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WHERE THE CONTINUOUS FUNCTIONS WITHOUT UNILATERAL DERIVATIVES ARE TYPICAL

BY

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ABSTRACT. An alternative proof of the existence of a Besicovitch function (i.e. a continuous function which has nowhere a unilateral derivative) is presented. The method consists in showing the residuality of Besicovitch functions in special subspaces of the Banach space of all continuous functions on [0, 1] and yields Besicovitch functions with additional properties of Morse or Hölder type. A way how to obtain functions with a similar behavior on normed linear spaces is briefly mentioned.

Introduction. We shall be interested in continuous functions on [0, 1] which have nowhere a unilaterial derivative (finite or infinite). In the sequel, we shall term them briefly Besicovitch functions. The first example is due to A. S. Besicovitch [**2**]. The studies of E. D. Pepper [**7**], A. N. Singh [**9**], R. L. Jeffery [**4**, pp. 172–181], and K. M. Garg [**3**] investigate further properties of the Besicovitch's function and try to clarify its rather complicated construction. Further examples of Besicovitch functions are given by A. P. Morse [**6**] and A. N. Singh [**10**].

In 1932, S. Saks [**8**] proved that the collection of all Besicovitch functions is of the first category in the space $\mathscr{C}$ of all continuous functions on [0, 1]. It seemed that the category method is not suitable to obtain existence results concerning Besicovitch functions. The purpose of the present paper is to defend the category approach in this question. In contrast to classical results saying that nowhere differentiable functions are residual in $\mathscr{C}$ (S. Banach [**1**], S. Mazurkiewicz [**5**]), here a certain difficulty consists in finding a convenient nontrivial space in which Besicovitch functions are typical. The construction part of the proof does not use the knowledge of the existence of a Besicovitch function and deals only with a finite partition of the interval [0, 1] (no "Cantor sets" are built). The method allows to construct Besicovitch functions with additional properties. Some applications of this kind are suggested.

1. The initial space. Let $f\colon [0,1] \to \mathbf{R}$ be a function. We denote

$$\|f\| = \sup\{|f(x)| : x \in [0,1]\},$$

$$\operatorname{Lip} f = \sup\left\{\left|\frac{f(x)-f(y)}{x-y}\right| : x, y \in [0,1], x \neq y\right\}$$

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and for $z \in [0,1)$

$$D_+f(x) = \liminf_{x \to z_+} \frac{f(x) - f(z)}{x - z}.$$

The Banach space of all continuous functions on $[0,1]$ with the norm $\| \ \|$ will be denoted by $\mathscr{C}$. We shall use the symbol K for the compact metric space of all functions $f \in \mathscr{C}$ with $f(0) = f(1) = 0$, $\operatorname{Lip} f \leqslant 1$. We introduce the "initial space" E as the collection of all sequences $u = \{u_n\} \in K^{\mathbf{N}}$ which satisfy

(e1) $u_1 \geqslant u_2 \geqslant \cdots \geqslant 0$,

(e2) for every interval $I \subset [0,1]$ and $n \in \mathbf{N}$, if u_{n+1} is strictly positive on I, then u_n is constant on I.

Obviously E is a closed subspace of the compact metrizable space $K^{\mathbf{N}}$ endowed by the product topology.

1. LEMMA. *If $u \in E$, then $\|u_n\| < 1/n$ for every $n \in \mathbf{N}$.*

PROOF. We have $u_n(0) = u_n(1) = 0$. Suppose there is $x_0 \in [0,1)$ such that $u_n(x_0) \geqslant 1/n$. Then there is $x_1 \in [0, x_0)$ such that $u_n(x_1) = 0$ and u_n is strictly positive on $(x_1, x_0]$. Since $\operatorname{Lip} u_n \leqslant 1$ we deduce $x_1 < x_0 - 1/n$ and (e2) says that u_{n-1} is constant on $[x_1, x_0]$, thus (using (e2)) $u_{n-1}(x_1) = u_{n-1}(x_0) \geqslant u_n(x_0) \geqslant 1/n$. By induction we can find points x_i, $i = 0, \ldots, n-1$ such that $x_0 > x_1 > \cdots > x_{n-1} \geqslant 0$, $x_i \leqslant x_0 - i/n$ and $u_{n-i}(x_i) \geqslant 1/n$. In particular, $x_{n-1} \leqslant x_0 - (n-1)/n < 1/n$ and $u_1(x_{n-1}) \geqslant 1/n$, $u_1(0) = 0$, so $\operatorname{Lip} u_1 > 1$, which is a contradiction.

