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AN INTEGRAL VERSION OF THE BROWN-GITLER SPECTRUM

BY

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ABSTRACT. In this paper, certain spectra $B_1(k)$ are studied whose behavior qualifies them as being integral versions of the Brown-Gitler spectra $B(k)$. The bulk of our work emphasizes the similarities between $B_1(k)$ and $B(k)$, shown mainly using the techniques of Brown and Gitler. In analyzing the homotopy type of $B_1(k)$, we provide a free resolution over the Steenrod algebra for its cohomology and study its Adams spectral sequence. We also list conditions which characterize it at the prime 2. The paper begins, however, on a somewhat different topic, namely, the construction of a configuration space model for $\Omega^2(S^3\langle 3\rangle)$ and other related spaces.

Introduction. In a paper published in 1973, E. H. Brown, Jr., and S. Gitler described a procedure by which one might conceivably find new characteristic classes for smooth n-dimensional manifolds [3]. Their approach was based on examining the way that certain cohomology operations act on the stable normal bundle. Much of the motivation for their work stemmed from the problem of immersing manifolds in Euclidean space. For example, suppose that M is an n-manifold and let U denote the mod 2 Thom class of its stable normal bundle. From standard properties of the Steenrod squares, it follows that if $\mathrm{Sq}^i U \neq 0$, then M cannot be immersed in R^{n+i-1}.

Brown and Gitler's idea was to take all those elements of the mod 2 Steenrod algebra which universally *vanish* on the Thom classes of n-manifolds and use these "primary" operations as the foundation for a coherent system of higher order cohomology operations which might then prove useful, say, to detect nonimmersions.

As it turns out, these higher order operations also universally vanish on n-manifolds, and the attempt to produce new characteristic classes failed. The story has a happy ending, though, as Brown and Gitler were able to use their analysis to deduce the existence of a family of spectra $B(k)$. Since then, the Brown-Gitler spectra have played an important part in the development of the immersion problem and have also appeared, startlingly, in certain constructions involving the May filtration of the double loop space $\Omega^2 S^3$.

This paper partially answers the questions of what happens if one tries to repeat Brown and Gitler's analysis, but only for Z-orientable n-manifolds. Our main results

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concern a family of spectra $B_1(k)$ having the following properties:

(0.1) $H^*(B_1(k)) = A/A\{\mathrm{Sq}^1, \chi(\mathrm{Sq}^i) | i > k\}$ as A-modules. (Here, and from now on, homology and cohomology should be taken with coefficients in Z_2, the integers mod 2, unless otherwise noted. Also, A denotes the mod 2 Steenrod algebra, and χ: $A \to A$ is the canonical antiautomorphism.)

(0.2) Let $Z_2^\wedge = \varprojlim Z_{2^i}$ denote the 2-adic integers. There is a map j: $B_1(k) \to K(Z_2^\wedge)$ such that, for any CW complex X, the induced map of homology theories j_*: $B_1(k)_n(X) \to H_n(X; Z_2^\wedge)$ is surjective, provided that $n < 2k + 2$.

(0.2′) All Z-orientable, closed n-manifolds are $B_1([n/2])$-orientable. That is, given such a manifold M, let U_Z: $T(\nu) \to K(Z_2^\wedge)$ represent the Thom class of its stable normal bundle. Then there exists U_B: $T(\nu) \to B_1([n/2])$ such that $jU_B = U_Z$.

We shall also prove that (0.1) and (0.2) characterize $B_1(k)$, up to homotopy 2-equivalence; further arguments show that (0.2) and (0.2′) are essentially equivalent. Moreover, $B_1([n/2])$ has the smallest possible mod 2 cohomology for any spectrum possessing the orientability property (0.2′), provided that n is not divisible by 4.

A definition for a family of spectra satisfying (0.1) was first proposed by Mahowald. We shall base our presentation on Mahowald's definition, but, in an appendix, we discuss another way in which $B_1(k)$ could be defined.

Our viewpoint is strongly influenced by a desire to display $B_1(k)$ as an integral version of the Brown-Gitler spectrum $B(k)$. Brown and Gitler originally constructed $B(k)$ by building a generalized Postnikov tower for it, or, really, the Pontrjagin dual of such a tower. Normally, one starts with a known spectrum and then builds a Postnikov tower for it. Brown and Gitler faced the difficulty of trying to use a Postnikov tower for $B(k)$ in order to prove that it actually exists. Their construction was quite complicated.

However, in a seemingly unrelated development, the theory of Thom spectra was attracting considerable attention, largely due to the techniques and examples of Mark Mahowald (see **[15]** for an overview of the subject). Proceeding on the basis of some of Mahowald's cohomological calculations, R. Cohen **[9]** and Brown and Peterson **[6]** made the discovery that the Brown-Gitler spectra could be realized as certain Thom spectra involving the May filtration of $\Omega^2 S^3$. Because of the complexity of Brown and Gitler's original work, it was surprising, but comforting, to find that the spectra $B(k)$ could be obtained by these more natural geometric means. Not surprisingly, it seems quite difficult to use these geometric realizations of $B(k)$ in direct proofs of the sort of deeper properties that Brown and Gitler obtained via their Postnikov analysis.

The spectra $B_1(k)$ that we shall study here will be defined geometrically, following Mahowald, as Thom spectra (completed at 2). The details of Mahowald's construction have not yet appeared in print, so, in the appendix, we describe an independent (non-Thom spectrum) approach to defining $B_1(k)$. At any rate, there is no question that $B_1(k)$ exists. Even so, one might try to mimic Brown and Gitler's analysis in order to obtain deeper homotopy results. Indeed, this is the approach we will take: a recurrent theme in this paper is to start with properties about the Brown-Gitler spectrum $B(k)$ and then prove suitable analogues for $B_1(k)$.

Brown and Gitler's starting point was the construction of an explicit A-free resolution of $H^*B(k)$. Correspondingly, we shall define A-modules C_q and differentials $d\colon C_q \to C_{q-1}$ such that

(0.3) $\cdots C_q \xrightarrow{d} C_{q-1} \to \cdots \to C_0 \to H^*B_1(k) \to 0$ is an A-free resolution of $H^*B_1(k)$.

Using this resolution and the Adams spectral sequence then yields:

(0.4a) $\pi_0(B_1(k)) = Z_2^{\hat{}}$.

(0.4b) If $1 \leqslant q \leqslant 4k+2$, then

$$\pi_q(B_1(2k)) = \pi_q(B_1(2k+1)) = (\Lambda\{\lambda_1, \lambda_3, \lambda_5, \ldots\}/\Lambda\{\lambda_1, \lambda_3, \ldots, \lambda_{2k-1}\})_q$$

as groups (and hence as Z_2-vector spaces). Here, Λ is the mod 2 algebra of A. K. Bousfield et al. [**1**]; we review some of its properties in §3.1.

In addition, we will define a map $i_k\colon B_1(k) \to B(k)$ and study how it relates the two spectra:

(0.5) $(i_k)^*\colon H^*B(k) \to H^*B_1(k)$ is the canonical projection.

(0.6) There exist cofibrations

$$B_1(2k+1) \xrightarrow{2} B_1(2k+1) \xrightarrow{i_{2k+1}} B(2k+1)$$

and

$$B_1(2k-1) \to B_1(2k) \xrightarrow{i_{2k}} B(2k).$$

Moreover, if M_2 denotes the Z_2 Moore spectrum, then $M_2 \wedge B_1(2k) \simeq B(2k+1)$.

(0.7) $(i_k)_*\colon \pi_q(B_1(k)) \to \pi_q(B(k))$ is injective for $1 \leqslant q \leqslant 2k$.

As mentioned above, the Brown-Gitler spectra $B(k)$ play a prominent role in shaping the statements and proofs of the preceding results. The style of exposition will be to recite facts about $B(k)$ as they are needed.

In §2, we define the spectra $B_1(k)$, compute $H^*B_1(k)$, and establish the cofibrations (0.6). §3 presents the aforementioned free resolution of $H^*B_1(k)$, along with some related resolutions of $H^*(K(Z)) = A/A\{\mathrm{Sq}^1\}$. Then, in §4, we analyze the Adams spectral sequence of $B_1(k)$. Lastly, §5 is devoted to proving the homology surjection (0.2) and its corollary, the orientability conclusion (0.2′); the method of proof will then be used to describe various means of characterizing $B_1(k)$, up to homotopy 2-equivalence.

Now, $B_1(k)$ will be defined as a Thom spectrum over a certain filtration of the space $\Omega^2(S^3\langle 3\rangle)$, where $S^3\langle 3\rangle$ is the 3-connective cover of S^3. In §1, we construct a configuration space model for $\Omega^2(S^3\langle 3\rangle)$ involving the space C_nX, May's homotopy approximation to $\Omega^n\Sigma^nX$. Actually, this model, described below, arises as a corollary to a property of a certain retraction $C_nG \to G$, where G is any abelian topological group.

(0.8) $\Omega^2(S^3\langle 3\rangle)$ is homotopy equivalent to

$$\left\{\langle c_1, \ldots, c_k; x_1, \ldots, x_k\rangle \in C_2S^1 \,\Big|\, \prod_{i=1}^{k} x_i = 1\right\}.$$

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Also, we have learned that the spectra $B_1(k)$, along with certain other related spectra, have been studied extensively by Paul Goerss in his thesis [**11**]; many of the results given here also appear in his work.

1. An approximation to $\Omega^2(S^3\langle 3\rangle)$ and related spaces. Let ω: $S^n \to K(Z, n)$ be a map representing a generator of $H^n(S^n; Z) = Z$. Then the homotopy-theoretic fibre of ω, denoted $S^n\langle n\rangle$, is called the n-connective cover of S^n.

At present, we are primarily interested in the double loop space $\Omega^2(S^3\langle 3\rangle)$. The reason is that the future sections in this paper are devoted to studying a certain family of spectra $B_1(k)$, and the space $\Omega^2(S^3\langle 3\rangle)$ plays an important, underlying role in the definition of these spectra. A priori, the geometry of $\Omega^2(S^3\langle 3\rangle)$ seems quite intangible. The material in this section was motivated by a desire to find a manageable geometric model for $\Omega^2(S^3\langle 3\rangle)$ along the lines of the James reduced product for $\Omega\Sigma X$ [**12**] or May's "little n-cubes monad" for $\Omega^n\Sigma^n X$ [**16**].

In Theorem (1.3), we present such a model, not just for $\Omega^2(S^3\langle 3\rangle)$, but for any $\Omega^n(S^{n+1}\langle n+1\rangle)$, $1 \leqslant n < \infty$. In fact, (1.3) is a consequence of a more general result involving the homotopy fibre of a certain special retraction. We should point out that the models which we end up with seem quite reasonable; unfortunately, it is not clear whether they possess the sort of combinatorial convenience that one might hope for.

The results of this section are not needed anywhere later in this work.

We begin by recounting some constructions of May. Suppose that $1 \leqslant n \leqslant \infty$. Then, given a space X with base point $*$, define

$$C_n X = \coprod_{k=1}^{\infty} F(R^n, k) \times_{\Sigma_k} X^k / \sim .$$

Here, $F(R^n, k) = \{(c_1, \ldots, c_k) \in (R^n)^k \mid c_i \neq c_j \text{ if } i \neq j\}$ is the configuration space of k distinct points in the Euclidean space R^n, and X^k is the k-fold Cartesian product. The symmetric group, Σ_k, acts on both $F(R^n, k)$ and X^k by permuting coordinates. Points of $F(R^n, k) \times X^k$ will be written $\langle c_1, \ldots, c_k; x_1, \ldots, x_k\rangle$, where $(c_1, \ldots, c_k) \in F(R^n, k)$ and $x_i \in X$. Lastly, the equivalence relation $\sim$ in $C_n X$ is generated by the identification

$$\langle c_1, \ldots, c_j, \ldots, c_k; x_1, \ldots, x_j, \ldots, x_k\rangle \sim \langle c_1, \ldots, \hat{c}_j, \ldots, c_k; x_1, \ldots, \hat{x}_j, \ldots, x_k\rangle$$

whenever $x_j = *$.

The significance of $C_n X$ is given by May's "approximation theorem":

THEOREM (1.1) [**16**]. *There is a map β: $C_n X \to \Omega^n\Sigma^n X$ which is a weak homotopy equivalence if X is path connected.* □

The key result of this chapter is the following

THEOREM (1.2). *Let G be an abelian topological group whose base point is the identity element* 0. *Define $\mu = \mu_{n,G}$: $C_nG \to G$ by the formula*

$$\mu(\langle c_1,\dots,c_k;\, g_1,\dots,g_k\rangle) = \sum_{i=1}^{k} g_i.$$

Let $Y = Y_{n,G}$ denote the homotopy fibre of μ. Then $Y \simeq \mu^{-1}(0)$. Explicitly,

$$Y \simeq \left\{\langle c_1,\dots,c_k;\, g_1,\dots,g_k\rangle \in C_nG \,\Big|\, \sum_{i=1}^{k} g_i = 0\right\}.$$

PROOF. The customary way of describing Y is as the pullback of the path fibration on G:

$$\begin{array}{ccc} Y & \to & \mathscr{P}G \\ \downarrow & & \downarrow e \\ C_nG & \underset{\mu}{\to} & G \end{array}$$

Here, $\mathscr{P}G = \{\alpha\colon (I,0) \to (G,0)\}$ is the path space of G and $e(\alpha) = \alpha(1)$. Thus, we shall regard Y as being

$$\left\{(\langle c_1,\dots,c_k;\, g_1,\dots,g_k\rangle, \alpha) \in C_nG \times \mathscr{P}G | \alpha(1) = \sum_{i=1}^{k} g_i\right\}.$$

Define a map f: $\mu^{-1}(0) \to Y$ by the formula

$$f(\langle c_1,\dots,c_k;\, g_1,\dots,g_k\rangle) = (\langle c_1,\dots,c_k;\, g_1,\dots,g_k\rangle, \kappa_0),$$

where κ_0 is the constant path at $0 \in G$.

We intend to show that f is a homotopy equivalence by constructing an explicit homotopy inverse. However, it takes some work to set this up.

Given $0 \leqslant t \leqslant 1$, define ϕ_t: $R \to R$ by $\phi_t(x) = (1 - t)x + (2t/\pi)\arctan(x)$. Note that $d(\phi_t(x))/dx = 1 - t + 2t/\pi(1 + x^2) > 0$, so ϕ_t is one-to-one. Hence, the maps ϕ_t describe an isotopy of R: observe that $\phi_0 = \mathrm{id}_R$ and that ϕ_1 is a homeomorphism of R onto the interval $-1 < x < 1$.

Next, define an isotopy ψ_t: $R^n \to R^n$ by the equation

$$\psi_t(x) = \begin{cases} \phi_t(|x|)x/|x| & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Since $|\psi_t(x)| = \phi_t(|x|)$, ψ_t is continuous at 0. Again, ψ_t is one-to-one. This time, $\psi_0 = \mathrm{id}_{R^n}$ and ψ_1 gives a homeomorphism of R^n onto $\{x \in R^n | \, |x| < 1\}$. As t goes from 0 to 1, one can envision ψ_t as taking points in R^n and radially retracting them inside the unit disk.

Let $\bar{2} \in R^n$ denote the point $(2, 0, 0, \dots, 0)$.

Finally, define h: $Y \to \mu^{-1}(0)$ by the formula

$$h(\langle c_1,\dots,c_k;\, g_1,\dots,g_k\rangle, \alpha) = \left\langle \psi_1 c_1,\dots,\psi_1 c_k, \bar{2};\, g_1,\dots,g_k, -\sum_{i=1}^{k} g_i\right\rangle.$$

The whole point is that f and h are homotopy inverses, and this is what we now show.

(i) $hf \simeq \mathrm{id}_{\mu^{-1}(0)}$. By tracing through the definitions, one computes that

$$\begin{aligned} hf(\langle c_1,\ldots,c_k;\, g_1,\ldots,g_k\rangle) &= \left\langle \psi_1 c_1,\ldots,\psi_1 c_k, \bar{2};\, g_1,\ldots,g_k, -\textstyle\sum g_i\right\rangle \\ &\sim \langle \psi_1 c_1,\ldots,\psi_1 c_k;\, g_1,\ldots,g_k\rangle \end{aligned}$$

since, by assumption, $\Sigma g_i = 0$.

There is an evident homotopy $G_t\colon \mu^{-1}(0) \to \mu^{-1}(0)$ between hf and $\mathrm{id}_{\mu^{-1}(0)}$, namely,

$$G_t(\langle c_1,\ldots,c_k;\, g_1,\ldots,g_k\rangle) = \langle \psi_t c_t,\ldots,\psi_t c_k;\, g_1,\ldots,g_k\rangle.$$

(ii) $fh \simeq \mathrm{id}_Y$. Now,

$$fh(\langle c_1,\ldots,c_k;\, g_1,\ldots,g_k\rangle, \alpha) = \left(\left\langle \psi_1 c_1,\ldots,\psi_1 c_k, \bar{2};\, g_1,\ldots,g_k, -\textstyle\sum g_i\right\rangle, \kappa_0\right).$$

Given $\alpha \in \mathscr{P}G$ and $0 \leqslant y \leqslant 1$, let α_y be the element of $\mathscr{P}G$ defined by $\alpha_y(s) = \alpha(sy)$. In particular, $\alpha_0 = \kappa_0$ and $\alpha_1 = \alpha$; also, $\alpha_y(1) = \alpha(y)$.

