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ON PAIRS OF RECURSIVELY ENUMERABLE DEGREES

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ABSTRACT. Lachlan and Yates proved that some, but not all, pairs of incomparable recursively enumerable (r.e.) degrees have an infimum. We answer some questions which arose from this situation. We show that not every nonzero incomplete r.e. degree is half of a pair of incomparable r.e. degrees which have an infimum, whereas every such degree is half of a pair without infimum. Further, we prove that every nonzero r.e. degree can be split into a pair of r.e. degrees which have no infimum, and every interval of r.e. degrees contains such a pair of degrees.

Lachlan [**5**] and, independently, Yates [**11**] have proved that the upper semilattice $(\mathbf{R}, \leqslant, \cup)$ of r.e. degrees is not a lattice: There are pairs of incomparable r.e. degrees which do not have an infimum. On the other hand, they have shown that for some incomparable r.e. degrees $\mathbf{a}_0$ and $\mathbf{a}_1$, $\mathbf{a}_0 \cap \mathbf{a}_1$ exists.

Here we answer some questions which arose from the situation that some, but not all, pairs of r.e. degrees have an infimum. By constructing an incomplete strongly noncappable degree we first positively answer a question of Soare [**10**], whether there is an r.e. degree $\neq \mathbf{0}, \mathbf{0}'$ which is not half of a pair of incomparable r.e. degrees which have an infimum. We then show that for every r.e. degree $\mathbf{a} \neq \mathbf{0}, \mathbf{0}'$ there is a strongly noncappable degree incomparable with $\mathbf{a}$, and thereby deduce that every nonzero incomplete r.e. degree is half of a pair of r.e. degrees without infimum. This answers a question of Jockusch [**4**].[1] In [**2**] we had obtained partial answers to these questions by extending Lachlan's nondiamond technique of [**5**].

From the construction of a strongly noncappable degree we extract a new easy construction of a pair of r.e. degrees without infimum. We combine this construction with Sacks' splitting and density theorems to show that every r.e. degree can be split into a pair of r.e. degrees without infimum, and every proper interval of r.e. degrees contains such a pair of degrees.

It is pointed out how the above results imply and extend a result of Cooper [**3**] on minimal upper bounds for ascending sequences of uniformly r.e. degrees.

0. Preliminaries. Our notation, with a few exceptions, is that of Soare [**10**]. Lower case letters denote elements of ω, the set of nonnegative integers; capital letters denote subsets of ω. The letters f, g, h stand for functions from ω to ω. A set and its characteristic function are identified; i.e., $x \in A$ iff $A(x) = 1$ and $x \notin A$ iff $A(x) = 0$. $A \upharpoonright x$ is the restriction of A to numbers less than x. $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ denote *recursively*

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[1] L. Harrington has independently answered these questions of Jockusch and Soare.

enumerable degrees. $u(e, A, x)$ is the use function of $\{e\}^A(x)$; i.e., if $\{e\}^A(x)\downarrow$, then the computation of $\{e\}^A(x)$ uses only numbers less than $u(e, A, x) > 0$, and if $\{e\}^A(x)\uparrow$, then $u(e, A, x) = 0$. The use function of $\{e\}^A_s(x)$ is denoted by $u(e, A, x, s)$. We assume that for all e, s, x and A,

$$\{e\}^A_s(x)\downarrow \rightarrow e, x, u(e, A, x, s) < s.$$

We fix a recursive 1-1 function $\langle x, y\rangle$ from $\omega \times \omega$ onto ω which is monotonic in both arguments. $(\langle x, y\rangle)_0 = x$ and $(\langle x, y\rangle)_1 = y$. For $n \geqslant 2$, $\langle x_0, \ldots, x_n\rangle = \langle x_0, \langle x_1, \ldots, x_n\rangle\rangle$.

$$A^{(x)} = \{y \in A : (y)_0 = x\},$$

$$A^{(x,y)} = \{w \in A : \exists z\,(w = \langle x, y, z\rangle)\}$$

(note that $A^{(x,y)} \subseteq A^{(x)}$),

$$A^{(<x)} = \bigcup_{y<x} A^{(y)}, \quad \text{etc.}$$

If in the following an r.e. set X (X_j) is constructed in infinitely many stages, X_s ($X_{j,s}$) denotes the set of numbers enumerated in X (X_j) by the end of stage s.

We assume the reader is familiar with Soare [**9**].

1. Strongly noncappable degrees.

DEFINITION. (a) An r.e. degree $\mathbf{a}$ is *non*-$\mathbf{b}$-*cappable* if

$$\forall \mathbf{c}\,(\mathbf{c} \nleqslant \mathbf{b} \rightarrow \exists \mathbf{d}\,(\mathbf{d} \leqslant \mathbf{a}, \mathbf{c} \textit{ and } \mathbf{d} \nleqslant \mathbf{b}));$$

otherwise $\mathbf{a}$ is $\mathbf{b}$-*cappable*.

(b) An r.e. degree $\mathbf{a}$ is *strongly noncappable* (*s.n.c.*) if $\mathbf{a} > \mathbf{0}$ and $\forall \mathbf{b} < \mathbf{a}$ ($\mathbf{a}$ is non-$\mathbf{b}$-cappable).

