $\exp I^{\omega_2}$  is not superadic where  $I$  is the closed unit interval. We decided to prove the result for  $\exp 2^{\omega_2}$  for reasons of simplicity. It then follows from a result of Efimov [8] that if  $X$  has a closed subspace  $F$  such that  $F \rightarrow 2^{\omega_2}$ , then  $\exp X$  is not superadic.

Can  $\exp 2^{\omega_2}$  be a continuous image of a space  $X$  with  $\text{cmpn } X < \infty$ ? We suspect not, but our technique breaks down because we do not have an example of a compact 0-dimensional space  $Y$  (in ZFC) of weight  $\omega_1$ , such that  $\exp Y$  (or for that matter  $Y$ ) is not a continuous image of any space  $X$  of finite compactness number. We mention that  $\beta\omega$  is such an example, but it has weight  $c$ .

Jan van Mill has made the following interesting deduction.

6.3. COROLLARY. *If  $X$  is a compact 0-dimensional space with  $s(X) > \omega_1$ , then  $\exp(\exp X)$  is not superadic.*

PROOF. Let  $D$  be a discrete subspace of  $X$  of size  $\omega_2$ . For each  $d \in D$  choose a clopen set  $C(d)$  such that  $C(d) \cap D = \{d\}$ . For each  $d \in D$ , set  $\tilde{d} = \{F \in \exp X : F \cap C(d) \neq \emptyset\}$ . Then  $\{\tilde{d} : d \in D\}$  is an independent collection of clopen subsets of  $\exp X$ . Consequently,  $\exp X \rightarrow 2^{\omega_2}$ . Hence,  $\exp(\exp X) \rightarrow \exp 2^{\omega_2}$  and thus,  $\exp(\exp X)$  is not superadic.

6.4. EXAMPLE. A 0-dimensional space  $X$  with  $\exp X$  superadic but  $\exp(\exp X)$  not superadic: Let  $X = \alpha\omega_2$ .  $\exp X$  is supercompact, see Example 3.5. Since  $s(X) = \omega_2$ ,  $\exp(\exp X)$  is not superadic.

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