Next, define a homotopy $H_t\colon Y \to Y$ by the equations

$$\begin{aligned} &H_t(\langle c_1,\ldots,c_k;\, g_1,\ldots,g_k\rangle, \alpha) \\ &\quad= \begin{cases} (\langle \psi_{2t} c_1,\ldots,\psi_{2t} c_k;\, g_1,\ldots,g_k\rangle, \alpha) & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ \left(\left\langle \psi_1 c_1,\ldots,\psi_1 c_k, \bar{2};\, g_1,\ldots,g_k, \alpha(2-2t) - \sum g_i\right\rangle, \alpha_{2-2t}\right) & \text{if } \frac{1}{2} \leqslant t \leqslant 1. \end{cases} \end{aligned}$$

Observe that H_t is well defined when $t = \frac{1}{2}$ by applying the identification

$$\begin{aligned} (\langle \psi_1 c_1,\ldots,\psi_1 c_k;\, g_1,\ldots,g_k\rangle, \alpha) &\sim (\langle \psi_1 c_1,\ldots,\psi_1 c_k, \bar{2};\, g_1,\ldots,g_k, 0\rangle, \alpha) \\ &= \left(\left\langle \psi_1 c_1,\ldots,\psi_1 c_k, \bar{2};\, g_1,\ldots,g_k, \alpha(1) - \textstyle\sum g_i\right\rangle, \alpha_1\right). \end{aligned}$$

Furthermore, $H_0 = \mathrm{id}_Y$ and $H_1 = fh$.

This completes the proof that Y and $\mu^{-1}(0)$ are homotopy equivalent. □

We shall apply (1.2) to the special case $G = S^1$. As usual, the group operation of S^1 will be written multiplicatively; the identity element will be denoted by 1.

Recall the fibration $S^{n+1}\langle n+1\rangle \to S^{n+1} \to K(Z, n+1)$ described at the beginning of this section. Looping this n times yields another fibration

$$\Omega^n\left(S^{n+1}\langle n+1\rangle\right) \xrightarrow{p} \Omega^n S^{n+1} \xrightarrow{m} \Omega^n K(Z, n+1) = K(Z,1) = S^1.$$

The map m above is of degree one in the sense that it represents a generator of $[\Omega^n S^{n+1}, S^1] = H^1(\Omega^n S^{n+1}; Z) = Z$.

Now, by May's approximation theorem, $C_n S^1 = \Omega^n \Sigma^n(S^1) = \Omega^n S^{n+1}$. In terms of the May model, the map $m\colon \Omega^n S^{n+1} \to S^1$ corresponds to the map $\mu_{n,S^1}\colon C_n S^1 \to S^1$ discussed in (1.2). (μ_{n,S^1} has degree one since it is split by the obvious map $x \mapsto \langle 0; x\rangle$.)

Thus, there is a map of fibrations

$$\begin{array}{ccc} Y_{n,S^1} & \xrightarrow{\beta'} & (S^{n+1}\langle n+1\rangle) \\ \downarrow & & \downarrow \\ C_n S^1 & \xrightarrow{\beta} & \Omega^n S^{n+1} \\ \mu\downarrow & & \downarrow m \\ S^1 & = & S^1 \end{array}$$

where β is May's equivalence and β' is the induced map of homotopy fibres. By the Five Lemma, β' induces an isomorphism on homotopy groups. Hence, (1.2) allows us to describe the homotopy type of $\Omega^n(S^{n+1}\langle n+1\rangle)$ in terms of $C_n S^1$.

THEOREM (1.3). $\Omega^n(S^{n+1}\langle n+1\rangle)$ *is homotopy equivalent to*

$$\mu^{-1}(1) = \left\{\langle c_1,\ldots,c_k;\, x_1,\ldots,x_k\rangle \in C_n S^1 \,\middle|\, \prod_{i=1}^{k} x_i = 1\right\}. \quad \square$$

REMARKS. (1) Of peculiar interest may be the case $n = 1$. Then (1.3) describes $\Omega(S^2\langle 2\rangle) = \Omega S^3$ as $Z = \{x_1 \cdots x_k \in J(S^1) \mid \prod_{i=1}^{k} x_i = 1\}$, where $J(S^1)$ is the James reduced product of S^1. (The expression $x_1 \cdots x_k$ is meant to denote a "word" of length k, not a multiplication in S^1.) From this point of view, the inclusion $Z \subset J(S^1)$ corresponds to the looped Hopf map $\Omega\eta\colon \Omega S^3 \to \Omega S^2$.

Of course, the usual model for ΩS^3 is the James reduced product $J(S^2)$.

(2) A version of (1.3) can also be formulated when $n = \infty$. Namely, let $m\colon \Omega^\infty\Sigma^\infty(S^1) \to S^1$ be a map of degree one. Then

$$\left\{\langle c_1,\ldots,c_k;\, x_1,\ldots,x_k\rangle \in C_\infty S^1 \,\middle|\, \prod_{i=1}^{k} x_i = 1\right\}$$

is a model for the homotopy fibre of m.

2. The spectra $B_1(k)$—elementary properties. We now introduce the spectra $B_1(k)$ whose study comprises the bulk of this work. These spectra are closely related to a family of spectra $B(k)$ first constructed by Brown and Gitler in **[3]**, and the last part of this section is devoted to establishing some simple connections between $B_1(k)$ and $B(k)$. For instance, thanks to a theorem of R. Cohen, the definition of $B_1(k)$ leads to an obvious map $i_k\colon B_1(k) \to B(k)$. We shall compute $H^*B_1(k)$ as a module over the mod 2 Steenrod algebra and then show that $(i_k)^*\colon H^*B(k) \to H^*B_1(k)$ is a certain natural projection. (It amounts to killing Sq^1.) In addition, we will show that $M_2 \wedge B_1(2k) \simeq B(2k+1)$, where M_2 is the Z_2 Moore spectrum. This will allow us to prove the existence of a cofibration of spectra $B_1(k-1) \to B_1(k) \xrightarrow{i_k} B(k)$. As a result, $B_1(k)$ can be realized as a cofibre; that is, there is a map $f\colon \Sigma^{-1}B(k) \to B_1(k-1)$ such that $B_1(k) \simeq B_1(k-1) \cup_f C(\Sigma^{-1}B(k))$.

The spectra $B_1(k)$ are defined in §2.2; up until then, we record some necessary background information.

2.1. *Preliminaries*. All homology and cohomology should be taken with coefficients in Z_2, the integers mod 2, unless otherwise noted.

Let A denote the mod 2 Steenrod algebra, and let $\chi\colon A \to A$ be the canonical antiautomorphism. In what follows, we will often be concerned with studying certain cyclic modules over A, as described by

DEFINITION (2.1). Given $k \geqslant 0$, let

(a) $M(k) = A/A\{\chi(\mathrm{Sq}^i) \mid i > k\}$; and

(b) $M_1(k) = A/A\{\mathrm{Sq}^1, \chi(\mathrm{Sq}^i) \mid i > k\}$.

REMARKS. (1) One can also write $M_1(k)$ as $M(k) \otimes_{A_0} Z_2$, where A_0 is the exterior subalgebra of A generated by Sq^1 and where the right action of A_0 on $M(k)$ is induced by right multiplication.

(2) Since $\chi(\mathrm{Sq}^{2k+1}) = \chi(\mathrm{Sq}^1\,\mathrm{Sq}^{2k}) = \chi(\mathrm{Sq}^{2k})\mathrm{Sq}^1$, one easily sees that $M_1(2k) = M_1(2k+1)$. Indeed, many of the definitions that we propose will contain this type of redundancy. However, we will continue to allow for both odd and even cases, as it will help make certain results (e.g., (2.15), (4.9)) easier to state.

The additive structures of $M(k)$ and $M_1(k)$ are not hard to determine:

THEOREM (2.2). (a) *$M(k)$ has an additive basis given by $\{\chi(\mathrm{Sq}^I) \mid \mathrm{Sq}^I$ is admissible, $I = (i_1,\ldots,i_t)$, and $i_1 \leqslant k\}$.*

(b) *$M_1(k)$ has an additive basis given by $\{\chi(\mathrm{Sq}^I) \mid \mathrm{Sq}^I$ is admissible, $I = (i_1,\ldots,i_t)$, $i_1 \leqslant k$, and $i_1 = 0 \pmod 2\}$.*

PROOF. (a) is a straightforward consequence of the Adem relations.

To prove (b), note first that if $i_1 = 1 \pmod 2$, then

$$\chi(\mathrm{Sq}^I) = \chi(\mathrm{Sq}^1\,\mathrm{Sq}^{i_1-1}\,\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_t}) = \chi(\mathrm{Sq}^{i_1-1}\,\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_t})\,\mathrm{Sq}^1 = 0$$

in $M_1(k)$. To see that the elements described in (b) are linearly independent, observe that $\chi(\mathrm{Sq}^{2i+1})\,\mathrm{Sq}^1 = \chi(\mathrm{Sq}^1\,\mathrm{Sq}^{2i+1}) = 0$, so that, in order to have $\chi(\mathrm{Sq}^I)$ in $A\{\mathrm{Sq}^1\}$, i_1 must be odd. This, along with the Adem relations, implies (b). $\square$

A good reason for studying the module $M(k)$ is supplied by the work of Brown and Gitler. They constructed a spectrum $B(k)$, now known as the "Brown-Gitler spectrum," one of whose properties is

THEOREM (2.3) [**3**]. *$H^*(B(k)) = M(k)$ as A-modules.* $\square$

$M(k)$ also made a rather unexpected appearance in certain constructions of Mahowald, which we now review.

Let $h\colon S^1 \to BO$ represent the generator of $\pi_1 BO = Z_2$. h admits a canonical extension $\Omega^2\Sigma^2 h\colon \Omega^2\Sigma^2 S^1 = \Omega^2 S^3 \to \Omega^2\Sigma^2 BO$, and, since BO is an infinite loop space, there is a retraction $r\colon \Omega^2\Sigma^2 BO \to BO$. Let $\gamma\colon \Omega^2 S^3 \to BO$ be the composite

$$\Omega^2 S^3 \xrightarrow{\Omega^2\Sigma^2 h} \Omega^2\Sigma^2 BO \xrightarrow{r} BO.$$

Mahowald [**14**, Corollary 4.5] made the striking observation that $T(\gamma) \simeq K(Z_2)$, where $T(\gamma)$ is the Thom spectrum associated to γ, and $K(Z_2)$ is the Z_2 Eilenberg-Mac Lane spectrum, both normalized to have bottom homology class in dimension zero.

Let C_2S^1 be May's configuration space model for Ω^2S^3, as described in §1. C_2S^1 has an obvious filtration, namely, in the notation of §1,

$$F_j(C_2S^1) = \coprod_{k=1}^{j} F(R^2, k) \times_{\Sigma_k} (S^1)^k / \sim .$$

By May's approximation theorem, $C_2S^1 \simeq \Omega^2S^3$, and hence we can also regard Ω^2S^3 as a filtered space. Abbreviate the jth stage of this filtration to F_j, so that $\Omega^2S^3 = \bigcup F_j$, $* = F_0 \subset F_1 \subset F_2 \subset \dots$.

Let $\gamma_k = \gamma|_{F_k}$, i.e., γ_k is the composite $F_k \subset \Omega^2S^3 \xrightarrow{\gamma} BO$. The possibility of realizing the Brown-Gitler spectrum $B(k)$ as the Thom spectrum $T(\gamma_k)$ was suggested by a calculation of Mahowald and then concluded by a proof of R. Cohen.

THEOREM (2.4) (MAHOWALD [**14**]). *$H^*(T(\gamma_k)) = M(k)$ as A-modules.* □

THEOREM (2.5) (R. COHEN [**9**]). *$T(\gamma_k)$ is homotopy 2-equivalent to $B(k)$.* □

(We should point out that, due to the manner in which it is constructed, $B(k)$ is trivial at odd primes.)

REMARKS. (1) For our purposes, two spectra will be called homotopy 2-equivalent if their 2-completions are homotopy equivalent. A suitable reference on completions is Bousfield and Kan [**2**, especially Chapters I, VI]. The 2-completion we shall use is what Bousfield and Kan refer to as the "Z_2-completion."

(2) In [**14**], Mahowald also studied the filtered quotients F_k/F_{k-1} of the May decomposition of Ω^2S^3. It is easy to see that F_k/F_{k-1} is the Thom space of the k-plane bundle

$$F(R^2, k) \times_{\Sigma_k} (R^1)^k \to F(R^2, k) \times_{\Sigma_k} (\text{pt.}).$$

Mahowald showed that $H^*(F_k/F_{k-1}) = \Sigma^k M([k/2])$. Brown and Peterson [**6**] then followed up on this by proving that F_k/F_{k-1} gives another realization of the Brown-Gitler spectrum, i.e., F_k/F_{k-1} is homotopy 2-equivalent to $\Sigma^k B([k/2])$.

Let $x_1 \in H_1(\Omega^2S^3)$ be the generator and set $x_i = (Q_1)^{i-1}(x_1)$ in the usual lower-index Dyer-Lashof notation. Given a monomial $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, define the weight of m, wt(m), to be $\sum_{j=1}^{n} \alpha_j 2^{j-1}$. The following theorem contains well-known results of May and F. Cohen.

THEOREM (2.6) [**7**, I, III]. (a) $H_*(\Omega^2S^3) = Z_2[x_1, x_2, \dots]$, *where* $x_i \in H_{2^i-1}(\Omega^2S^3)$.
(b) *$H_*(F_k)$ has a Z_2-basis consisting of all monomials m such that* wt$(m) \leqslant k$. □

Now recall the fibration $\Omega^2S^3\langle 3\rangle \xrightarrow{p} \Omega^2S^3 \xrightarrow{m} S^1$ of §1, where $S^3\langle 3\rangle$ is the 3-connective cover of S^3. If α: $S^1 \to \Omega^2S^3$ represents a generator of $\pi_1(\Omega^2S^3) = Z$, then it is easy to check that $(\alpha + p)$: $S^1 \times \Omega^2S^3\langle 3\rangle \to \Omega^2S^3$ induces an isomorphism on homotopy groups and hence is a homotopy equivalence. ("+" above denotes the loop sum in Ω^2S^3.) Thus, $H_*(\Omega^2S^3) = H_*S^1 \otimes H_*(\Omega^2S^3\langle 3\rangle)$. Moreover, the map p: $\Omega^2S^3\langle 3\rangle \to \Omega^2S^3$ is a double loop map so that

$$p_*: H_*(\Omega^2S^3\langle 3\rangle) \to H_*(\Omega^2S^3)$$

is a ring homomorphism which commutes with Q_1. In light of (2.6), this establishes

THEOREM (2.7). $H_*(\Omega^2 S^3\langle 3\rangle) = Z_2[x_1^2, x_2, x_3, \ldots] \subset H_*(\Omega^2 S^3)$. □

Note that the set of all monomials of even weight forms an additive basis for $H_*(\Omega^2 S^3\langle 3\rangle)$.

One outgrowth of (2.6) is that $H_*(\Omega^2 S^3)$ and the Steenrod algebra A are isomorphic as graded vector spaces; the link between them is provided by Milnor's analysis of the dual of the Steenrod algebra [**17**]. Milnor's isomorphism will come in handy later, and so now we briefly recall how it is defined.

Let $\mathscr{M}$ be the set of all sequences of nonnegative integers having finitely many nonzero entries: $J = (\alpha_1, \ldots, \alpha_n, 0, 0, \ldots)$. Let $\mathscr{N}$ be the set of all sequences I such that Sq^I is admissible in A. Define $\theta\colon \mathscr{M} \to \mathscr{N}$ by $\theta(\alpha_1, \ldots, \alpha_n, 0, 0, \ldots) = (i_1, \ldots, i_n)$, where $i_q = \sum_{j=q}^{n} 2^{j-q}\alpha_j$; for instance, $i_1 = \alpha_1 + 2\alpha_2 + 4\alpha_3 + \cdots + 2^{n-1}\alpha_n$.

LEMMA (2.8). *$\theta\colon \mathscr{M} \to \mathscr{N}$ is a bijection of sets.*

PROOF. θ has an inverse $\phi\colon \mathscr{N} \to \mathscr{M}$ defined by

$$\phi(i_1, \ldots, i_n) = (i_1 - 2i_2, \ldots, i_{n-1} - 2i_n, i_n, 0, 0, \ldots). \quad \square$$

Next, given a monomial $m = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, define $\theta_*(m) = \chi(\mathrm{Sq}^{\theta(\alpha_1, \ldots, \alpha_n, 0, 0, \ldots)})$. Extending this by Z_2-linearity, one obtains a vector space homomorphism $\theta_*\colon H_*(\Omega^2 S^3) \to A$.

LEMMA (2.9). (a) *θ_* is an isomorphism of graded vector spaces.*
(b) *If $\theta_*(m) = \chi(\mathrm{Sq}^I)$ as above and $I = (i_1, \ldots, i_n)$, then $i_1 = \mathrm{wt}(m)$.*

PROOF. Checking that θ_* preserves graded dimension is a triviality. Also, (2.8) implies that θ induces a one-to-one correspondence between basis elements, and this proves (a). (b) follows directly from the definition of θ. □

We now return to the study of Thom spectra.