SNC denotes the set of strongly noncappable degrees. Instead of non-$\mathbf{0}$-cappable and $\mathbf{0}$-cappable we usually say *noncappable* (*n.c.*) and *cappable*, respectively. A nonzero r.e. degree $\mathbf{a}$ is cappable iff it is half of a *minimal pair*, i.e. if there is a $\mathbf{b}\,|\,\mathbf{a}$ such that $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$. The existence of minimal pairs was shown in Lachlan [**5**] and Yates [**11**], that of incomplete noncappable degrees in Yates [**11**]. It is easily shown that the n.c. degrees form a filter, i.e. for n.c. $\mathbf{a}$ and $\mathbf{b} \geqslant \mathbf{a}$, $\mathbf{b}$ is n.c., and for n.c. degrees $\mathbf{a}_0, \ldots, \mathbf{a}_n$ s.t. $\mathbf{a}_0 \cap \cdots \cap \mathbf{a}_n$ exists, $\mathbf{a}_0 \cap \cdots \cap \mathbf{a}_n$ is n.c. More facts about non-$\mathbf{b}$-cappable degrees can be found in [**2**]. Obviously $\mathbf{0}'$ is s.n.c. and every s.n.c. degree is n.c. There are n.c. degrees, however, which are not s.n.c. This follows from the existence of a branching n.c. degree [**1**, Theorem 1.1], i.e. an n.c. degree $\mathbf{a}$ s.t. $\mathbf{a} = \mathbf{a}_0 \cap \mathbf{a}_1$ for r.e. degrees $\mathbf{a}_0, \mathbf{a}_1 > \mathbf{a}$. Note that if $\mathbf{a}$ is s.n.c. and $\mathbf{a}\,|\,\mathbf{b}$, then $\mathbf{a} \cap \mathbf{b}$ does not exist. This is the crucial property of s.n.c. degrees which will be used in the following.

THEOREM 1. *There is an incomplete strongly noncappable degree.*

An immediate consequence of Theorem 1 is that not every nonzero incomplete r.e. degree is half of a pair of incomparable r.e. degrees which have an infimum.

COROLLARY 1. *There is an r.e. degree* $\mathbf{a} \neq \mathbf{0}, \mathbf{0}'$ *such that for all* $\mathbf{b}\,|\,\mathbf{a}$, $\mathbf{a} \cap \mathbf{b}$ *does not exist.* □

PROOF OF THEOREM 1. By a finite injury priority argument we construct an r.e. set A such that $\mathbf{a} = \deg A$ is incomplete and s.n.c.

To make A nonrecursive and incomplete we meet the requirements

$$\mathrm{P}_e\colon A^{(0)} \neq \{e\} \quad \text{and} \quad \mathrm{N}_e\colon C \neq \{e\}^A$$

for a fixed nonrecursive r.e. set C and all numbers e. These requirements are handled by standard methods. To satisfy P_e we wait for a stage s and a number $x \in \omega^{(0,e)}$ s.t. $\{e\}_s(x) = 0$. Then—provided no higher priority requirement restrains x from A—we put x into $A^{(0)}$ at stage $s + 1$, thus establishing a disagreement between $A^{(0)}$ and $\{e\}$. To meet N_e we use Sacks' preservation strategy. Given an effective enumeration $\langle C_s\colon s < \omega\rangle$ of C, we define

$$l(e, s) = \max\{x\colon \forall y < x\,(C_s(y) = \{e\}_s^{A_s}(y))\} \quad (\textit{length function})$$

and

$$r(e, s) = \max\{u(e, A_s, x, s)\colon x \leqslant l(e, s)\} \quad (\textit{restraint function}).$$

By standard arguments N_e is met and $\lim_s r(e, s) < \omega$ exists provided that the *injury set*

$$I_e = \{x\colon \exists s\,(x \in A_{s+1} - A_s \text{ and } x < r(e, s))\}$$

is finite.

The main task of the construction is to satisfy

$$\forall \mathbf{v} < \mathbf{a}\, \forall \mathbf{w} \nleqslant \mathbf{v}\, \exists \mathbf{e}\,(\mathbf{e} \leqslant \mathbf{a}, \mathbf{w} \text{ and } \mathbf{e} \nleqslant \mathbf{v}), \tag{1}$$

which, together with the nonrecursiveness of A, implies $\mathbf{a}$ is s.n.c. Since, for $\mathbf{v}$ and $\mathbf{w}$ as in (1) s.t. $\mathbf{w} \leqslant \mathbf{a}$, $\mathbf{w} \leqslant \mathbf{a}, \mathbf{w}$ and $\mathbf{w} \nleqslant \mathbf{v}$, (1) may be replaced by the weaker condition

$$\forall \mathbf{v} < \mathbf{a}\, \forall \mathbf{w} \nleqslant \mathbf{a}\, \exists \mathbf{e}\,(\mathbf{e} \leqslant \mathbf{a}, \mathbf{w} \text{ and } \mathbf{e} \nleqslant \mathbf{v}). \tag{2}$$

To satisfy (2), for all $e = \langle e_0, e_1, e_2\rangle$ we construct r.e. sets E_e such that

$$(W_{e_0} = \{e_1\}^A, A \nleqslant_T W_{e_0} \text{ and } W_{e_2} \nleqslant_T A) \to (E_e \leqslant_T A, W_{e_2} \text{ and } E_e \nleqslant_T W_{e_0}). \tag{3}$$

The premise $W_{e_0} = \{e_1\}^A$ is used to control the enumeration of W_{e_0} by controlling that of A. We ensure (3) by meeting the requirements

$$\mathrm{R}_{\langle e,i\rangle}\colon (W_{e_0} = \{e_1\}^A, A \nleqslant_T W_{e_0} \text{ and } W_{e_2} \nleqslant_T A) \to (E_e \neq \{i\}^{W_{e_0}})$$

and satisfying the conditions

$$E_e \leqslant_T W_{e_2}, \tag{4}$$

$$W_{e_0} = \{e_1\}^A \to E_e \leqslant_T A \tag{5}$$

for all $e = \langle e_0, e_1, e_2\rangle$ and i.

