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THE EQUIVARIANT DOLBEAULT LEMMA

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ABSTRACT. A form of the Dolbeault Lemma is obtained for circular domains $D \subset \mathbf{C}^n$, which is equivariant for the subgroup of $\mathrm{GL}(n; \mathbf{C})$ that stabilizes D.

1. Introduction and statement. In our work on representation theory we have had occasion to use holomorphic triviality arguments in an equivariant context. Since the appropriate form of the Dolbeault Lemma is generally useful but is not in the literature, we decided to write it up separately.

Let D be a domain in $\mathbf{C}^n$ that is circular in the sense $tD \subset D$ whenever t is a complex number with $|t| \leqslant 1$. $G = \{g \in \mathrm{GL}(n; \mathbf{C}) : gD = D\}$ is its stabilizer in the general linear group. Fix a Fréchet space F and denote

$$C^q(D; F): \quad \text{the space of all } F\text{-valued } C^\infty \ (0, q)\text{-forms on } D,$$
$$Z^q(D; F): \quad \text{the subspace of } C^q(D; F) \text{ given by } \bar\partial\omega = 0.$$

Here, of course, $C^q(D; F)$ is a Fréchet space in the C^∞ topology and $Z^q(D; F)$ is a closed subspace. Our result is the

EQUIVARIANT DOLBEAULT LEMMA. *There exist continuous operators*

$$H: C^{q+1}(D; F) \to C^q(D; F), \qquad 0 \leqslant q < n,$$

such that

(i) $H\bar\partial\omega + \bar\partial H\omega = \omega$ *for every* $\omega \in C^q(D; F)$, $q \geqslant 1$;

(ii) *if* $g \in G$ *then* $g^* H\omega = Hg^*\omega$ *for every* $\omega \in C^q(D; F)$;

(iii) *if* $b: F \to F$ *is a continuous linear operator then* $b \cdot H\omega = H(b \cdot \omega)$ *for every* $\omega \in C^q(D; F)$.

In particular, $H: Z^{q+1}(D; F) \to C^q(D; F)$ *is a continuous* $G \times BL(F)$ *equivariant inverse to* $\bar\partial: C^q(D; F) \to Z^{q+1}(D; F)$ *for* $q \geqslant 1$.

2. The one variable formula. Suppose $n = 1$, so either $D = \mathbf{C}^1$ or D is a disk $\{z \in \mathbf{C} : |z| < \rho\}$ in $\mathbf{C}^1$. If $\phi: F \to \mathbf{C}$ is a continuous linear functional, we define

$$h_\phi: C^1(D; F) \to C^0(D; \mathbf{C})$$

as follows. On the ϕ^{-1}-image of the polynomial (0, 1)-forms we define

$$\text{if } \phi \cdot \omega = \sum a_{rs} z^r \bar z^s \, d\bar z \text{ then } h_\phi(\omega) = \sum \frac{a_{rs}}{s+1} z^r \bar z^{s+1}.$$

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Now h_ϕ extends by continuity to $C^1(D; F)$. By construction, $\bar\partial h_\phi(\omega) = \phi \cdot \omega$. A short calculation with monomials gives the formal properties

if $a \in \mathbf{C}$ then $aD \subset D$, and $m_a\colon z \mapsto az$ satisfies
$m_a^* h_\phi(\omega) = h_\phi(m_a^*\omega)$;
if m is a positive integer and $f \in C^0(D; \mathbf{C})$ then
$h_\phi(\bar\partial(\bar z^m f)) = \bar z^m \phi(f)$.

Let F_1' and F_2' be subspaces of the continuous dual F' of F, and suppose that we have continuous linear maps

$$h_i\colon C^1(D; F) \to C^0\big(D; F/(F_i')^\perp\big), \qquad i = 1, 2,$$

such that

$$\text{if } \phi \in F_i' \text{ then } \phi \cdot h_i(\omega) = h_\phi(\omega) \text{ for all } \omega \in C^1(D; F).$$

Then there is a unique continuous linear map

$$h_3\colon C^1(D; F) \to C^0\big(D; F/(F_1' + F_2')^\perp\big)$$

that yields h_1 and h_2 on projection of function values from $F/(F_1' + F_2')^\perp$ to $F/(F_1')^\perp$ and $F/(F_2')^\perp$, and that satisfies $\phi \cdot h_3(\omega) = h_\phi(\omega)$ for all $\phi \in F_1' + F_2'$. Now an easy transfinite induction give us

LEMMA. *There is a unique continuous linear map h: $C^1(D; F) \to C^0(D; F)$ such that*

(2.1) *if $\phi \in F'$ and $\phi \cdot \omega = \sum a_{rs} z^r \bar z^s\, d\bar z$, then $\phi \cdot h(\omega) = \sum \dfrac{a_{rs}}{s+1} z^r \bar z^{s+1}$.*