Let $g\colon \Omega^2 S^3\langle 3\rangle \to BO$ denote the composition $\Omega^2 S^3\langle 3\rangle \xrightarrow{p} \Omega^2 S^3 \xrightarrow{\gamma} BO$. Mahowald [**15**] recognized that the Thom spectrum $T(g)$ was another familiar object:

THEOREM (2.10). *$T(g)$ is homotopy 2-equivalent to the Eilenberg-Mac Lane spectrum $K(Z)$.*

PROOF. Since $H^1(\Omega^2 S^3\langle 3\rangle) = 0$, the map g lifts to $\tilde{g}\colon \Omega^2 S^3\langle 3\rangle \to BSO$, i.e., $T(g)$ has an integral Thom class. Let $U\colon T(g) \to K(Z)$ represent this class. We shall prove that U is a 2-equivalence by showing that it induces an isomorphism in mod 2 cohomology.

The map $p\colon \Omega^2 S^3\langle 3\rangle \to \Omega^2 S^3$ induces a map on the level of Thom spectra $T(p)\colon T(g) \to T(\gamma) \simeq K(Z_2)$ which is nontrivial; thus, $T(p)$ represents the mod 2 Thom class. This gives a commutative diagram

$$\begin{tikzcd} & K(Z) \arrow[d, "r"] \\ T(g) \arrow[ur, "U"] \arrow[r, "T(p)"'] & K(Z_2) \end{tikzcd}$$

where r represents reduction mod 2. Hence, there is a diagram in cohomology:

$$\begin{tikzcd} & H^*(K(Z)) = A/A\{\mathrm{Sq}^1\} \arrow[dl, "U^*"'] \\ H^*(T(g)) & H^*(K(Z_2)) = A \arrow[l, "T(p)^*"] \arrow[u, "r^*"'] \end{tikzcd}$$

Now, $p_*: H_*(\Omega^2 S^3\langle 3\rangle) \to H_*(\Omega^2 S^3)$ is injective, so, after applying the Thom isomorphism, $T(p)^*$ must be surjective in cohomology. This in turn implies that U^*: $A/A\{\mathrm{Sq}^1\} \to H^*(T(g))$ is surjective. Thus, to prove that U^* is an isomorphism, it suffices to show that $A/A\{\mathrm{Sq}^1\}$ and $H^*(T(g))$ have the same rank over Z_2 in each graded dimension.

By the Thom isomorphism, $H^*(T(g)) = H_*(\Omega^2 S^3\langle 3\rangle)$ as graded vector spaces. Recall that $H_*(\Omega^2 S^3\langle 3\rangle)$ has a basis consisting of monomials of even weight. On the other hand, by (2.2)(b) when $k = \infty$, $A/A\{\mathrm{Sq}^1\}$ has a basis consisting of $\chi(\mathrm{Sq}^{(i_1,\ldots,i_n)})$ with i_1 even. A one-to-one dimension-preserving correspondence between these basis elements is provided by (2.9), and this completes the proof. □

The idea of using (2.8) in conjunction with the Thom isomorphism for proofs of this type seems to be due to F. Cohen, May and Taylor [**8**, p. 105].

2.2. *The spectra* $B_1(k)$. We now commence the study of the spectra $B_1(k)$. The definition to be given was originally presented by Mahowald [**15**, p. 554]. (See the appendix for an alternative, non-Thom spectrum definition.)

In view of the homotopy equivalence $\Omega^2 S^3 \simeq S^1 \times \Omega^2 S^3\langle 3\rangle$, Mahowald allows the May filtration on $\Omega^2 S^3$ to induce a filtration on $\Omega^2 S^3\langle 3\rangle$. Let $W_{2n} \subset \Omega^2 S^3\langle 3\rangle$ denote the stage of this induced filtration such that $W_{2n} \subset F_{2n}$ and $H_* W_{2n} = H_* F_{2n} \cap H_*(\Omega^2 S^3\langle 3\rangle)$. Also, define W_{2n+1} to be equal to W_{2n}. In this case, by (2.6)(b) and (2.7), the preceding homological equation is still true, i.e., $H_* W_{2n+1} = H_* F_{2n+1} \cap H_*(\Omega^2 S^3\langle 3\rangle)$. Either way, $H_* W_k$ has an additive basis comprised of those monomials m in the x_i which satisfy $\mathrm{wt}(m) \leqslant k$ and $\mathrm{wt}(m) = 0 \pmod 2$.

Note that, for any k, there are "canonical" inclusions $W_k \subset F_k$, namely, either $W_{2n} \subset F_{2n}$ or $W_{2n+1} = W_{2n} \subset F_{2n} \subset F_{2n+1}$.

Finally, let g_k: $W_k \to BO$ denote the composite

$$W_k \subset F_k \xrightarrow{\gamma_k} BO.$$

DEFINITION (2.11). The spectrum $B_1(k)$ is defined to be the Thom spectrum $T(g_k)$, completed at 2.

REMARKS. (1) $B_1(2k) = B_1(2k+1)$. (The definition was not meant to disguise this.)

(2) The spectra $B_1(k)$ played an essential role in Mahowald's presentation of *bo*-resolutions in [**13**], as well as in a subsequent elaboration of that exposition by Davis, Gitler and Mahowald [**10**].

Induced over the inclusion $W_k \subset F_k$ is a map of Thom spectra $T(g_k) \to T(\gamma_k)$. Completing this at 2 and then using R. Cohen's Theorem (2.5), one obtains a map i_k: $B_1(k) \to B(k)$.

In view of the relationship between $B(k)$ and $K(Z_2)$ as Thom spectra, one can look at the realization of $K(Z)$ in (2.10) and then begin to regard $B_1(k)$ as being what an integral version of the Brown-Gitler spectrum should look like, at least as far as 2-primary information is concerned. Such an interpretation is enhanced by

THEOREM (2.12). *$H^*(B_1(k)) = M_1(k)$ as A-modules. Furthermore, $(i_k)^*$: $H^*(B(k)) \to H^*(B_1(k))$ is the obvious projection π: $M(k) \to M_1(k)$.*

PROOF. The arguments here are similar in spirit to those given in the proof of (2.10).

The inclusion $W_k \subset F_k$ induces a homomorphism $H_* W_k \to H_* F_k$ which is injective. Thus, by dualizing and translating to the Thom spectrum level via the Thom isomorphism, we conclude that $(i_k)^*$: $H^*B(k) \to H^*B_1(k)$ is surjective. Moreover,

$$\begin{aligned} H^1 B_1(k) &= H^1 W_k \quad \text{(by the Thom isomorphism)} \\ &= (H_1 W_k)^* = 0, \end{aligned}$$

so that $(i_k)^* \operatorname{Sq}^1 = 0$. This implies that $(i_k)^*$ factors

$$\begin{tikzcd} & M_1(k) \arrow[dr, "w"] & \\ H^*B(k) = M(k) \arrow[ur, "\pi"] \arrow[rr, "(i_k)^*"'] & & H^*B_1(k), \end{tikzcd}$$

where w must also be surjective.

As shown in (2.2)(b), $M_1(k)$ has a Z_2-basis given by those $\chi(\operatorname{Sq}^I)$ with $i_1 \leqslant k$ and $i_1 = 0 \pmod 2$. On the other hand, $H^*B_1(k) = H_* W_k$ as vector spaces by the Thom isomorphism, and we know that $H_* W_k$ has a basis given by monomials m with $\operatorname{wt}(m) \leqslant k$ and $\operatorname{wt}(m) = 0 \pmod 2$. Thus, (2.9) shows that $M_1(k)$ and $H^*B_1(k)$ have the same rank over $\mathbf{Z}_2$ in each graded dimension, and this implies that w must be an isomorphism.

The assertion in the theorem concerning $(i_k)^*$ is forced by the fact that $(i_k)^*$ is A-linear. □

The next lemma is easily verified.

LEMMA (2.13). *The following sequence of A-modules is exact*:

$$0 \to M_1(k-1) \xrightarrow{\beta} M(k) \xrightarrow{\pi} M_1(k) \to 0,$$

where $\beta(1) = \operatorname{Sq}^1$ and $\pi(1) = 1$.

PROOF. Use the description of the bases for these modules given in (2.2). □

We shall show that this exact sequence can be realized by a cofibration. To do this, let M_2 denote the stable Z_2 Moore complex $S^0 \cup_2 e^1$.

LEMMA (2.14). $M_2 \wedge B_1(2k) \simeq B(2k+1)$.

PROOF. This fact seems first to have been noticed by Mahowald [**15**].

Consider the composition c: $S^1 \times W_{2k} \subset F_1 \times F_{2k} \to F_{2k+1}$, the last map coming from the multiplicative properties of the May filtration of $\Omega^2 S^3$ [**16**]. It is easy to check that c induces an isomorphism in homology.

Next, there is a commutative diagram

$$\begin{tikzcd} S^1 \times W_{2k} \arrow[rr, "c"] \arrow[dr, "h + g_{2k}"'] & & F_{2k+1} \arrow[dl, "\gamma_{2k+1}"] \\ & BO & \end{tikzcd}$$

where $h\colon S^1 \to BO$ represents the nontrivial element of $\pi_1 BO$. The Thom spectrum $T(h)$ is the Moore spectrum M_2. Thus, c induces a map $T(c)\colon M_2 \wedge B_1(2k) \to B(2k+1)$ which is an isomorphism in homology and, consequently, must be an equivalence. $\square$

THEOREM (2.15). *There is a cofibration*

$$B_1(k-1) \to B_1(k) \xrightarrow{i_k} B(k)$$

whose long exact sequence in cohomology realizes (2.13).

PROOF. *Case* 1. $k = 2n+1$. Let $S^0 \xrightarrow{2} S^0 \xrightarrow{j} M_2$ be the usual cofibration. Since cofibrations are preserved by smash products, one obtains a cofibration

$$S^0 \wedge B_1(2n) \xrightarrow{2 \wedge \mathrm{id}} S^0 \wedge B_1(2n) \xrightarrow{j \wedge \mathrm{id}} M_2 \wedge B_1(2n),$$

which can be rewritten

$$B_1(2n) \xrightarrow{2} B_1(2n+1) \xrightarrow{j \wedge \mathrm{id}} B(2n+1)$$

using (2.14). (Recall that $B_1(2n) = B_1(2n+1)$.) Tracing through the arguments given in (2.14), it is easy to check that $j \wedge \mathrm{id}$ is the same as the standard map i_{2n+1}, as desired.

Case 2. $k = 2n$. Let X denote the cofibre of $i_{2n}\colon B_1(2n) \to B(2n)$. Since $(i_{2n})^*$ is surjective, the long exact sequence in cohomology of $B_1(2n) \xrightarrow{i_{2n}} B(2n) \to X$ breaks up into a series of short exact sequences

$$0 \to H^q X \to H^q B(2n) \xrightarrow{(i_{2n})^*} H^q B_1(2n) \to 0.$$

Thus, by (2.13), $H^*X = \Sigma M_1(2n-1)$.

Over the obvious maps of base spaces

$$\begin{array}{ccc} W_{2n-1} & \subset & F_{2n-1} \\ \cap & & \cap \\ W_{2n} & \subset & F_{2n} \end{array}$$

there are maps of Thom spectra:

$$\begin{tikzcd} B_1(2n-1) \arrow[r, "i_{2n-1}"] \arrow[d] & B(2n-1) \arrow[r] \arrow[d, "\zeta"] & \Sigma B_1(2n-2) \arrow[d, "\bar{\zeta}"] \\ B_1(2n) \arrow[r, "i_{2n}"] & B(2n) \arrow[r] & X \end{tikzcd}$$

Here, $\bar{\zeta}$ is the induced map of cofibres, the cofibre of i_{2n-1} having been identified as $\Sigma B_1(2n-2)$ in Case 1. We will prove that $\bar{\zeta}$ is an equivalence by showing that it induces an isomorphism in cohomology.

Since $H^*X = \Sigma M_1(2n-1) = \Sigma M_1(2n-2) = H^*\Sigma B_1(2n-2)$ and since these modules are cyclic over A, it suffices by A-linearity to check that $\bar{\zeta}^*$: $H^1X \to H^1\Sigma B_1(2n-2)$ is an isomorphism, i.e., that $\bar{\zeta}^*(1) = 1$. But, by using the Thom isomorphism, it is easy to see that ζ^*: $H^1B(2n) \to H^1B(2n-1)$ satisfies $\zeta^*(\mathrm{Sq}^1) = \mathrm{Sq}^1$. The desired result follows from the exact sequences (2.13).

The upshot of all this is that $X \simeq \Sigma B_1(2n-2) = \Sigma B_1(2n-1)$, so that there is a cofibration

$$B_1(2n) \xrightarrow{i_{2n}} B(2n) \to \Sigma B_1(2n-1).$$

This completes the proof. □

REMARKS. (1) The proof of Case 1 shows that, when k is odd, the cofibration of (2.15) should really be written

$$B_1(2n+1) \xrightarrow{2} B_1(2n+1) \to B(2n+1).$$

(2) One implication of (2.15) is that the Brown-Gitler spectrum $B(k)$ can be realized as a cofibre. Namely, there is a map g: $B_1(k-1) \to B_1(k)$ of degree 2 on the bottom cell such that $B(k) \simeq B_1(k) \cup_g C(B_1(k-1))$.

Brown and Peterson [6] obtained $B(k)$ as a cofibre in a different way. Specifically, they proved that there exist cofibrations of the form $\Sigma^{k-1}B([k/2]) \xrightarrow{h} B(k-1) \to B(k)$ and hence

$$B(k) \simeq B(k-1) \cup_h C(\Sigma^{k-1}B([k/2])).$$

3. Some families of subcomplexes of the Λ-algebra. In this section, we shall construct explicit free resolutions of the A-modules $M_1(k)$. These will be used later in the Adams spectral sequence and Postnikov analyses of $B_1(k)$. In addition, we will describe a family of closely related resolutions of $A/A\{\mathrm{Sq}^1\}$ which are needed in some of the naturality arguments in §5.

3.1. *A resolution of $M_1(k)$.* Each of the resolutions to be discussed in this section will be a subcomplex of a certain fixed A-resolution of Z_2, the so-called "Λ-algebra" resolution. To recall, let $\bar{\Lambda}$ be the free associative Z_2-algebra with unit generated by λ_i, $i = -1, 0, 1, \ldots$, modulo the relations

$$\lambda_i\lambda_j = \sum \binom{s-1}{2s-(j-2i)} \lambda_{i+s}\lambda_{j-s} \quad \text{whenever } 2i < j. \tag{3.1}$$

Then let $\Lambda = \bar{\Lambda}/\bar{\Lambda}\{\lambda_{-1}\}$; Λ is the algebra introduced by Bousfield et al. in [1].

Grade Λ by setting $\dim \lambda_i = i$.

If $I = (i_1, i_2, \ldots, i_q)$, let $\lambda_I = \lambda_{i_1}\lambda_{i_2} \cdots \lambda_{i_q}$. Also, define $l(I) = q$ and $t(I) = i_q$. The sequence I is called admissible if $t(I) \geq 0$ and $2i_j \geq i_{j+1}$ for $j = 1, 2, \ldots, q-1$. (By convention, if $I = (\ )$, then $\lambda_I = 1$, $l(I) = 0$, $t(I) = \infty$, and I is considered admissible.) In [1], it is shown that

LEMMA (3.2). *$\{\lambda_I | I$ admissible$\}$ is an additive basis for Λ.* □

Next, let $\Lambda^* = \mathrm{Hom}_{Z_2}(\Lambda, Z_2)$ be the graded vector space dual and let $\{\lambda^I\}$ denote the dual basis with respect to the basis of (3.2). On the free A-module $A \otimes \Lambda^*$, there is an A-linear differential $d\colon A \otimes \Lambda^* \to A \otimes \Lambda^*$ which acts on basis elements according to the formula

$$d(1 \otimes \lambda^I) = \sum \lambda^I(\lambda_j \lambda_J)\chi(\mathrm{Sq}^{j+1}) \otimes \lambda^J. \tag{3.3}$$

The summation on the right runs over all $j \geqslant -1$ and all basis elements λ^J. From now on, given $a \in A$ and $\lambda \in \Lambda^*$, the symbol $a \otimes \lambda$ will be abbreviated to $a\lambda$.

LEMMA (3.4). *If $\lambda^I(\lambda_j\lambda_J) \neq 0$, then $t(J) \geqslant t(I)$ and $l(J) = l(I) - 1$.*

PROOF. This follows by induction on $l(I)$, using the relations (3.1). □

For the remainder of this section, let $k \geqslant 0$ be a fixed integer.

We now recall the Brown-Gitler resolution of $M(k) = A/A\{\chi(\mathrm{Sq}^i) | i > k\}$. Let

$$D_q\left(= D_q(k)\right) = A \otimes \{\lambda^I | I \text{ admissible}, l(I) = q, t(I) \geqslant k\},$$

viewed as an A-submodule of $A \otimes \Lambda^*$. In particular, $D_0 = A \otimes \{\lambda^{()}\} = A$. From (3.4), it follows immediately that $d(D_q) \subset D_{q-1}$. Brown and Gitler proved

THEOREM (3.5) [3]. *The sequence*

$$\cdots \to D_q \xrightarrow{d} D_{q-1} \to \cdots \to D_0 \xrightarrow{\varepsilon} M(k) \to 0$$

is exact, where $\varepsilon(\lambda^{()}) = 1$. □

Note that $M(0) = Z_2$ so that, when $k = 0$, (3.5) gives an A-free resolution of Z_2. The resolution we will construct for $M_1(k)$ lies between this full Λ-algebra resolution of Z_2 and the Brown-Gitler resolution of $M(k)$.