The basic idea to meet requirement $\mathrm{R}_{\langle e,i\rangle}$ is that for meeting P_e: We wait for a number $x \in \omega^{(e+1,i)}$ and a stage s such that $\{i\}_s^{W_{e_0,s}}(x) = 0$. If we then put x into A $\mathrm{R}_{\langle e,i\rangle}$ is met unless the computation $\{i\}_s^{W_{e_0,s}}(x)$ is not W_{e_0}-correct, i.e.

$$W_{e_0,s} \restriction u(i, W_{e_0,s}, x, s) \neq W_{e_0} \restriction u(i, W_{e_0,s}, x, s).$$

To prevent W_{e_0} from changing below $u(i, W_{e_0,s}, x, s)$ after stage s, we attack $R_{\langle e,i \rangle}$ with x only if

$$(6) \qquad W_{e_0,s} \upharpoonright u(i, W_{e_0,s}, x, s) = \{e_1\}_s^{A_s} \upharpoonright u(i, W_{e_0,s}, x, s).$$

In this case we impose a restraint on A to preserve the right side of (6). Then either the left side of (6) does not change and, therefore, $\{i\}^{W_{e_0}}(x) = 0$, or it is not correct, which implies $W_{e_0} \neq \{e_1\}^A$. In either case, requirement $R_{\langle e,i \rangle}$ is met.

This procedure to meet $R_{\langle e,i \rangle}$ is restricted by the need to satisfy (4) and (5). Condition (4) is ensured by the permitting method: A number x is allowed to enter E_e only at such stages $s + 1$ for which a $y < x$ exists such that $y \in W_{e_2,s+1} - W_{e_2,s}$. A similar action to satisfy (5), i.e. to put a number $y \leqslant x$ into A whenever x is put into E_e, is in conflict with the necessity to preserve the right side of (6), which requires us to restrain certain numbers from A. To solve this conflict, for e such that $W_{e_0} = \{e_1\}^A$, we define an A-recursive function $f_e(x)$ and a recursive monotonic approximation $f_e(x, s)$ to $f_e(x)$ such that for sufficiently many x and s, $f_e(x, s)$ is greater than the restraint required to preserve the right side of (6). Only for such x and s we will put x into $E_{e,s+1}$, and in this case we enumerate a new number less than or equal to $f_e(x, s)$ in A_{s+1}. It follows that

$$\forall x, s\,(A_s \upharpoonright f_e(x) + 1 = A \upharpoonright f_e(x) + 1 \rightarrow E_{e,s} \upharpoonright x + 1 = E_e \upharpoonright x + 1),$$

whence (5) is satisfied.

To define f_e, for $e = \langle e_0, e_1, e_2 \rangle$, we first inductively define the *standard marker* (*function*) γ of A (with respect to $\langle A_s\colon s < \omega \rangle$):

$$\gamma(x,0) = 0, \quad \gamma(x, s + 1) = \begin{cases} s + 1 & \text{if } A_{s+1} \upharpoonright x \neq A_s \upharpoonright x, \\ \gamma(x, s) & \text{otherwise.} \end{cases}$$

$\gamma(x, s)$ is recursive, nondecreasing in both arguments, and for all x, $\lim_s \gamma(x, s) < \omega$ exists. We denote $\lim_s \gamma(x, s)$ by $\gamma(x)$. Then $\gamma(x)$ is an A-recursive function. The crucial property of standard marker functions is stated in the following lemma.

LEMMA 1. *Let A be an r.e. set, γ a standard marker of A with respect to some effective enumeration $\langle A_n\colon n < \omega \rangle$ of A, B an infinite set, and h a function such that $A \not\leqslant_T B \oplus h$. Then there are infinitely many $x \in B$ such that $\gamma(x) > h(x)$.*

PROOF. Assume, for a contradiction, there are only finitely many $x \in B$ s.t. $\gamma(x) > h(x)$, say x_0 is the greatest such x. Then

$$\forall x\,(x > x_0 \text{ and } x \in B \rightarrow A_{h(x)} \upharpoonright x = A \upharpoonright x).$$

Since B is infinite this implies $A \leqslant_T B \oplus h$. □

Based on the definition of γ we define, for $e = \langle e_0, e_1, e_2 \rangle$ and $x = \langle x_0, x_1, x_2 \rangle$,

$$f_e(x, s) = \langle x_0, x_1, x_2, \max\{u(e_1, A_s, y, t)\colon y \leqslant \gamma(x, s) \text{ and } t \leqslant s\} \rangle.$$

The binary function f_e is recursive and nondecreasing in both arguments. For e such that $\{e_1\}^A$ is total, $\lim_s f_e(x, s) = \sup_s f_e(x, s) < \omega$ exists for all x. For such e we set $f_e(x) = \lim_s f_e(x, s)$. If defined, $f_e(x)$ is a total A-recursive function. Note that $x \leqslant f_e(x, s)$ and for $x \in \omega^{(i,j)}$, $f_e(x, s)$ is an element of $\omega^{(i,j)}$ too.

