ON THE PROPER HOLOMORPHIC EQUIVALENCE
FOR A CLASS OF PSEUDOCONVEX DOMAINS

BY

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Abstract. A complete and explicit description of the holomorphic proper mappings between weakly pseudoconvex domains of the class $\Delta_p$ (see (•) below) is given.

The study of proper holomorphic mappings (i.e. holomorphic mappings which antitransform compact sets into compact sets),

$$F: D_1 \to D_2, \quad D_1, D_2 \subset C^n \ (n \geq 1),$$

between bounded pseudoconvex domains with smooth boundary has had, recently, a considerable impulse and has been applied, mainly, to investigate

(a) under which hypothesis there is no obstruction for the existence of such an $F$,

(b) the regularity up to the boundary of (1),

(c) for which domains proper means, in effect, biholomorphic.

The first class of problems has been taken into consideration, for instance, in [8, 5, 4] and a typical result is, for example, that a strictly pseudoconvex domain cannot be mapped properly onto a weakly pseudoconvex domain.

About (b), see, for instance, [4, 2, 3] and the bibliography there included.

The property enunciated in (c) has been proved in the case $D_1 = D_2 =$ unit ball (see [1 and 9]), when $D_1$ and $D_2$ are strictly pseudoconvex domains with, in addition, $D_2$ simply connected (see [7 and 4]), and in the case $D_1 = D_2$ and strictly pseudoconvex (see [7]). The aim of this note is to look at questions of type (a) and (c), above, posed for the following class of weakly pseudoconvex domains,

$$(\cdot) \quad \Delta_p = \left\{ (z_1, \ldots, z_n) : |z_1|^{2p_1} + \cdots + |z_n|^{2p_n} < 1 \right\}, \quad (p_1, \ldots, p_n) \in (\mathbb{Z}^+)^n.$$

Precisely, we prove

Theorem. In order for a proper holomorphic mapping from $\Delta_p$ onto $\Delta_q$ to exist it is necessary and sufficient that

$$(\star \star) \quad \frac{p}{q} = \left( \frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n} \right) \in (\mathbb{Z}^+)^n.$$

Furthermore, assuming (\star \star), the only proper holomorphic mapping between $\Delta_p$ and $\Delta_q$ is, up to biholomorphisms of $\Delta_q$, $F(z_1, \ldots, z_n) = (z_1^{p_1/q_1}, \ldots, z_n^{p_n/q_n})$.

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COROLLARY. Every proper self-mapping of $\Delta_p$ is a biholomorphism.

1. In this section we find some holomorphic automorphisms of $\Delta_p$; in §2, as a corollary of the theorem there stated, it will be proved that they form the whole automorphism group of $\Delta_p$.

Let us suppose $\Delta_p$, in order to make easier the writing of what follows, to be of the form

$$\Delta_p = \{ (z_1, \ldots, z_n) : |z_1|^2 + \cdots + |z_k|^2 + |z_{k+1}|^{2p_{k+1}} + \cdots + |z_n|^{2p_n} < 1 \}, \quad k \geq 1,$$

where the $p_i$'s, for $i = k + 1, \ldots, n$, are assumed bigger than 1. Furthermore, let us denote

$$B_k(0, 1) = B_k = \{ (z_1, \ldots, z_k) : |z_1|^2 + \cdots + |z_k|^2 < 1 \}$$

and $z^k = (z_1, \ldots, z_k)$, $z = (z^k, z_{k+1}, \ldots, z_n)$. For a fixed point $z_0 \in B_k$, let us consider a holomorphic automorphism of $B_k$, say $T_{z_0}$, which carries the point $z_0$ into the origin of $C^k$. Then we have

**Proposition 1.1.** Every holomorphic mapping $S: \Delta_p \to C^n$ of the form

$$S(z) = \left( T_{z_0}(z^k), c_{k+1}z_{k+1} \left[ \frac{1 - |z_0^k|^2}{(1 - \langle z^k, z_0^k \rangle)^2} \right]^{1/2p_{k+1}}, \ldots, c_nz_n \left[ \frac{1 - |z_0^n|^2}{(1 - \langle z_n, z_0^n \rangle)^2} \right]^{1/2p_n} \right)$$

where $\langle , \rangle$ is the canonical hermitian scalar product in $C^k$ and $|c_j| = 1$ for $j = k + 1, \ldots, n$, is for every choice of $z_0 \in B_k$ an automorphism of $\Delta_p$.

