

## FACTORIZING COMPACT AND WEAKLY COMPACT OPERATORS THROUGH REFLEXIVE BANACH LATTICES<sup>1</sup>

BY

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**ABSTRACT.** When does a weakly compact operator between two Banach spaces factor through a reflexive Banach lattice?

This paper provides some answers to this question. One of the main results: If an operator between two Banach spaces factors through a Banach lattice with weakly compact factors, then it also factors through a reflexive Banach lattice. In particular, the square of a weakly compact operator on a Banach lattice factors through a reflexive Banach lattice.

Similar results hold for compact operators. For instance, the square of a compact operator on a Banach lattice factors with compact factors through a reflexive Banach lattice.

**1. Preliminaries.** For terminology concerning Riesz spaces and Banach lattices, we follow [2 and 17]. In this section, we briefly review a few basic results concerning Banach lattices.

Let  $X$  be a Riesz space. Then the symbol  $X^+$  will denote the positive cone of  $X$ , i.e.,  $X^+ = \{x \in X: x \geq 0\}$ . The sets of the form

$$[-x, x] := \{y \in X: -x \leq y \leq x\}, \quad x \in X^+,$$

are called the *order intervals* of  $X$ . A subset  $A$  of  $X$  is said to be *solid* whenever  $|x| \leq |y|$  in  $X$  and  $y \in A$  imply  $x \in A$ . The *solid hull* of a set  $A$  is the smallest solid set that contains  $A$  and is precisely the set  $\text{sol}(A) := \{x \in X: \exists y \in A \text{ with } |x| \leq |y|\}$ . A solid vector subspace of  $X$  is referred to as an *ideal* of  $X$ .

In this paper the word "operator" is synonymous with "linear operator." An operator  $T: X \rightarrow Z$  between two Riesz spaces is said to be *positive*, in symbols  $0 \leq T$ , whenever  $x \in X^+$  implies  $T(x) \in Z^+$ .

A norm  $\|\cdot\|$  on a Riesz space is said to be a *lattice norm* whenever  $|x| \leq |y|$  implies  $\|x\| \leq \|y\|$ . A *Banach lattice* is a Riesz space equipped with a lattice norm under which it is a Banach space. A Banach lattice is said to have *order continuous norm* whenever  $x_\alpha \downarrow 0$  implies  $\|x_\alpha\| \downarrow 0$ . The closed unit ball of an arbitrary Banach space  $Z$  will be denoted by  $B_Z$ ; i.e.,  $B_Z := \{z \in Z: \|z\| \leq 1\}$ .

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For our work, we shall need the following characterization of Banach lattices with order continuous norms. (We shall denote the norm dual of an arbitrary Banach space  $Z$  by  $Z'$  and its second dual by  $Z''$ .)

**THEOREM 1.1.** *For a Banach lattice  $X$  the following statements are equivalent:*

1.  $X$  has order continuous norm.
2. Each order interval of  $X$  is weakly compact.
3.  $X$  is an ideal of  $X''$ .
4. Every relatively weakly compact subset of  $X^+$  has a relatively weakly compact solid hull.

**PROOF.** For the equivalence of (1), (2), and (3) see [17, Theorem 5.10, p. 89].

(3)  $\Rightarrow$  (4) Let  $A \subseteq X^+$  be relatively weakly compact, and let  $x'' \in X''$  be a  $w^*$ -closure point of  $\text{sol}(A)$ . Pick a net  $\{x_\alpha\} \subseteq \text{sol}(A)$  with  $x_\alpha \xrightarrow{w^*} x''$ . For each  $\alpha$ , choose some  $y_\alpha \in A$  with  $-y_\alpha \leq x_\alpha \leq y_\alpha$ . By passing to a subnet, we can assume that  $y_\alpha \xrightarrow{w} y$  holds in  $X$ . This implies  $-y \leq x'' \leq y$  in  $X''$ , and so,  $x'' \in X$ . Therefore,  $\text{sol}(A)$  is relatively weakly compact. (For an alternate proof of this implication see [18, Proposition 2.1, p. 241].)

(4)  $\Rightarrow$  (2) Note that for each  $x \in X^+$  we have  $\text{sol}\{x\} = [-x, x]$  and that  $[-x, x]$  is weakly compact.  $\square$

It should be noted that in a Banach lattice with order continuous norm the solid hull of a relatively weakly compact set need not be relatively weakly compact [16, p. 308]. However, there are some Banach lattices in which the solid hull of a relatively weakly compact set remains relatively weakly compact. These are the KB-spaces, and they will be needed for our work.

