

LINEARIZATION AND MAPPINGS ONTO PSEUDOCIRCLE DOMAINS

BY

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ABSTRACT. We demonstrate the existence of linearizations for groups of conformal and anticonformal homeomorphisms of Riemann surfaces. The finitely generated groups acting on plane domains are classified in terms of specific linearizations. This extends Maskit's work in the directly conformal case.

As an application we prove that there exist conformal representations of finite genus open Riemann surfaces for which accessible boundary points are either isolated or lie on circular arcs of pseudocircular boundary components. In many cases these are actually circle domains. Along the way we extend the applicability of Carathéodory's boundary correspondence theorem for prime ends.

Introduction. Given a plane domain D it is natural to look for conformal representations of D by domains which highlight certain of its structural properties. In this paper we examine a connection between two problems of this type: we seek a conformal representation D' of D for which either (i) the group G of conformal and anticonformal self-maps of D' extend to the entire Riemann sphere, or (ii) the boundary of D' consists of circles and isolated points.

A striking example of this connection is the following immediate corollary to Theorem 8 of §4. *Let D be a plane domain, and G the group of conformal and anticonformal self-maps of D . If the quotient D/G is a topologically finite surface then D is conformally equivalent to a plane domain D' bounded by circles and isolated points. Moreover, the group G' of conformal and anticonformal self-maps of D' is the restriction to D' of a group of extended Möbius transformations.* If G is a conformal group whose action gives a conformally finite quotient then the corollary follows easily from some deep theorems of Maskit (see Lemma 1).

In the case where G contains only directly conformal maps problem (i) was successfully resolved by Maskit [7]. In later papers [8, 9] he went on to give a construction-oriented description of the Kleinian groups which arise when the group G is finitely generated. In §§1 and 2 we will extend Maskit's results to groups G with anticonformal elements. This includes a resolution of problem (i) as well as a description of the groups arising in the finitely generated case.

In §3 we define a notion of prime end for domains of arbitrary connectivity and finite genus. We will show that each conformal homeomorphism f from a domain D

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to a domain D' induces a homeomorphism of the “isolated” prime ends extensions of D and D' . This generalizes a well-known theorem of Carathéodory which gives the correspondence when D is simply connected.

Problem (ii) was introduced by Koebe and solved classically for domains of finite connectivity. More recent work has been done by K. Strebel and R. J. Sibner (see [12]). We will also be concerned with the more general question as to whether a finite genus Riemann surface can be embedded in a closed Riemann surface in such a way that the boundary of its image consists of isolated points and circles which are defined in the natural metric of constant curvature of the ambient surface. This problem does not appear to have been considered in the literature.

In §4 we will distinguish two classes of domains, called pseudocircle domains and pseudocurve domains, and show that every domain of finite genus is conformally equivalent to one of these. This implies most of the known solutions to problem (ii) and some new ones as well. It also solves the analogous problem for many finite genus Riemann surfaces and Klein surfaces. We note that pseudocircle domains have also been studied by R. J. Sibner (unpublished).

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1. Linearization.

1.1. Let $M\ddot{o}b$ be the group of transformations of the Riemann sphere $\hat{\mathbb{C}}$ of the form $z \mapsto (az + b)/(cz + d)$ and $z \mapsto (a\bar{z} + b)/(c\bar{z} + d)$, with $ad - bc \neq 0$. If G is a subgroup of $M\ddot{o}b$ then G^+ will denote the subgroup of G whose elements preserve orientation. The elements of $M\ddot{o}b^+ = M\ddot{o}b$ are usually referred to as Möbius transformations.

A discontinuous subgroup of $(M\ddot{o}b) M\ddot{o}b$ is called (extended) Kleinian. The set of points at which G acts discontinuously is $\Omega(G)$, the regular set. The complement of $\Omega(G)$ in $\hat{\mathbb{C}}$ is $\Lambda(G)$, the limit set.

More generally we may consider a discontinuous group G of conformal and anticonformal self-maps of an open subset D of a Riemann surface S . The orbit space D/G^+ of D under the action of the orientation-preserving subgroup G^+ of G carries a natural conformal structure for which the projection $\pi: D \rightarrow D/G^+$ is a holomorphic branched regular covering. If G^+ is not all of G then it is an index two subgroup, and there is a unique anticonformal involution γ of the orbit space D/G^+ satisfying $\pi \circ g = \gamma \circ \pi$ for all $g \in G - G^+$.

1.2. Let G be a group of conformal and, perhaps, anticonformal self-maps of a plane domain D . We say that G can be linearized if there is a conformal homeomorphism f mapping D into the plane so that fGf^{-1} is a subgroup of $M\ddot{o}b$.

In [6] Maskit proves that a freely acting group of conformal self-maps of a domain D for which D/G is a finite Riemann surface can be linearized. Recall that a finite Riemann surface is one that embeds conformally in a compact Riemann surface missing only a finite set of values. We will prove the following generalization.

THEOREM 1. *Let D be a plane domain and let G be a group of conformal and anticonformal homeomorphisms of D onto itself so that G^+ acts freely on D and $S = D/G^+$ is a finite Riemann surface. Then G can be linearized.*

The techniques employed by Maskit in [7] may be used to extend Theorem 1 in two ways:

THEOREM 2. *Let D be a plane domain and let G be a group of conformal and anticonformal homeomorphisms of D onto itself. Then G can be linearized.*

THEOREM 3. *Let S be a Riemann surface of genus g . Then there is a closed Riemann surface S^* of genus g and a conformal embedding of S into S^* so that, under embedding, every conformal and anticonformal self-map of S is the restriction of a conformal or anticonformal self-map of S^* .*

In this setting, a Riemann surface of genus g is one that embeds conformally in a compact surface of genus g but not in one of lower genus.

We omit the proofs of Theorems 2 and 3, as they simply entail verifying that Maskit's arguments in [7] go through with only slight modification using Theorem 1. The new result here is Theorem 1 and the remainder of this section will be devoted to its proof.

