JORDAN DOMAINS AND
THE UNIVERSAL TEICHMÜLLER SPACE

BY

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Abstract. Let \( L \) denote the lower half plane and let \( B(L) \) denote the Banach space of analytic functions \( f \) in \( L \) with \( \| f \|_L < \infty \), where \( \| f \|_L \) is the supremum over \( z \in L \) of the values \( |f(z)|(\Im z)^2 \). The universal Teichmüller space, \( T \), is the subset of \( B(L) \) consisting of the Schwarzian derivatives of conformal mappings of \( L \) which have quasiconformal extensions to the extended plane. We denote by \( J \) the set
\[
\{ S_f : f \text{ is conformal in } L \text{ and } f(L) \text{ is a Jordan domain} \},
\]
which is a subset of \( B(L) \) contained in the Schwarzian space \( S \). In showing \( S - T \neq \emptyset \), Gehring actually proves \( S - J \neq \emptyset \). We give an example which demonstrates that \( J - T \neq \emptyset \).

1. Introduction. If \( D \) is a simply connected domain of hyperbolic type in \( \mathbb{C} \), then the hyperbolic metric in \( D \) is given by
\[
\rho_D(z) = \frac{2|g'(z)|}{1 - |g(z)|^2}, \quad z \in D,
\]
where \( g \) is any conformal mapping of \( D \) onto the unit disk \( \Delta = \{ z : |z| < 1 \} \). If \( f \) is a locally univalent meromorphic function in \( D \), the Schwarzian derivative of \( f \) is given by
\[
S_f = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2
\]
at finite points of \( D \) which are not poles of \( f \). The definition of \( S_f \) is extended to all of \( D \) by means of inversions. We let \( B(D) \) denote the Banach space of Schwarzian derivatives of all such functions \( f \) in a fixed domain \( D \) for which the norm
\[
\| S_f \|_D = \sup_{z \in D} |S_f(z)| \rho_D(z)^{-2}
\]
is finite.

In the case that \( D \) is the lower half plane \( L = \{ z : \Im z < 0 \} \) certain subsets of \( B(L) \) are of special interest. We let
\[
S = \{ S_f : f \text{ is conformal in } L \},
\]
B. B. FLINN

$J = \{ S_f \in S : f(L) \text{ is a Jordan domain} \}$,
$T = \{ S_f \in S : \partial f(L) \text{ is a quasicircle} \}$.

$T$ is called the universal Teichmüller space: it is known that $T$ is open, $S$ is closed and $T = \text{Int}(S)$ (see [1, 3]).

In [4], Gehring shows $S - T \neq \emptyset$, but his proof actually gives $S - \bar{J} \neq \emptyset$. We will show that $J - \bar{T} \neq \emptyset$.

We recall the key result and construction in [4]. Let $a > 0$, and set

$\beta = \{ \pm i e^{(-a+i)t} : t \in (-\infty, \infty) \} \cup \{0, \infty\}$,
$\gamma = \beta \cap \bar{\Delta}$,
$\Omega = \mathbb{C} - \gamma$.

**Theorem 1 (Gehring).** If $a \in (0, 1/8\pi)$, then there exists $\delta = \delta(a) > 0$ such that if $f$ is conformal in $\Omega$ with $\|S_f\|_{\phi} < \delta$, then $\partial f(\Omega)$ is not a quasicircle.

That $S - \bar{T} \neq \emptyset$ is an immediate corollary of Theorem 1 and the transformation law for the Schwarzian derivative, $S_{f\circ g} = (S_f \circ g) g^2 + S_g$, which implies $\|S_{f\circ g} - S_g\|_L = \|S_f\|_{\phi(L)}$. Now let $g$ be a conformal mapping of $L$ onto $\Omega$ and let $h$ be a conformal mapping of $L$ with $\|S_h - S_g\|_L < \delta$; then $f = h \circ g^{-1}$ is a conformal mapping of $\Omega$ with $\|S_f\|_{\phi} < \delta$. By Theorem 1, $\partial f(\Omega) = \partial h(L)$ is not a quasicircle; consequently, $S_h \notin T$ and $S_g \in S - \bar{T}$.