Following the Russian literature, we say that a Banach lattice  $X$  is a *KB-space* (Kantorovič-Banach space) whenever every increasing norm bounded sequence of  $X^+$  is norm convergent. Clearly, every KB-space has order continuous norm.

**THEOREM 1.2.** *The norm dual  $X'$  of a Banach lattice  $X$  is a KB-space if and only if  $X'$  has order continuous norm.*

**PROOF.** Assume that  $X'$  has order continuous norm. Let  $0 \leq x'_n \uparrow$  hold in  $X'$  with  $\sup\{\|x'_n\|\} < \infty$ . Then  $x'(x) = \lim x'_n(x)$  exists in  $\mathbf{R}$  for each  $x \in X$ , and moreover, this formula defines a positive linear functional on  $X$ . Thus,  $x' \in X'$ , and clearly,  $x'_n \uparrow x'$  holds in  $X'$ . Now note that  $\lim\|x'_n - x'\| = 0$ .  $\square$

Recall that if  $X$  is a Riesz space, then the symbol  $X_n \widetilde{\sim}$  denotes the *order continuous dual* of  $X$ , i.e., the Riesz space of all linear functionals  $f$  on  $X$  such that  $x_\alpha \downarrow 0$  in  $X$  implies  $\lim|f(x_\alpha)| = 0$ .

The next result presents some characterizations of KB-spaces. For a proof see [2, p. 159; 17, p. 95; or 16, p. 309].

**THEOREM 1.3.** *For a Banach lattice  $X$  the following statements are equivalent:*

1.  $X$  is a KB-space.
2.  $X$  is a band of  $X''$ .
3.  $X = (X')_n \widetilde{\sim}$ .

In a KB-space a relatively weakly compact set has a relatively weakly compact solid hull. This result appeared in [1]; see also [17, Exercise 30, p. 153].

**THEOREM 1.4.** *In a KB-space the convex solid hull of a relatively weakly compact set remains relatively weakly compact.*

**PROOF.** Let  $A$  be a relatively weakly compact subset of  $X$ , and consider  $A$  as a subset of  $(X')_n^\sim$ . By [2, Corollary 20.12, p. 140] we know that  $A$  is relatively  $\sigma((X')_n^\sim, X')$ -compact if and only if its convex solid hull is also relatively  $\sigma((X')_n^\sim, X')$ -compact. By Theorem 1.3 we have  $X = (X')_n^\sim$ , and our conclusion follows.  $\square$

For constructing reflexive Banach spaces from weakly compact sets, we shall need the following basic result of [7] (see also [8, p. 250]).

**THEOREM 1.5 (DAVIS-FIGIEL-JOHNSON -PEŁCZYŃSKI).** *Let  $X$  be a Banach space with closed unit ball  $B = \{x \in X: \|x\| \leq 1\}$ , and let  $W$  be a convex, symmetric, norm bounded subset of  $X$ . For each  $n$  put  $U_n := 2^n W + 2^{-n} B$ , and denote by  $\|\cdot\|_n$  the Minkowski functional of  $U_n$ , i.e.,*

$$\|x\|_n := \inf\{\alpha > 0: x \in \alpha U_n\}.$$

Set  $Y = \{x \in X: \| \|x\| \| = (\sum_{n=1}^{\infty} \|x\|_n^2)^{1/2} < \infty\}$ , and let  $J: Y \rightarrow X$  denote the natural inclusion. Then:

1.  $(Y, \| \cdot \|)$  is a Banach space and  $J$  is continuous;
2.  $W$  is a subset of the closed unit ball of  $(Y, \| \cdot \|)$ ;
3.  $J'': Y'' \rightarrow X''$  is one-to-one; and
4.  $(Y, \| \cdot \|)$  is reflexive if and only if  $W$  is a relatively weakly compact subset of  $X$ .

Another important property associated with the Banach space  $(Y, \| \cdot \|)$ , needed for our work, is described in the next lemma.