2. Special images of the initial space. Let $\varphi\colon [0,1] \to \mathbf{R}$ be a continuous increasing function, $\varphi(0) = 0$. We introduce the operator $A_\varphi\colon E \to \mathscr{C}$ by the formula $A_\varphi u = \sum_{n=1}^{\infty} (-1)^{n+1} \varphi \circ u_n$.

2. PROPOSITION. *A_φ is continuous.*

PROOF. Fix $\varepsilon > 0$. Find $m \in \mathbf{N}$ and $\delta > 0$ such that $\varphi(1/m) < \varepsilon$ and if $s, t \in [0,1]$, $|s - t| < \delta$, then $|\varphi(s) - \varphi(t)| < \varepsilon/m$. Suppose $u, v \in E$, $\|u_i - v_i\| \leqslant \delta$ for every $i = 1, \ldots, m$. By (e1),

$$\varphi \circ u_1 \geqslant \varphi \circ u_2 \geqslant \cdots \geqslant 0, \qquad \varphi \circ v_1 \geqslant \varphi \circ v_2 \geqslant \cdots \geqslant 0,$$

according to Lemma 1

$$\|\varphi \circ u_{m+1}\| \leqslant \varphi(1/(m+1)) < \varepsilon, \qquad \|\varphi \circ v_{m+1}\| \leqslant \varphi(1/(m+1)) < \varepsilon.$$

Hence

$$\begin{aligned} \|A_\varphi u - A_\varphi v\| &\leqslant \left\| \sum_{i=1}^{m} (-1)^{i+1} (\varphi \circ u_i - \varphi \circ v_i) \right\| \\ &\quad + \left\| \sum_{i=m+1}^{\infty} (-1)^{i+1} \varphi \circ u_i \right\| + \left\| \sum_{i=m+1}^{\infty} (-1)^{i+1} \varphi \circ v_i \right\| \\ &\leqslant \sum_{i=1}^{m} \frac{\varepsilon}{m} + \|\varphi \circ u_{m+1}\| + \|\varphi \circ v_{m+1}\| \leqslant 3\varepsilon. \end{aligned}$$

3. COROLLARY. *$A_\varphi(E)$ is a compact subspace of $\mathscr{C}$.*

PROOF. By Proposition 2, $A_\varphi(E)$ is a continuous image of the compact space E.

4. PROPOSITION. *Suppose that φ has a finite derivative at every point of $(0,1)$ and $D_+\varphi(0) < +\infty$. If $f \in A_\varphi(E)$, then f does not have an infinite right derivative at any point.*

PROOF. Let $u \in E$, $A_\varphi u = f$. Fix $z \in [0,1)$. We shall distinguish three cases.

(1) Assume $u_n(z) > 0$ for every $n \in \mathbf{N}$. Put

$$b_n = \inf\{x \in (z,1] : u_n(x) = 0\}.$$

Then by (e1), $\{b_n\}$ is a nonincreasing sequence. From (e2) it follows that every $u_{n-1},\dots,u_1$ is constant on $[z, b_n]$. For every $n \in \mathbf{N}$ we have

$$f(z) \geqslant \sum_{i=1}^{2n} (-1)^{i+1}\varphi(u_i(z)) = \sum_{i=1}^{2n} (-1)^{i+1}\varphi(u_i(b_{2n+1})) = f(b_{2n+1})$$

and similarly $f(z) \leqslant b_{2n}$. Hence if $\lim b_n = z$, then $D_+f(z) < +\infty$, $D_+(-f)(z) < +\infty$. If $\lim b_n > z$, then u_n are constant on $[z, \lim b_n]$ and thus $f'_+(z) = 0$.

(2) Assume $u_1(z),\dots,u_{m-1}(z) > 0$, $u_m(z) = u_{m+1}(z) = \cdots = 0$, $u_1,\dots,u_{m-1}$ are constant on a right neighborhood I of z. Then for every $x \in I$ we have $|f(x) - f(z)| \leqslant \varphi(u_m(x)) \leqslant \varphi(x - z)$ because $u_m(x) = u_m(x) - u_m(z) \leqslant (\operatorname{Lip} u_m)(x - z) \leqslant x - z$. Hence

$$\liminf_{x \to z_+} \frac{|f(x) - f(z)|}{x - z} \leqslant \liminf_{x \to z_+} \frac{\varphi(x - z)}{x - z} < +\infty.$$