The crux of Gehring's argument is showing if $\|S_f\|_{\phi} \leq \delta$, and if $\Omega_j$, $j = 1, 2$, denotes the component of $\Omega - \beta$ containing $a_j$, where

$\alpha_1 = \{ e^{(-a+i)t} : t \in (0, \infty) \}$,
$\alpha_2 = \{-z : z \in \alpha_1\}$,

then the mappings $f|_{\Omega_j}$ have the same limit as $z$ tends to 0 on $a_j$. Thus $f(\Omega_j)$ is not even a Jordan domain, and it follows that $S - \bar{J} \neq \emptyset$. For what remains, we fix $a \in (0, 1/8\pi)$ and so fix $\gamma$, $\Omega$, $\alpha_1$ and $\alpha_2$. Our aim is to establish the following result.

**Theorem 2.** There exists a Jordan domain $D$ and a constant $d = d(a) > 0$ such that if $f$ is conformal in $D$ and $\|S_f\|_D \leq d$ then $\partial f(D)$ is not a quasicircle.

**Corollary.** $J - \bar{T} \neq \emptyset$.

The Corollary follows from Theorem 2 in the same manner that $S - \bar{T} \neq \emptyset$ follows from Theorem 1.

We construct a candidate Jordan domain $D$ whose boundary consists of a line with a countable number of spiral-like wrinkles in it: the wrinkles are Jordan arcs resembling $\gamma$. We show how to find the appropriate value of $d$ using methods like those in [4], but the proof that $D$ and $d$ satisfy Theorem 2 requires a different argument. In this case, $\partial f(D)$ may be a Jordan curve for $\|S_f\|_D < d$, so we use normal families and a geometric characterization of quasicircles to show that $\partial f(D)$ is not a quasicircle.

2. **Construction of the candidate domain.** It is simplest to describe $D$ by giving its complement. For this, we first construct a sequence of closed Jordan regions $E_m$ with $\bar{\Delta} \supset E_1 \supset E_2 \supset \cdots$ and $\bigcap_{m=1}^\infty E_m = \gamma$, and then attach a translation of each $E_m$ to the closed half plane $H = \{ x + iy : y \leq -1 \} \cup \{ \infty \}$. More precisely, let $\sigma_m = (\pi/8)^m$. 
$\tau_m = e^{-2\pi m}$, and set $E_m = R_m \cup P_m$ where

\[ P_m = \left\{ e^{i\sigma} z : z \in \gamma, -\sigma_m \leq \sigma \leq \sigma_m \right\} \cup \left\{ z : |z| \leq \tau_m \right\}, \]

\[ R_m = \left\{ x + iy : |x| \leq \sin \sigma_m, -1 \leq y \leq -\cos \sigma_m \right\} - \Delta. \]

Let $V$ denote the translation $V(z) = z + 8$ and set

\[ D = \overline{C} - \left( H \cup \bigcup_{m=1}^{\infty} V^m(E_m) \right). \]

To see that $\gamma$ is a Jordan curve, we note that $\gamma_m = \partial D \cap \{ x + iy : -4 \leq x - 8m < 4 \}$ is a half-open Jordan arc from $-4 + 8m$ to $4 + 8m$ for $m = 1, 2, \ldots$, and that $\partial D$ may be written as the union of pairwise disjoint components,

\[ \partial D = \bigcup_{m=1}^{\infty} \gamma_m \cup (-\infty,4) \cup \{ \infty \}. \]

Another way to see that $\partial D$ is a simple, closed curve in $\overline{C}$ is to consider its image under the Möbius transformation $z \rightarrow (z + 2i)^{-1}$.

Throughout the proof of Theorem 2 we will refer to a sequence of domains $D_m$ with $D_m \subset V^{-m}(D)$. Let $A$ denote the open region

\[ A = \{ x + iy : y > 1 \} \cup \{ x + iy : |x| < 4, -1 < y < 1 \} \]

and set

\[ D_m = A - E_m, \quad m = 1, 2, \ldots, \quad D_\infty = A - \gamma. \]

Note that $D_m \subset D_{m+1}$ for all $m$ and $\bigcup_{m=1}^{\infty} D_m = D_\infty = A \cap \emptyset$. For each $m$, including $m = \infty$, and for $j = 1, 2$, we let $D_{m,j}$ denote the component of $D_m - \Gamma$ containing $D_m \cap \alpha_j$ where $\Gamma = \{ iy : y > 1 \}$.