**LEMMA 1.6.** *Let  $W$  be a convex, symmetric, norm bounded subset of a Banach space  $X$ , and let  $(Y, \| \cdot \|)$  be the Banach space of Theorem 1.5 determined by  $W$ . Then a subset of  $W$  is totally bounded in  $X$  if and only if it is totally bounded in  $Y$ .*

*In particular, if a compact operator  $T: Z \rightarrow X$  satisfies  $T(B_Z) \subseteq W$ , then  $T$  considered as an operator from  $Z$  into  $Y$  is also compact.*

**PROOF.** Let  $D$  be a subset of  $W$  which is totally bounded in  $X$ , and let  $0 < \varepsilon < 1$ . Since  $x, y \in W$  implies

$$x - y \in W + W = 2W = 2^{1-n}(2^n W) \subseteq 2^{1-n}U_n,$$

we see that  $\|x - y\|_n \leq 2^{1-n}$  holds for all  $n$ , and so, there exists some  $k$  satisfying  $\sum_{n=k}^{\infty} \|x - y\|_n^2 < \varepsilon$  for all  $x, y \in W$ . Pick  $x_1, \dots, x_m \in D$  such that

$$D \subseteq \{x_1, \dots, x_m\} + \varepsilon 2^{-2k} B,$$

where  $B$  is the closed unit ball of  $X$ .

Now let  $x \in D$  be fixed. Choose some  $x_i$  with  $\|x - x_i\| < \varepsilon 2^{-2k}$ . Then for  $1 \leq n \leq k$  we have

$$x - x_i \in \varepsilon 2^{-2k} B \subseteq \varepsilon 2^{-k} 2^{-n} B \subseteq \varepsilon 2^{-k} U_n,$$

and so,  $\|x - x_i\|_n \leq \epsilon 2^{-k}$  holds for all  $1 \leq n \leq k$ . Therefore,

$$\| \|x - x_i\| \|^2 \leq \sum_{n=1}^k \|x - x_i\|_n^2 + \sum_{n=k}^{\infty} \|x - x_i\|_n^2 < k\epsilon^2 2^{-2k} + \epsilon < \epsilon + \epsilon = 2\epsilon$$

holds, from which it follows that  $D$  is also a totally bounded subset of  $Y$ .  $\square$

Recall that a positive operator  $T: Z \rightarrow X$  between two Riesz spaces is said to be:

- (a) *interval preserving*, whenever  $T[0, x] = [0, Tx]$  holds for all  $x \in Z^+$ ; and
- (b) *a lattice homomorphism*, whenever  $T(x \vee y) = T(x) \vee T(y)$  holds for all  $x, y \in Z$ .

In the sequel, the expression “an operator  $T: Z \rightarrow X$  between two Banach spaces is positive” will mean that  $Z$  and  $X$  are both Banach lattices and that  $T$  is a positive operator (i.e.,  $z \geq 0$  implies  $T(z) \geq 0$ ).

When  $X$  is a Banach lattice and  $W$  is also a solid set, then the Banach space  $(Y, \| \cdot \|)$  of Theorem 1.5 is itself a Banach lattice.

**THEOREM 1.7.** *Let  $W$  be a convex, solid, and norm bounded subset of a Banach lattice  $X$ . If  $(Y, \| \cdot \|)$  is the Banach space of Theorem 1.5 determined by  $W$ , then we have:*

- 1.  $(Y, \| \cdot \|)$  is a Banach lattice and  $Y$  is an ideal of  $X$ ;
- 2. the operator  $J: Y \rightarrow X$  is an interval preserving lattice homomorphism; and
- 3.  $J': X' \rightarrow Y'$  is also an interval preserving lattice homomorphism.

**PROOF.** (1) Let  $|x| \leq |y|$  hold in  $X$  with  $y \in Y$ . If  $y \in \alpha U_n$ , then (in view of the solidness of  $U_n = 2^n W + 2^{-n} B$ ) we see that  $x \in \alpha U_n$ , and so,  $\|x\|_n \leq \|y\|_n$ . Therefore,  $\| \|x\| \| \leq \| \|y\| \|$  holds. This shows that  $Y$  is an ideal of  $X$ , and that  $(Y, \| \cdot \|)$  is a Banach lattice.

(2) Obvious.

(3) It follows immediately from [15, Proposition 1.2, p. 89].  $\square$

**2. Factoring weakly compact operators.** Let  $T: Z \rightarrow X$  be a continuous operator between two Banach spaces. Recall that  $T$  is said to *factor* through a Banach space  $Z_1$  whenever there exist continuous operators  $Z \xrightarrow{S} Z_1 \xrightarrow{R} X$  satisfying  $T = RS$ . (The operators  $R$  and  $S$  are called factors of  $T$ .) By [7] we know that every weakly compact operator factors through a reflexive Banach space. Also, by [10 and 14] we know that every compact operator factors with compact factors through a reflexive Banach space. In this section we show how a weakly compact operator can be factored through a reflexive Banach lattice.

We start our discussion with a theorem that will be the basis for our results.