1.3. We need to establish several definitions before proceeding with the proofs. The square of the trace of a Möbius transformation is defined by taking a representative of the transformation in $SL(2, \mathbb{C})$ via the surjective homomorphism $\phi: SL(2, \mathbb{C}) \rightarrow \text{Möb}$, $\phi\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (az + b)/(cz + d)$.

An isomorphism $\phi: G \rightarrow G^*$ between Kleinian groups is called *type preserving* if:

1. ϕ preserves the square of the trace of elliptic elements, and
2. ϕ and ϕ^{-1} take parabolic elements to parabolic elements.

An *invariant component* of a group G is a connected component of $\Omega(G)$ which is mapped onto itself by each element of G . A *Fuchsian group* is a Kleinian group with an invariant component which is either a disc or a half-plane.

We denote by C_1 (C_1^*) the set of finitely generated (extended) Kleinian group with invariant components.

Let G and G^* be groups in C_1 with invariant components Δ and Δ^* . An orientation-preserving homeomorphism $f: \Delta \rightarrow \Delta^*$ which induces an isomorphism f_* of G onto G^* by $g \mapsto f \circ g \circ f^{-1}$ is called a *weak similarity*. If f_* is type preserving then f is a *similarity*.

Let g be a parabolic element of a group G in C_1 . g is called *accidental* if there is a weak similarity ϕ between G and another group G^* in C_1 so that $\phi \circ g \circ \phi^{-1}$ is not parabolic.

A group G in C_1 is called *basic* if:

1. the invariant component Δ of G is simply connected, and
2. G contains no accidental parabolic elements.

A basic group G is *degenerate* if the invariant component $\Delta = \Omega(G)$.

Let G be a group in C_1 with invariant component Δ . A *structure* subgroup H of G is a subgroup which satisfies the following.

1. H is a basic group.
2. If a fixed point of a parabolic element $g \in G$ lies in $\Lambda(H)$ then $g \in H$.
3. H is a maximal subgroup of G satisfying (1) and (2).

C_0 is the set of groups in C_1 none of whose structure subgroups are degenerate. A group G in C_1 is called a *Koebe group* if every structure subgroup of G is either Fuchsian or elementary. Clearly all Koebe groups are in the set C_0 . An *extended Koebe group* is an extended Kleinian group G with an invariant component and with G^+ a Koebe group.

1.4.

LEMMA 1. *Let D be a plane domain with noncyclic fundamental group, and let H be a finitely generated group of conformal homeomorphisms of D onto itself. Then there is a conformal homeomorphism ϕ of D onto a domain D' so that the group $H' = \phi H \phi^{-1}$ is a Koebe group containing no accidental parabolic elements.*

PROOF. Due to restrictions on the topology of D we may infer, by a standard argument from the theory of Fuchsian groups, that H acts discontinuously on D . By Maskit [7, Theorem A], H can be linearized; so there is no loss of generality in assuming that H is a Kleinian group and that D is contained in a maximal invariant component Δ of $\Omega(H)$.

Let G be a Kleinian group in C_1 with invariant component Δ^* . According to [8, Theorem 5] there is a conformal similarity f mapping Δ^* into the plane so that $fGf^{-1} = G'$ is a Koebe group. Since f is a similarity, it is type preserving. In particular, accidental parabolics of G must correspond, via f , exactly to accidental parabolics of G' .

In light of the above, the lemma is proven if we demonstrate the existence of a Kleinian group G with an invariant component Δ^* so that G is without accidental parabolics and there is a weak conformal similarity $\phi: \Delta \rightarrow \Delta^*$ with $\phi H \phi^{-1} = G$.

Although Maskit never explicitly states the fact, the methods employed in [6 and 7] produce a linearized group which is without accidental parabolics. This is because the group combination methods employed in “regluing” the Fuchsian and elementary pieces cannot produce accidental parabolics.

Rather than assume this implicit result we shall demonstrate how accidental parabolics may be removed using the methods of [8 and 9]. These techniques will come in handy again later on.

In [9] Maskit defines the signature of a finitely generated Kleinian group with an invariant component (also see [10]). Our group H has a signature (g, K) where g is the genus of the surface Δ/H and K is a 2-complex technically defined in terms of a set of homotopically distinct simple disjoint loops a_1, \dots, a_n on Δ/H , called structure loops, with associated indices α_i , $1 \leq i \leq n$. These loops divide the surface into subsurfaces each of which possesses a signature, as for a Fuchsian group [5], cataloguing the genus of the subsurface and the branch structure of the induced covering.

A loop a_k with $\alpha_k < \infty$ has the property that $a_k^{\alpha_k}$ lifts to a closed curve on the covering surface Δ . The set of all such loops along with the information about

branching completely characterizes the topology of the covering $\pi: \Delta \rightarrow \Delta/H$, i.e., this is the highest branched regular covering for which the loops lift to the indicated powers and the branching is as specified [10 and 11].

A loop a_k with $\alpha_k = \infty$ represents a homotopy class of curves whose corresponding deck transformations have been realized as parabolic elements of the group. Since the stabilizer of a lift of such a loop to Δ is infinite cyclic the loop cannot lift to a closed loop in Δ .

The accidental parabolics in H correspond exactly to those loops a_k with $\alpha_k = \infty$. By definition, an accidental parabolic can be made loxodromic if the group is altered by a weak similarity. Such a map, being an isomorphism of groups and a homeomorphism of invariant components, induces an isomorphism of induced coverings. Consequently, two groups whose signatures differ only in their loops of infinite index determine isomorphic coverings.

Let (g, K_1) be the signature containing the same information as the signature (g, K) except for the loops of ∞ index. The new signature (g, K_1) satisfies the conditions for admissibility; hence, by [9, Theorem 1] there is a group H_1 in C_1 with invariant component Δ_1 and signature (g, K_1) . By the above considerations the covering $\pi_1: \Delta_1 \rightarrow \Delta_1/H_1$ must be isomorphic to the original. In other words there is a weak similarity $w: \Delta_1 \rightarrow \Delta$. As Bers notes in [3] we can assume that w is quasiconformal.