Crucial to our argument is the fact that $\partial D_{m,j}$ is a $K_0$-quasicircle for some fixed $K_0 = K_0(a) \in (1, \infty)$ and for all values of $m$ and $j$. We sketch the proof of this fact: the idea is to find, for each $m$, a $K_0$-quasiconformal mapping $F_m$ of $\overline{C}$ which maps $D_{m,2}$ onto $A \cap \{ z : \Re z < 0 \}$. A similar construction yields a quasiconformal mapping $G_m$ of $\overline{C}$ with $G_m(D_{m,1}) = A \cap \{ z : \Re z > 0 \}$ and $K(G_m) \leq (a + 2/a)^2 K_0$. Both $A \cap \{ z : \Re z < 0 \}$ and $A \cap \{ z : \Re z > 0 \}$ are $K$-quasidisks for some finite $K$, so our claim is established with $K_0 = K \cdot K_0 \cdot (a + 2/a)^2$.

The mappings $F_m$ are compositions of three basic types of mappings, each of which is the identity mapping outside either a disk or a rectangle. First consider the $(a + 2/a)$-quasiconformal mapping $h_a$ of $\overline{C}$ which fixes 0 and $\infty$ and satisfies

\[ h_a(re^{i\theta}) = r^{2a}e^{(\theta - \log r)}, \quad r \in (0, \infty) \]

(see [4]). We use $h_a^{-1}$ to define an $(a + 2/a)$-quasiconformal mapping $h$ of $\overline{C}$ which fixes every point outside $\Delta$; namely,

\[ h(z) = \begin{cases} z, & z \notin \Delta, \\ h_a^{-1}(z), & z \in \Delta. \end{cases} \]

We may take $G_{\infty} = F_{\infty} = h$, and although we must continue for finite $m$ our task is simplified because $h(D_{m,2})$ is bounded by circular arcs and line segments.
The second function we use is a composition of the “quasiconformal foldings” described in [6, Lemma 13]. Briefly, if \( r > 0 \) and \( \theta \in (0, \pi) \), \( f([r, \theta; \varphi]) \) is a \( \pi/(\pi - \theta) \)-quasiconformal mapping of \( \mathbb{C} \) which maps the arc \( \{re^{i\varphi}: \varphi \in [\theta - \pi/2, \theta + \pi/2]\} \) onto the line segment with the same endpoints, and fixes every point outside the disk whose boundary is orthogonal to \( \{z: |z| = r\} \) at those endpoints. More precisely, \( f([r, \theta; \varphi]) \) is the conjugation by a Möbius transformation of the mapping \( F \) which fixes 0 and \( \infty \) and satisfies \( F(re^{i\varphi}) = re^{i\varphi} \) for \( r > 0 \) and \( \varphi \in [-\pi/2, 3\pi/2] \). We define \( g \) to be the continuous function which is linear on each of the intervals \([-\pi/2, \pi/2], [\pi/2, \pi], [\pi, 3\pi/2]\) and satisfies \( g(-\pi/2) = -\pi/2, g(\pi/2) = \pi/2, g(\pi) = -\theta/2 \) and \( g(3\pi/2) = 3\pi/2 \). We set

\[
    f_m = f([r, \theta; \varphi]) \circ f([r, \theta; -\varphi])
\]

where \( r = r_m^{1/\alpha}, \theta = (\pi - 2\sigma_m)/2 \) and \( \varphi = (3\pi + 2\sigma_m)/4 \). Note that \( K(f_m) \leq 4 \) for all \( m \).

The third type of mapping is guaranteed by the lemma below: it fixes every point outside a rectangle and maps a cross-cut of the rectangle onto the segment with the same endpoints.