**Proof.** To prove the statement it is sufficient, after having observed that $S$ is well defined as $|z_0^k| < 1$, to show that

$$S(z) \in \Delta_p \text{ if } z \in \Delta_p, \quad \text{and} \quad S(z) \in \partial \Delta_p \text{ iff } z \in \partial \Delta_p.$$

Because of the special form of $S_{k+1}, \ldots, S_n$ (i.e. the last $n - k$ components of $S$) and the fact that $T_{z_0}$ is, by assumption, an automorphism of the ball in $C^k$, we have

$$|S_1(z)|^2 + \cdots + |S_k(z)|^2 + |S_{k+1}(z)|^{2p_{k+1}} + \cdots + |S_n(z)|^{2p_n}$$

$$= |T_{z_0}|^2 + \frac{1 - |z_0^k|^2}{(1 - \langle z^k, z_0^k \rangle)^2} (|z_{k+1}|^{2p_{k+1}} + \cdots + |z_n|^{2p_n}) < 1$$

where the last inequality is deduced from (see [10, p. 26])

$$|T_{z_0}|^2 = 1 - \frac{(1 - |z_0^k|^2)(1 - |z^k|^2)}{|1 - \langle z^k, z_0^k \rangle|^2}. $$
Remark (a). For every point of \( \Delta_p \) lying in the linear subspace \( z_{k-1} = \cdots = z_n = 0 \) there exists an automorphism of \( \Delta_p \) which carries this point into the origin, and obviously vice-versa.

Remark (b). The hypothesis that \( p_1, \ldots, p_k = 1 \) (that is, the first \( k \) are ones, see (\( \ast \))) is completely unessential. It is sufficient to make a formal change of indices to get a general statement of the proposition. In the case \( k = 0 \) the biholomorphisms taken into consideration will be, obviously, \( S(z) = (c_1z_1, \ldots, c_nz_n) \) with \( |c_i| = 1, \ i = 1, \ldots, n \).

2. This section is mainly devoted to the proof of the theorem stated in the introduction. We shall start by observing a geometric property which has to be satisfied by a proper holomorphic mapping between domains of the class (\( \ast \)). Then, let \( F = (F_1, \ldots, F_n) : \Delta_p \to \Delta_q \) and let us denote
\[
A^*_p = \{ i \in \{ p_1, \ldots, p_n \} \text{ s.t. } i > 1 \}
\]
and \( A^*_q \) the analogous set for \( q \in (\mathbb{Z}^+)^n \) we have

**Proposition 2.1.** There exists a function \( j: A^*_q \to A^*_p \) such that
\[
F^{-1}\left( \{ z_i = 0 \} \cap \Delta_p \right) = \{ z_i = 0 \} \cap \Delta_q, \quad i \in A^*_q.
\]
Furthermore, such a function is 1-1.

**Proof.** \( F \) has an analytic extension to a neighbourhood of \( \Delta_p \) [3 Remark B, p. 112]; moreover [3, Theorem 2], calling \( H_1(p) = \prod_{i \in \{ z_i = 0 \} \cap \Delta_p} \), we have
\[
\partial \left( F^{-1}(H^*_p(q)) \right) \subseteq \partial H^*_p(p)
\]
where \( \partial \) means “boundary of”. This implies \( F^{-1}(H^*_q(q)) \subseteq H^*_p(p) \) or, in other words,

\[
\{ z \in \Delta_p : F_i(z) = 0 \} \subseteq H^*_p(p), \quad i \in A^*_q.
\]

The relation (2.2) implies (see for instance [6, p. 60]) that for every \( i \in A^*_q \) there exists an index belonging to \( A^*_p \), say \( k \), such that on \( \Delta_p \) we have \( F_i |_{z_k = 0} \equiv 0 \). Fix, once and for all, the correspondence \( i \to k \) and call it \( j \). The function \( j \) is injective, otherwise it might happen for \( i_1 \neq i_2 \),

\[
F_i |_{z_k = 0} = F_{i_2} |_{z_k = 0} = 0
\]
and hence, since \( F \) is a proper mapping between \( \Delta_p \cap \{ z_k = 0 \} \) and its image, this is impossible because this last set is contained in a coordinate subspace of dimension \( n - 2 \).

**Remark.** The injectivity of the function \( j \) says that a necessary condition for the proper equivalence between \( \Delta_p \) and \( \Delta_q \) is that \( \#(A^*_p) \geq \#(A^*_q) \).