**THEOREM 2.1.** *Let  $W$  be the convex solid hull of a relatively weakly compact subset of a Banach lattice  $X$ , and let  $(Y, \| \cdot \|)$  be the Banach lattice as defined in Theorem 1.5. Then  $Y'$  has order continuous norm (and hence, it is a KB-space).*

**PROOF.** By Theorem 1.7,  $J': X' \rightarrow Y'$  is interval preserving, and so,  $J'(X')$  is an ideal of  $Y'$ . On the other hand, it is also true that  $J'(X')$  is weakly dense in  $Y'$ . Indeed, if some  $y'' \in Y''$  satisfies  $y''(J'x') = 0$  for all  $x' \in X'$ , then  $J''y''(x') = 0$

holds for all  $x' \in X'$ , and so,  $J''y'' = 0$ . Since  $J'': Y'' \rightarrow X''$  is one-to-one, we get  $y'' = 0$ . Thus,  $J'(X')$  is weakly dense in  $Y'$ , and hence, norm dense in  $Y'$ . Now if  $J'(X')$  has order continuous norm, then its norm closure  $Y'$  will also have order continuous norm; see [2, Theorems 17.9 and 10.6]. Therefore, in order to establish that  $Y'$  has order continuous norm, it is enough to show that  $J'x'_n \downarrow 0$  in  $J'(X')$  implies  $\|J'x'_n\| \downarrow 0$ .

To this end, let  $J'x'_n \downarrow 0$  in  $J'(X')$ . Since  $J'x'_n = |J'x'_n| = J'|x'_n|$  holds, we can assume that  $x'_n \geq 0$  for each  $n$ . Now let  $\epsilon > 0$ . Fix some  $k$  with  $2^{-k}\|x'_1\| < \epsilon$ . Since  $W$  is the convex solid hull of a relatively weakly compact subset of  $X$ , it follows from [2, Theorems 20.9 and 20.6] that there exists some  $u \geq 0$  in the ideal generated by  $W$  in  $X$  satisfying

$$x'_1(|w| - u)^+ < 2^{-k}\epsilon \quad \text{for all } w \in W.$$

From  $W \subseteq Y$  and the fact that  $Y$  is an ideal of  $X$ , we see that  $u \in Y$ . Next choose some  $m$  with

$$2^k J'x'_n(u) < \epsilon \quad \text{for all } n \geq m.$$

Now if  $y \in Y$  satisfies  $\|y\| \leq 1$ , then  $\|y\|_k \leq 1$  also holds, and so,

$$y \in 2(2^k W + 2^{-k} B).$$

Write  $y = 2(2^k w + 2^{-k} v)$  with  $w \in W$  and  $v \in B$ . Clearly,  $v \in Y$ , and moreover,

$$\begin{aligned} |J'x'_n(y)| &\leq 2[2^k J'x'_n(|w|) + 2^{-k} J'x'_n(|v|)] \\ &\leq 2[2^k J'x'_n(|w| - u)^+ + 2^k J'x'_n(u) + 2^{-k} x'_n(|v|)] \\ &\leq 2[2^k x'_1(|w| - u)^+ + 2^k J'x'_n(u) + 2^{-k} \|x'_1\|] \\ &\leq 2[2^k \cdot 2^{-k}\epsilon + \epsilon + \epsilon] = 6\epsilon \end{aligned}$$

holds for all  $n \geq m$ . This implies  $\|J'x'_n\| \leq 6\epsilon$  for all  $n \geq m$ , and hence,  $\|J'x'_n\| \downarrow 0$ , as desired.  $\square$

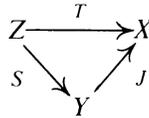
The next theorem deals with the factorization of a weakly compact operator whose range is in a Banach lattice.

**THEOREM 2.2.** *Let  $T: Z \rightarrow X$  be a weakly compact operator from a Banach space  $Z$  into a Banach lattice  $X$ . If the solid hull of  $T(B_Z)$  is relatively weakly compact, then  $T$  factors through a reflexive Banach lattice.*

*If, in addition,  $T$  is a positive operator, then the factors can be taken to be positive operators.*

**PROOF.** Denote by  $W$  the convex hull of the solid hull of  $T(B_Z)$ , i.e.,  $W = \text{co}(\text{sol}(T(B_Z)))$ . By [2, Theorem 1.3, p. 4] the set  $W$  is a solid set, and by our hypothesis, it follows that  $W$  is also relatively weakly compact. Let  $(Y, \|\cdot\|)$  be the reflexive Banach lattice corresponding to  $W$  (Theorems 1.5 and 1.7), and let

$S: Z \rightarrow Y$  be the operator defined by  $S(z) = T(z)$  for all  $z \in Z$ . Now a glance at the diagram



completes the proof of the theorem.  $\square$

The following is an immediate consequence of the preceding result and Theorem 1.4.