**Lemma 1.** Let \( 0 < y_1 < y_2, \alpha \in (0, \pi/2), \) and suppose \( f: [x_1, x_2] \to [y_1, y_2] \) is a piecewise differentiable function with \( f(x_1) = f(x_2) = y_1 \) and, for all \( x, x' \in [x_1, x_2] \),

\[
    |f(x) - f(x')| \leq |x - x'| \tan \alpha.
\]

Then there exists a \((1 + k)/(1 - k)\)-quasiconformal mapping \( g \) of \( \mathbb{C} \) which maps every vertical line onto itself, fixes every point outside \( R = \{x + iy: x_1 < x < x_2, 0 < y < y_1 + y_2\} \), and satisfies \( g(x + iy(x)) = x + iy(x) \) for \( x \in (x_1, x_2) \), where

\[
    k = \left(1 - \frac{4}{4 + \tan^2 \alpha} \left(\frac{y_1}{y_2}\right)^2\right)^{1/2}.
\]

**Proof.** An easy check shows the mapping \( g \) defined by

\[
    g(x + iy) = \begin{cases} 
        x + iy, & x + iy \notin R, \\
        x + i \left(\frac{y_1 + y_2}{y_1 + y_2 - f(x)} \right), & x + iy \in R, y \geq f(x), \\
        x + iyy/f(x), & x + iy \in R, y \leq f(x),
    \end{cases}
\]

is a homeomorphism of \( \mathbb{C} \) fixing each point of \( \mathbb{C} - R \), mapping vertical lines onto themselves, and satisfying \( g(x + iy(x)) = x + iy(x) \). One also computes that \( g \) is ACL on \( \mathbb{C} \) with \( |g_z| \leq k |g_\bar{z}| \) almost everywhere.

We define \( g_m \) to be a mapping of the type in Lemma 1 which takes

\[
    \{ire^{i\alpha_m}: 1 - 2 \sin(\sigma_m/2) \leq r \leq 1\} \cup \{ie^{i\theta}: 0 \leq \theta \leq \sigma_m\}
\]

onto the segment from \( i(1 - 2 \sin(\sigma_m/2))e^{i\alpha_m} \) to \( i \). Finally, we set

\[
    F_m = w_m \circ g_m \circ f_m \circ h, \quad G_m = r \circ F_m \circ h^{-1} \circ r \circ h,
\]

where \( r \) denotes reflection in the imaginary axis and \( w_m \) denotes another mapping of the type in Lemma 1. We take \( w_m \) to fix every point outside \([-2, 1] \times [-3, 1] \), to map
horizontal lines onto themselves, and to take the arc of $A \cap \partial(g_m \circ f_m \circ h(D_m, z))$ from $-i - \sin \sigma_m$ to $i$ onto the line segment from $-i$ to $i$. The definitions of $\tau_m$ and $\sigma_m$ give uniform bonds for the dilatations of $g_m$ and $w_m$; therefore, we obtain a uniform bound, $K_0(a)$, for $K(F_m)$. Thus, by the definition of $G_m$, $K(G_m) \leq (a + 2/a)^2 K_0(a)$, and our claim is established.

Remark. Since $\partial D_{m,j}$ is a $K_0(a)$-quasicircle for all $m$ and $j$, we are guaranteed the existence of $d_1 = d_1(a) > 0$ such that the following holds for every $m$ (see Lemma 6 of [4]). If $f$ is a conformal mapping of $D_m$ and if $\|S_f\|_{D_m} \leq d_1$, then for $j = 1, 2$ the mapping $f_j = f\vert_{D_{m,j}}$ has a $K$-quasiconformal extension $g_j$ to $\overline{C}$ with $K \leq (1 - c\|S_f\|_{D_m})^{-1}$ and $c = c(a)$. In this case $f_j$ has a homeomorphic extension to $D_{m,j}$ which we also denote by $f_j$. If $z \in \Gamma$ then $f_j(z) = f(z) = f_2(z)$, and the continuity of $f_1$ and $f_2$ implies $f_1(i) = f_2(i)$ and $f_1(\infty) = f_2(\infty)$. These two common values will be denoted $f(i)$ and $f(\infty)$, respectively.