Given sequences $I = (i_1, \ldots, i_q)$ and $J = (j_1, \ldots, j_r)$, let $(I, J) = (i_1, \ldots, i_q, j_1, \ldots, j_r)$ and $\lambda^I\lambda^J = \lambda^{(I,J)}$.

DEFINITION (3.6). The sequence I is called k-acceptable if $I = (I', Z)$ where $l(I') \geqslant 0$, $t(I') \geqslant k - 1$ if k is even, $t(I') \geqslant k$ if k is odd, $Z = (0, \ldots, 0)$, and $l(Z) \geqslant 1$.

Now define certain free A-modules $C_q = C_q(k)$ as follows:

$$\begin{aligned} C_q = A \otimes \{\lambda^I | \, &I \text{ admissible}, l(I) = q, \text{ and} \\ &\text{either } t(I) \geqslant k \text{ or } I \text{ is } k\text{-acceptable}\}. \end{aligned} \tag{3.7}$$

We view C_q as a submodule of $A \otimes \Lambda^*$. In terms of the Brown-Gitler resolution above,

$$C_q(k) = D_q(k) \oplus \left(A \otimes \{\lambda^I | I \text{ admissible}, l(I) = q, I\ k\text{-acceptable}\}\right);$$

in particular, $D_q(k) \subset C_q(k)$. Also, $C_0 = D_0 = A \otimes \{\lambda^{()}\} = A$.

The main point of this section is that the C_q's, together with the differential d above, form a resolution of $M_1(k)$. This will be proven in Theorem (3.10); first, however, some calculations are required.

LEMMA (3.8). *Suppose that $I = (i_1, \ldots, i_q)$. Then*

$$d\lambda^{(I,0)} = (d\lambda^I)\lambda^0 + i_q\lambda^{(i_1, \ldots, i_{q-1}, i_q+1)}.$$

PROOF. Write

$$d\lambda^{(I,0)} = \sum_{t(J)>0} \lambda^{(I,0)}(\lambda_j\lambda_J)\chi(\mathrm{Sq}^{j+1})\lambda^J + \sum_{t(J)=0} \lambda^{(I,0)}(\lambda_j\lambda_J)\chi(\mathrm{Sq}^{j+1})\lambda^J.$$

Suppose that $t(J) > 0$. By the relations (3.1), if $2i < m$, then $\lambda_i\lambda_m = \Sigma\lambda_{I_\alpha}$, where each I_α is admissible and $t(I_\alpha) \geqslant 2i + 1$. Thus, in order to get $\lambda^{(I,0)}(\lambda_j\lambda_J) \neq 0$, j must equal -1. Let $J = (j_1, \ldots, j_q)$, $j_q > 0$. Once again using (3.1),

$$\lambda_{-1}\lambda_J = \lambda_{j_1}\lambda_{-1}\lambda_{j_2}\cdots\lambda_{j_q} + \left(\sum\lambda_{J_\alpha}\right)\lambda_{j_2}\cdots\lambda_{j_q},$$

where $t(J_\alpha) \geqslant 0$. The second term can be expanded in terms of admissible monomials λ_{T_γ} with $t(T_\gamma) \geqslant 1$. Consequently,

$$\lambda^{(I,0)}(\lambda_{-1}\lambda_J) = \lambda^{(I,0)}\left(\lambda_{j_1}\lambda_{-1}\lambda_{j_2}\cdots\lambda_{j_q}\right).$$

Repeating this argument, it follows that

$$\lambda^{(I,0)}(\lambda_{-1}\lambda_J) = \lambda^{(I,0)}\left(\lambda_{j_1}\lambda_{j_2}\cdots\lambda_{j_{q-1}}\lambda_{-1}\lambda_{j_q}\right).$$

Now, $\lambda_{-1}\lambda_{j_q} = (j_q - 1)\lambda_{j_q-1}\lambda_0 + \Sigma\lambda_{U_\alpha}$, where $t(U_\alpha) \geqslant 1$. Thus,

$$\lambda^{(I,0)}(\lambda_{-1}\lambda_J) = (j_q - 1)\lambda^{(I,0)}\left(\lambda_{j_1}\cdots\lambda_{j_{q-1}}\lambda_{j_q-1}\lambda_0\right).$$

This will be nonzero if and only if $J = (i_1, \ldots, i_{q-1}, i_q + 1)$ and $i_q = 1 \pmod 2$. Hence,

$$\sum_{t(J)>0} \lambda^{(I,0)}(\lambda_j\lambda_J)\chi(\mathrm{Sq}^{j+1})\lambda^J = i_q\lambda^{(i_1, i_2, \ldots, i_{q-1}, i_q+1)}.$$

Next, assume that $t(J) = 0$, that is, $J = (J', 0)$. Observe that if $\lambda_j\lambda_{J'} = \Sigma\lambda_{J_\alpha}$, where each J_α is admissible, then $\lambda_j\lambda_{(J',0)} = \Sigma\lambda_{(J_\alpha,0)}$, where each $(J_\alpha, 0)$ is admissible. Thus, $\lambda^{(I,0)}(\lambda_j\lambda_{(J',0)}) \neq 0$ if and only if $\lambda^I(\lambda_j\lambda_{J'}) \neq 0$. As a result,

$$\sum_{t(J)=0} \lambda^{(I,0)}(\lambda_j\lambda_J)\chi(\mathrm{Sq}^{j+1})\lambda^J = \sum\lambda^I(\lambda_j\lambda_{J'})\chi(\mathrm{Sq}^{j+1})\lambda^{(J',0)} = (d\lambda^I)\lambda^0.$$

Combining the preceding two paragraphs, we obtain (3.8). □

LEMMA (3.9). $d(C_q) \subset C_{q-1}$.

PROOF. It suffices to show that $d\lambda^I \in C_{q-1}$ for each basis element λ^I in C_q. For those λ^I with $t(I) \geqslant k$, this follows from (3.4). For those λ^I with I k-acceptable, it follows by iterating (3.8) and then applying (3.4). □

We are now prepared to prove that the C_q's give a free resolution of $M_1(k)$.

THEOREM (3.10). *The sequence*

$$\cdots \to C_q \xrightarrow{d} C_{q-1} \to \cdots \to C_0 \xrightarrow{\varepsilon} M_1(k) \to 0$$

is exact, where $\varepsilon(\lambda^{()}) = 1$.

PROOF. The following arguments are modelled quite closely on Brown and Gitler's proof of (3.5).

Exactness at C_0: $d\lambda^i = \chi(\mathrm{Sq}^{i+1})\lambda^{()}$, and $\{\lambda^i | i \geqslant k \text{ or } i = 0\}$ is a basis for C_1.

Exactness at C_q, $q > 0$: This resolution is a subcomplex of the full Λ-algebra resolution, so $d^2 = 0$.

Suppose that $u \in C_q$ and that $u \in \ker d$. We need to show that $u = dv$ for some $v \in C_{q+1}$.

Order all sequences of length q lexicographically from the right, that is, if $I = (i_1,\dots,i_q)$ and $J = (j_1,\dots,j_q)$, then $I > J$ if there exists $t \geqslant 1$ such that $i_t > j_t$ and $i_s = j_s$ for $s > t$. Observe that $(0,\dots,0)$, the sequence of q zeros, is the minimum element.

Given any inadmissible sequence K, one can use (3.1) and induction on $l(K)$ to show that $\lambda_K = \Sigma\lambda_{K_\alpha}$, where each K_α is admissible and $K_\alpha < K$. It then follows from the definition of d, (3.3), that, if $I = (i, J)$ is admissible, then

$$d\lambda^I = \chi(\mathrm{Sq}^{i+1})\lambda^J + \sum_{J'>J} b_{J'}\lambda^{J'}. \tag{3.11}$$

Next, express $u = \Sigma a_\alpha \lambda^{I_\alpha}$, $a_\alpha \neq 0$, where each I_α is admissible, and let $I_0 = \min I_\alpha$ with respect to the above ordering.

Case 1. $I_0 > (0,\dots,0)$. (Thus, $I_0 \geqslant (k-1,0,\dots,0)$ if k is even and $I_0 \geqslant (k,0,\dots,0)$ if k is odd.) Write $I_0 = (i, J)$. Then $u = \Sigma_{j\geqslant i} a_j\lambda^{(j,J)} + \Sigma c_{I'}\lambda^{I'}$, where $I' = (i', J')$ and $J' > J$. Hence, by (3.11),

$$0 = du = \sum_{j\geqslant i} a_j\chi(\mathrm{Sq}^{j+1})\lambda^J + \sum_{J'>J} b_{J'}\lambda^{J'}.$$

In particular, $\Sigma_{j\geqslant i} a_j\chi(\mathrm{Sq}^{j+1}) = 0$. Suppose that $a_i = \Sigma\chi(\mathrm{Sq}^{L_t})$, where each L_t is admissible in the usual Steenrod algebra sense. Then

$$\begin{aligned} 0 &= \sum_t \chi(\mathrm{Sq}^{L_t})\chi(\mathrm{Sq}^{i+1}) + \sum_{j>i} a_j\chi(\mathrm{Sq}^{j+1}) \\ &= \sum_t \chi\left(\mathrm{Sq}^{(i+1,L_t)}\right) + \sum_{j>i}\chi\left(\mathrm{Sq}^{j+1}a_j'\right), \end{aligned}$$

where $a_j' = \chi(a_j)$. According to the Adem relations, if Sq^B is admissible, then $\mathrm{Sq}^{(a,B)}$ is either admissible or can be written as a sum of Sq^{B_j}'s where the first entry of B_j is greater than a. Consequently, after expressing each a_j' in terms of admissible monomials, one sees that, in the preceding sum, the $(i+1, L_t)$ cannot all be admissible. Say that $(i+1, L_0)$ is not, so that $L_0 = (j+1, K)$, where $i \leqslant 2j$. Note that

$$d\lambda^{(j,i,J)} = d\lambda^{(j,I_0)} = \chi(\mathrm{Sq}^{j+1})\lambda^{I_0} + \sum_{T>I_0} c_T\lambda^T,$$

so adding $d(\chi(\mathrm{Sq}^K)\lambda^{(j,I_0)})$ to u replaces $\chi(\mathrm{Sq}^{L_0})\lambda^{I_0}$ by terms involving λ^T, $T > I_0$. Moreover, $\lambda^{(j,I_0)} \in C_{q+1}$. (Note the use of $I_0 > (0,\dots,0)$ here.) Continuing in this way, we may express $u = dv$ + terms in λ^T with T arbitrarily large. But $\dim\lambda^T$ increases with T. Hence, $u = dv$, as desired.

Case 2. $I_0 = (0,\dots,0)$. We will find $w \in C_q$ such that $w = u + dv$ for some $v \in C_{q+1}$ and such that w satisfies the conditions for Case 1.

Let $u = a_0\lambda^{I_0} + \Sigma_{T>I_0} a_T\lambda^T$.

Since $\chi(\mathrm{Sq}^{2i+1}) = \chi(\mathrm{Sq}^1\,\mathrm{Sq}^{2i}) = \chi(\mathrm{Sq}^{2i})\mathrm{Sq}^1$, we may write $a_0 = b\,\mathrm{Sq}^1 + \Sigma\chi(\mathrm{Sq}^{(m_t,J_t)})$, where each (m_t, J_t) is admissible and $m_t = 0 \pmod 2$. Now, by (3.8),

$d\lambda^{(0,\ldots,0)} = \mathrm{Sq}^1 \lambda^{(0,\ldots,0)}$. Let $u_1 = u + d(b\lambda^{(0,\ldots,0)})$. Then

$$u_1 = \sum \chi(\mathrm{Sq}^{(m_t, J_t)})\lambda^{I_0} + \sum_{T > I_0} a_T \lambda^T.$$

Also,

$$\begin{aligned} 0 = du_1 &= \sum \chi(\mathrm{Sq}^{(m_t, J_t)})\,\mathrm{Sq}^1 \lambda^{(0,\ldots,0)} + \sum_{T > I_0} a_T d\lambda^T \\ &= \sum \chi(\mathrm{Sq}^{(m_t+1, J_t)})\lambda^{(0,\ldots,0)} + \sum_{T > I_0} a_T d\lambda^T. \end{aligned}$$

Each of the sequences $(m_t + 1, J_t)$ is admissible, and hence each $\chi(\mathrm{Sq}^{(m_t+1,J_t)})\lambda^{(0,\ldots,0)}$ must be cancelled by a term coming from some $a_T d\lambda^T$. Choose a particular $(m_t + 1, J_t)$ and suppose that $\chi(\mathrm{Sq}^{(m_t+1,J_t)})\lambda^{(0,\ldots,0)}$ is cancelled by a term coming from $a_T d\lambda^T$. According to (3.11), in order to contribute to such a cancellation, T must be of the form $(j, 0, \ldots, 0)$:

$$\begin{aligned} d\left(a_T \lambda^{(j,0,\ldots,0)}\right) &= a_T \chi(\mathrm{Sq}^{j+1})\lambda^{(0,\ldots,0)} + \sum_{K > (0,\ldots,0)} c_K \lambda^K \\ &= \chi(\mathrm{Sq}^{j+1} a'_T)\lambda^{(0,\ldots,0)} + \sum_{K > (0,\ldots,0)} c_K \lambda^K, \end{aligned}$$

where $a'_T = \chi(a_T)$. Using the Adem relations, one can see that $j + 1 \leqslant m_t + 1$, that is, $j \leqslant m_t$. But, by definition of C_q,

$$j \geqslant \begin{cases} k - 1 & \text{if } k \text{ is even,} \\ k & \text{if } k \text{ is odd.} \end{cases}$$

Since m_t is even, this implies that

$$m_t \geqslant \begin{cases} k & \text{if } k \text{ is even,} \\ k + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Therefore, $\lambda^{(m_t - 1, 0, \ldots, 0)} \in C_{q+1}$.

Now,

$$\begin{aligned} d\left(\sum \chi(\mathrm{Sq}^{J_t})\lambda^{(m_t - 1, 0,\ldots,0)}\right) &= \sum \chi(\mathrm{Sq}^{J_t})\chi(\mathrm{Sq}^{m_t})\lambda^{(0,\ldots,0)} + \sum_{K > (0,\ldots,0)} e_K \lambda^K \\ &= \sum \chi(\mathrm{Sq}^{(m_t, J_t)})\lambda^{I_0} + \sum_{K > I_0} e_K \lambda^K. \end{aligned}$$

Let

$$\begin{aligned} w &= u_1 + d\left(\sum \chi(\mathrm{Sq}^{J_t})\lambda^{(m_t - 1, 0,\ldots,0)}\right) \\ &= u + d\left(b\lambda^{(0,\ldots,0)} + \sum \chi(\mathrm{Sq}^{J_t})\lambda^{(m_t - 1, 0,\ldots,0)}\right). \end{aligned}$$

Then $w = \sum_{T > (0,\ldots,0)} x_T \lambda^T$, which reduces the problem to Case 1, thereby completing the proof. □

3.2. *Some resolutions of* $A/A\{\mathrm{Sq}^1\}$. As mentioned earlier, the modules C_q of (3.7) can be decomposed into two pieces, one piece being the modules D_q that Brown and Gitler use for $M(k)$ and the other piece involving the "k-acceptable" basis elements. Inasmuch as one passes from $M(k)$ to $M_1(k)$ by killing Sq^1, the k-acceptable piece can be regarded as accounting for the relation $\mathrm{Sq}^1 = 0$ and other higher order relations associated to it. In §5, we shall need to use this idea. More precisely, we shall require A-free resolutions of $A/A\{\mathrm{Sq}^1\}$ which include the k-acceptable basis elements; the remainder of this section is devoted to constructing such resolutions.

Define certain free A-modules $E_q = E_q(k)$ as follows:

$$E_q = E_q(k) = A \otimes \{\lambda^I \mid I \text{ admissible}, l(I) = q, \text{ and either (i) } t(I) \geqslant k \text{ and } t(I) = 0 \ (\mathrm{mod}\, 2) \text{ or (ii) } I \text{ is } k\text{-acceptable}\}. \tag{3.12}$$

As usual, we regard E_q as a submodule of $A \otimes \Lambda^*$. Note that $E_q \subset C_q$ and that $C_q = D_q + E_q$ (though not a direct sum). Also, $E_0 = C_0 = A \otimes \{\lambda^{(\ )}\} = A$. In Theorem (3.15), we will prove that the E_q's with the differential d form a resolution of $A/A\{\mathrm{Sq}^1\}$, but first there are some preliminaries.

LEMMA (3.13). *Suppose that I and J are admissible sequences such that $t(I) = 0$ (mod 2) and $t(J) = 1$ (mod 2). Then $\lambda^I(\lambda_j\lambda_J) = 0$.*

PROOF. It follows from (3.1) and standard arguments involving mod 2 binomial coefficients (e.g., [18, Lemma 2.6]) that, if J' is inadmissible and $t(J') = 1$ (mod 2), then $\lambda_{J'} = \Sigma\lambda_{J_\alpha}$, where J_α is admissible and $t(J_\alpha) = 1$ (mod 2). The lemma is an immediate consequence. □

REMARK. Let $\Lambda^*_{\mathrm{even}}$ be the subspace of Λ^* with basis $\{\lambda^I \mid I \text{ admissible}, t(I) = 0 \ (\mathrm{mod}\, 2)\}$. The content of the lemma is that $d(A \otimes \Lambda^*_{\mathrm{even}}) \subset A \otimes \Lambda^*_{\mathrm{even}}$.