**Proof of the Theorem.** The proof is divided into two parts: in the first, we prove the Theorem in the case \( q_1 = q_2 = \cdots = q_n = 1 \); in the second part, we prove the general statement exploiting the result in the particular case. Suppose then, that
$\Delta_q = B = \text{unit ball}$. In this situation, the only fact to prove is that the proper mappings $F: \Delta_p \to B$ are, up to automorphisms of the ball, the mapping

$$\left( z_1, \ldots, z_n \right) \mapsto \left( z_1^{p_1}, \ldots, z_n^{p_n} \right).$$

Consider a convenient open neighbourhood $U$ of

$$z^0 = \left( z_1^0, \ldots, z_n^0 \right) \in \partial \Delta_p, \quad z_1^0, \ldots, z_n^0 \neq 0,$$

and the mapping

$$\left( z_1, \ldots, z_n \right) \mapsto \left( z_1^{p_1}, \ldots, z_n^{p_n} \right).$$

For suitable $U$, (2.3) is invertible. Denote by $\Psi: V \to U$ a fixed local inverse of (2.3). On the other hand (shrinking $U$ if necessary), $F$ is invertible on $U$, $z^0$ being a strictly pseudoconvex point of $\partial \Delta_p$, so the composite mapping,

$$G = F \circ \Psi: V \to F(U)$$

is holomorphic, 1-1, and maps, by construction,

$$V \cap \partial B \to F(U) \cap \partial B.$$

Hence [1], $G$ is the restriction to $V$ of an automorphism $\Theta$ of the ball. This means

$$F(z_1, \ldots, z_n) = \Theta(z_1^{p_1}, z_2^{p_2}, \ldots, z_n^{p_n}),$$

and the statement of the theorem then follows in this particular case.

Consider, now, the general case. First of all we can assume, without loss of generality and combining Propositions 2.1 and 1.1, that

$$F(0, \ldots, 0) = (0, \ldots, 0).$$

In fact, by Proposition 2.1, we have that

$$F_i(0, \ldots, 0) = 0, \quad i \in A_q^*,$$

and hence Proposition 1.1 allows us to consider (2.4) satisfied.

Now take into consideration the following holomorphic mapping:

$$E = \left( F_1^{q_1}, \ldots, F_n^{q_n} \right): \Delta_p \to B.$$

The first part of the statement, already proved, implies then, because of 2.4, that

$$E(z_1, \ldots, z_n) = A \left( \begin{array}{c} z_1^{q_1} \\ \vdots \\ z_n^{q_n} \end{array} \right)$$

where $A$ is a unitary matrix, $A = (a_{ij})$, and hence

$$\delta_{ik} = \sum_s a_{is} \bar{a}_{ks}.$$

Let $i \in A_q^*$: by Proposition 2.1 there exists $j(i) \in A_p^*$ such that $F_i|_{z_{j(i)} = 0} \equiv 0$ which by (2.5) forces us to have

$$a_{ir} = 0, \quad r \neq j(i), \quad i \in A_q^*.$$
The relation (2.5) written down for \( i \in A_q^s \), taking into account (2.7), actually gives

\[
|a_{j(j')}|^2 = 1, \quad a_{k(j')} = 0, \quad k \neq i, i \in A_q^s.
\]

Let us now consider the square matrix (\( j \) is injective)

\[
B = (a_{rs}), \quad r, s \in A_q^s.
\]

\( B \), by virtue of (2.8), is unitary as

\[
\sum_{s \in j(A_q^s)} a_{is} \bar{a}_{ks} = \sum_{s=1}^n a_{is} \bar{a}_{ks} = \delta_{ik}, \quad i, k \in A_q^s.
\]

This completes the proof, observing that by composing \( F \) with the automorphism of \( \Delta_q \), which operates on the \( z_i \) variables, \( i \notin A_q^s \), as the linear unitary mapping carried by \( B^{-1} \), and on the other variables by multiplication by \( a_{j(i)}^{-1} \), \( F \) can be seen to satisfy

\[
(F_1, \ldots, F_n) = (z_1, \ldots, z_n).
\]

A remark of some interest is that, having used in the proof only the automorphisms given by Proposition 1.1, we get

**Corollary.** The automorphism group of \( \Delta_p \) is the set given in Proposition 1.1 (see also [11]).

During the submission of this note, the author came to know about a generalisation (to any smooth pseudoconvex domain with real analytic boundary) of the corollary of the theorem which, therefore, he includes in the bibliography [12].

**References**