3. A mapping property of $D_\infty$. Our next step is to show that a conformal mapping of $D_\infty$ with sufficiently small Schwarzian norm is fairly rigid. The first part of the following lemma gives the value of $d$ for Theorem 2 and states that Theorem 1 holds for $D_\infty$ and $d$ in place of $\delta$ and $\delta$. The second part gives an estimate we use in proving $f(D)$ is not a quasidisk when $\|S_f\|_D < d$.

Lemma 2. There exists $d = d(a) \in (0, d_1]$ such that whenever $f$ is a conformal mapping of $D_\infty$ with $\|S_f\|_{D_\infty} \leq d$, then $f(D_\infty)$ is not a Jordan domain. In fact, if $f_j = f\vert_{D_{\infty,j}}$, then $f_j(0) = f_j(0)$.

If, moreover, $f$ fixes $-1, -3$ and $\infty$ then

$$|f(0) - f(i)| \geq 1/3$$

where $f(0)$ denotes the common value of $f_1(0)$ and $f_2(0)$.

Before proving Lemma 2 we state three propositions that are analogues of Lemmas 7, 8 and 9 in [4], respectively. We prove only Proposition 3 since the proofs of the first two propositions are identical to those of the corresponding lemmas.

Proposition 1. For each $\eta > 0$ there exists $K_1 = K_1(\eta) \in (1, \infty)$ such that if $g$ is a sense-preserving $K_1$-quasiconformal mapping of $\overline{C}$ with $g(\infty) = \infty$ and if $z_1$ and $z_2$ are distinct points in $C$, then

$$\left| \frac{g(z) - g(z_2)}{g(z_1) - g(z_2)} \frac{z - z_2}{z_1 - z_2} \right| \leq \eta$$

for all $z \in \overline{C}$ with $|z - z_2| < |z_1 - z_2|$. In particular, if $g$ fixes $z_1$ and $z_2$ then $|g(z) - z| \leq \eta |z_1 - z_2|$.

Proposition 2. There exists $d_2 = d_2(a) \in (0, d_1]$ such that whenever $f$ is a conformal mapping of $D_\infty$ with $\|S_f\|_{D_\infty} \leq d_2$ and $f(\infty) = \infty$, then for $j = 1, 2, f(a_j)$ is a $b$-spiral onto $f_j(0)$ with $b \in (1, 2)$.

Proposition 3. Given $\epsilon > 0$ there exists $d_3 = d_3(a, \epsilon) \in (0, d_1]$ with the following property. If $f$ is a conformal mapping of $D_\infty$ with $\|S_f\|_{D_\infty} \leq d_3$ and if $f$ fixes $-1, 1$ and $\infty$, then $|f_1(0)| < \epsilon$ and $|f_2(0)| < \epsilon$. 
Proof. Let \( \eta = \min(1/8, \epsilon/(5 + \epsilon)) \) and choose \( d_3 \in (0, d_1] \) so that \( (1 - cd_3)^{-2} \leq K_1 \), where \( c = c(a) \) and \( K_1 = K_1(\eta) \) are as in the Remark and Proposition 1.

If \( g_j \) is a \( K_1^{1/2} \)-quasiconformal extension of \( f_j \) to \( \overline{C} \) then \( g_2^{-1} \circ g_1 \) is \( K_1 \)-quasiconformal in \( \overline{C} \) and fixes each point of \( \overline{\Gamma} \). In particular, \( g_2^{-1} \circ g_1 \) fixes \( i, 3i \) and \( \infty \). From Proposition 1 we obtain, with \( z_1 = i \) and \( z_2 = 3i \),

\[
|g_2^{-1}(1) - 1| = |g_2^{-1} \circ g_1(1) - 1| \leq 2\eta \leq 1/4
\]

and hence

\[
\frac{1 - g_2^{-1}(1)}{-1 - g_2^{-1}(1)} \leq \eta(1 - \eta)^{-1} < 1.
\]