LEMMA (3.14). *$d(E_q) \subset E_{q-1}$.*

PROOF. It suffices to show that $d\lambda^I \in E_{q-1}$ for each basis element λ^I in E_q. For those λ^I with $t(I) \geqslant k$ and $t(I) = 0$ (mod 2), this follows from (3.4) and (3.13). For those λ^I with I k-acceptable, it follows by iterating (3.8) and then applying (3.4). □

THEOREM (3.15). *For each k, the sequence*

$$\cdots \to E_q \xrightarrow{d} E_{q-1} \to \cdots \to E_0 \xrightarrow{\varepsilon} A/A\{\mathrm{Sq}^1\} \to 0$$

is exact, where $\varepsilon(\lambda^{()}) = 1$.

PROOF. *Exactness at E_0*: Note that

$$E_1 = A \otimes \{\lambda^i \mid i = 0 \text{ or } i = 2j \geqslant k\}.$$

For λ^i of this type, $d\lambda^i = \chi(\mathrm{Sq}^{i+1})\lambda^{(\ )} = \chi(\mathrm{Sq}^1\mathrm{Sq}^i)\lambda^{(\ )} = \chi(\mathrm{Sq}^i)\mathrm{Sq}^1\lambda^{(\ )}$, and hence $d(E_1) = (A\{\mathrm{Sq}^1\})\lambda^{()}$.

Exactness of E_q, $q > 0$: The arguments are identical to those used in the proof of (3.10). □

REMARKS. (1) A series of closely related resolutions of $A/A\{\mathrm{Sq}^1\}$ can be constructed by tacking a fixed number of zeros onto the ends of all the basis elements of the E_q's. That is, let $Z(t) = (0,\ldots,0)$ be the sequence consisting of t zeros, and let $E_q\lambda^{Z(t)} = A \otimes \{\lambda^{(I,Z(t))} \mid \lambda^I \in E_q\}$. Then the arguments that went into the proof of

(3.15) can easily be extended to show that the following subcomplex of the Λ-algebra resolution is also exact:

$$\cdots \to E_q\lambda^{Z(t)} \to E_{q-1}\lambda^{Z(t)} \to \cdots \to E_0\lambda^{Z(t)} \to A \otimes \{\lambda^{Z(t-1)}\} \to$$
$$\cdots \to A \otimes \{\lambda^0\} \to A \otimes \{\lambda^{()}\} \to A/A\{\mathrm{Sq}^1\} \to 0.$$

(2) Observe that (3.13) implies that the differential d induces a map d': $A \otimes (\Lambda^*/\Lambda^*_{\text{even}}) \to A \otimes (\Lambda^*/\Lambda^*_{\text{even}})$. (See the remark following (3.13).) We have learned that Paul Goerss has used this idea to obtain resolutions of the modules $M_1(k)$ in the following manner [**11**].

One can think of $\{\lambda^I | I$ admissible, $t(I) = 1 \pmod 2\}$ as an additive basis for $\Lambda^*/\Lambda^*_{\text{even}}$. Define

$$C'_q\ (= C'_q(k)) = A \otimes \{\lambda^I | I \text{ admissible}, l(I) = q, t(I) \geqslant k, \text{ and } t(I) = 1 \pmod 2\},$$

regarded as a submodule of $A \otimes (\Lambda^*/\Lambda^*_{\text{even}})$. Then Goerss shows that the sequence

$$\cdots \to C'_q \xrightarrow{d'} C'_{q-1} \to \cdots \to C'_1 \to A/A\{\mathrm{Sq}^1\} \to M_1(k) \to 0$$

is exact.

4. On the homotopy groups of $B_1(k)$. In this section, we use the Adams spectral sequence to compute the homotopy groups of $B_1(k)$ up through a dimension roughly equal to $2k$. By calculating in the Λ-algebra via the resolution of $H^*B_1(k) = M_1(k)$ constructed in §3, we obtain the E_2 term. Then, to pass to E_∞ and solve the subsequent extensions, we translate the problem to the Brown-Gitler spectrum $B(k)$, where the corresponding problems have already been solved.

More precisely, the following results are obtained:

(a) $\pi_0(B_1(k)) = Z_2^\wedge$ ($Z_2^\wedge = \varprojlim Z_{2^t}$, the 2-adic integers).

(b) All elements of $\pi_q(B_1(2k)) = \pi_q(B_1(2k+1))$, $1 \leqslant q \leqslant 4k+2$, are of order 2. A Z_2-basis for these groups is given by

$$\{\lambda_I | I \text{ admissible}, \dim \lambda_I \leqslant 4k+2, t(I) \geqslant 2k+1, \text{ and } t(I) = 1 \pmod 2\}.$$

(c) $(i_k)_*$: $\pi_q(B_1(k)) \to \pi_q(B(k))$ is injective for $1 \leqslant q \leqslant 2k$.

To begin, given a spectrum X, $\{(E_r(X), d_r)\}$ will denote the Adams spectral sequence of X. The bigrading of $E_r(X)$ is the usual one so that a nonzero element $x \in E_r^{s,t}(X)$ which survives to $E_\infty^{s,t}(X)$ contributes to an extension in $\pi_{t-s}(X)$. $t-s$ is called the total dimension of x.

As explained in §3, the free A-module $A \otimes \Lambda^*$ has a differential d which gives rise to an A-free resolution of $H^*S^0 = Z_2$, where S^0 is the sphere spectrum. (This is Theorem (3.5) for $k = 0$.) Thus, $(E_1(S^0), d_1) = (\mathrm{Hom}_A(A \otimes \Lambda^*, Z_2), d^*)$. Now, the primary appeal of using the Λ-algebra is that this E_1 term is tractable [**1**]; namely, for this resolution, $(E_1(S^0), d_1) = (\Lambda, \partial)$, where ∂: $\Lambda \to \Lambda$ gives Λ the structure of a graded differential algebra and satisfies

$$\text{(4.1)} \qquad \partial\lambda_i = \sum \binom{s-1}{2s-i-2} \lambda_{s-1}\lambda_{i-s}.$$

The action of ∂ is identical to multiplication on the left by λ_{-1}. We remark that, as an element of $E_1(S^0)$, λ_I has total dimension equal to the dimension of λ_I in Λ.

We next summarize some of the Adams spectral sequence properties of the Brown-Gitler spectrum $B(k)$. Let $\mathscr{D} = \mathscr{D}(k)$ denote the resolution of $H^*B(k) = M(k)$ described in Theorem (3.5):

$$\mathscr{D}\colon \cdots \to D_q \xrightarrow{d} D_{q-1} \to \cdots \to D_0 \xrightarrow{\varepsilon} M(k) \to 0.$$

Recall that $D_q = A \otimes \{\lambda^I | I \text{ admissible, } l(I) = q, t(I) \geqslant k\}$. For this resolution, $E_1(B(k)) = \mathrm{Hom}_A(\mathscr{D}, Z_2)$. Since $\mathscr{D}$ is a subcomplex of $A \otimes \Lambda^*$, $E_1(B(k))$ can be thought of as a quotient of Λ; specifically, $E_1(B(k)) = \Lambda/\Lambda\{\lambda_0, \ldots, \lambda_{k-1}\}$. From this point of view, $E_1(B(k))$ has a Z_2-basis consisting of those λ_I with I admissible and $t(I) \geqslant k$, and $d_1\colon E_1(B(k)) \to E_1(B(k))$ can be computed by using ∂, followed by passing to the quotient.

The following theorem is a reasonably straightforward consequence of Brown and Gitler's construction of $B(k)$ in [3]; details have since been written up by Brown and Peterson.

THEOREM (4.2) [4, THEOREM 5.1]. (a) $E_\infty^{s,t}(B(k)) = E_1^{s,t}(B(k))$ *provided that* $t - s \leqslant 2k$ (*i.e., the spectral sequence collapses in this range*).

(b) *All elements of* $\pi_q(B(k))$, $q < 2k$, *are of order* 2 (*i.e., there are no nontrivial extensions in this range*). □

REMARK. Observe that, by using (4.2)(a), one can compute the order of $\pi_{2k}(B(k))$, at least in principle; however, the extensions have not been completely determined. It is known that elements of order greater than 2 do exist. For instance, Brown and Peterson [4] have used a map $f \in \pi_{2^{j+1}}(B(2^j))$ of order 2^{j+2} which is represented by $\lambda_{2^{j+1}}$ in order to shorten Mahowald's construction of a nontrivial element $\eta_{j+2} \in \pi_{2^{j+2}}(S^0)$ [14].

We now begin to analyze the Adams spectral sequence of the spectrum $B_1(k)$. Let $\mathscr{C} = \mathscr{C}(k)$ denote the resolution of $H^*B_1(k) = M_1(k)$ given in Theorem (3.10):

$$\mathscr{C}\colon \cdots \to C_q \xrightarrow{d} C_{q-1} \to \cdots \to C_0 \xrightarrow{\varepsilon} M_1(k) \to 0.$$

Recall that $C_q = A \otimes \{\lambda^I | I \text{ admissible, } l(I) = q, \text{ and either } t(I) \geqslant k \text{ or } I \text{ is } k\text{-acceptable}\}$. Like $\mathscr{D}$, the resolution $\mathscr{C}$ is a subcomplex of $A \otimes \Lambda^*$. As a result, $E_1(B_1(k)) = \mathrm{Hom}_A(\mathscr{C}, Z_2)$ can be regarded as a quotient of Λ and $d_1\colon E_1(B_1(k)) \to E_1(B_1(k))$ is the Z_2-quotient map induced by ∂. In this setting,

$$\{\lambda_I | I \text{ admissible and either } t(I) \geqslant k \text{ or } I \text{ is } k\text{-acceptable}\}$$

is a Z_2-basis for $E_1(B_1(k))$.

The following lemma was proven by Brown and Gitler in [3] and in fact is implicit in the statement of (4.2)(a).

LEMMA (4.3). *Suppose that* $\dim \lambda_I \leqslant 2k + 1$. *Then* $\partial\lambda_I \in \Lambda\{\lambda_0, \ldots, \lambda_{k-1}\}$.

PROOF. As defined in §3, $\overline{\Lambda}$ is the free associative Z_2-algebra generated by λ_i, $i = -1, 0, 1, \ldots$, modulo the relations (3.1), and $\Lambda = \overline{\Lambda}/\overline{\Lambda}\{\lambda_{-1}\}$. Let $J_n \subset \overline{\Lambda}$ be the left ideal $\overline{\Lambda}\{\lambda_{-1}, \lambda_0, \ldots, \lambda_{n-1}\}$.

We will show by induction on $l(I)$ that

$$J_n \lambda_I \subset J_s, \tag{4.4}$$

where $s = [(2n + \dim \lambda_I)/2]$. Since, in Λ, $\partial\lambda_I = \lambda_{-1}\lambda_I$, the lemma will follow from (4.4) by setting $n = 0$.

Suppose $l(I) = 1$, i.e., $I = (i)$. Let $j < n$ be given. If $2j \geqslant i$, then $\lambda_j \lambda_i \in J_{i+1} \subset J_{[(2n+i)/2]}$. On the other hand, if $2j < i$, then $\lambda_j\lambda_i \in J_{[(2n+i)/2]}$ by (3.1). Next, suppose that $I = (i, I')$, $l(I') > 0$, and that (4.4) is true for I'. Then

$$J_n\lambda_I = (J_n\lambda_i)\lambda_{I'} \subset J_{[(2n+i)/2]}\lambda_{I'} \subset J_s.$$

This proves (4.4) and, as mentioned, implies the lemma. □

LEMMA (4.5). *Suppose that $I = (i_1, \ldots, i_q)$ is admissible, $i_q > 0$. Then $\partial\lambda_I = (i_q - 1)\lambda_{(i_1,\ldots,i_{q-1},i_q-1,0)} + \Sigma\lambda_{I_\alpha}$, where I_α is admissible and $t(I_\alpha) \geqslant 1$.*

PROOF. This calculation was carried out as part of the proof of (3.8). □

A rather technical computation now yields $E_2(B_1(k))$ in total dimension less than or equal to $2k$, as follows.

LEMMA (4.6). *For $t - s \leqslant 2k$, $E_2^{s,t}(B_1(k))$ has a Z_2-basis consisting of $\lambda_{(0,\ldots,0)} = \lambda_0^j$, $j \geqslant 0$, and λ_I, where I is admissible, $t(I) \geqslant k$, and $t(I) = 1 \pmod 2$.*

PROOF. The result is trivial if $k = 0$, so assume $k > 0$.

As remarked earlier, if $\partial\lambda_I = \Sigma\varepsilon_\alpha\lambda_{I_\alpha}$, where each I_α is admissible, then, in $E_1(B_1(k))$, $d_1\lambda_I = \Sigma\varepsilon_\alpha\lambda_{I_\alpha}$, where this latter sum runs only over those λ_{I_α} in $E_1(B_1(k))$. This procedure will now be used to identity cycles and boundaries of $E_1(B_1(k))$ in the desired range of dimensions.

For the rest of this proof, all sequences I should be assumed admissible with $\dim \lambda_I \leqslant 2k + 1$. Let $Z(t) = (0, \ldots, 0)$, the sequence of t zeros.

Suppose that $I = (i_1, \ldots, i_q)$, $i_q > 0$. By (4.3) and (4.5), we may write

$$\partial\lambda_I = (i_q - 1)\lambda_{(i_1,\ldots,i_{q-1},i_q-1,0)} + \sum\lambda_{(I_\alpha,k-1)} + \sum\lambda_{J_\beta},$$

where $0 < t(J_\beta) < k - 1$. Moreover, since $\partial\lambda_0 = 0$ and ∂ is a derivation,

$$\begin{aligned}\partial\lambda_{(I,Z(t))} = (\partial\lambda_I)\lambda_{Z(t)} &= (i_q - 1)\lambda_{(i_1,\ldots,i_{q-1},i_q-1,Z(t+1))} \\ &\quad + \sum\lambda_{(I_\alpha,k-1,Z(t))} + \sum\lambda_{(J_\beta,Z(t))}.\end{aligned}$$

This yields the following formulas in $E_1(B_1(k))$:

(4.7a) If $t(I) = i_q \geqslant k$, then $d_1\lambda_I = (i_q - 1)\lambda_{(i_1,\ldots,i_{q-1},i_q-1,0)}$.

(4.7b) If $(I, Z(t))$ is k-acceptable, $l(I) \geqslant 1$, and $t(I) = i_q > 0$, then

$$d_1\lambda_{(I,Z(t))} = (i_q - 1)\lambda_{(i_1,\ldots,i_{q-1},i_q-1,Z(t+1))} + (k-1)\sum\lambda_{(I_\alpha,k-1,Z(t))}.$$

A careful inspection of these formulas (see the following note) reveals that, for $t - s \leqslant 2k$, the $E_1(B_1(k))$-cycles have a basis consisting of those λ_I for which either:

(i) $I = Z(t)$, $t \geqslant 0$;

(ii) $t(I) \geqslant k$, $t(I) = 1 \pmod 2$; or

(iii) $I = (I', Z(t))$ is k-acceptable, $t(I') = 1 \pmod 2$;

and that the $E_1(B_1(k))$-boundaries are generated by precisely those elements of type (iii). This produces the E_2 term stated in the lemma.

(*Note on proof*. When k is even, the analysis given above requires some special care because sequences of the form $I = (I', k-1, Z(t))$ are then k-acceptable. This accounts for the presence of the second term on the right-hand side of formula (4.7b). However, one can show by induction on t that, for sequences I of this form, λ_I is a boundary, using (4.7a) to start the induction and (4.7b) to continue. It then follows directly from (4.7b) that all basis elements of type (iii) are boundaries; in fact, this is probably the easiest way to see that such elements are cycles. Clearly, (4.7a) and (4.7b) imply that type (iii) elements generate all possible boundaries, and it is not hard to check that (i)–(iii) account for all possible cycles.) □

We next show that $E_r^{s,t}(B_1(k))$ collapses from E_2 on (still assuming $t-s \leqslant 2k$). This will be accomplished through a comparison with the Adams spectral sequence for $B(k)$.

Recall from (2.12), that, for the map $i_k\colon B_1(k) \to B(k)$, the induced homomorphism $(i_k)^*\colon H^*B(k) \to H^*B_1(k)$ is the canonical projection $M(k) \to M_1(k)$. Thus, $(i_k)^*$ is covered by a map of resolutions $\mathscr{D} \to \mathscr{C}$:

$$\begin{array}{ccccccccccc}
\cdots & \to & D_q & \to & D_{q-1} & \to & \cdots & \to & D_0 & \to & M(k) \to 0 \\
 & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow (i_k)^* \\
\cdots & \to & C_q & \to & C_{q-1} & \to & \cdots & \to & C_0 & \to & M_1(k) \to 0
\end{array}$$

(By definition, $D_q \subset C_q$, and the maps $D_q \to C_q$ above are the inclusions.) This in turn induces a map of spectral sequences $(i_k)_*\colon E_r(B_1(k)) \to E_r(B(k))$. Lemma (4.6) identifies $E_2^{s,t}(B_1(k))$ as a subcomplex of $E_2^{s,t}(B(k))$ when $1 \leqslant t-s \leqslant 2k$; that is, $(i_k)_*\colon E_2^{s,t}(B_1(k)) \to E_2^{s,t}(B(k))$ is injective in this range of total dimensions. But, according to (4.2)(a), when $1 \leqslant t-s \leqslant 2k$, $E_2^{s,t}(B(k)) = E_\infty^{s,t}(B(k))$, i.e, the spectral sequence collapses. As a subcomplex, $E_r^{s,t}(B_1(k))$ must behave in the same way, and this enables us to write down $E_\infty^{s,t}(B_1(k))$.