Let μ be the Beltrami coefficient of w . Extend μ to all of \hat{C} by defining it to be 0 in the complement of Δ_1 . The mapping theorem of Ahlfors and Bers [2] asserts the existence of a quasiconformal homeomorphism f_μ solving the Beltrami equation $\partial f_\mu / \partial \bar{z} = \mu(z) \partial f_\mu / \partial z$ on \hat{C} .

As w is a similarity, the H_1 invariance of its distortion may be expressed by the equation $(w \circ h) / \partial \bar{z} = \mu(z)(w \circ h) / \partial z$ for all $h \in H$. From this follows the invariance $\mu \circ h(z) \overline{h'(z)} / h'(z) = \mu(z)$ for the Beltrami coefficient. Extending μ to all \hat{C} as we did allows us to extend the invariance of the Beltrami coefficient to all \hat{C} . A consequence is that the distortion of f_μ is H_1 invariant and thus $f_\mu \circ h \circ f_\mu^{-1} \in \text{Möb}$ for all $h \in H_1$. We conclude that f_μ is a quasiconformal similarity.

The Kleinian group $H_2 = f_\mu H_1 f_\mu^{-1}$ is without accidental parabolics since H_1 is without accidental parabolics and f_μ is a similarity.

Computation of the \bar{z} derivative reveals that the map $\phi = f_\mu \circ w$ is a conformal homeomorphism. It maps Δ into the plane conjugating H into H_2 . Since H_2 is without accidental parabolics the desired end is achieved.

1.5. In the proof of Theorem 1 we will show that the linearization of the orientation-preserving part of the group given by Lemma 1 forces the linearization of the entire group. It may be observed that the generic linearization of the orientation-preserving part does not have this desirable side effect.

PROOF OF THEOREM 1. As Maskit observes in [7] this result is well known when the fundamental group of D is cyclic. If D is simply connected then the Riemann mapping theorem solves the problem. If D has an infinite cyclic fundamental group then it can be mapped conformally onto either the punctured plane, the punctured disc or an annulus. In all three cases the group is linearized by such a mapping.

For the remainder of this argument we shall assume that D does not have a cyclic fundamental group.

We first show that G^+ is finitely generated. Since G^+ acts freely, the projection $p: D \rightarrow S$ is a regular covering, and the induced homomorphism $p_*: \pi_1(D) \rightarrow \pi_1(S)$ is a monomorphism. G^+ acts as the group of deck transformations for the regular covering and is therefore isomorphic to the quotient group $\pi_1(S)/p_*(\pi_1(D))$. Since S is a finite Riemann surface, the group $\pi_1(S)$ is finitely generated; hence, $\pi_1(S)/p_*(\pi_1(D)) \simeq G^+$ is finitely generated.

By Lemma 1 there is a conformal homeomorphism $\phi: D \rightarrow D'$ in \mathbf{C} so that $\Gamma^+ = \phi G^+ \phi^{-1}$ is a Koebe group, Γ^+ does not contain accidental parabolic elements, and D' is contained in the invariant component Δ of $\Omega(\Gamma^+)$.

The anticonformal maps in $\Gamma = \phi G \phi^{-1}$ act a priori only on D' , but this action extends naturally to all of Δ . To see this, notice that since $D' \subset \Delta$, D'/Γ^+ is a subset of Δ/Γ^+ . One hypothesis of the theorem is that D'/Γ^+ is a finite Riemann surface: meaning that it is conformally equivalent to a compact Riemann surface from which a finite set of points have been removed. Consequently, the complement of D'/Γ^+ in Δ/Γ^+ consists of at most a finite set of points in Δ/Γ^+ . This set is the image of $\Delta - D'$ under the projection map. The points in $\Delta - D'$ are therefore isolated from one another. An isolated singularity is removable for an anticonformal homeomorphism; hence, all of the anticonformal maps in Γ extend to homeomorphism of Δ .

Let γ be an anticonformal element of Γ . The proof will be complete if we show that γ is an element of $M\ddot{o}b$.

γ induces an automorphism $\gamma_*: \Gamma^+ \rightarrow \Gamma^+$ by $\gamma_*(g) = \gamma \circ g \circ \gamma^{-1}$. Since Γ^+ does not contain accidental parabolic elements, γ_* and γ_*^{-1} must take parabolic elements to parabolic elements. In order to see that γ_* is type preserving we must also show that it preserves the square of the trace of elliptic elements, or equivalently, that it preserves the minimal geometric generators of finite cyclic subgroups of Γ^+ . The minimal geometric generators of a finite cyclic group are those elements which in normal form look like $z \rightarrow e^{i\sigma}z$ where $|\sigma|$ is minimal over the cyclic group. The fact that minimal generators are preserved is a well-known consequence of γ_* having been induced by a homeomorphism of Δ .

Let $j: \hat{\mathbf{C}} \rightarrow \hat{\mathbf{C}}$ be complex conjugation; that is, $j(z) = \bar{z}$. j induces, by conjugation, an isomorphism j_* taking Γ^+ onto another Koebe group Γ^* with invariant component $\Delta^* = j(\Delta)$. Since j_* is induced by a global homeomorphism it is type preserving.

The map $j \circ \gamma = \phi: \Delta \rightarrow \Delta^*$ is a conformal homeomorphism. It induces an isomorphism $\phi_*: \Gamma^+ \rightarrow \Gamma^*$ by $\phi_*(g) = \phi \circ g \circ \phi^{-1}$. ϕ_* is type preserving, since it may be written as $\phi_* = j_* \circ \gamma_*$, a composition of type-preserving isomorphisms; hence, ϕ is a conformal similarity. Consequently, by [8, Theorem 4] ϕ is a linear fractional transformation, and $\gamma = j \circ \phi$ is an extended linear fractional transformation. \square

2. Signatures.

2.1. In this section we prove an expanded version of Maskit’s Signature Theorem. The reader is referred to the abbreviated definition of a signature given in the proof of Lemma 1. For a detailed explanation of Maskit’s definitions see [9].