Since \( g_2 \) fixes \( -1 \) and \( \infty \), another application of Proposition 1, with \( z_1 = -1 \) and \( z_2 = g_2^{-1}(1) \), yields

\[
\left| \frac{g_2(1) - 1}{2} - \frac{1 - g_2^{-1}(1)}{-1 - g_2^{-1}(1)} \right| \leq \eta,
\]

and we conclude that

\[
|g_2(1) - 1| \leq 2\eta(1 + (1 - \eta)^{-1}).
\]

Finally, we consider the mapping

\[
h(z) = \frac{2g_2(z) - g_2(1) + 1}{g_2(1) + 1},
\]

which is \( K_1 \)-quasiconformal in \( \overline{C} \) and fixes \( -1, 1 \) and \( \infty \). Proposition 1 implies \( |h(0)| \leq 2\eta \), so by (3.2) and our choice of \( \eta \) we find

\[
|f_2(0)| = |g_2(0)| < 5\eta(1 - \eta)^{-1} \leq \epsilon.
\]

Similarly, we find \( |f_3(0)| < \epsilon. \)

Proof of Lemma 2. Let \( d = \min(d_2(a), d_3(a, 1/5)) \) and suppose \( f \) is a conformal mapping of \( D_\infty \) with \( \|f\|_{D_\infty} \leq d \). To prove \( f(0) = f_2(0) \), one argues in exactly the same way as in the proof of Theorem 2 in [4], using Propositions 1, 2 and 3 in place of Lemmas 7, 8 and 9.

Now suppose \( d \) and \( f \) are as above and suppose \( f \) fixes \( -1, -3 \) and \( \infty \). Let \( f(0) \) denote the common value of \( f_1(0) \) and \( f_2(0) \). Choose \( \eta \) as in Proposition 3 with \( \epsilon = 1/5 \) and let \( g_2 \) denote a \( K_1(\eta)^{1/2} \)-quasiconformal extension of \( f_2 \). Since \( g_2 = f_2 \) on \( \overline{D_\infty} \), Proposition 1 implies

\[
\left| \frac{f(0) - f(i)}{f_2(-i) - f(i)} - \frac{1}{2} \right| \leq \eta < \frac{1}{6};
\]

therefore,

\[
|f(i) - f(0)| \geq \frac{1}{3}|f_2(-i) - f(i)|.
\]

Because \( g_2 \) fixes \( -1, -3 \) and \( \infty \), two more applications of Proposition 1 yield

\[
|f(i) - i| \leq 2\eta, \quad |f_2(-i) + i| \leq 2\eta.
\]

Then \( |f_2(-i) - f(i)| \geq 2 - 4\eta > 1 \), and (3.1) now follows from (3.3).
4. Proof of Theorem 2. Let $d$ be as in Lemma 2 and let $f$ be a conformal mapping of $D$ with $\|S_f\|_D \leq d$. We may assume $\partial f(D)$ is a Jordan curve, and we will denote the homeomorphic extension of $f$ to $\overline{D}$ by $\tilde{f}$. As well, we may further assume $f(\infty) = \infty$, so that $\infty \in \partial f(D)$. With these assumptions, in order to show $\partial f(D)$ is not a quasicircle we need only exhibit for each $\lambda > 0$ three points $z_1, z_2, z_3$ on $\partial D - \{\infty\}$ such that $z_2$ separates $z_1$ and $z_3$ and such that

$$\|f(z_2) - f(z_3)\| > \lambda \|f(z_1) - f(z_3)\|$$

(see [1]).

Fix $\lambda > 0$. We will show that for some $m$, the following triple on $\partial D$ satisfies (4.1):

$$z_1 = V'(\tau_m), \quad z_2 = V'(i), \quad z_3 = V'(\tau_m),$$

where $V'(z) = z + 8m$, as before. For this we construct a sequence of conformal mappings $f_m$ from the restrictions of $f$ to the $V'(D_m)$. We show that the $f_m$ converge to a mapping of $D_\infty$ which, by Lemma 2, nearly preserves the ratio $|z_2 - z_3|/|z_1 - z_3|$. This fact and the nature of the convergence imply (4.1) for the triple (4.2) when $m$ is large.