We now turn to the formal description of the construction. The priority ordering of the requirements is given by

$$\mathrm{N}_0 > \mathrm{R}_0 > \mathrm{P}_0 > \cdots > \mathrm{N}_n > \mathrm{R}_n > \mathrm{P}_n > \mathrm{N}_{n+1} > \mathrm{R}_{n+1} > \mathrm{P}_{n+1} > \cdots.$$

For any $\langle e, i\rangle$ and s, $R(\langle e, i\rangle, s)$ denotes the restraint imposed by requirement $\mathrm{R}_{\langle e,i\rangle}$ at the end of stage s; it is defined at stage s of the construction below. The requirement $\mathrm{R}_{\langle e,i\rangle}$, where $e = \langle e_0, e_1, e_2\rangle$, *requires attention at stage* $s+1$ if there is an $x \in \omega^{\langle e+1, i\rangle}$ such that the following conditions hold:

(7.1) $R(\langle e, i\rangle, s) = 0$,

(7.2) $\{i\}_s^{W_{e_0,s}}(x) = 0$ and $x \notin E_{e,s}$,

(7.3) $\gamma(x, s) \geq u$, where $u = u(i, W_{e_0,s}, x, s)$,

(7.4) $W_{e_0,s} \restriction u = \{e_1\}_s^{A_s} \restriction u$,

(7.5) $f_e(x, s) \geq \max(\{r(k, s)\colon k \leq \langle e, i\rangle\} \cup \{R(k, s)\colon k < \langle e, i\rangle\})$,

and

(7.6) $$\exists y < x\left(y \in W_{e_2,s+1} - W_{e_2,s}\right).$$

Requirement P_e *requires attention* at stage $s+1$ if

(8.1) $$\forall y \in \omega^{(0,e)}(\{e\}_s(y)\downarrow \rightarrow A_s(y) = \{e\}_s(y))$$

and there is an $x \in \omega^{(0,e)}$ such that

(8.2) $\{e\}_s(x) = 0$ and $x \geq \max(\{r(k, s)\colon k \leq e\} \cup \{R(k, s)\colon k \leq e\})$.

CONSTRUCTION.

Stage 0. For all x and e, set $R(x, 0) = 0$ and define $l(x, 0)$, $r(x, 0)$, $\gamma(x, 0)$, and $f_e(x, 0)$ as described above.

Stage $s+1$. Choose the highest priority requirement which requires attention.

Case 1. P_e is this requirement. Then choose the least $x \in \omega^{(0,e)}$ which satisfies (8.2). Put x into A and for all k set

$$R(k, s+1) = \begin{cases} R(k, s) & \text{if } k \leq e, \\ 0 & \text{if } k > e. \end{cases}$$

Case 2. $\mathrm{R}_{\langle e,i\rangle}$, $e = \langle e_0, e_1, e_2\rangle$, is this requirement. Then choose the least $x \in \omega^{(e+1,i)}$ for which (7.1)–(7.6) hold. Put x into E_e and $f_e(x, s)$ into A. For $k < \omega$, define

$$R(k, s+1) = \begin{cases} R(k, s) & \text{if } k < \langle e, i\rangle, \\ \max\left\{u(e_1, A_s, y, s)\colon y < u\left(i, W_{e_0,s}, x, s\right)\right\}, & \text{if } k = \langle e, i\rangle, \\ 0 & \text{if } k > \langle e, i\rangle. \end{cases}$$

Case 3. There is no requirement which requires attention. Then set $R(k, s+1) = R(k, s)$ for all k.

In any case define $l(x, s+1)$, $r(x, s+1)$, $\gamma(x, s+1)$ and $f_e(x, s+1)$ as described above. If Case 1 (2) holds, we say $\mathrm{P}_e(\mathrm{R}_{\langle e,i\rangle})$ *receives atttention* or is *active* at stage $s+1$.

This completes the construction. The construction is effective. Hence the functions $R(x, s)$, $l(x, s)$, $r(x, s)$, $\gamma(x, s)$ and $f_e(x, s)$ are recursive, and the sets A and E_e are recursively enumerable. Note that a number $x \in \omega^{(0,e)}$ enters A only at a stage at which P_e is active; a number $x \in \omega^{(e+1,i)}$ enters E_e or A only at a stage at which $\mathrm{R}_{\langle e,i \rangle}$ is active. To verify that the constructed sets have the desired properties, we prove a series of lemmas.