Let $\sigma = (g, K)$ be a signature. Corresponding to the complex K is a surface S of genus g upon which is distinguished a finite collection of homotopically distinct loops a_1, \dots, a_n with associated indices $\alpha_1, \dots, \alpha_n$. The a_i divide S into subsurfaces each of which has specified a genus and indexed branch points.

An involution γ of the complex K is defined to be an orientation-reversing period two diffeomorphism of the surface S .

An involution γ is *admissible* if it is homotopic to an involution γ' for which indexed branch points are mapped to branch points of the same index and for each $i, 1 \leq i \leq n$, there is a $j, 1 \leq j \leq n$, so that $\gamma'(a_i) = a_j$ and $\alpha_i = \alpha_j$.

Those familiar with the definitions in [9] will see that an admissible γ induces an actual self-map of the complex K . Note also that a substantially simpler but equivalent combinatorial topological definition of admissible involution could be given directly in terms of K .

An *extended signature* is a collection $\sigma^* = (g, K, \gamma)$ where $\sigma = (g, K)$ is a signature in the usual sense, and γ is an involution of K .

Let G be a group in C_1^* with an invariant component Δ . σ^* is the signature of G if σ is the signature of G^+ , as usual, and the anticonformal elements of G project to an involution whose action on Δ/G^+ is homotopic to the action of γ on S .

An extended signature σ^* is *admissible* if γ is an admissible involution and σ is an admissible signature (in the sense of [9]).

2.2. The methods employed in the last section give a partial generalization of [9, Theorem 1].

THEOREM 4. *Every admissible extended signature is the signature of a group in C_1^* .*

PROOF. Let (g, K, γ) be an admissible extended signature. Maskit's theorem says that there is a group H in C_1 with the signature (g, K) . Let Δ be the invariant component of H . We may suppose that γ has been chosen in its homotopy class to preserve the structure loops and branch points on the surface S associated to the complex K . Since (g, K) is the signature of G^+ there is a natural identification of S with the Riemann surface Δ/G^+ .

From a geometric point of view the conformal structure on S is determined by an equivalence class of Riemannian metrics. Choose such a metric η representing the conformal structure induced on S by identification with the surface Δ/G^+ . The average of η with respect to the involution γ is a new metric $\rho = \frac{1}{2}(\eta + \gamma_*\eta)$ on S . Let S^* be the Riemann surface with the conformal structure defined by the metric ρ . Since γ is an isometry of the metric ρ , it is an anticonformal map of S^* .

If S and S^* are viewed as distinct Riemann surface whose conformal structures are defined on the same ambient topological surfaces then the identity map is a quasiconformal homeomorphism $f: S \rightarrow S^*$ with Beltrami differential μ . We may lift μ a.e. to a Beltrami differential $\tilde{\mu}$ on Δ . Extend $\tilde{\mu}$ to all of \hat{C} by defining it to be identically 0 outside of Δ . By [2] there is a quasiconformal homeomorphism \tilde{f} of \hat{C} solving the Beltrami equation with coefficient $\tilde{\mu}$. It is clear from the invariance properties of \tilde{f} that $\tilde{f}H\tilde{f}^{-1}$ is a Kleinian group H' and that $\tilde{f}(\Delta)/H'$ is holomorphically equivalent to S^* . Set $\tilde{f}(\Delta) = \Delta'$. γ now lifts to an anticonformal homeomorphism of Δ' . Let G' be the extension of H' by the lifts of γ . Notice that while G' is

defined on Δ' , H' acts on all of \hat{C} . We also observe that since f is a similarity H' has signature (g, K) .

Let h be a conformal similarity mapping Δ' onto a domain Δ'' so that $H'' = hH'h^{-1}$ is a Koebe group. Set $G'' = hG'h^{-1}$. From the invariance of the complex K under the involution γ we may infer that the isomorphism of H'' induced by a lift of γ to Δ'' is type preserving. Proceeding as in the proof of Theorem 1 the proof is completed. \square

Although we will not prove the converse here we note that it follows easily from an involution invariant version of the Planarity Theorem which shall appear elsewhere.

2.3. In proving Theorem 4 we have also generalized a uniqueness theorem for Koebe groups [8].

COROLLARY 1. *Let G be a finitely generated extended Kleinian group with an invariant component. There is a unique extended Koebe group G^* , and a unique conformal similarity between G and G^* (unique up to elements of Möb).*

PROOF. All that remains to be proven is the uniqueness. Let Δ be the invariant component of G . Suppose that G_1 and G_2 are extended Koebe groups with invariant components Δ_1 and Δ_2 , conformally similar to G by similarities ϕ_1 and ϕ_2 . By [8] G_1^+ is conjugate to G_2^+ in Möb, and the maps ϕ_1 and ϕ_2 differ by a Möbius transformation. We may therefore assume that $G_1^+ = G_2^+$ and $\phi_2 = \phi_1$.

It remains to be shown that $G_1 = G_2$. This is clear since the orbit spaces Δ/G^+ , Δ_1/G_1^+ and Δ_2/G_2^+ all represent the same Riemann surface S , and the anticonformal parts of the groups all project to the same anticonformal involution γ of S . The set of all lifts of γ to $\Delta_1 = \Delta_2$ gives the unique extension of $G_1^+ = G_2^+$ to $G_1 = G_2$. \square

3. Prime ends.

3.1. In this section we will present a theory of prime ends for domains of arbitrary connectivity and for Riemann surfaces embedded in compact Riemann surfaces of finite genus. For a certain class of prime ends there is a natural generalization of the boundary correspondence theorem due to Carathéodory [4]. These results enable us to pin down the “isolated” pieces of the boundary.