LEMMA (4.8). *$E_2^{s,t}(B_1(k)) = E_\infty^{s,t}(B_1(k))$ for $t-s \leqslant 2k$.* □

COROLLARY (4.9). *$(i_k)_*\colon \pi_q(B_1(k)) \to \pi_q(B(k))$ is injective if $1 \leqslant q \leqslant 2k$.*

PROOF. The preceding arguments show that $(i_k)_*\colon E_\infty^{s,t}(B_1(k)) \to E_\infty^{s,t}(B(k))$ is injective for $1 \leqslant t-s \leqslant 2k$. The corollary follows by an easy induction over the Adams filtration. □

We now close this discussion by listing some of the homotopy groups of $B_1(k)$.

Let $Z_{\hat{2}} = \varprojlim Z_{2^r}$ denote the 2-adic integers.

THEOREM (4.10). (a) $\pi_0(B_1(k)) = Z_{\hat{2}}$.

(b) *All elements of $\pi_q(B_1(2k)) = \pi_q(B_1(2k+1))$, $1 \leqslant q \leqslant 4k+2$, are of order 2. A Z_2-basis for these groups is in one-to-one correspondence with $\{\lambda_I \mid I$ admissible, $\dim \lambda_I \leqslant 4k+2$, $t(I) \geqslant 2k+1$, $t(I) = 1 \pmod 2\}$.*

PROOF. Recall that in the Λ-algebra, left multiplication by λ_0 corresponds to precomposition with the degree 2 map $S^0 \to S^0$, i.e., $\lambda_0 = h_0$ in the usual Adams spectral sequence notation. Thus, a nontrivial left multiplication by λ_0 in E_∞ indicates the presence of a nontrivial extension.

Statement (a) of the theorem now follows directly by looking at the tower of elements λ_0^j, $j \geqslant 0$, of total dimension 0 in $E_\infty(B_1(k))$.

To prove (b), we use the cofibration of spectra

$$B_1(2k) = B_1(2k+1) \xrightarrow{2} B_1(2k+1) \xrightarrow{i_{2k+1}} B(2k+1),$$

the existence of which was verified in (2.15). This yields a long exact sequence of homotopy groups

$$\cdots \to \pi_q B_1(2k+1) \xrightarrow{2} \pi_q B_1(2k+1) \xrightarrow{(i_{2k+1})_*} \pi_q B(2k+1)$$
$$\to \pi_{q-1} B_1(2k+1) \to \cdots.$$

By (4.9), $(i_{2k+1})_*\colon \pi_q(B_1(2k+1)) \to \pi_q(B(2k+1))$ is injective for $1 \leqslant q \leqslant 4k+2$. Hence, $2\pi_q(B_1(2k+1)) = 0$ by exactness, i.e, all elements have order 2.

The statement concerning a Z_2-basis for these groups is a consequence of the description of $E_\infty^{s,t}(B_1(k)) = E_2^{s,t}(B_1(k))$ given in (4.6). $\square$

REMARKS. (1) Recall that the group structure of $\pi_{2k}(B(k))$ has not been completely determined. Property (b) of (4.1) identifies a rather large subgroup, namely, $(i_k)_*(\pi_{2k}(B_1(k)))$, as consisting entirely of elements of order 2.

(2) Outside the range of dimensions handled in (4.10), the groups $\pi_q(B_1(k))$ will have elements of order greater than 2. For instance, let $f \in \pi_{2^{j+1}}(B(2^j))$ be a map represented by $\lambda_{2^{j+1}}$, as discussed in the remark following (4.2).

Consider the cofibration

$$B_1(2^j - 1) \to B_1(2^j) \xrightarrow{i_{2^j}} B(2^j).$$

Since $\lambda_0^n \lambda_{2^{j+1}} = \lambda_{2^{j-n+1}}(\lambda_{2^{j-n+1}} \lambda_{2^{j-n+2}} \cdots \lambda_{2^j})$ in $E_\infty(B(2^j))$, the spectral sequence calculations given earlier show that neither f nor any of its multiples can lie in the image of

$$(i_{2^j})_*\colon \pi_{2^{j+1}}(B_1(2^j)) \to \pi_{2^{j+1}}(B(2^j)).$$

Thus, from the long exact sequence

$$\cdots \to \pi_q(B_1(2^j - 1)) \to \pi_q(B_1(2^j)) \xrightarrow{(i_{2^j})_*} \pi_q(B(2^j)) \xrightarrow{\delta} \pi_{q-1}(B_1(2^j - 1)) \to \cdots,$$

one sees that $\delta f \in \pi_{2^{j+1}-1}(B_1(2^j - 1))$ has order 2^{j+2}.

5. $B_1(k)$ representability of homology classes. Given spectra E and X, $E_n(X)$ and $E^n(X)$ will denote the nth generalized homology and cohomology groups of X with respect to E, e.g., $E_n(X) = \pi_n(E \wedge X)$.

Recall from §2 the commutative diagram:

$$\begin{tikzcd} W_k \arrow[dr, "g_k"'] & \subset & \Omega^2 S^3\langle 3\rangle \arrow[dl, "g"] \\ & BO & \end{tikzcd}$$

The inclusion $W_k \subset \Omega^2 S^3\langle 3\rangle$ induces a map of Thom spectra $T(g_k) \to T(g)$. Completing this at 2 and using (2.10) then yields a map: $j\colon B_1(k) \to K(Z_{\hat{2}})$ representing the Thom class of $B_1(k)$.

The object of this section is to prove

THEOREM (5.1). *For any CW complex X,*

$$j_*\colon B_1(k)_n(X) \to H_n(X; Z_{\hat{2}})$$

is surjective, provided that $n < 2k + 2$.

The proof will be given in §5.2. The arguments in this proof can then be easily used to discover conditions which characterize $B_1(k)$, and this process is carried out in §5.3.

An immediate consequence of (5.1) is

COROLLARY (5.2). *Suppose that M is a Z-orientable, closed n-manifold. Let* ν *be its stable normal bundle, denote the Thom spectrum by* $T(\nu)$, *and let* $U_Z\colon T(\nu) \to K(Z_{\hat{2}})$ *represent the Thom class. Then M is* $B_1([n/2])$*-orientable in the sense that there is a class* $U_B \in B_1([n/2])^0(T(\nu)) = [T(\nu), B_1([n/2])]$ *such that* $j_* U_B = U_Z$, *i.e., such that the diagram*

$$\begin{tikzcd} & B_1([n/2]) \arrow[d, "j"] \\ T(\nu) \arrow[ur, "U_B"] \arrow[r, "U_Z"'] & K(Z_{\hat{2}}) \end{tikzcd}$$

commutes.

PROOF. It is well known that the suspension spectrum of M^+ is S-dual to $T(\nu)$. (M^+ means M plus a disjoint base point.) Now, by (5.1), $j_*\colon B_1[n/2]_n(M^+) \to H_n(M^+; Z_{\hat{2}})$ is surjective so that, under S-duality, $j_*\colon B_1[n/2]^0(T(\nu)) \to H^0(T(\nu); Z_{\hat{2}})$ must be surjective, too. The result follows by pulling back the Thom class U_Z. $\square$

REMARK. Let $U_{\nu_M} \in H^0(T(\nu))$ denote the mod 2 Thom class, and let $I_n = \{a \in A \mid aU_{\nu_M} = 0$ for all Z-orientable, closed n-manifolds $M\}$. For instance, $\mathrm{Sq}^1 \in I_n$ for any n. A fairly straightforward calculation, utilizing the connection between S-duality and the canonical antiautomorphism $\chi\colon A \to A$, shows that

$$A\{\mathrm{Sq}^1, \chi(\mathrm{Sq}^i) \mid i > [n/2]\} \subset I_n.$$

Thus, for any Z-orientable, closed n-manifold, there is always an algebraic factorization of U_Z^*

$$\begin{tikzcd} & M_1[n/2] \arrow[dr, "v"] & \\ A/A\{\mathrm{Sq}^1\} \arrow[ur, "j^*"] \arrow[rr, "U_Z^*"] & & H^*(T(\nu)) \end{tikzcd}$$

where j^* is the projection. (5.2) then says that one can geometrically realize the homomorphism v above by a $B_1[n/2]$ Thom class U_B.

Furthermore, I_n has been determined explicitly by Brown and Peterson [**5**], and, when $n \neq 0 \pmod 4$, their calculations show that $I_n = A\{\mathrm{Sq}^1, \chi(\mathrm{Sq}^i) | i > [n/2]\}$. Thus, for such n, $B_1[n/2]$ has the smallest possible cohomology for any spectrum possessing the orientability property of (5.2), meaning that, in this sense, (5.2) is best possible.

As usual, the results here are analogous to properties of the Brown-Gitler spectrum $B(k)$. This time, the corresponding theorems, due to Brown and Gitler, are

THEOREM (5.3) [**3**]. *If* α: $B(k) \to K(Z_2)$ *is the map representing* $1 \in H^*B(k) = M(k)$, *then, for any CW complex* X,

$$\alpha_*: B(k)_n(X) \to H_n X$$

is surjective, provided that $n < 2k + 2$. $\square$

COROLLARY (5.4) [**3**]. *Suppose that* M *is any closed* n*-manifold and* U: $T(\nu) \to K(Z_2)$ *represents the* (mod 2) *Thom class. Then* M *is* $B[n/2]$*-orientable in the sense that there is a class* $\tilde{U} \in B[n/2]^0(T(\nu))$ *such that* $\alpha_* \tilde{U} = U$. $\square$

5.1. *Background and notation.* In this subsection, we organize the material needed to prove (5.1).

First of all, Brown and Gitler actually proved a result which is slightly stronger than (5.3), and this stronger version is what we shall use in the proof of (5.1). To state it, we need to review a few details about the way that $B(k)$ is constructed.

Let $\mathscr{D} = \mathscr{D}(k)$ denote the A-free resolution of $H^*B(k)$ described in (3.5):

$$\mathscr{D}: \cdots \to D_q \xrightarrow{d} D_{q-1} \to \cdots \to D_0 \xrightarrow{\varepsilon} M(k) \to 0.$$

Based on this resolution, Brown and Gitler explicitly constructed a generalized Postnikov tower $\mathscr{D}_T$ for $B(k)$:

$$\mathscr{D}_T: \quad \begin{array}{c} \vdots \\ \downarrow \\ Y_q \xrightarrow{h_{q+1}} L_{q+1} \\ \alpha_q \downarrow \qquad\qquad \\ Y_{q-1} \xrightarrow{h_q} L_q \\ \downarrow \\ \vdots \\ \downarrow \\ Y_0 = L_0 \xrightarrow{h_1} L_1 \end{array}$$

That is, the Y_q are Ω-spectra, and the L_q are generalized Eilenberg-Mac Lane spectra such that $\pi_* L_q$ is a graded vector space over Z_2 and $H^* L_q = D_q$. Also, α_q: $Y_q \to Y_{q-1}$ is the fibration with fibre L_q induced by h_q from the contractible fibration over L_q. (h_q is a morphism of degree one, i.e., $(h_q)_t$: $(Y_{q-1})_t \to (L_q)_{t+1}$.) If ε_q: $L_q \to Y_q$ denotes the inclusion of the fibre, then $(h_{q+1}\varepsilon_q)^*$: $H^* L_{q+1} \to H^* L_q$ is the same as the differential d: $D_{q+1} \to D_q$. Note that the base spectrum of the tower, L_0, satisfies $H^* L_0 = D_0 = A$, and hence $L_0 = K(Z_2)$.

With this machinery in place, $B(k)$ is defined as the limit space

$$B(k) = Y_\infty = \varprojlim Y_q.$$

The construction of the tower $\mathscr{D}_T$ is carefully controlled to possess the following property.

LEMMA (5.5). *For any CW complex X,*

$$(\alpha_q)_*\colon (Y_q)_n(X) \to (Y_{q-1})_n(X)$$

is surjective, provided that $n < 2k + 2$.

PROOF. Given an abelian group G, define $\mathrm{ch}(G) = \mathrm{Hom}(G, R/Z)$. In [3], Brown and Gitler discuss a functor χ on spectra, the Pontrjagin duality functor, whose main property is that, for any spectrum h and CW complex X,

$$\chi(h)^n(X) = \mathrm{ch}(h_n(X)) \tag{5.6}$$

(assuming that $h_n(S^0)$ is finite for all n).

Applying χ to the fibration $L_q \to Y_q \xrightarrow{\alpha_q} Y_{q-1}$ then produces another fibration $\chi(Y_{q-1}) \xrightarrow{\chi(\alpha_q)} \chi(Y_q) \to \chi(L_q)$ induced by $\chi(h_q)$: $\chi(L_q) \to \chi(Y_{q-1})$. An important detail in Brown and Gitler's work is that $\chi(h_q)_n$: $\chi(L_q)_n \to \chi(Y_{q-1})_{n+1}$ is zero when $n < 2k + 1$ (see [3, Theorem 5.1(iv)]). Therefore, $\chi(\alpha_q)_*$: $\chi(Y_{q-1})^n(X) \to \chi(Y_q)^n(X)$ is injective for $n < 2k + 2$. Equivalently, after applying (5.6), $(\alpha_q)_*$: $(Y_q)_n(X) \to (Y_{q-1})_n(X)$ is surjective for $n < 2k + 2$. $\square$

(5.5) can also be stated more directly in terms of Postnikov towers. Namely, consider the complex

$$\mathscr{D} \otimes H^*X\colon \cdots \to D_q \otimes H^*X \xrightarrow{d \otimes \mathrm{id}} D_{q-1} \otimes H^*X \to \cdots$$
$$\to D_0 \otimes H^*X \xrightarrow{\varepsilon \otimes \mathrm{id}} M(k) \otimes H^*X \to 0.$$

This is an A-free resolution of $H^*(B(k) \wedge X) = M(k) \otimes H^*X$. (Here, each $D_q \otimes H^*X$ has the diagonal A-module structure, that is, $a(x \otimes y) = \Sigma a_i' x \otimes a_i'' y$, and an argument involving the formula $\chi(a)\lambda^I \otimes y = \Sigma \chi(a_i')(\lambda^I \otimes a_i'' y)$ shows that these modules are in fact free over A.) Associated to this resolution is a Postnikov tower $\mathcal{D}_T \wedge X$ with limit space $B(k) \wedge X$:

$$\begin{array}{llll} & B(k) \wedge X & & \\ \mathcal{D}_T \wedge X: & \vdots & & \\ & \downarrow & & \\ & Y_q \wedge X & \xrightarrow{h_{q+1} \wedge \mathrm{id}} & L_{q+1} \wedge X \\ & \alpha_q \wedge \mathrm{id} \downarrow & & \\ & Y_{q-1} \wedge X & \xrightarrow{h_q \wedge \mathrm{id}} & L_q \wedge X \\ & \downarrow & & \\ & \vdots & & \\ & \downarrow & & \\ & Y_0 \wedge X & \xrightarrow{h_1 \wedge \mathrm{id}} & L_1 \wedge X \end{array}$$

The content of (5.5) is that any map $u: S^n \to Y_{q-1} \wedge X$, $n < 2k+2$, lifts up the tower to $Y_q \wedge X$ (and hence, by induction, all the way up to $B(k) \wedge X$). Of course, if one happens to start with $u: S^n \to Y_0 \wedge X = K(Z_2) \wedge X$, the result is (5.3).

The proof of (5.1) will make use of the tower $\mathcal{D}_T \wedge X$, as well as similar towers for $B_1(k) \wedge X$ and $K(Z_2^\wedge) \wedge X$.

The following notation of modules and spectra shall be preserved for the rest of the section.

First, let $\mathcal{C} = \mathcal{C}(k)$ denote the free resolution for $M_1(k)$ described in (3.10):

$$\mathcal{C}: \cdots \to C_q \xrightarrow{d} C_{q-1} \to \cdots \to C_0 \xrightarrow{\varepsilon} M_1(k) \to 0;$$

and let $\mathcal{C}_T$ denote a Postnikov tower associated to it with limit space $B_1(k)$ (recall that $B_1(k)$ was defined as a 2-completion):

$$\begin{array}{llll} & B_1(k) & & \\ & \vdots & & \\ \mathcal{C}_T: & \downarrow & & \\ & X_q & \xrightarrow{k_{q+1}} & K_{q+1} \\ & \downarrow & & \\ & X_{q-1} & \xrightarrow{k_q} & K_q \\ & \downarrow & & \\ & \vdots & & \\ & \downarrow & & \\ & X_0 = K_0 & \xrightarrow{k_1} & K_1 \end{array}$$

As with $\mathcal{D}_T$, at the bottom of the tower, one has $X_0 = K(Z_2)$.