(3) Assume $u_1(z),\dots,u_{m-1}(z) > 0$, $u_m(z) = u_{m+1}(z) = \cdots = 0$, u_{m-1} is not constant on any right neighborhood of z. By (e2) there are $x_j \searrow z$ such that $u_m(x_j) = 0$ $(= u_{m+1}(x_j) = \cdots)$ and $u_i(x_j) = u_i(z)$ for any $i < m - 1$ (because $u_{m-1}(z) > 0$, and thus $u_{m-2},\dots,u_1$ are constant on a neighborhood of z). We obtain

$$\begin{aligned}\limsup_{j \to \infty} \frac{|f(x_j) - f(z)|}{x_j - z} &= \limsup_{j \to \infty} \frac{|\varphi(u_{m-1}(x_j)) - \varphi(u_{m-1}(z))|}{x_j - z} \\ &\leqslant \varphi'(u_{m-1}(z)) \cdot \limsup_{j \to \infty} \frac{|u_{m-1}(x) - u_{m-1}(z)|}{x_j - z} \\ &\leqslant \varphi'(u_{m-1}(z)) \operatorname{Lip} u_{m-1} < +\infty.\end{aligned}$$

5. THEOREM. *Let φ, ψ be continuous increasing functions on $[0,1]$, ψ is concave, $\varphi(0) = \psi(0) = 0$. If $\limsup_{x \to 0_+} \varphi(x)/\psi(x) = +\infty$, then the set*

$$M = \left\{ f \in A_\varphi(E) \text{: there is } z \in [0,1) \text{ such that } \limsup_{x \to z_+} \frac{|f(x) - f(z)|}{\psi(x - z)} < +\infty \right\}$$

is of the first category in $A_\varphi(E)$.

PROOF. Obviously $M \subset \bigcup_{k=1}^{\infty} M_k$, where

$$M_k = \Big\{ f \in A_\varphi(E)\colon \text{there is } z \in [0, 1 - 1/k] \text{ such that}$$
$$|f(x) - f(z)| \leqslant k\psi(x - z) \text{ for every } x \in [z, z + 1/k] \Big\}$$

are closed sets in $A_\varphi(E)$. We shall prove that the M_k are nowhere dense, Fix $u \in E$, $k \in \mathbf{N}$ and $\varepsilon \in (0, 1/2)$. Let $f = A_\varphi u$. We shall form "better" $u^* \in E$ near to u in several steps. Let $m \in \mathbf{N}$ be with $1/m < \varepsilon$, define $v = \{v_n\}$ by $v_n = (1 - 2\varepsilon)\max(0, u_n - nm^{-2})$ (if $n > m$, then $v_n = 0$ by Lemma 1). Then $v \in E$ and moreover, $\operatorname{Lip} v_n \leqslant 1 - 2\varepsilon$. It is easy to see that $\|v_n - u_n\| \leqslant 2\varepsilon + 1/m < 3\varepsilon$ for every $n \in \mathbf{N}$. Since by the concavity of ψ

$$\limsup_{q \in \mathbf{N}} \frac{\varphi(\varepsilon/q)}{\psi(2/q)} \geqslant \limsup_{x \to 0_+} \frac{\psi(\varepsilon x/4)}{\psi(x)} \geqslant \frac{1}{4}\varepsilon \limsup_{x \to 0_+} \frac{\varphi(\varepsilon x/4)}{\psi(\varepsilon x/4)} = +\infty,$$

we find $q \in \mathbf{N}$ such that $q \geqslant k$, $\varepsilon/q < 1/m^2$, $i/q < \varepsilon$, and $8k\psi(2/q) \leqslant \varphi(\varepsilon/q)$. Consider the partition

$$0 = b_0 < a_1 < c_1 < b_1 < a_2 < c_2 < b_2 < \cdots < a_p < c_p < b_p = 1$$

of $[0, 1]$ where for every $j = 1, \dots, p$, $b_j = j/q$, $c_j = (j - \varepsilon)/q$, $a_j = (j - 2\varepsilon)q$. Let $\lambda\colon [0, 1] \to [0, 1]$ be the function which is for every $j = 1, \dots, p$ linear on $[b_{j-1}, a_j]$, constant on $[a_j, b_j]$ and $\lambda(0) = 0$, $\lambda(b_j) = b_j$. Obviously

$$\operatorname{Lip} \lambda = \frac{b_j - b_{j-1}}{a_j - b_{j-1}} = \frac{1}{1 - 2\varepsilon}.$$

Put $w = \{w_n\}$ where $w_n = v_n \circ \lambda$. Then for any $n \in \mathbf{N}$

$$\operatorname{Lip} w_n \leqslant \operatorname{Lip} v_n \operatorname{Lip} \lambda \leqslant 1,$$