If $f \in C^0(D; F)$ and $\omega \in C^1(D; F)$ then, from the corresponding properties of the h_ϕ,

(2.2) $\bar\partial h(\omega) = \omega$;

(2.3) $m_a^* h(\omega) = h(m_a^*\omega)$ *when $a \in \mathbf{C}$ with $aD \subset D$; and*

(2.4) $h \cdot \bar\partial(\bar z^m f) = \bar z^m f$ *for all integers $m > 0$.*

3. Several variables. Now $D \subset \mathbf{C}^n$, $n \geqslant 1$. If $1 \leqslant j \leqslant n$ then $i(\partial/\partial \bar z_j)$ and $e(d\bar z_j)$ denote interior and exterior product, respectively. D is assumed stable under the multiplications $m_t\colon z \mapsto tz$, $|t| \leqslant 1$, and of course if $z \in D$ we have $\varepsilon > 0$ depending on z such that m_t maps a neighborhood of z into D whenever $|t| < 1 + \varepsilon$. By (2.3) we may thus define continuous linear maps

$$H\colon C^{q+1}(D, F) \to C^q(D, F) \quad \text{for } 0 \leqslant q < n$$

by the formula

$$H\omega = \sum_{j=1}^{n} \bar z_j h_t \left(m_t^* \left\{ i\left(\frac{\partial}{\partial \bar z_j}\right)\omega \right\} \wedge d\bar t \right)\Bigg|_{t=1}. \tag{3.1}$$

Here h_t is the operator of the one variable lemma in §2, for the complex variable t.

Compute

$$H\,\bar\partial\omega + \bar\partial H\omega = \sum_{j=1}^{n} \bar z_j h_t\left(m_t^*\left\{ i\left(\frac{\partial}{\partial \bar z_j}\right)\bar\partial\omega\right\} \wedge d\bar t\right)\Bigg|_{t=1} \tag{3.2}$$

$$+ \sum_{j=1}^{n} e(d\bar z_j)\cdot h_t\left(m_t^*\left\{ i\left(\frac{\partial}{\partial \bar z_j}\right)\omega\right\} \wedge d\bar t\right)\Bigg|_{t=1}$$

$$+ \sum_{j=1}^{n} \bar z_j \bar\partial_z\left[h_t\left(m_t^*\left\{ i\left(\frac{\partial}{\partial \bar z_j}\right)\omega\right\} \wedge d\bar t\right)\Bigg|_{t=1}\right].$$

To prove (3.3) below, it suffices to consider the case $\omega = f\,d\bar z_1 \wedge \cdots \wedge d\bar z_k$, $k = q + 1 \geqslant 1$. In (3.2) the second summand is equal to

$$\sum_{j=1}^{n} h_t\left(\left\{ d\bar z_j \wedge \bar t^{-1} i\left(\frac{\partial}{\partial \bar z_j}\right) f(tz)\bar t^{k}\, d\bar z_1 \wedge \cdots \wedge d\bar z_k\right\} \wedge d\bar t\right)\Bigg|_{t=1}$$

$$= \sum_{j=1}^{n} h_t\left(\left\{ \bar t^{-1} e(d\bar z_j)\cdot i\left(\frac{\partial}{\partial \bar z_j}\right) m_t^*\omega\right\} \wedge d\bar z\right)\Bigg|_{t=1}$$

$$= k h_t\left(\bar t^{-1}\{m_t^*\omega\} \wedge d\bar t\right)\big|_{t=1}.$$

The first and third summands of (3.2) add up to

$$\sum_{j=1}^{n} h_t\left(\bar z_j\left[m_t^*\left\{ i\left(\frac{\partial}{\partial \bar z_j}\right)\bar\partial_z\omega\right\} + \bar\partial_z m_t^*\left\{ i\left(\frac{\partial}{\partial \bar z_j}\right)\omega\right\}\right] \wedge d\bar t\right)\Bigg|_{t=1}$$

$$= \sum_{j=1}^{n} h_t\left(\bar z_j m_t^*\left\{ i\left(\frac{\partial}{\partial \bar z_j}\right)\cdot\bar\partial_z\omega + \bar\partial_z\cdot i\left(\frac{\partial}{\partial \bar z_j}\right)\omega\right\} \wedge d\bar z\right)\Bigg|_{t=1}$$

$$= \sum_{j=1}^{n} h_t\left(\bar z_j m_t^*\left\{ \frac{\partial f}{\partial \bar z_j} d\bar z_1 \wedge \cdots \wedge d\bar z_k\right\} \wedge d\bar t\right)\Bigg|_{t=1}$$

$$= h_t\left(\left\{ \bar t^{k}\frac{\partial}{\partial \bar t} f(tz)\, d\bar z_1 \wedge \cdots \wedge d\bar z_k\right\} \wedge d\bar t\right)\Bigg|_{t=1}.$$