3.2. Let D be a domain in the plane. A *cross cut* in D is a Jordan arc in \bar{D} with interior in D and both endpoints lying in the same connected component of the boundary of D , ∂D . The boundary component containing the endpoints of a cross cut is the *base* for that cross cut. A *chain* in D is a sequence $\{q_i\}$ of cross cuts in D satisfying:

1. There is a single connected component b of ∂D which is the base for each cross cut q_i .
2. $q_i \cap q_j = \emptyset$ for $i \neq j$.
3. q_n separates D into two open sets. One of these contains q_{n-1} , the other contains q_{n+1} .
4. Measured in the spherical metric $\lim_{n \rightarrow \infty} (\text{diam } q_n) = 0$.

Of the two domains determined by q_n one, which we shall call d_n , contains all q_i for $i > n$. Two chains $\{q_i\}$ and $\{q'_i\}$ are equivalent if for all positive integers n the

domains d_n contain all but a finite number of the cross cuts q'_n and the domains d'_n contain all but a finite number of the cross cuts q_n .

A *prime end* is the equivalence class of a chain in D . The impressions of a prime end P is the set $I(P) = \bigcap_{n=1}^{\infty} \bar{d}_n$.

A sequence of points $\{z_i\}_{i=1}^{\infty}$ in D is said to *converge to a prime end* P if given any chain $\{q_n\}_{n=1}^{\infty}$ in P , then each domain d_n contains all but a finite number of the z_i .

A prime end P is *isolated* if there is a chain $\{q_i\}_{i=1}^{\infty}$ in P based at a boundary component b so that for each d_n , $\partial d_n \subset b \cup q_n$. Equivalently, the d_n are simply connected.

3.3.

THEOREM 5. *Let $f: D \rightarrow D'$ be a conformal homeomorphism between plane domains. f induces a one-to-one correspondence between the isolated prime ends of D and the isolated prime ends of D' .*

The correspondence between prime ends P and P' induced by f may be described as follows: a sequence of points $\{z_i\}_{i=1}^{\infty}$ in D converges to P if and only if the sequence $\{f(z_i)\}_{i=1}^{\infty}$ in D' converges to P' .

As with prime ends of simply connected domains one could define isolated prime end partial compactifications of D and D' . Then the theorem asserts that f extends to a homeomorphism of the partially compactified domains.

PROOF. We begin by observing that the homeomorphism f induces a one-to-one correspondence between the boundary components of D and D' . The idea of this correspondence is that a sequence $\{z_i\}_{i=1}^n$ accumulates exactly at a boundary component b of D if and only if the corresponding sequence $\{f(z_i)\}_{i=1}^n$ accumulates exactly at a boundary component b' of D' . This is a consequence of a strong form of the Jordan Curve Theorem which says that two boundary components b_1 and b_2 of D may be separated by a Jordan curve lying entirely in D [13, p. 35]. If the sequence $\{f(z_i)\}_{i=1}^n$ were to accumulate at distinct boundary components b'_1 and b'_2 then we could find a Jordan curve σ in D' separating b'_1 and b'_2 . $f^{-1}(\sigma)$ separates D and infinitely many of the z_i must lie on both sides of the curve; therefore, the sequence $\{z_i\}_{i=1}^n$ could not accumulate at exactly one boundary component of D .

Let b be a boundary component of D which serves as the base for an isolated prime end P , and let B be the component in the complement of D in $\hat{\mathbf{C}}$ containing b . Since b supports a prime end it cannot be a single point boundary component. Choose a Riemann map $\phi: \hat{\mathbf{C}} - B \rightarrow U$, where U denotes the upper half-plane. The prime end P may also be viewed as a prime end of the simply connected domain $\hat{\mathbf{C}} - B$. In this capacity P corresponds under ϕ to a point on $\mathbf{R} \cup \{\infty\} = \hat{\mathbf{R}}$ which, since P is isolated, is not an accumulation point of bounded boundary components of $\phi(D)$. We shall call such a point on a boundary component b which is not an accumulation point of other boundary components *isolated*. Let I denote the set of isolated points. I is clearly open in $\hat{\mathbf{R}}$.

It is easy to see that a boundary component b which is the base for an isolated prime end can never correspond to a single point boundary component b' under f . For suppose this were to happen and let b' be the origin. Then $h(z) = f \circ \phi^{-1}(z)$ is a conformal homeomorphism, taking $\phi(D)$ onto D' , which extends continuously to I

by setting h to be zero there. Applying the reflection principle, h extends to a conformal map H on the domain $\phi(D) \cup I \cup \overline{\phi(D)}$ which is zero on I . That is impossible.

Let b' be the boundary component of D' corresponding to b under f , and let B' be the connected component of $\hat{C} - D'$ containing b' . Since b' is not a single point we choose a Riemann map $\psi: \hat{C} - B' \rightarrow U$. Denote by I' the set of isolated points on the boundary component \hat{R} of $\psi(D')$.

Consider the conformal homeomorphism $g = \psi \circ f \circ \phi^{-1}$ taking $\phi(D)$ onto $\psi(D')$. Again, invoking the reflection principle, we see that g extends to a conformal homeomorphism G mapping $\phi(D) \cup I \cup \overline{\phi(D)}$ onto $\psi(D') \cup I' \cup \overline{\psi(D')}$. In particular, g extends to a homeomorphism between $\phi(D) \cup I$ and $\psi(D') \cup I'$.

By the Carathéodory extension theorem the map ϕ induces a one-to-one correspondence between the prime ends of the simply connected domain $\hat{C} - B$ and the points of \hat{R} . As we observed this establishes a correspondence between the isolated prime ends of D based at b and the points of I . Similarly, ψ induces a correspondence between the isolated prime ends of D' based at b' and the points of I' . Writing $f(z) = \psi^{-1} \circ (\psi \circ f \circ \phi^{-1}) \circ \phi(z)$ we see that a prime end P based at b corresponds to a point in I under ϕ , which corresponds to a point in I' under $\psi \circ f \circ \phi^{-1}$, which in turn corresponds to a prime end based at b' under ψ^{-1} . Since P is based at an arbitrary boundary component b , the theorem is proved. \square

3.4. Let S be a Riemann surface of finite genus g , and let S^* be a closed Riemann surface of genus g so that there is a conformal embedding $i: S \rightarrow S^*$. We would like to extend the notion of prime end in order to describe the boundary behavior of $i(S)$ under conformal mappings. This is accomplished by modifying the definitions in §3.2 as follows:

1. D is a subsurface, as above, of a finite genus Riemann surface of the same genus. In this way closure and boundary are defined as usual.
2. Diameter is measured in the natural metric of constant curvature, 1, -1 or 0 on the ambient surface.