For each $m$, we choose $U_m$ to be the Möbius transformation such that

$$f_m(z) = U_m \circ f \circ V'(z), \quad z \in D_m,$$

fixes $-1, -3$ and $\infty$. Then $f_m$ is a conformal mapping of $D_m$ onto a Jordan domain, and

$$\|S_{f_m}\|_{D_m} = \|S_f \circ V'(z)\|_{D_m} = \|S_f \|_{V'(D_m)} \approx \|S_f\|_{D}.$$

Because $U_m$ fixes $\infty$ and preserves cross-ratios, proving (4.1) for the triple (4.2) is equivalent to showing

$$|f_m(i) - f_m(\tau_m)| > \lambda |f_m(-\tau_m) - f_m(\tau_m)|.$$  

(4.4)

By (4.3), for some fixed $K'$ and all $m$, $f_m|_{D_{m,j}}$ has a $K'$-quasiconformal extension $g_{m,j}$ to $\overline{C}$. The family $\{g_{m,2}\}_{m=1}^{\infty}$ fixes $-1, -3$ and $\infty$ while the family $\{g_{m,1}^{-1} \circ g_{m,2}\}_{m=1}^{\infty}$ fixes each point of $\Gamma$. Thus both families are normal families, and we conclude that there exists an increasing sequence of integers $m(k)$ such that both $\{g_{m(k),1}\}_{k}^{\infty}$ and $\{g_{m(k),2}\}_{k}^{\infty}$ converge uniformly in the chordal metric on $\overline{C}$ to $K'$-quasiconformal mappings [5]. The limit mappings will be denoted $g_{\infty,1}$ and $g_{\infty,2}$, respectively. The mappings $f_{m(k)}$ of $D_{m(k)}$ likewise converge uniformly on compact subsets of $D_\infty$ to the conformal mapping $f_\infty$ of $D_\infty$ satisfying

$$f_{\infty}|(D_{\infty,j} \cup \Gamma) = g_{\infty,j}, \quad j = 1, 2.$$

We claim that $f_{\infty}$ satisfies the hypotheses of both parts of Lemma 2. Clearly $f_{\infty}$ fixes $-1$ and $-3$; moreover, (4.3) implies $\|S_{f_{\infty}}\|_{D_{\infty}} \leq d$ since $S_{f_{m(k)}}(z)$ and $\rho_{D_{m(k)}}(z)^{-1}$ converge to $S_{f_{\infty}}(z)$ and $\rho_{D_{\infty}}(z)^{-1}$ as $k$ tends to $\infty$, for $z \in D_{\infty}$ [2]. Consequently, the Remark applies to $f_{\infty}$, and we deduce that $f_{\infty}(\infty) = \infty$ and that Lemma 2 is applicable. According to (3.1) we may choose $\mu \in (0, 1/3)$ so that

$$|f_{\infty}(i) - f_{\infty}(0)| - \mu > \lambda \mu.$$  

(4.5)
Next we appeal to the equicontinuity and uniform convergence of the extensions $g_{m(k),j}$. We may first choose $s \in (0, 1)$ so that for $z, w \in \overline{\Delta}$ with $|z - w| < s$,

$$|g_{m(k),j}(z) - g_{m(k),j}(w)| < \frac{\mu}{4}$$

for all $k$ and for $j = 1, 2$. We may then choose $k$ large enough so that $r_{m(k)} < s$ and

$$|g_{m(k),j}(z) - g_{\infty,j}(z)| < \frac{\mu}{4}, \quad j = 1, 2,$$

whenever $z \in \overline{\Delta}$.

Because $g_{m(k),j} = f_{m(k)}$ on $D_{m(k),j}$, (4.4) follows with $m = m(k)$ from (4.5) and the inclusions

$$i, \tau_{m(k)} \in \overline{\Delta} \cap D_{m(k),1}, \quad i, -\tau_{m(k)} \in \overline{\Delta} \cap D_{m(k),2}.$$

As we noted, (4.4) is equivalent to (4.1) for the triple (4.2). Since $\lambda$ was an arbitrary positive number, we conclude that the Jordan curve $\partial f(D)$ is not a $K$-quasicircle for any $K$.

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REFERENCES


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