LEMMA 2. (a) *Every requirement requires attention at most finitely often.*
(b) *For all e,* $\lim_s R(e, s) < \omega$ *exists.*

PROOF. Part (a) is shown by induction on the priority of the requirement. Given a requirement R, by inductive hypothesis fix a stage s_0 after which no higher priority requirement requires attention. Assume R requires attention at stage $s_1 > s_0$. Then R receives attention at stage s_1. If R is requirement P_e, a number $x \in \omega^{(0,e)}$ such that $\{e\}_{s_1}(x) = 0$ is put into A at stage s_1, whence by (8.1) P_e does not require attention after stage s_1. If R is requirement $\mathrm{R}_{\langle e,i \rangle}$, then $R(\langle e,i \rangle, s_1) > 0$, and by choice of s_0 and $s_1 > s_0$, $R(\langle e, i \rangle, s) = R(\langle e,i \rangle, s_1)$ for all $s \geqslant s_1$. Hence by (7.1), $\mathrm{R}_{\langle e,i \rangle}$ does not require attention after stage s_1. This completes the proof of (a). Part (b) follows from (a), since $R(e, s+1) \neq R(e, s)$ implies R_e or a higher priority requirement is active at stage $s + 1$. □

LEMMA 3. *For every e, N_e is met and* $\lim_s r(e, s) < \omega$ *exists.*

PROOF. For any e and s, a number $x < r(e, s)$ can enter A at stage $s + 1$ only for the sake of a requirement P_k or R_k where $k < e$. Hence by Lemma 2 the injury set I_e is finite. As in Soare [**9**, Lemmas 1.1 and 1.2] we may conclude Lemma 3 holds. □

We set $r(e) = \sup_s r(e, s)$ and $R(e) = \lim_s R(e, s)$.

LEMMA 4. *For all e, P_e is met.*

PROOF. For a contradiction assume $A^{(0)} = \{e\}$. Then P_e never receives attention, since if P_e is active at stage $s + 1$ there is an $x \in \omega^{(0,e)}$ such that

$$1 = A^{(0)}(x) = A^{(0)}_{s+1}(x) \neq \{e\}_s(x) = \{e\}(x) = 0.$$

Since a number $x \in \omega^{(0,e)}$ can enter A only if P_e is active, it follows that no $x \in \omega^{(0,e)}$ is in A, i.e. by assumption

$$\forall x \in \omega^{(0,e)} \big(A(x) = \{e\}(x) = 0 \big). \tag{9}$$

This implies that (8.1) holds for all s. By Lemmas 2 and 3, choose s_0 and $x_0 \in \omega^{(0,e)}$ such that no requirement of higher priority than P_e requires attention after stage s_0 and such that

$$\forall s \forall k \leqslant e \big(r(k) \leqslant x_0 \text{ and } R(k, s) \leqslant x_0 \big).$$

By (9) there is a stage $s_1 > s_0$ such that $\{e\}_{s_1}(x_0) = 0$, so P_e requires and receives attention at stage $s_1 + 1$, contradiction. □

LEMMA 5. *For all e and i, $\mathrm{R}_{\langle e,i \rangle}$ is met.*

PROOF. Fix $e = \langle e_0, e_1, e_2 \rangle$ and i, and for a contradiction assume $\mathrm{R}_{\langle e,i \rangle}$ is not met, i.e.

$$W_{e_0} = \{e_1\}^A, \tag{10}$$

$$A \not\leqslant_T W_{e_0}, \tag{11}$$

$$W_{e_2} \not\leqslant_T A, \tag{12}$$

$$E_e = \{i\}^{W_{e_0}}. \tag{13}$$

We distinguish the following two cases.

Case 1. $R(\langle e, i \rangle) > 0$. Choose the least s such that

$$\forall t > s\,(R(\langle e, i \rangle, t) > 0). \tag{14}$$

Then $\mathrm{R}_{\langle e,i \rangle}$ is active at stage $s + 1$, i.e. there is an $x \in \omega^{(e+1,i)}$ which satisfies (7.1)–(7.6) and which is put into E_e at stage $s + 1$. Moreover,

$$R(\langle e, i \rangle, s + 1) = \max\Big\{u(e_1, A_s, y, s) : y < u\big(i, W_{e_0,s}, x, s\big)\Big\}. \tag{15}$$

By (14) neither $\mathrm{R}_{\langle e,i \rangle}$ nor a requirement of higher priority is active after stage $s + 1$. Hence

$$\forall t > s\,(R(\langle e, i \rangle, t) = R(\langle e, i \rangle, s + 1))$$

and no number less than $R(\langle e, i \rangle, s + 1)$ enters A after stage $s + 1$. Since at stage $s + 1$ only $f_e(x, s)$ enters A, which by (7.3) is greater than or equal to $R(\langle e, i \rangle, s + 1)$, it follows that

$$A_s \restriction R(\langle e, i \rangle, s + 1) = A \restriction R(\langle e, i \rangle, s + 1). \tag{16}$$

From (7.4), (15) and (16) we can deduce that for $u = u(i, W_{e_0,s}, x, s)$,

$$W_{e_0,s} \restriction u = \{e_1\}_s^{A_s} \restriction u = \{e_1\}^A \restriction u.$$

Hence by (10), $W_{e_0} \restriction u = W_{e_0,s} \restriction u$ and, therefore, by (7.2),

$$\{i\}^{W_{e_0}}(x) = \{i\}_s^{W_{e_0,s}}(x) = 0.$$

Since $x \in E_e$ this implies $\mathrm{R}_{\langle e,i \rangle}$ is met, contrary to our assumption.