**COROLLARY 2.3.** *Every weakly compact operator  $T: Z \rightarrow X$  from a Banach space into a KB-space factors through a reflexive Banach lattice.*

*Note.* Theorem 2.2 and Corollary 2.3 can be derived also from [11, Theorem 3.1, p. 403].

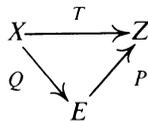
For positive operators the following version holds.

**COROLLARY 2.4.** *Let  $T: Z \rightarrow X$  be a positive weakly compact operator between two Banach lattices. If  $X$  has order continuous norm, then  $T$  factors (with positive factors) through a reflexive Banach lattice.*

**PROOF.** Note that  $\text{sol}(T(B_Z)) = \text{sol}(T(B_Z^+))$  holds. Taking into account Theorem 1.1(4), we see that  $T(B_Z)$  has a relatively weakly compact solid hull in  $X$ . Now apply Theorem 2.2.  $\square$

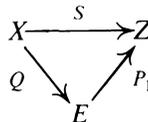
The following result describes a factorization property of weakly compact operators whose domains are Banach lattices.

**THEOREM 2.5.** *Let  $T: X \rightarrow Z$  be a weakly compact operator between two Banach spaces. If  $X$  is a Banach lattice such that  $X'$  has order continuous norm, then there exist a reflexive Banach lattice  $E$  and a factorization of  $T$*



with the factor  $Q$  a lattice homomorphism and the factor  $P$  positive if  $T$  is positive.

Moreover, if another operator  $X \xrightarrow{S} Z$  satisfies  $0 \leq S \leq T$ , then  $S$  admits a factorization through the reflexive Banach lattice  $E$



with  $0 \leq P_1 \leq P$ .

**PROOF.** Clearly,  $T': Z' \rightarrow X'$  is weakly compact, and by Theorem 1.2, the Banach lattice  $X'$  is a KB-space. Thus, by Theorem 1.4, if  $W$  denotes the convex solid hull of  $T'(B_{Z'})$ , then  $W$  is a relatively weakly compact subset of  $X'$ . Let  $(Y, \|\cdot\|)$  be the reflexive Banach lattice of Theorems 1.5 and 1.7 determined by  $W$  (where  $Y$  is also

an ideal of  $X'$ ). Next, define the operator  $R: Z' \rightarrow (Y, \|\cdot\|)$  by  $R(z') = T'(z')$ , and note that  $R$  is continuous (and that  $R$  is positive if  $T$  is positive). Thus, we have the diagrams:



Since  $J'(X'')$  is dense in  $Y'$  and  $T''(X'') \subseteq Z$ , we see that  $R'(Y') \subseteq Z$ . Now consider the reflexive Banach lattice  $E = Y'$  and the continuous operators  $X \xrightarrow{Q} E \xrightarrow{P} Z$  where  $P(u) = R'(u)$  for  $u \in E$  and  $Q(x) = J'(x)$  for  $x \in X$ . By Theorem 1.7,  $Q$  is a lattice homomorphism, and clearly,  $P$  is positive if  $T$  is positive. Now note that  $T = PQ$  holds.

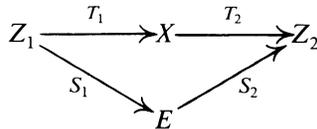
Now assume  $0 \leq S \leq T$ . Then  $S'(Z') \subseteq Y$  holds. On the other hand, by [4, Theorem 7]  $S$  is also weakly compact, and hence,  $S''(X'') \subseteq Z$  also holds. As above, this implies that the operator  $S_1: Z' \rightarrow Y$  defined by  $S_1(z') = S'(z')$  satisfies  $S'_1(Y') \subseteq Z$ . Define  $P_1: Y' \rightarrow Z$  by  $P_1(y') = S'_1(y')$ , and note that  $0 \leq P_1 \leq P$  and  $S = P_1Q$  hold. The proof of the theorem is now complete.  $\square$

Consider two positive operators  $S, T: X \rightarrow X$  on a Banach lattice satisfying  $0 \leq S \leq T$ . By [4, Example 1] we know that if  $T$  is weakly compact then  $S$  need not be weakly compact. However, we proved in [4] that if  $T$  is weakly compact, then  $S^2$  is necessarily weakly compact. The next theorem is a generalization of this result.