(e1), (e2) are also trivially satisfied, hence $w \in E$. For any $x \in [0, 1]$ and $n \in \mathbf{N}$

$$|v_n(x) - w_n(x)| = |v_n(x) - v_n(\lambda(x))| \leqslant \operatorname{Lip} v_n |x - \lambda(x)| \leqslant 1/q < \varepsilon.$$

Finally we define $u^* = \{u_n^*\}$: if $j \in \{1, \dots, p\}$ and n is the smallest positive integer for which $v_n(b_j) < \varepsilon/q$, then put

$$u_n^*(x) = \begin{cases} v_n(b_j) + \dfrac{\varepsilon}{q} - |x - c_j| & \text{if } \varphi\big(v_n(b_j)\big) \leqslant 4k\psi(2/q), \\ v_n(b_j) - \min\big(v_n(b_j), (\varepsilon/q) - |x - c_j|\big) & \text{if } \varphi\big(v_n(b_j)\big) > 4k\psi(2/q) \end{cases}$$

for $x \in [a_j, b_j]$; otherwise and elsewhere put $u_n^*(x) = w_n(x)$. Obviously $\operatorname{Lip} u_n^* \leqslant 1$. From the definition of v we see that for every $n \in \mathbf{N}$ and $x \in [0, 1]$ either $v_{n+1}(x) = 0$ or $v_n(x) > v_{n+1}(x) + m^{-2}$. Using the inequality $\varepsilon/q < m^{-2}$, if $j \in \{1, \dots, p\}$ and n is the smallest positive integer for which $v_n(b_j) < \varepsilon/q$, then $u_{n+1}^* = 0 \leqslant u_n^*$ on $[a_j, b_j]$ and, if moreover $n > 1$, $u_n^* \leqslant u_{n-1}^*$ on $[a_j, b_j]$. Hence (e1) and (e2) are satisifed, $u^* \in E$. Clearly $\|u_n^* - w_n\| \leqslant \varepsilon/q < \varepsilon$ for every $n \in \mathbf{N}$. Put $f^* = A_\varphi u^*$. We shall prove $f^* \notin M_k$. Fix $z \in [0, 1 - 1/k]$. Find $j \in \{2, \dots, p\}$ such that $b_{j-2} \leqslant z < b_{j-1}$ (this is possible because $1/q < 1/k$) and the smallest $n \in \mathbf{N}$ for which $v_n(b_j) < \varepsilon/q$. Then $\varphi(u_n^*(b_j)) = \varphi(v_n(b_j))$ and if $\varphi(v_n(b_j)) \leqslant 4k\psi(2/q)$, then

$$\varphi\big(u_n(c_j)\big) = \varphi\big(v_n(b_j) + \varepsilon/q\big) \geqslant \varphi(\varepsilon/q) \geqslant 8k\psi(2/q),$$

else $\varphi(u_n(c_j)) = 0$. In either case

$$\left|\varphi\left(u_n^*(c_j)\right) - \varphi\left(u_n^*(b_j)\right)\right| \geqslant 4k\psi(2/q).$$

Since u_i^* are constant on $[a_j, b_j]$ for every $i \neq n$, we have $|f^*(c_j) - f^*(b_j)| \geqslant 4k\psi(2/q)$. Thus there is $x \in \{c_j, b_j\}$ such that

$$\left|\frac{f^*(x) - f^*(z)}{\psi(x - z)}\right| \geqslant \frac{2k\psi(2/q)}{\psi(2/q)} = 2k$$

and we conclude, z being arbitrary, $f^* \notin M_k$. For every $n \in \mathbf{N}$ we have $\|u_n^* - u_n\| \leqslant \|u_n^* - w_n\| + \|w_n - v_n\| + \|v_n - u_n\| \leqslant 5\varepsilon$. Since we can construct $u^* \in E \setminus A_\varphi^{-1}(M_k)$ in any neighborhood of given $u \in E$ and A_φ is continuous, M_k are nowhere dense subsets of $A_\varphi(E)$.

3. Applications. (A) *Morse-Besicovitch functions.* A. P. Morse **[6]** constructed a continuous function on [0, 1] without infinite unilateral derivatives such that

$$\limsup_{x \to z_-}\left|\frac{f(x) - f(z)}{x - z}\right| = +\infty$$

for every $z \in (0, 1]$ and

$$\limsup_{x \to z_+}\left|\frac{f(x) - f(z)}{x - z}\right| = +\infty$$

for every $z \in [0, 1)$. The category method yields functions with this Morse property, which, of course, implies the Besicovitch one. (Note that the original Besicovitch's function does not satisfy the Morse property.)