On the other hand, since $k > 0$ we can use (2.4) to see

$$\begin{aligned} h_t(\bar\partial_t m_t^*\omega) &= h_t\left(\bar\partial_t\left[\bar t^{k} f(tz)\, d\bar z_1 \wedge \cdots \wedge d\bar z_k\right]\right) \\ &= \bar t^{k} f(tz)\, d\bar z_1 \wedge \cdots \wedge d\bar z_k \\ &= m_t^*\omega \end{aligned}$$

so

$$\begin{aligned} \omega &= m_t^*|_{t=1}\omega \\ &= h_t(\bar\partial_t m_t^*\omega)|_{t=1} \\ &= h_t\left(\bar\partial_t\{\bar t^{k} f(tz)\}\, d\bar z_1 \wedge \cdots \wedge d\bar z_k\right)|_{t=1} \\ &= k h_t\left(\bar t^{-1}\{m_t^*\omega\} \wedge d\bar t\right)|_{t=1} + h_t\left(\left\{ \bar t^{k}\frac{\partial}{\partial \bar t} f(tz)\, d\bar z_1 \wedge \cdots \wedge d\bar z_k\right\} \wedge d\bar t\right)\Bigg|_{t=1}. \end{aligned}$$

In conclusion,

$$H\,\bar{\partial}\omega + \bar{\partial}H\omega = \omega \quad \text{for all } \omega \in C^k(D;\,F),\, k \geqslant 1. \tag{3.3}$$

In the formula (3.1) for H, we can replace the multipliers $\bar{z}_j$ by any independent set of conjugate linear functionals $l_1, \ldots, l_n$ on $\mathbf{C}^n$, provided that we replace the $\partial/\partial\bar{z}_j$ by the dual basis of $\overline{\mathbf{C}}^n$. Thus, if $g \in \mathrm{GL}(n;\mathbf{C})$ preserves D, then (3.1) is equivalent to

$$H\omega = \sum_{j=1}^{n} g^*(\bar{z}_j)h_t\left(m_t^*\left\{i\left(g_*\frac{\partial}{\partial\bar{z}_j}\right)\omega\right\} \wedge d\bar{t}\right)\Bigg|_{t=1}.$$

Now

$$\begin{aligned} H(g^*\omega) &= \sum_{j=1}^{n} g^*(\bar{z}_j)h_t\left(m_t^*\left\{i\left(g_*\frac{\partial}{\partial\bar{z}_j}\right)\cdot g^*\omega\right\} \wedge d\bar{t}\right)\Bigg|_{t=1} \\ &= g^*\sum_{j=1}^{n} \bar{z}_j h_t\left(m_t^*\left\{i\left(\frac{\partial}{\partial\bar{z}_j}\right)\omega\right\} \wedge d\bar{t}\right)\Bigg|_{t=1} \qquad \text{because } g^*m_t^* = m_t^*g^* \text{ and } (2.3) \\ &= g^*H(\omega). \end{aligned}$$

We have just verified that

$$g^*H\omega = Hg^*\omega \quad \text{for all } g \in G,\, \omega \in C^{q+1}(D;\,F). \tag{3.4}$$

Finally, let b: $F \to F$ be a continuous linear map. It acts on the values of ω and $H\omega$. If b has rank 1 then we have $\phi \in F'$ and $v \in F$ such that

$$b(u) = \phi(u)v \quad \text{for all } u \in F$$

and then

$$\begin{aligned} b\cdot H(\omega) &= \sum_{j=1}^{n} \bar{z}_j\phi\left(h_t\left(m_t^*\left\{i\left(\frac{\partial}{\partial\bar{z}_j}\right)\omega\right\} \wedge d\bar{t}\right)\Bigg|_{t=1}\right)v \\ &= \sum_{j=1}^{n} z_j h_t\left(m_t^*\left\{i\left(\frac{\partial}{\partial\bar{z}_j}\right)(\phi\cdot\omega)\right\} \wedge d\bar{t}\right)\Bigg|_{t=1}\cdot v \\ &= H(b\cdot\omega). \end{aligned}$$

Now an extension argument along the lines of that §2 gives a maximal subspace of F such that $b \cdot H(\omega) = H(b \cdot \omega)$ whenever b has range in the subspace, and that subspace must be all of F because we can always add the range of a rank 1 operator to it. Thus

$$b\cdot H(\omega) = H(b\cdot\omega) \quad \text{for all } b \in BL(F). \tag{3.5}$$

The Equivariant Dolbeault Lemma is obtained by combining (3.1) and (3.3)–(3.5).

Note that, given a net $\{\phi_\alpha : \alpha \in A\} \subset F'$ such that the ϕ_α are linearly independent and F' is their weak* closed span, the definition (3.1) is constructive. This applies in particular if $\dim F < \infty$. This constructiveness is the key point at which our argument differs from the standard, somewhat combinatorial, proof of the Dolbeault Lemma. Note also that our proof is loosely modeled on the standard proof of the Poincaré Lemma.

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