In this setting Theorem 5 generalizes to give

THEOREM 6. *Let S be a Riemann surface of finite genus g , and let $i_1: S \rightarrow S_1$ and $i_2: S \rightarrow S_2$ be conformal embeddings of S into closed Riemann surfaces of genus g . A conformal homeomorphism $f: i_1(S) \rightarrow i_2(S)$ induces a one-to-one correspondence between the isolated prime ends of $i_1(S)$ and the isolated prime ends of $i_2(S)$.*

PROOF. We may suppose that S_1 and S_2 are not planar. Let S_1 and S_2 be the holomorphic universal covers of S_1 and S_2 where $\pi_1: S_1 \rightarrow S_1$ and $\pi_2: S_2 \rightarrow S_2$ are the covering projections. Since all surfaces involved are of the same finite genus it is clear from standard covering theory that f lifts to a map $\tilde{f}: \tilde{S}_1 \rightarrow \tilde{S}_2$ where $\tilde{S}_1 = \pi_1^{-1}(i_1(S))$ and $\tilde{S}_2 = \pi_2^{-1}(i_2(S))$.

For a prime end P of $i_1(S)$ and a chain $\{q_i\}$ representing P , the arcs q_i are, by definition, separating curves in $i_1(S)$. Since the genus is finite, there must be an integer $N > 0$ so that for $i > N$ the domains d_i are planar—equivalently, the d_i lie in a simply connected subset of S_1 . Thus, if P is an isolated prime end, then for i sufficiently large the domains d_i are simply connected. Since S_1 is certainly planar

we may lift all but a finite part of any representative chain in a simply connected subregion. Apply Theorem 5 to the composition of the map \tilde{f} and the projection π_2 on this subregion to get the correspondence. \square

4. Pseudocircle domains.

4.1. Let D be a genus g subsurface of a closed Riemann surface S of genus g . A boundary component b of D is *spacious* if b is the base for an isolated prime end of D . b is a *pseudocircle* (*pseudocurve*) if there is a circle (analytic Jordan curve) C_b and an open disc Δ_b in S bounded by C_b so that:

1. $\bar{D} \cap \Delta_b = \emptyset$ and
2. if P is an isolated prime end of D based at b , then $I(P)$ lies on C_b .

For $g > 0$, a circle is defined in the natural metric of constant curvature on S . The impression of an isolated prime end based on a pseudocircle (pseudocurve) b is an isolated point on C_b . It is clear from Theorem 6 that spaciousness is a property of D which is independent of the ambient surface S . On the other hand, the pseudocircularity of a boundary component depends on the embedding.

A boundary component which is neither spacious nor a single point is a *limit* boundary component.

A surface D is a *pseudocircle* (*pseudocurve*) *domain* if its boundary consists of pseudocircles (pseudocurves), single point boundary components, and limit boundary components—in other words, every spacious boundary component is a pseudocircle (pseudocurve).

We are now equipped to state the main results of this section.

THEOREM 7. *Every plane domain is conformally equivalent to a pseudocircle domain.*

As a special case we get

COROLLARY 2 (STREBEL AND SIBNER). *Every plane domain is conformally equivalent to a domain whose isolated boundary components are either circles or points.*

THEOREM 8. *Let D be a Riemann surface of genus g . Then there is a closed Riemann surface S of genus g , and conformal embedding $i: D \rightarrow S$ so that (1) if $g = 1$ then $i(D)$ is a pseudocircle domain, and (2) if $g > 1$ then $i(D)$ is a pseudocurve domain where any chosen finite set of spacious boundary components correspond to pseudocircles.*

4.2. The argument begins with the evidently necessary

LEMMA 2. *A finite genus Riemann surface D has at most a countable collection of spacious boundary components.*

PROOF. If D is not planar, then we may assume that it is embedded in a closed surface S of the same genus. Let \tilde{S} be the universal cover of S and \tilde{D} the preimage of D in \tilde{S} under the covering projection. It clearly suffices to show that \tilde{D} has countably many spacious boundary components; hence we may argue the lemma for planar D .

Let B be the set of spacious boundary components. For each $b \in B$ let $P(b)$ be an isolated prime end based at b . By [4, p. 171] there is a chain $\{q_n(b)\}_{n=1}^\infty$ in $P(b)$ with the cross cuts lying on concentric circles about a point $Z(b)$. Since $P(b)$ is isolated we may further stipulate that the domains $d_n(b)$ determined by the cross cuts $q_n(b)$ have their boundaries in $q_n(b) \cup b$.

Let $\delta(b)$ be the radius of the circle containing $q_1(b)$. Choose a cross cut $q_k(b)$ which lies on a circle of radius less than $\delta(b)/4$. We call this cross cut $q(b)$ and the associated domain $d(b)$. In this way we have associated to each spacious boundary component b an open set $d(b)$. By the way these sets were chosen they must be disjoint. Since only a countable number of disjoint open sets are possible, the lemma is proven. \square

4.3. Starting with a plane domain D we will construct a planar surface \mathfrak{D} which contains the original and is naturally built up from it. If D is already a pseudocircle domain then \mathfrak{D} is the orbit of D , with its isolated boundary arcs adjoined, under the group generated by reflection in the spacious boundary circles. The reader is referred back to the proof of Theorem 5 in which a similar construction was employed.