Let φ: $[0, 1] \to \mathbf{R}$ be a continuous increasing function such that $\varphi(0) = 0$, φ' exists finite everywhere on $(0, 1)$ and

$$\liminf_{x \to 0_+} \frac{\varphi(x)}{x} < \limsup_{x \to 0_+} \frac{\varphi(x)}{x} = +\infty.$$

According to Proposition 4, the functions from $A_\varphi(E)$ do not have infinite right derivatives on $[0, 1)$ and by Theorem 5 (we put $\psi(x) = x$) there is a residual set $S^+ \subset A_\varphi(E)$ such that every function $f \in S^+$ satisfies

$$\limsup_{x \to z_+}\left|\frac{f(x) - f(z)}{x - z}\right| = +\infty$$

at any $z \in [0, 1)$. From the symmetry reason the elements of $A_\varphi(E)$ do not have an infinite left derivative at any $z \in (0, 1]$ and there is a residual set $S^- \subset A_\varphi(E)$ such that every $f \in S^-$ satisifies

$$\limsup_{x \to z_-}\left|\frac{f(x) - f(z)}{x - z}\right| = +\infty$$

for any $z \in (0, 1]$. Hence every element of $S^+ \cap S^-$ has the Morse property.

(B) *Hölder continuity*. Let μ be an increasing continuous function on $[0,1]$, $\mu(0)=0$. For $f\in\mathscr{C}$ denote

$$H_\mu(f)=\sup_{x\neq y}\frac{|f(x)-f(y)|}{\mu(|x-y|)}.$$

We say that f is μ-Hölder if $H_\mu(f)<+\infty$. The most usual choice of μ is $\mu(t)=t^\alpha$, $\alpha\in(0,1)$; such μ-Hölder functions are, among others, uniform sums of their Fourier series (see e.g. [**11**]). The authors of the cited papers do not examine Hölder continuity of their examples of Besicovitch functions, although, for instance, it is possible to see that the Morse's function [**6**] is μ-Hölder for $\mu(t)=\sqrt{t}$. The verification of Hölder continuity is very easy if we produce Besicovitch functions by the category method. The following theorem enables us to obtain μ-Hölder Besicovitch functions provided $\mu'_+(0)=+\infty$.

6. THEOREM. *Let φ be a continuous increasing function on* $[0,1]$, $\varphi(0)=0$. *If $H_\mu(\varphi)\leqslant 1$, then $H_\mu(f)\leqslant 2$ for any $f\in A_\varphi(E)$.*

PROOF. Let $u\in E, f=A_\varphi u$; $x, y\in[0,1], f(x)\neq f(y)$. Put

$$m=\min\{n\in\mathbf{N}\colon u_n(x)\neq u_n(y)\}.$$

According to (e2), there is $z\in[x,y]$ such that $u_{m+1}(z)=0$. Suppose m is odd. Then

$$f(y)\leqslant\sum_{i=1}^{m}(-1)^{i+1}\varphi(u_i(y)),\qquad f(x)\geqslant\sum_{i=1}^{m+1}(-1)^{i+1}\varphi(u_i(x)).$$

Since $u_i(x)=u_i(y)$ for $i=1,\dots,m-1$ and $\operatorname{Lip}u_m\leqslant 1$, $\operatorname{Lip}u_{m+1}\leqslant 1$,

$$\begin{aligned}f(y)-f(x)&\leqslant\varphi(u_m(y))-\varphi(u_m(x))+\varphi(u_{m+1}(x))-\varphi(u_{m+1}(z))\\&\leqslant\mu(|u_m(y)-u_m(x)|)+\mu(|u_{m+1}(x)-u_{m+1}(z)|)\\&\leqslant 2\mu(|y-x|).\end{aligned}$$

Similarly $f(x)-f(y)\leqslant 2\mu(|y-x|)$ and the proof is analogous if m is even.

(C) *Besicovitch functions on normed linear spaces*. Up to now, we have applied Theorem 5 only with $\psi(x)=x$. We sketch the significance of its formulation with a general function ψ. Indeed, if $(X,|\ \ |)$ is a uniformly convex normed linear space, then a suitable choice of ψ in Theorem 5 enables us to obtain a function $f\colon[0,1]\to\mathbf{R}$ such that $F=f\circ|\ \ |$ has the following "Besicovitch type" property: F is continuous and if $x,h\in X$, $|x|\leqslant 1$, $|h|=1$, then the limit $\lim_{t\to 0_+}(F(x+th)-F(x))/t$ does not exist (neither finite, nor infinite). Using a continuous linear one-to-one embedding into a uniformly convex (e.g. Hilbert) space we can construct a function of "Besicovitch type" on every separable Banach space. We leave here a detailed discussion of this matter being rather forced in the context of this paper.

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