**THEOREM 2.6.** *Consider the scheme of weakly compact operators*

$$Z_1 \xrightarrow{T_1} X \xrightarrow{T_2} Z_2$$

*between Banach spaces. If  $X$  is a Banach lattice, then there exist a reflexive Banach lattice  $E$  and a factorization of  $T_2T_1$*

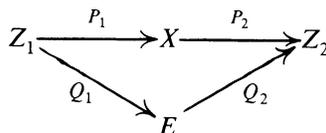


*such that  $S_i$  ( $i = 1, 2$ ) is positive if  $T_i$  is positive.*

*Moreover, if another scheme of operators*

$$Z_1 \xrightarrow{P_1} X \xrightarrow{P_2} Z_2$$

*satisfies  $0 \leq P_i \leq T_i$  ( $i = 1, 2$ ), then there exists a factorization of  $P_2P_1$  through the reflexive Banach lattice  $E$*



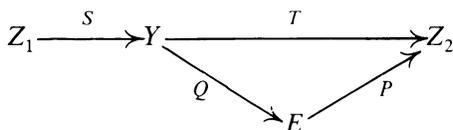
*such that  $0 \leq Q_i \leq S_i$  holds for each  $i$ .*

PROOF. Consider two weakly compact operators  $Z_1 \xrightarrow{T_1} X \xrightarrow{T_2} Z_2$  between Banach spaces with  $X$  a Banach lattice. Let  $W$  be the convex solid hull of the relatively weakly compact set  $T_1(B_{Z_1})$  and consider the Banach lattice  $(Y, \|\cdot\|)$  of Theorems 1.5 and 1.7. By Theorem 2.1 the Banach lattice  $Y'$  has order continuous norm.

Now if  $S: Z_1 \rightarrow Y$  is defined by  $S(z) = T_1(z)$ , then  $S$  is continuous. On the other hand, if  $T$  denotes the restriction of  $T_2$  to  $Y$ , then (since every norm bounded subset of  $Y$  is also norm bounded in  $X$ ) the operator  $T: Y \rightarrow Z_2$  is weakly compact. Clearly, the operators  $Z_1 \xrightarrow{S} Y \xrightarrow{T} Z_2$  satisfy  $T_2T_1 = TS$ .

If  $0 \leq P_i \leq T_i$  ( $i = 1, 2$ ) holds, then clearly  $P_1(Z_1) \subseteq Y$ , and if  $Z_1 \xrightarrow{R_1} Y \xrightarrow{R_2} Z_2$  are defined by  $R_1(z) = P_1(z)$  and  $R_2(z) = P_2(z)$ , then  $0 \leq R_1 \leq S$ ,  $0 \leq R_2 \leq T$ , and  $P_2P_1 = R_2R_1$  hold.

By Theorem 2.5 there exists a factorization of  $T$  through a reflexive Banach lattice  $E$



such that  $Q$  is positive and with  $P$  positive if  $T$  is positive. Moreover, in case  $0 \leq P_2 \leq T_2$  holds, there exists an operator  $Q_2: E \rightarrow Z_2$  satisfying  $0 \leq Q_2 \leq P$  and  $R_2 = Q_2Q$ . To finish the proof, take  $S_1 = QS$ ,  $S_2 = P$ , and  $Q_1 = QR_1$ .  $\square$

An important special case of Theorem 2.6 is the following

**COROLLARY 2.7.** *If  $T: X \rightarrow X$  is a weakly compact operator on a Banach lattice, then  $T^2$  factors (with positive factors if  $T$  is positive) through a reflexive Banach lattice.*

*Note.* After submitting this paper, we learnt of the work of N. Ghoussoub and W. B. Johnson [12] which has some overlapping with ours.

We close this section with two problems related to the question in the abstract.

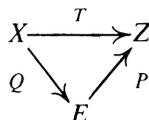
*Problem 1.* Does a weakly compact operator between two Banach lattices factor through a reflexive Banach lattice?

*Problem 2.* Does a positive weakly compact operator between two Banach lattices factor (if possible, with positive factors) through a reflexive Banach lattice?

**3. Factoring compact operators.** In this section we shall study when a compact operator has a factorization with compact factors through a reflexive Banach lattice. Some results of this type were obtained in [6].

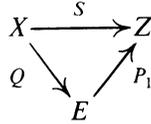
We start the section with the ‘‘compact operator’’ analogue of Theorem 2.5.

**THEOREM 3.1.** *Let  $T: X \rightarrow Z$  be a compact operator from a Banach lattice  $X$  into a Banach lattice space  $Z$ . If  $X'$  has order continuous norm, then  $T$  admits a factorization*



through a reflexive Banach lattice  $E$  with the factor  $Q$  a lattice homomorphism and the factor  $P$  compact. In case  $T$  is also positive, then  $P$  can be chosen to be a positive operator.