Similarly, let $\mathscr{E} = \mathscr{E}(k)$ denote the "k-acceptable" resolution of $H^*(K(Z_{\hat{2}})) = A/A\{\mathrm{Sq}^1\}$ given by (3.15):

$$\mathscr{E}\colon \cdots \to E_q \xrightarrow{d} E_{q-1} \to \cdots \to E_0 \xrightarrow{\varepsilon} A/A\{\mathrm{Sq}^1\} \to 0;$$

and let $\mathscr{E}_T$ denote a Postnikov tower based on this resolution having as limit space the 2-complete spectrum $K(Z_{\hat{2}})$:

$$\mathscr{E}_T\colon\quad \begin{array}{ccc} K(Z_{\hat{2}}) & & \\ \vdots & & \\ \downarrow & & \\ V_q & \xrightarrow{r_{q+1}} & R_{q+1} \\ \downarrow & & \\ V_{q-1} & \xrightarrow{r_q} & R_q \\ \downarrow & & \\ \vdots & & \\ \downarrow & & \\ V_0 = R_0 & \xrightarrow{r_1} & R_1 \end{array}$$

In particular, $V_0 = K(Z_2)$.

In the previous resolution, one can tensor all modules with H^*X and differentials with the identity in order to obtain new resolutions, denoted by $\mathscr{C} \otimes H^*X$ and $\mathscr{E} \otimes H^*X$. These in turn give rise to new Postnikov towers, denoted $\mathscr{C}_T \wedge X$ and $\mathscr{E}_T \wedge X$, obtained by smashing all spectra with X and all maps with the identity.

Now, recall from (2.12) that the map $i_k\colon B_1(k) \to B(k)$ produces a homomorphism in cohomology, $(i_k)^*\colon H^*B(k) \to H^*B_1(k)$, which is the obvious projection $M(k) \to M_1(k)$. Thus, $(i_k)^* \otimes \mathrm{id}$ is covered by a map of resolutions $\mathscr{D} \otimes H^*X \to \mathscr{C} \otimes H^*X$:

$$\begin{array}{ccccccccc} \cdots \to & D_q \otimes H^*X & \to & D_{q-1} \otimes H^*X & \to \cdots \to & D_0 \otimes H^*X & \to & M(k) \otimes H^*X & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow (i_k)^* \otimes \mathrm{id} & \\ \cdots \to & C_q \otimes H^*X & \to & C_{q-1} \otimes H^*X & \to \cdots \to & C_0 \otimes H^*X & \to & M_1(k) \otimes H^*X & \to 0 \end{array}$$

(Recall that, by definition, $D_q \subset C_q$, and the vertical maps are the inclusions.) Then, by naturality, there is an associated map of Postnikov towers $(i_k)_T \wedge \mathrm{id}\colon \mathscr{C}_T \wedge X \to \mathscr{D}_T \wedge X$:

$$\begin{tikzcd}
B_1(k) \wedge X \arrow[r, "i_k \wedge \mathrm{id}"] \arrow[d, dotted] & B(k) \wedge X \arrow[d, dotted] \\
X_q \wedge X \arrow[r] \arrow[d] & Y_q \wedge X \arrow[d] \\
X_{q-1} \wedge X \arrow[r] \arrow[d, dotted] & Y_{q-1} \wedge X \arrow[d, dotted] \\
X_0 \wedge X \arrow[r] & Y_0 \wedge X
\end{tikzcd}$$

Similarly, for the map $j\colon B_1(k) \to K(Z_2^{\wedge})$, the induced homomorphism $j^*\colon H^*(K(Z_2^{\wedge})) \to H^*B_1(k)$ is the projection $A/A\{\mathrm{Sq}^1\} \to M_1(k)$. Thus, there is an obvious map of resolutions $\mathscr{E} \otimes H^*X \to \mathscr{C} \otimes H^*X$ covering $j^* \otimes \mathrm{id}$. (By definition, $E_q \subset C_q$, so that $\mathscr{E}$ maps to $\mathscr{C}$ by inclusion.) And, once again, one winds up with an associated map of Postnikov towers $j_T \wedge \mathrm{id}\colon \mathscr{C}_T \wedge X \to \mathscr{E}_T \wedge X$.

The idea is that all this notation can be molded into a sensible proof of (5.1).

5.2. *Proof of Theorem* (5.1). To begin the proof of (5.1), let $w \in H_n(X; Z_2^{\wedge})$ be given, $n < 2k + 2$. Represent w as a map $w\colon S^n \to K(Z_2^{\wedge}) \wedge X$. Then w can be regarded as a coherent sequence of maps $w_q\colon S^n \to V_q \wedge X$ up the tower $\mathscr{E}_T \wedge X$:

$$\begin{tikzcd}
 & K(Z_2^{\wedge}) \wedge X \arrow[d, dotted] \\
 & V_q \wedge X \arrow[d] \\
 & V_{q-1} \wedge X \arrow[d, dotted] \\
S^n \arrow[r, "w_0"] \arrow[ur, "w_{q-1}"] \arrow[uur, "w_q"] \arrow[uuur, "w_\infty = w"] & V_0 \wedge X
\end{tikzcd}$$

Note that $w_0\colon S^n \to V_0 \wedge X = K(Z_2) \wedge X$ represents the mod 2 reduction of the homology class w.

We want to find $u \in B_1(k)_n(X)$ (i.e., $u\colon S^n \to B_1(k) \wedge X$) such that $j_* u = w$. The strategy will be to show inductively that w_0 lifts up the tower $\mathscr{C}_T \wedge X$ all the way to $B_1(k) \wedge X$ in a manner compatible with the class w. That is, we shall complete the diagram:

$$
\begin{tikzcd}
 & B_1(k)\wedge X \arrow[rr, "j\wedge \mathrm{id}"] \arrow[d, dotted] & & K(\hat{Z}_2)\wedge X \arrow[d, dotted] \\
 & X_q\wedge X \arrow[rr] \arrow[d] & & V_q\wedge X \arrow[d] \\
 & \vdots \arrow[d] & & \vdots \arrow[d] \\
S^n \arrow[r, "w_0"'] \arrow[uuur, dashed, "u"] \arrow[uuurrr, "w"] & X_0\wedge X \arrow[r, phantom, "="] & K(Z_2)\wedge X \arrow[r, phantom, "="] & V_0\wedge X
\end{tikzcd}
$$

Inductive assumption. Assume that there exist maps $u_0, u_1, \ldots, u_q$, where u_i: $S^n \to X_i \wedge X$, which make the following diagrams (P_q) and (Q_q) commute:

(P_q):

$$
\begin{tikzcd}
 & B_1(k)\wedge X \arrow[d, dotted] \\
 & X_q\wedge X \arrow[d] \\
 & X_{q-1}\wedge X \arrow[d] \\
 & \vdots \arrow[d] \\
S^n \arrow[r, "u_0"] \arrow[uuur, "u_q"] \arrow[uur, "u_{q-1}"'] & X_0\wedge X
\end{tikzcd}
$$

(Q_q):

$$
\begin{tikzcd}
 & B_1(k)\wedge X \arrow[r, "j\wedge \mathrm{id}"] \arrow[d, dotted] & K(Z_2)\wedge X \arrow[d, dotted] \\
 & X_q\wedge X \arrow[r, "j_q\wedge \mathrm{id}"] & V_q\wedge X \\
S^n \arrow[ur, "u_q"] \arrow[urr, "w_q"'] & &
\end{tikzcd}
$$

Begin the induction by taking $u_0 = w_0$.

Next, assume that $u_0, \ldots, u_{q-1}$ have been found satisfying (P_{q-1}) and (Q_{q-1}). We shall describe how to choose u_q so that (P_q) and (Q_q) hold.

(P_q) Showing that u_{q-1} lifts to $X_q \wedge X$ is equivalent to verifying that $k'_q u_{q-1} = 0$, where $k'_q = k_q \wedge \mathrm{id}$ is the k-invariant for $X_q \wedge X \to X_{q-1} \wedge X$ in the tower $\mathscr{C}_T \wedge X$:

$$\begin{tikzcd}
 & B_1(k) \wedge X \arrow[d, dotted] & \\
 & X_q \wedge X \arrow[d] & \\
S^n \arrow[r, "u_{q-1}"] & X_{q-1} \wedge X \arrow[r, "k'_q"] & K_q \wedge X
\end{tikzcd}$$

Since $K_q \wedge X$ is a generalized Z_2 Eilenberg-Mac Lane spectrum, this is the same as showing that $(k'_q u_{q-1})^* = 0$ in cohomology, i.e., that $(k'_q u_{q-1})^*(\lambda^I \otimes x) = 0$ for all A-basis elements $\lambda^I \in H^* K_q = C_q$ and all $x \in H^* X$.

Recall that C_q has two types of basis elements, namely, λ^I with $l(I) = q$ such that either $t(I) \geqslant k$ or I is k-acceptable.

First, suppose that $t(I) \geqslant k$. Then, as part of the mapping of towers $(i_k)_T \wedge \mathrm{id}$: $\mathscr{C}_T \wedge X \to \mathscr{D}_T \wedge X$, there is a commutative diagram involving the k-invariants k'_q and $h'_q = h_q \wedge \mathrm{id}$:

$$\begin{tikzcd}
 & B_1(k) \wedge X \arrow[rr, "i_k \wedge \mathrm{id}"] \arrow[dd, dotted] & & B(k) \wedge X \arrow[dd, dotted] & \\
 & & K_q \wedge X \arrow[rr, "e_q"] & & L_q \wedge X \\
S^n \arrow[r, "u_{q-1}"'] & X_{q-1} \wedge X \arrow[ur, "k'_q"] \arrow[rr, "(i_k)'"'] & & Y_{q-1} \wedge X \arrow[ur, "h'_q"'] &
\end{tikzcd}$$

where $(e_q)^*$: $H^*(L_q \wedge X) \to H^*(K_q \wedge X)$ is the inclusion $D_q \otimes H^* X \to C_q \otimes H^* X$. Now, by (5.5), $(i_k)' u_{q-1}$ lifts to $Y_q \wedge X$. Hence, $h'_q (i_k)' u_{q-1} = 0$, so that

$$0 = \left(e_q k'_q u_{q-1}\right)^*(\lambda^I \otimes x) = \left(k'_q u_{q-1}\right)^* e_q^*(\lambda^I \otimes x) = \left(k'_q u_{q-1}\right)^*(\lambda^I \otimes x),$$

as desired.

Secondly, suppose that I is k-acceptable. Then, as part of the mapping of towers $j_T \wedge \mathrm{id}$: $\mathscr{C}_T \wedge X \to \mathscr{E}_T \wedge X$, there is a commutative diagram involving the k-invariants k'_q and $r'_q = r_q \wedge \mathrm{id}$:

$$\begin{tikzcd}
 & B_1(k) \wedge X \arrow[rr, "j \wedge \mathrm{id}"] \arrow[dd, dotted] & & K(Z_{\hat{2}}) \wedge X \arrow[dd, dotted] & \\
 & & K_q \wedge X \arrow[rr, "t_q"] & & R_q \wedge X \\
 & X_{q-1} \wedge X \arrow[ur, "k'_q"] \arrow[rr, "j'_{q-1}"'] & & V_{q-1} \wedge X \arrow[ur, "r'_q"'] & \\
S^n \arrow[ur, "u_{q-1}"] \arrow[urrr, "w_{q-1}"'] & & & &
\end{tikzcd}$$

where $(t_q)^*\colon H^*(R_q \wedge X) \to H^*(K_q \wedge X)$ is the inclusion $E_q \otimes H^*X \to C_q \otimes H^*X$ and $j'_{q-1} = j_{q-1} \wedge \mathrm{id}$. (Note that $w_{q-1} = j'_{q-1}u_{q-1}$ by induction.) Since w_{q-1} lifts to $V_q \wedge X$ as w_q, it follows that $r'_q j'_{q-1} u_{q-1} = r'_q w_{q-1} = 0$. Therefore,

$$0 = \left(t_q k'_q u_{q-1}\right)^*(\lambda^I \otimes x) = \left(k'_q u_{q-1}\right)^* t_q^*(\lambda^I \otimes x) = \left(k'_q u_{q-1}\right)^*(\lambda^I \otimes x).$$

These calculations show that $k'_q u_{q-1} = 0$. Consequently, there exists $z_q\colon S^n \to X_q \wedge X$ satisfying property (P_q).

(Q_q) Consider the commutative diagram involving $j_T \wedge \mathrm{id}\colon \mathscr{C}_T \wedge X \to \mathscr{E}_T \wedge X$:

$$\begin{tikzcd}
 & B_1(k) \wedge X \arrow[r, "j \wedge \mathrm{id}"] \arrow[d, dotted] & K(Z_{\hat{2}}) \wedge X \arrow[d, dotted] \\
 & X_q \wedge X \arrow[r, "j'_q"] \arrow[d] & V_q \wedge X \arrow[d] \\
 & X_{q-1} \wedge X \arrow[r, "j'_{q-1}"] & V_{q-1} \wedge X \\
S^n \arrow[ur, "u_{q-1}"] \arrow[uurr, "w_q"] \arrow[urr, "w_{q-1}"'] & &
\end{tikzcd}$$

Recall that $K_q \wedge X$ denotes the fibre of $X_q \wedge X \to X_{q-1} \wedge X$ and $R_q \wedge X$ denotes the fibre of $V_q \wedge X \to V_{q-1} \wedge X$.

We would like to check if $j'_q z_q$ and w_q are equal. If not, then as two liftings of w_{q-1}, they must differ by a map ε into the fibre $R_q \wedge X$, i.e., $w_q = j'_q z_q + \varepsilon$ for some $\varepsilon\colon S^n \to R_q \wedge X$. But since $H^*(R_q \wedge X) = E_q \otimes H^*X \subset C_q \otimes H^*X = H^*(K_q \wedge X)$, such an ε would factor as $S^n \xrightarrow{\varepsilon'} K_q \wedge X \xrightarrow{t_q} R_q \wedge X$. Let ε'' denote the composition of ε' with the inclusion of the fiber $K_q \wedge X \to X_q \wedge X$. Then by letting $u_q = z_q + \varepsilon''$, one obtains a map satisfying property (Q_q).

We should remark that this map, u_q, will still satisfy property (P_q), since z_q and u_q differ by a map into $K_q \wedge X$ and the composition $K_q \wedge X \to X_q \wedge X \to X_{q-1} \wedge X$ is zero.

This completes the induction.

Set $u = u_\infty$, and it is clear that $j_* u = w$. □

5.3. *Homotopy characterizations of* $B_1(k)$. Our goal in this subsection is to find necessary and sufficient conditions which determine whether a given spectrum is homotopy 2-equivalent to the spectrum $B_1(k)$. In **[6]**, Brown and Peterson formulated various sets of conditions of this type for the Brown-Gitler spectrum $B(k)$. For example, they proved that a spectrum Y is homotopy 2-equivalent to $B(k)$ if and only if $H^*Y = M(k)$ and the homology surjectivity of (5.3) remains valid with Y in place of $B(k)$. The proof of this result, as well as other results like it, depended on properties of the Postnikov tower $\mathscr{D}_T$ together with a rather exotic condition on manifolds related to Brown and Peterson's earlier work on characteristic classes **[5]**.

Our approach here will be to apply Brown and Peterson's methods to the tower $\mathscr{C}_T$ —the arguments involved are quite similar to those used in the proof of (5.1). In the end, this will lead us to the desired characterizations of $B_1(k)$.

We begin with a definition motivated by the work of Brown and Peterson [**6**, p. 289].

DEFINITION (5.7). Suppose that N is a closed n-manifold, ν is its stable normal bundle, $T(\nu)$ is the Thom spectrum of ν, and $v \in H^p(T(\nu))$. Then (N, v) is said to be adapted to $M_1(k)$ if $n - p < 2k + 2$ and

$$0 \to A\{\mathrm{Sq}^1, \chi(\mathrm{Sq}^i) | i > k\} \to A \xrightarrow{v^*} H^*(T(\nu))$$

is exact, where $v^*(a) = av$.

The notion of adapted manifolds is somewhat unmotivated at this point. However, we can at least show that it is not vacuous.

LEMMA (5.8). *There exists a Z-orientable, closed $(2k + 1)$-manifold Q_k such that, if $U_{\nu_{Q_k}} \in H^0(T(\nu_{Q_k}))$ is the Thom class, then $(Q_k, U_{\nu_{Q_k}})$ is adapted to $M_1(k)$.*

PROOF. Let $I_n = \{a \in A | aU_{\nu_M} = 0$ for all Z-orientable, closed n-manifolds $M\}$. Brown and Peterson [**5**] showed that

$$I_{2k+1} = A\{\mathrm{Sq}^1, \chi(\mathrm{Sq}^i) | i > k\}.$$

Let $\{v_\alpha\}$ denote a Z_2-basis for $A/I_{2k+1} = M_1(k)$. For each v_α, choose a Z-orientable, closed $(2k + 1)$-manifold N_α such that $v_\alpha U_{\nu_{N_\alpha}} \neq 0$. Then let $Q_k = \#_\alpha N_\alpha$. $\square$

Now, for some notation:

Let ρ_q: $V_\infty = K(\hat{Z}_2) \to V_q$ denote the projection down the tower $\mathscr{E}_T$. For instance, ρ_0: $K(\hat{Z}_2) \to V_0 = K(Z_2)$ represents reduction mod 2.