Case 2. $R(\langle e, i \rangle) = 0$. Then by Lemmas 2 and 3 we can choose s_0 and x_0 such that $\mathrm{R}_{\langle e,i \rangle}$ does not require attention after stage s_0,

$$\forall s \geqslant s_0\,(R(\langle e, i \rangle, s) = 0), \tag{17}$$

$$\forall k \geqslant \langle e, i \rangle\, \forall s\,(r(k) \leqslant x_0 \text{ and } R(k, s) \leqslant x_0), \tag{18}$$

and no number greater than x_0 is put into E_e for the sake of requirement $\mathrm{R}_{\langle e,i \rangle}$, i.e.

$$\forall x \in \omega^{(e+1,i)}\,(x > x_0 \to E_e(x) = 0)$$

and, consequently, by (13),

$$\forall x \in \omega^{(e+1,i)}\Big(x > x_0 \to E_e(x) = \{i\}^{W_{e_0}}(x) = 0\Big).$$

Since by (13) $u(i, W_{e_0}, x)$ is a total W_{e_0}-recursive function, it follows from (11) and Lemma 1 that the set

$$G = \Big\{x \in \omega^{(e+1,i)} : x > x_0 \text{ and } \gamma(x) > u\big(i, W_{e_0}, x\big)\Big\}$$

is infinite. Obviously,

$$G \leqslant_T \lambda x(\gamma(x)) \oplus \lambda x\big(u(i, W_{e_0}, x)\big) \leqslant_T A \oplus W_{e_0},$$

i.e. by (10), $G \leqslant_T A$. For $x \in G$,

(19.1) $\{i\}_s^{W_{e_0,s}}(x) = 0$ via a W_{e_0}-correct computation, i.e.
$W_{e_0,s} \upharpoonright u\big(i, W_{e_0,s}, x, s\big) = W_{e_0} \upharpoonright u\big(i, W_{e_0,s}, x, s\big)$,

(19.2) $\gamma(x, s) > u\big(i, W_{e_0,s}, x, s\big)$, and

(19.3) $W_{e_0,s} \upharpoonright u\big(i, W_{e_0,s}, x, s\big) = \{e_1\}_s^{A_s} \upharpoonright u\big(i, W_{e_0,s}, x, s\big)$ via A-correct computations, i.e. $A_s \upharpoonright v = A \upharpoonright v$, where
$v = \max\big\{u(e_1, A_s, y, s) : y < u\big(i, W_{e_0,s}, x, s\big)\big\}$

are true for almost all s. Moreover, the function

$$s(x) = \begin{cases} \mu s > s_0 \ (s \text{ satisfies } (19.1)\text{–}(19.3)) & \text{if } x \in G, \\ 0 & \text{otherwise,} \end{cases}$$

is A-recursive and for all $x \in G$ and $s \geqslant s(x)$, (19.1)–(19.3) hold. Together with (17) and (18) this implies that for all $x \in G$ and $s \geqslant s(x)$, conditions (7.1)–(7.5) are satisfied. Since $R_{\langle e,i\rangle}$ does not require attention after stage s_0, for no such x and s, (7.6) can hold, i.e.

$$\forall x \in G\big(W_{e_2,s(x)} \upharpoonright x = W_{e_2} \upharpoonright x\big).$$

Since G is infinite, and G and $\lambda x(s(x))$ are recursive in A, this implies that W_{e_2} is recursive in A, a contradiction to (12). □

LEMMA 6. *For all* $e = \langle e_0, e_1, e_2\rangle$,
(a) $E_e \leqslant_T W_{e_2}$,
(b) $W_{e_0} = \{e_1\}^A \to E_e \leqslant_T A$.

PROOF. (a) By (7.6) a number is put into E_e at stage $s + 1$ only if there is a smaller number in $W_{e_2,s+1} - W_{e_2,s}$. Hence

$$\forall s, x\big(W_{e_2,s} \upharpoonright x = W_{e_2} \upharpoonright x \to E_{e,s} \upharpoonright x = E_e \upharpoonright x\big),$$

which implies $E_e \leqslant_T W_{e_2}$.

(b) Assume $W_{e_0} = \{e_1\}^A$. Then $\{e_1\}^A$ is total and thus the unary function f_e is total and A-recursive, as we have pointed out above. Since for all e, e', x, y, s and t, $x \neq y \to f_e(x, s) \neq f_{e'}(y, t)$ and $E_e \subseteq \omega^{(e+1)}$, it follows from the construction that

$$\forall x \in \omega^{(e+1)} \forall s\big(x \in E_{e,s+1} - E_{e,s} \to f_e(x, s) \in A_{s+1} - A_s\big).$$

Hence

$$\forall x, s\big(A_s \upharpoonright f_e(x) + 1 = A \upharpoonright f_e(x) + 1 \to E_{e,s} \upharpoonright x + 1 = E_e \upharpoonright x + 1\big),$$

which implies $E_e \leqslant_T A$. □

As shown above, Lemmas 3–6 imply deg A is s.n.c. and incomplete, which completes the proof of Theorem 1. ■

The proof of Theorem 1 actually shows that for any $\mathbf{c} > \mathbf{0}$ there is an s.n.c. degree $\mathbf{a} \not\geq \mathbf{c}$. Moreover, the constructed set is low (see Soare [**9**, Remarks 4.4 and 4.5]).

We now combine the splitting technique with the s.n.c. degree construction to show that for every $\mathbf{c} \neq \mathbf{0}, \mathbf{0}'$ there is a (low) s.n.c. degree incomparable with $\mathbf{c}$.