Moreover, if another operator  $S: X \rightarrow Z$  satisfies  $0 \leq S \leq T$ , then  $S$  admits a factorization through the reflexive Banach lattice  $E$



with  $0 \leq P_1 \leq P$ .

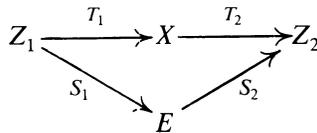
PROOF. Repeat the proof of Theorem 2.5, and use Lemma 1.6 to see that  $R: Z' \rightarrow Y$  is a compact operator which implies that  $P$  is a compact operator.  $\square$

The next theorem extends Theorem 2.6 and is the major factorization result of the paper.

THEOREM 3.2. Consider the scheme of operators

$$Z_1 \xrightarrow{T_1} X \xrightarrow{T_2} Z_2$$

between Banach spaces with each  $T_i$  ( $i = 1, 2$ ) compact or weakly compact. If  $X$  is a Banach lattice, then there exists a reflexive Banach lattice  $E$  and a factorization of  $T_2T_1$

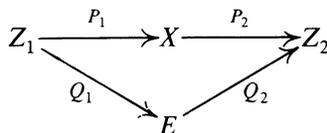


such that  $S_i$  ( $i = 1, 2$ ) has the same compactness property as  $T_i$  and is positive if  $T_i$  is positive.

Moreover, if another scheme of operators

$$Z_1 \xrightarrow{P_1} X \xrightarrow{P_2} Z_2$$

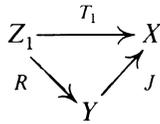
satisfies  $0 \leq P_i \leq T_i$  ( $i = 1, 2$ ), then there exists a factorization of  $P_2P_1$  through the reflexive Banach lattice  $E$



such that  $0 \leq Q_i \leq S_i$  holds for each  $i$ .

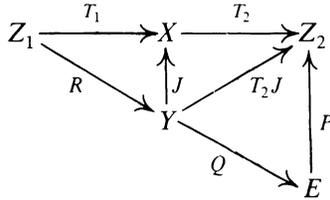
PROOF. Assume that  $T_1$  is compact or weakly compact, and let  $W$  denote the convex solid hull of the relatively weakly compact set  $T_1(B_{Z_1})$ . If  $Y$  denotes the Banach lattice determined by  $W$ , then (by Theorem 2.1)  $Y'$  has order continuous

norm, and  $T_1$  admits the factorization



where  $R(z) = T_1(z)$  for all  $z \in Z_1$ . Note that  $J$  is a positive operator and that  $R$  is positive if  $T_1$  is positive.

The proof will be based upon the following diagram:



The factorization  $T_2 J = P Q$  will be through an appropriate reflexive Banach lattice  $E$  that will be explained below. In all cases the desired factors will be  $S_1 = Q R$  and  $S_2 = P$ . As we shall see,  $Q$  always will be a positive operator, and  $P$  will be positive if  $T_2$  is positive.

*Case 1.  $T_1$  and  $T_2$  are both weakly compact.*

The factorization of  $T_2 J$  through a reflexive Banach lattice  $E$  follows from Theorem 2.5.

*Case 2.  $T_1$  is weakly compact and  $T_2$  is compact.* By Theorem 3.1 the compact operator  $T_2 J$  factors through a reflexive Banach lattice  $E$  with  $P$  compact.

*Case 3.  $T_1$  is compact and  $T_2$  is weakly compact.* By Lemma 1.6 the operator  $R$  is compact, and by Theorem 2.5 the weakly compact operator  $T_2 J$  factors through a reflexive Banach lattice  $E$ .

*Case 4.  $T_1$  and  $T_2$  are both compact.* As above,  $R$  is compact, and by Theorem 3.1 the compact operator  $T_2 J$  factors through a reflexive Banach lattice  $E$  with  $P$  compact. Therefore,  $S_1$  and  $S_2$  are both compact.

Finally, note that  $S_i$  is positive if  $T_i$  is positive, and the rest of the proof follows from Theorem 2.6.  $\square$

An immediate consequence of the preceding theorem is the following.