Also, let j_q: $X_q \to V_q$ denote the qth stage of the mapping of towers j_T: $\mathscr{C}_T \to \mathscr{E}_T$.

DEFINITION (5.9). Suppose that X is a spectrum and let z: $X \to K(\hat{Z}_2)$ be given. Then a map v: $X \to X_q$ is called a k-acceptable q-lifting of $\rho_0 z$ if the following diagram commutes:

$$\begin{tikzcd}
 & & K(\hat{Z}_2) \arrow[d, "\rho_q"] \\
X \arrow[urr, "z"] \arrow[r, "v"] \arrow[dr, "\rho_0 z"'] & X_q \arrow[r, "j_q"] \arrow[d, dotted] & V_q \arrow[d, dotted] \\
 & X_0 \arrow[r, equal, "j_0"'] & V_0
\end{tikzcd}$$

Recall that, in the Postnikov tower $\mathscr{C}_T$ for $B_1(k)$, if k_q: $X_{q-1} \to K_q$ denotes the k-invariant and i: $K_{q-1} \to X_{q-1}$ is the inclusion of the fibre, then $(k_q i)^*$: $H^*K_q \to H^*K_{q-1}$ realizes the differential d: $C_q \to C_{q-1}$ of the resolution $\mathscr{C}(k)$. We next show that this fact, along with the existence of adapted manifolds, actually characterizes the k-invariant k_q.

THEOREM (5.10). *Suppose that N is a closed n-manifold, $z \in H^p(T(\nu_N); Z_2^\wedge)$, and $n - p < 2k + 2$. Let $v_{q-1}\colon T(\nu_N) \to X_{q-1}$ be a k-acceptable $(q-1)$-lifting of $\rho_0 z$. Then*:

(a) *v_{q-1} lifts to a k-acceptable q-lifting $v_q\colon X \to X_q$ of $\rho_0 z$.*

(b) *Furthermore, if $(N, \rho_0 z)$ is adapted to $M_1(k)$, then k_q is the unique map such that $(k_q i)^* = d$ and $k_q v_{q-1} = 0$.*

PROOF. The proof of (a) becomes identical to the proof of (5.1) after applying S-duality. Hence, we shall only sketch the necessary steps.

First, we may regard v_{q-1} as an element of $(X_{q-1})^p(T(\nu_N))$. Let $u_{q-1}\colon S^{n-p} \to X_{q-1} \wedge N^+$ represent a class in $(X_{q-1})_{n-p}(N^+)$ S-dual to v_{q-1}. Similarly, let $w \in H_{n-p}(N^+; Z_2^\wedge)$ be S-dual to z. Now, the proof of (5.1) shows that u_{q-1} lifts to a map $u_q\colon S^{n-p} \to X_q \wedge N^+$ in a way compatible with both the homology class w and the mapping of towers $j_T \wedge \mathrm{id}\colon \mathscr{C}_T \wedge N^+ \to \mathscr{E}_T \wedge N^+$. Let $v_q\colon T(\nu_N) \to X_q$ represent a cohomology class in $(X_q)^p(T(\nu_N))$ S-dual to u_q. Then v_q satisfies the conclusion of (a).

To prove (b), note first that $k_q v_{q-1} = 0$ by (a). Since the tower

$$\mathscr{C}_T\colon \cdots \to X_q \to X_{q-1} \to X_{q-2} \to \cdots$$

is constructed from an acyclic resolution of $M_1(k)$, the image of H^*X_{q-2} in H^*X_{q-1} is $M_1(k)$; in fact, $H^*X_{q-1} = M_1(k) \oplus \operatorname{im} k_q^*$. Thus, the sequence

$$0 \to M_1(k) \to H^*X_{q-1} \xrightarrow{i^*} H^*K_{q-1}$$

is exact. Since $k_q v_{q-1} = 0$, $v_{q-1}^*\colon H^*X_{q-1} \to H^*(T(\nu_N))$ factors through $M_1(k)$, splitting the above exact sequence, i.e., there is a commutative diagram of the form:

$$\begin{tikzcd}
0 \arrow[r] & M_1(k) \arrow[r] \arrow[d, equal] & H^*X_{q-1} \arrow[r, "i^*"] \arrow[dl] \arrow[d, "(v_{q-1})^*"] & H^*K_{q-1} \\
0 \arrow[r] & M_1(k) \arrow[r, "\tau"'] & H^*(T(\nu_N)) &
\end{tikzcd}$$

(Note that the homomorphism labelled τ above must be injective because $(N, \rho_0 z)$ is adapted to $M_1(k)$.) Consequently, $(k_q)^*\colon H^*K_q \to H^*X_{q-1}$, and hence k_q, is uniquely determined by the conditions that $k_q v_{q-1} = 0$ and $(k_q i)^* = d$. This proves (b). □

Finally, we will use (5.10) to give two characterizations of $B_1(k)$.

COROLLARY (5.11). *Suppose that Y is a spectrum such that $H^*Y = M_1(k)$. In addition, suppose there is a map $z\colon Y \to K(Z_2^\wedge)$ such that $\rho_0 z\colon Y \to K(Z_2)$ represents $1 \in H^*Y$. If, for some (N, v) adapted to $M_1(k)$, there is a map $\tilde{v}\colon T(\nu_N) \to Y$ for which $\rho_0 z \tilde{v} = v$, then Y is homotopy 2-equivalent to $B_1(k)$.*

PROOF. We will construct k-acceptable q-liftings of $\rho_0 z$ by induction on q. Assume by induction that $f\colon Y \to X_{q-1}$ is a k-acceptable $(q-1)$-lifting of $\rho_0 z$, yielding the following commutative diagram:

$$\begin{tikzcd}
 & Y \arrow[r, "f"] \arrow[d, "\rho_0 z"] & X_{q-1} \arrow[r, "k_q"] \arrow[d, dotted] & K_q \\
T(\nu_N) \arrow[ur, "\tilde{v}"] \arrow[r, "v"'] & K(Z_2) \arrow[r, equal] & X_0 &
\end{tikzcd}$$

Since $f\tilde{v}$ is a k-acceptable $(q-1)$-lifting of $v = \rho_0(z\tilde{v})$, (5.10)(a) implies that $k_q f\tilde{v} = 0$. Moreover, since (N, v) is adapted to $M_1(k)$, $\tilde{v}^*\colon H^*Y \to H^*(T(\nu_N))$ is injective. Hence, $k_q f = 0$, and, therefore, f lifts to $\tilde{f}\colon Y \to X_q$. A priori, this lifting need not be k-acceptable. However, by using the arguments described in the proof of property (Q_q) of (5.1), one can alter $\tilde{f}$, if necessary, to obtain a k-acceptable q-lifting of $\rho_0 z$. This completes the induction.

As a result, we may find a map $F\colon Y \to B_1(k)$ which induces an isomorphism in Z_2-cohomology and is therefore a homotopy 2-equivalence. □

COROLLARY (5.12). *Suppose that Y is a spectrum satisfying $H^*Y = M_1(k)$ such that Theorem* (5.1) *remains true with Y in place of $B_1(k)$. Then Y is homotopy* 2*-equivalent to $B_1(k)$.*

PROOF. By assumption, there exists $j\colon Y \to K(\hat{Z}_2)$ such that $\rho_0 j\colon Y \to K(Z_2)$ represents $1 \in H^*Y$. According to (5.8), one can find $(Q_k, U_{\nu_{Q_k}})$ adapted to $M_1(k)$, where Q_k is a $(2k+1)$-dimensional manifold. Since Y satisfies (5.1), $j_*\colon Y_{2k+1}(Q_k^+) \to H_{2k+1}(Q_k^+; \hat{Z}_2)$ is surjective. Thus, by S-duality, $j_*\colon Y^0(T(\nu_{Q_k})) \to H^0(T(\nu_{Q_k}); \hat{Z}_2)$ is surjective, also. Let $U_Z\colon T(\nu_{Q_k}) \to K(\hat{Z}_2)$ represent the Thom class, and choose a $\tilde{U}_{\nu_{Q_k}}\colon T(\nu_{Q_k}) \to Y$ such that $j\tilde{U}_{\nu_{Q_k}} = U_Z$. Then $\rho_0 j\tilde{U}_{\nu_{Q_k}} = \rho_0 U_Z = U_{\nu_{Q_k}}$. The result now follows directly from (5.11). □

Appendix. An inductive definition of $B_1(k)$. As mentioned in §2, the spectra $B_1(k)$ were originally defined by Mahowald. In this appendix, we shall propose an alternative, inductive definition based on the techniques of §5. The spectra that are obtained by this induction will satisfy the properties heretofore ascribed to $B_1(k)$, so, in particular, by (5.12), they will be homotopy 2-equivalent to $B_1(k)$.

THEOREM (A.1). *For each $k \geqslant 0$, there exists a spectrum $\bar{B}(k)$ satisfying the following properties*:

(a) $H^*\bar{B}(k) = M_1(k)$.

(b) *There is a map $\bar{i}_k\colon \bar{B}(k) \to B(k)$ fitting into a cofibration*

$$\bar{B}(k-1) \to \bar{B}(k) \xrightarrow{\bar{i}_k} B(k)$$

whose long exact sequence in cohomology realizes (2.13).

(c) *Let $\rho_0\colon K(\hat{Z}_2) \to K(Z_2)$ represent reduction* mod 2. *Then there exists a map $\bar{j}_k$: $\bar{B}(k) \to K(\hat{Z}_2)$ such that $\rho_0 \bar{j}_k\colon \bar{B}(k) \to K(Z_2)$ represents* $1 \in H^*\bar{B}(k)$.

PROOF. We shall proceed by induction on k.

If $k = 0$, define $\bar{B}(0)$ to be the sphere spectrum, completed at 2. Then $\bar{B}(0) = B(0)$, so let $\bar{i}_0 =$ identity. Also, define $\bar{j}_0$: $\bar{B}(0) \to K(Z_{\hat{2}})$ to be the 2-completion of a map representing a generator of $H^0(S^0; Z) = Z$.

Next, assume that a spectrum $\bar{B}(k-1)$ has been defined satisfying (A.1). The following analysis relies on the fact that the proofs of §5 depend only on those properties listed in (A.1). Thus, we may assume that the results of §5 are valid for $\bar{B}(k-1)$.

Let β: $M_1(k-1) \to M(k)$ be the map of A-modules defined by $\beta(1) = \mathrm{Sq}^1$. The following lemma is the key step in the proof of (A.1).

LEMMA (A.2). *There is a map* b: $B(k) \to \bar{B}(k-1)$ *such that* b^*: $H^*\bar{B}(k-1) \to H^*B(k)$ *realizes* β.

PROOF OF (A.2). Let δ: $K(Z_2) \to K(Z_{\hat{2}})$ denote the (degree 1) Bockstein operation associated to the exact sequence of groups

$$0 \to Z_{\hat{2}} \xrightarrow{2} Z_{\hat{2}} \xrightarrow{\rho_0} Z_2 \to 0.$$

Now, suppose α: $B(k) \to K(Z_2)$ represents $1 \in H^*B(k)$ and define $z \in H^1(B(k); Z_{\hat{2}})$ by the equation $z = \delta\alpha$. Observe that $\rho_0 z$: $B(k) \to \bar{X}_0 = K(Z_2)$ represents $\mathrm{Sq}^1 \in H^*B(k)$.

Based on the resolution $\mathscr{C}(k-1)$ of $M_1(k-1)$, there is a Postnikov tower $\bar{\mathscr{C}}_T$ with limit space $\bar{B}(k-1)$, denoted as follows:

$$\begin{array}{llll}
\bar{\mathscr{C}}_T: & \bar{B}(k-1) & & \\
& \vdots & & \\
& \downarrow & & \\
& \bar{X}_q & \to & \bar{K}_{q+1} \\
& \downarrow & & \\
& \bar{X}_{q-1} & \to & \bar{K}_q \\
& \downarrow & & \\
& \vdots & & \\
& \downarrow & & \\
& \bar{X}_0 & \to & \bar{K}_1
\end{array}$$

We will show that, in the language of §5, $\rho_0 z$ lifts all the way up the tower to $\bar{B}(k-1)$ through $(k-1)$-acceptable q-liftings. To do this, we again rely on some results of Brown and Peterson.

Let $T_n = \{a \in A | aU_{\nu_M} = 0$ for all closed n-manifolds $M\}$. Brown and Peterson [5] proved that $T_{2k} = A\{\chi(\mathrm{Sq}^i) | i > k\}$. Thus, by arguments identical to those of (5.8), one can find a $2k$-dimensional manifold N such that, if $v \in H^0(T(\nu_N))$ is the Thom class, then

$$0 \to A\{\chi(\mathrm{Sq}^i) | i > k\} \to A \xrightarrow{v^*} H^*(T(\nu_N))$$

is exact. Next, according to (5.4), there exists a map $\tilde{v}$: $T(\nu_N) \to B(k)$ such that $\alpha\tilde{v} = v$. Thus, we have the following commutative diagram:

$$\begin{tikzcd}
 & & & \bar{B}(k-1) \arrow[d, dotted] \\
 & B(k) \arrow[d, "\alpha"] \arrow[dr, "\rho_0 z"] & & \bar{X}_1 \arrow[d] \\
T(\nu_N) \arrow[r, "v"'] \arrow[ur, "\tilde{v}"] & K(Z_2) \arrow[r, "Sq^1"'] & K(Z_2) \arrow[r, equal] & \bar{X}_0
\end{tikzcd}$$

By induction, there are maps $\bar{i}_{k-1}$: $\bar{B}(k-1) \to B(k-1)$ and $\bar{j}_{k-1}$: $\bar{B}(k-1) \to K(Z_{\hat{2}})$ which induce the obvious maps in cohomology and therefore yield maps of Postnikov towers $(\bar{i}_{k-1})_T$: $\mathscr{C}_T \to \mathscr{D}_T$ and $(\bar{j}_{k-1})_T$: $\mathscr{C}_T \to \mathscr{E}_T$. Hence, to prove that $\rho_0 z$ lifts to $\bar{B}(k-1)$, one can simply repeat the proof of (5.11) and thereby obtain the desired map b: $B(k) \to \bar{B}(k-1)$. □

Returning to the proof of (A.1), define $\bar{B}(k)$ to be the fibre of b, and let $\bar{i}_k$: $\bar{B}(k) \to B(k)$ be the inclusion of the fibre. Then properties (a) and (b) follow immediately from the exact sequence (2.13). (Recall that, in the stable category, fibrations and cofibrations are equivalent.)

Note that the map b of (A.2) fits into the following commutative diagram:

$$\begin{tikzcd}
 & & K(Z_{\hat{2}}) \arrow[d, "\rho_0"] \\
\bar{B}(k) \arrow[r, "\bar{i}_k"] & B(k) \arrow[r, "\alpha"] \arrow[d, "b"'] \arrow[dr, "z"] & K(Z_2) \arrow[d, "\delta"] \\
 & \bar{B}(k-1) \arrow[r, "\bar{j}_{k-1}"'] & K(Z_{\hat{2}})
\end{tikzcd}$$

Since $b\bar{i}_k = 0$, $\alpha\bar{i}_k$ lifts to a map $\bar{j}_k$: $\bar{B}(k) \to K(Z_{\hat{2}})$ such that $\rho_0\bar{j}_k = \alpha\bar{i}_k$. Then $\bar{j}_k$ satisfies property (c).

This completes the induction and thus proves (A.1). □

REMARK. Using the lifting techniques of (5.11) and (A.2), one can show that there is a map h: $\bar{B}(2k) \to \bar{B}(2k+1)$ inducing an isomorphism in cohomology; therefore, $\bar{B}(2k) \simeq \bar{B}(2k+1)$.

For completeness, we include one final result.

Let M_2 denote the Z_2 Moore spectrum. Note that, for purely algebraic reasons, $B(1) = M_2$.

LEMMA (A.3). $M_2 \wedge \bar{B}(2k) \simeq B(2k+1)$.

PROOF. Recall that $\{F_j\}_{j=0}^{\infty}$ denotes the May filtration of $\Omega^2 S^3$. The structure of this filtration includes products $F_m \times F_n \to F_{m+n}$ which, when translated to Thom spectra, yield maps $\mu_{m,n}$: $B(m) \wedge B(n) \to B(m+n)$.

Now consider the composition

$$f: M_2 \wedge \bar{B}(2k) \xrightarrow{\mathrm{id} \wedge \bar{i}_{2k}} B(1) \wedge B(2k) \xrightarrow{\mu_{1,2k}} B(2k+1).$$

It is easy to check that f induces an isomorphism in cohomology and hence must be a homotopy equivalence. □

Upon careful review, one finds that our prior findings about $B_1(k)$ really depended only on naturality arguments involving the maps $i_k\colon B_1(k) \to B(k)$ and $j\colon B_1(k) \to K(Z_2^{\wedge})$; also, the computation of $\pi_* B_1(k)$ made use of the fact that $M_2 \wedge B_1(2k) \simeq B(2k+1)$. But, according to (A.1) and (A.3), the spectra $\bar{B}(k)$ possess completely analogous properties. Hence, our results on $B_1(k)$ all go over to $\bar{B}(k)$ with exactly the same proofs. In other words, we could have initially treated $\bar{B}(k)$ as our basic object of study without any loss.

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