Let $B = \{b_i\}_{i=1}^K$, where K is a positive integer or ∞ , be an ordered set of all spacious boundary components of D . Each b_i is contained in a component B_i of $\hat{C} - D$. For each integer n , $1 \leq n \leq K$, choose a Riemann map $\phi_n: \hat{C} - B_n \rightarrow U$. Let I_n be the set of isolated boundary points of $\phi_n(D)$ on $\hat{\mathbf{R}}$. I_n is a countable union of open intervals on $\hat{\mathbf{R}}$. Although it does not affect our argument, it can be shown that since the ϕ_n are unique up to conformal self-maps of the disc, the construction is independent of our choice of Riemann maps.

\mathfrak{D} is constructed by gluing together pieces of the form $D^* = D \cup (\cup_{n=1}^K I_n)$ and $\bar{D}^* = j(D) \cup (\cup_{n=1}^K I_n)$. (Here j denotes complex conjugation.) The pieces are first indexed by elements of a group which corresponds to the group generated by “reflections” in the spacious boundary components.

Let A be a set. A *reduced word* on the elements of A is a finite sequence $\{a_i\}_{i=1}^N$ of elements of the set A with the additional property that $a_j \neq a_{j+1}$ for all $j = 1, \dots, N - 1$. We include the empty sequence as a word.

Let W be the set of reduced words on B , the set of spacious boundary components. In order to avoid confusion, when elements of B are used in this context they shall be represented by Greek letters: β_i being the i th boundary component with the ordering as before. We define a binary operation on W which makes it into a group which we shall call G . The empty sequence will serve as the identity element and so we denote it by 1. Suppose w_1 and w_2 are two reduced words in W , $w_1 = \beta_{j_0}, \dots, \beta_{j_n}$, $w_2 = \beta_{k_0}, \dots, \beta_{k_m}$. Let l be the smallest nonnegative integer for which $\beta_{j_{n-l}} \neq \beta_{k_l}$. Then

$$w_1 \circ w_2 = \begin{cases} 1 & \text{if } n = m = l, \\ \beta_{k_l} \beta_{k_{l+1}} \cdots \beta_{k_m} & \text{if } n = l < m, \\ \beta_{j_0} \cdots \beta_{j_{n-l}} & \text{if } m = l < n, \\ \beta_{j_0} \cdots \beta_{j_{n-l}} \beta_{k_l} \cdots \beta_{k_m} & \text{otherwise.} \end{cases}$$

This corresponds to successively cancelling pairs of adjacent letters, which are the same, in the sequence gotten by following w_1 by w_2 .

The group G is isomorphic to a free product of \mathbf{Z}_2 's indexed by the set B . We will continue, where convenient, to represent elements of G by reduced words in W . For

each $w \in G$ define

$$D_w = \begin{cases} D^* \times \{w\} & \text{if } w = 1 \text{ or if the number of letters in } w \text{ is even,} \\ \bar{D}^* \times \{w\} & \text{otherwise.} \end{cases}$$

D_w is to be thought of as a copy of one of D^* or \bar{D}^* with gluing information appended.

Let $\delta = \bigcup_{w \in G} D_w$. We define an equivalence relation on δ which accomplishes the gluing together of the pieces D_w along the “isolated” boundary curves $I_j \times \{w\}$. Two points (x, w) and (y, w') in δ are equivalent if and only if there is a positive integer n for which $x = y \in I_n$ and $w = \beta_n \circ w'$. We denote the equivalence by \sim .

Let \mathfrak{D} be the set of equivalence classes of δ with respect to this equivalence relation.

4.4.

LEMMA 3. (i) *There is a naturally defined conformal structure on \mathfrak{D} which makes it into a planar Riemann surface.* (ii) *G acts as a group of conformal and anticonformal homeomorphisms of \mathfrak{D} onto itself.*

PROOF. (i) A point d in \mathfrak{D} which has a unique representative in δ of the form (z, w) has a neighborhood homeomorphic to a neighborhood, depending on w , of either z in D or \bar{z} in $j(D)$. This gives a chart about (z, w) . If d has two distinct representatives in δ they must have the form (x, w) and $(x, \beta_n \circ w)$ where $n \in \mathbf{Z}^+$, $x \in I_n$ and $w \in G$. A chart about $(x, w)/\sim$ is gotten by taking the projection onto a neighborhood of x in $w_n = \phi_n(D) \cup I_n \cup j \circ \phi_n(D)$.

To see that \mathfrak{D} is planar we take two Jordan curves α and γ in \mathfrak{D} with finite intersection number and show that this number is even.

Clearly, α and γ lie in a connected open subset of \mathfrak{D} which is contained in the image of a finite number of D_w in δ ; that is, there exist distinct words w_1, \dots, w_n so that $\alpha \cup \gamma \subset (\bigcup_{i=1}^n D_{w_i})/\sim$. This is a consequence of the compactness of $\alpha \cup \gamma$.

It is evident that any subset of \mathfrak{D} , constructed as above out of a finite number of the D_w , may be assembled in the plane. Then $(\bigcup_{i=1}^n D_{w_i})/\sim$ is planar; hence, γ and α must have even intersection number.

(ii) We define the action of G on generators, and points in δ .

$$\beta_n(z, w) = \begin{cases} (\bar{z}, w \circ \beta_n) & \text{for } z \in D \text{ or } j(D), \\ (z, w \circ \beta_n) & \text{for } z \in I_K \text{ for a positive integer } K. \end{cases}$$

This is well defined on \mathfrak{D} since if $(x, w) \sim (x, \beta_K \circ w)$ for $x \in I_K$ then $(x, w \circ \beta_n) \sim (x, \beta_K \circ w \circ \beta_n)$. The way \mathfrak{D} was defined it is evident that β_n acts as an anticonformal involution leaving fixed the points in $I_n \times \{1\}/\sim$. \square

4.5. We now complete the proof of Theorem 7. By Theorem 2 there is a conformal homeomorphism $f: \mathfrak{D} \rightarrow \hat{\mathbf{C}}$ so that $fGf^{-1} = G^*$ is a subgroup of $M\ddot{o}b$. Set $f \circ \beta_n \circ f^{-1} = \beta_n^*$. Normalize so that for some $\beta_k, \infty \in f(D \times \{\beta_k\})$. For a given n , the map β_n^* leaves the set $f(I_n \times \{1\})$ pointwise fixed. β_n^* must therefore be a reflection in some circle C_n in \mathbf{C} , with $f(I_n \times \{1\}) \subset C_n$. Clearly, $f(D \times \{1\})$ will lie in one of the components in the complement of C_n .