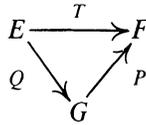
**COROLLARY 3.3.** *If  $T: X \rightarrow X$  is a compact operator on a Banach lattice, then  $T^2$  factors with compact factors through a reflexive Banach lattice.*

*If  $T$  is also positive, then the factors can be taken to be positive compact operators.*

It was shown in [9] that if  $E'$  and  $F$  have order continuous norms, then every positive compact operator from  $E$  into  $F$  dominated by a compact operator is itself compact. (Recall that an operator  $S$  is dominated by another operator  $T$  whenever  $S \leq T$  holds.) Next, we shall present a proof of this result using factorization.

**THEOREM 3.4 (DODDS - FREMLIN).** *Let  $E$  and  $F$  be two Banach lattices such that  $E'$  and  $F$  have order continuous norms. If a positive operator  $T: E \rightarrow F$  is dominated by a compact operator, then  $T$  is a compact operator.*

PROOF. By Theorem 3.1 there exists a reflexive Banach lattice  $G$  and a factorization of  $T$



such that  $P$  is positive dominated by a compact operator. By [5, Theorem 2.6, p. 234],  $P$  is a Dunford-Pettis operator, and hence, a compact operator. Thus,  $T$  is a compact operator.  $\square$

The next result generalizes [3, Theorem 2.5, p. 296].

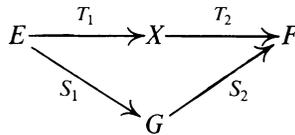
THEOREM 3.5. Consider the scheme of positive operators

$$E \xrightarrow{T_1} X \xrightarrow{T_2} F$$

between Banach lattices. If  $F$  has order continuous norm,  $T_1$  is dominated by a weakly compact operator, and  $T_2$  is dominated by a compact operator, then  $T_2T_1$  is a compact operator.

Dually, if  $E'$  has order continuous norm,  $T_1$  is dominated by a compact operator, and  $T_2$  is dominated by a weakly compact operator, then  $T_2T_1$  is a compact operator.

PROOF. By Theorem 3.2 there exist a reflexive Banach lattice  $G$  and a factorization of  $T_2T_1$



with  $S_2$  positive dominated by a compact operator. By Theorem 3.4,  $S_2$  is compact, and hence,  $T_2T_1$  is also compact.  $\square$

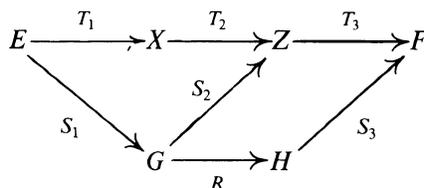
The following theorem generalizes [3, Theorem 2.4, p. 296].

THEOREM 3.6. Consider the scheme of positive operators

$$E \xrightarrow{T_1} X \xrightarrow{T_2} Z \xrightarrow{T_3} F$$

between Banach lattices such that  $T_1$  is dominated by a weakly compact operator and  $T_2$  by a compact operator. If  $T_3$  is dominated either by a weakly compact or by a Dunford-Pettis operator, then  $T_3T_2T_1$  is a compact operator.

PROOF. The proof is based upon the following diagram:



According to Theorem 3.2 the scheme  $E \xrightarrow{T_1} X \xrightarrow{T_2} Z$  factors through a reflexive Banach lattice  $G$  with  $S_2$  positive dominated by a compact operator.

Assume first that  $T_3$  is dominated by a Dunford-Pettis operator. Then by [5, Theorem 3.3, p. 236] the operator  $T_3 S_2$  is Dunford-Pettis, and so,  $T_3 T_2 T_1 = (T_3 S_2) S_1$  is a compact operator.

Now assume that  $T_3$  is dominated by a weakly compact operator. Then by Theorem 3.2 the scheme  $G \xrightarrow{S_2} Z \xrightarrow{T_3} F$  factors through a reflexive Banach lattice  $H$  with the factor  $R$  positive dominated by a compact operator. By Theorem 3.4,  $R$  is compact, and so,  $T_3 T_2 T_1 = S_3 R S_1$  is also compact.  $\square$

In the preceding theorem the case when  $T_1$  and  $T_3$  were dominated by weakly compact operators was established in [13] by a different method.

Finally, it should be mentioned that W. B. Johnson has pointed out to us that his techniques in [14] yield also the following result: *If  $T: Z \rightarrow X$  is a compact operator and  $X$  is a Banach lattice with the approximation property, then  $T$  factors with compact factors (which can be taken positive if  $T$  is positive) through a reflexive Banach space with an unconditional basis.*

We close the paper with the corresponding open problems for compact operators.

**Problem 3.** Does a compact operator between two Banach lattices factor with compact factors through a reflexive Banach lattice?

**Problem 4.** Does a positive compact operator between two Banach lattices factor with compact factors (if possible, with positive compact factors) through a reflexive Banach lattice?

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