THEOREM 2. *Let $\mathbf{c} > \mathbf{0}$ be given. Then there are low s.n.c. degrees $\mathbf{a}_0$ and $\mathbf{a}_1$ such that $\mathbf{c} \not\leq \mathbf{a}_0, \mathbf{a}_1$ and $\mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{0}'$.*

COROLLARY 2. *For every $\mathbf{c} \neq \mathbf{0}, \mathbf{0}'$ there is a low s.n.c. degree incomparable with $\mathbf{c}$.*

PROOF. Given $\mathbf{c} \neq \mathbf{0}, \mathbf{0}'$, by Theorem 2 split $\mathbf{0}'$ into low s.n.c. degrees $\mathbf{a}_0$ and $\mathbf{a}_1$ s.t. $\mathbf{c} \not\leq \mathbf{a}_0, \mathbf{a}_1$. Then $\mathbf{c} | \mathbf{a}_0$ or $\mathbf{c} | \mathbf{a}_1$. □

COROLLARY 3. *For every r.e. degree $\mathbf{c} \neq \mathbf{0}, \mathbf{0}'$ there is an r.e. degree $\mathbf{a}$ such that $\mathbf{c} \cap \mathbf{a}$ does not exist.* □

PROOF OF THEOREM 2. We combine the proof of Theorem 1 with that of the splitting theorem (see, e.g., Soare [**9**, Theorem 1.2]).

Fix $\mathbf{c} > \mathbf{0}$, $C \in \mathbf{c}$, r.e. and an effective enumeration $\langle C_s : s < \omega \rangle$ of C, and choose a complete set $A \subseteq \omega^{(0)}$ and a recursive function g which enumerates A without repetitions. We construct r.e. sets A_0 and A_1 such that $\mathbf{a}_0 = \deg A_0$ and $\mathbf{a}_1 = \deg A_1$ satisfy the theorem.

To ensure $\mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{0}'$, at stage $s + 1$ of the construction we put $g(s)$ into A_0 or A_1 and no other numbers are enumerated in $A_0^{(0)}$ and $A_1^{(0)}$. Then $A = A_0^{(0)} \cup A_1^{(0)}$, which implies $A \leq_T A_0 \oplus A_1$.

As in the preceding proof we use the preservation strategy to make C nonrecursive in A_0 and A_1. For all e and for $j \leq 1$ we meet the requirements

$$\mathrm{N}_{2e+j}\colon C \neq \{e\}^{A_j}.$$

The length function $l(2e + j, s)$, the restraint function $r(2e + j, s)$ and the injury set I_{2e+j} are obtained from the definitions of $l(e, s)$, $r(e, s)$ and I_e in the proof of Theorem 1 by replacing A by A_j.

To make $\mathbf{a}_0$ and $\mathbf{a}_1$ s.n.c. it suffices to construct r.e. sets E_e^j such that for all $e = \langle e_0, e_1, e_2 \rangle$, $i < \omega$ and $j \leq 1$, the requirements

$$\mathrm{R}_{\langle 2e+j,i \rangle}\colon \left(W_{e_0} = \{e_1\}^{A_j}, A_j \not\leq_T W_{e_0} \text{ and } W_{e_2} \not\leq_T A_j\right) \rightarrow E_j^e \neq \{i\}^{W_{e_0}},$$

and conditions

(20) $$E_e^j \leq_T W_{e_2}$$

(21) $$W_{e_0} = \{e_1\}^{A_j} \rightarrow E_e^j \leq_T A_j$$

are satisfied.

Let γ_j be the standard marker of A_j with respect to $\langle A_{j,s} : s < \omega \rangle$, and define the functions f_e^j by replacing A and γ by A_j and γ_j, respectively, in the definition of f_e in the foregoing proof ($j = 0, 1$).

The requirement $\mathrm{R}_{\langle 2e+j,i \rangle}$, where $e = \langle e_0, e_1, e_2 \rangle$, *requires attention* at stage $s + 1$ if there is an $x \in \omega^{(2e+j+1,i)}$ such that

(22.1) $$R(\langle 2e + j, i \rangle, s) = 0,$$

(22.2) $\{i\}_s^{W_{e_0,s}}(x) = 0,$

(22.3) $\gamma_j(x, s) \geqslant u,$ where $u = u(i, W_{e_0,s}, x, s),$

(22.4) $W_{e_0,s} \upharpoonright u = \{e_1\}_s^{A_{j,s}} \upharpoonright u,$

(22.5) $f_e^j(x, s) \geqslant \max(\{r(k, s): k \leqslant \langle 2e + j, i \rangle\} \cup \{R(k, s): k < \langle 2e + j, i \rangle\}),$ and

(22.6) $\exists y < x (y \in W_{e_2,s+1} - W_{e_2,s}).$

We say k *needs protection* at stage $s + 1$ if $g(s) < \max\{r(k, s), R(k, s)\}$.

CONSTRUCTION.

Stage 0. For $x, e < \omega$ and $j \leqslant 1$, set $R(x,0) = 0$ and define $l(x,0)$, $r(x,0)$, $\gamma_j(x,0)$ and $f_e^j(x,0)$ as described above.