Let $i: D \rightarrow \mathfrak{D}$ be the conformal embedding given by $i(z) = (z, 1)/\sim$. The composition $f \circ i: D \rightarrow \mathbb{C}$ induces a one-to-one correspondence between the isolated prime ends of D and those of $f \circ i(D)$. In particular, let b be a spacious boundary component of D , B the component of $\hat{\mathbb{C}} - D$ containing b , and $\phi: \hat{\mathbb{C}} - B \rightarrow U$ the Riemann map selected in §4.3. Then $i \circ \phi^{-1}: \phi(D) \rightarrow \mathfrak{D}$ is a conformal homeomorphism that extends, as in the proof of Theorem 5, to a homeomorphism of $\phi(D) \cup I_n$ onto $(D \times \{1\} \cup I_n \times \{1\})/\sim$ in \mathfrak{D} . The isolated prime ends based at b correspond to isolated points on the boundary $\hat{\mathbf{R}}$ of U ; hence, they also correspond to isolated points on the boundary of $i(D)$ in \mathfrak{D} . Since these points lie in the interior of \mathfrak{D} , this correspondence is preserved by f . \square

4.6. Just a little more work will give Theorem 8.

LEMMA 4. *Let D be a plane domain and H a group of conformal homeomorphisms of D onto itself. Let $i: D \rightarrow \mathfrak{D}$ be the conformal embedding with $i(z) = (z, 1)/\sim$. Then iHi^{-1} extends to a group of conformal homeomorphisms of \mathfrak{D} onto itself.*

PROOF. The action is extended inductively. Once defined on $D \times \{w\}$ or $j(D) \times \{w\}$ it may be extended to an adjacent domain and along their shared boundary by applying the reflection principle. \square

PROOF OF THEOREM 8. We vary the notation from the statement of the theorem to avoid confusion. Let S be a Riemann surface of genus g identified with its embedded image in a closed surface S^* of genus g . We may suppose that $g > 0$. Let Δ be the universal cover of S^* , H the group of covering transformations, and $\Pi: \Delta \rightarrow S^*$ the covering projection. We know by the Uniformization Theorem that Δ is planar. The preimage of S under the covering projection $\Pi^{-1}(S) = D$ is connected, and H restricted to D is a group of conformal self-maps with $D/H = S$.

Let \mathfrak{D} be the planar surface constructed from D as in §4.3. The group G of §4.3, generated by reflections, and the group H , as in Lemma 4, both act on \mathfrak{D} . Let $\mathfrak{G} = \langle G, H \rangle$, the group generated by G and H . It is clear that \mathfrak{G} acts discontinuously on \mathfrak{D} , and the subgroup of \mathfrak{G} that leaves $(D \times \{1\})/\sim$ invariant is H .

By Theorem 2 there is a conformal homeomorphism $f: \mathfrak{D} \rightarrow \hat{\mathbb{C}}$ so that $\mathfrak{G}' = f\mathfrak{G}f^{-1}$ is an extended Kleinian group. Set $fHf^{-1} = H'$. H' has maximal invariant component Ω_1 of its domain of discontinuity containing $f(\mathfrak{D})$. By Theorem 5 it is clear that the limit set of H' lies on exactly one boundary component of $f(\mathfrak{D})$. It follows that Ω_1 is simply connected, and Ω_1/H' is a surface of genus g . Let $\Pi: \Omega_1 \rightarrow \Omega_1/H'$ be the covering projection, and let $i: D \rightarrow \mathfrak{D}$ be the inclusion $i(z) = (z, 1)/\sim$. Then $\Pi \circ f \circ i: D \rightarrow \Omega_1/H'$ projects to a conformal embedding $\phi: S \rightarrow \Omega_1/H'$. The analysis for Theorem 7 shows that $f \circ i(D)$ is a pseudocircle domain; hence, the quotient $\phi(S)$ is a pseudocurve domain.

When $g = 1$, H' is a rank two parabolic group, and we may suppose that its fixed points are at ∞ . $f(D)$ is a pseudocircle domain. Since these are circles with respect to the Euclidean metric on \mathbb{C} the quotient is a pseudocircle domain.

When $g > 1$ we must show that a given finite set $\{b_1, \dots, b_k\}$ of spacious boundary components of S can be made to correspond to pseudocircles under the embedding ϕ . Let $\beta_1^*, \dots, \beta_k^*$ be the elements of \mathfrak{G}' which reflect in the circles

C_{b_1}, \dots, C_{b_k} in \mathbf{C} corresponding to the given boundary components. Let $F = \langle \beta_1^*, \dots, \beta_k^*, H' \rangle$ be the subgroup of \mathcal{G}' generated by H' and these reflections.

F is a finitely generated extended Kleinian group with a maximal invariant component A to its set of discontinuity, which is contained in Ω_1 . By Corollary 1 there is a conformal similarity $h: A \rightarrow \mathbf{C}$ so that $\mathcal{F} = hFh^{-1}$ is an extended Koebe group.

We may suppose that $\mathcal{H} = hH'h^{-1}$ is a Fuchsian group fixing the upper half-plane U . The possibility of accidental parabolic elements occurring may be disposed of by applying Lemma 1. The circles C_{b_1}, \dots, C_{b_k} are circles in the Poincaré metric on U ; hence, they project to hyperbolic circles on U/\mathcal{H} . As above, S embeds in this quotient giving the desired result. \square

COROLLARY 3. *Every topologically finite Riemann surface is conformally equivalent to a subdomain of a surface of the same genus whose boundary consists of circles and points.*

It should be observed that the arguments all go through if H is taken to be a group of conformal and anticonformal self-maps of D . Thus, the analogous theorem for Klein surfaces follows by the same proof.

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