Stage $s + 1$. If there is a number y such that y needs protection or R_y requires attention at stage $s + 1$, then let $\langle 2e + j, i \rangle$, where $e = \langle e_0, e_1, e_2 \rangle$ and $j \leqslant 1$, be the least such number and distinguish the two cases below. Otherwise put $g(s)$ into A_0 and set $R(k, s + 1) = R(k, s)$ for all k.

Case 1. $R_{\langle 2e+j,i \rangle}$ requires attention. Then choose the least $x \in \omega^{\langle 2e+j+1,i \rangle}$ which satisfies (22.1)–(22.6). Put x into E_e^j, $f_e^j(x, s)$ into A_j, and $g(s)$ into A_{1-j}. Define

$$R(k, s + 1) = \begin{cases} R(k, s) & \text{if } k < \langle 2e + j, i \rangle, \\ \max\{u(e_1, A_{j,s}, y, s) : y < u(i, W_{e_0,s}, x, s)\} & \text{if } k = \langle 2e + j, i \rangle, \\ 0 & \text{if } k > \langle 2e + j, i \rangle. \end{cases}$$

Case 2. Otherwise. Then put $g(s)$ into A_{1-j} and define

$$R(k, s + 1) = \begin{cases} R(k, s) & \text{if } k \leqslant \langle 2e + j, i \rangle, \\ 0 & \text{if } k > \langle 2e + j, i \rangle. \end{cases}$$

In any case, define $l(x, s + 1)$, $r(x, s + 1)$, $\gamma_j(x, s)$ and $f_e^j(x, s)$ for all $x, e < \omega$ and $j \leqslant 1$ as described above.

Obviously the construction is effective and $A = A_0^{(0)} \cup A_1^{(0)}$. The latter implies $\mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{0}'$. To verify that $\mathbf{c} \nleqslant \mathbf{a}_0, \mathbf{a}_1$ and $\mathbf{a}_0, \mathbf{a}_1$ are low and s.n.c., we first prove

LEMMA 7. *For all numbers k the following hold*:

(i) *The injury set* I_k *is finite.*

(ii) *Requirement* N_k *is met.*

(iii) $\lim_s l(k, s)$ *and* $\lim_s r(k, s)$ *exist and are finite.*

(iv) *Requirement* R_k *requires attention at most finitely often.*

(v) $\lim_s R(k, s)$ *exists and is finite.*

(vi) *k needs protection at at most finitely many stages.*

The claims of Lemma 7 are proven in the given order by a simultaneous induction. The inductive step is like the proofs of Lemmas 2 and 3. For the proof of (i) and (iv) we have only to note that for $k = \langle 2e + j, i \rangle$, $j \leqslant 1$, there is a stage s_0 such that for all $s > s_0$,

$$g(s) \notin I_k, \quad \text{i.e. } g(s) < r(k, s) \to g(s) \notin A_{j,s+1}$$

and

$$g(s) < R(k, s) \to g(s) \notin A_{j,s+1}.$$

By construction, a stage s_0 has this property if after stage s_0 for no $k' < k$, $\mathrm{R}_{k'}$ requires attention or k' needs protection, and by inductive hypotheses (iv) and (vi) such an s_0 exists.

(vi) is an immediate consequence of (iii) and (v).

Lemma 7(ii) implies $\mathbf{c} \not\leq \mathbf{a}_0, \mathbf{a}_1$, and from Lemma 7(iii) we can conclude, as in Soare [**9**, Remarks 4.4 and 4.5], that $\mathbf{a}_0$ and $\mathbf{a}_1$ are low. Finally, that the requirements $\mathrm{R}_{\langle 2e+j,i\rangle}$ and conditions (20) and (21) are satisfied for all $e = \langle e_0, e_1, e_2\rangle$, $i < \omega$ and $j \leqslant 1$ follows from Lemma 7 as Lemmas 5 and 6 follow from Lemmas 2 and 3 in the proof of Theorem 1.

This completes the proof of Theorem 2. ■

The proof of Theorem 2 can be combined with a technique of Robinson [**8**] for handling low oracle sets to do the construction of $\mathbf{a}_0$ and $\mathbf{a}_1$ above any low r.e. degree $\mathbf{d} \not\geq \mathbf{c}$. This implies

(23) $$\forall \mathbf{d}\ \text{low}\ \forall \mathbf{c} \not\leq \mathbf{d}\ \exists \mathbf{a}\ \text{low}\ (\mathbf{a} \text{ is s.n.c.}, \mathbf{d} < \mathbf{a} \text{ and } \mathbf{c} \not\leq \mathbf{a}).$$

An easy consequence of (23) is that the s.n.c. degrees form an automorphism basis, i.e. any two order automorphisms of **R** which agree on **SNC** must be identical. The s.n.c. degrees do not generate **R** (under $\cup$ and $\cap$), however, since they are contained in the proper filter of noncappable degrees.

The set **SNC** does not have interesting closure properties. Trivially, **SNC** is closed under finite infima (if they exist), since a finite set of s.n.c. degrees has an infimum iff it has a least element. **SNC** is not closed downwards in **R** or $\mathbf{R} - \{\mathbf{0}\}$, however, since every s.n.c. degree is n.c., and every n.c. degree bounds a nonzero cappable degree. That **SNC** is not closed under finite suprema, and therefore is not closed upwards, can be shown as follows: The proof of Theorem 2 can be combined with that of Lachlan's nondiamond theorem [**5**, Theorem 5] to prove

(24) $$\forall \mathbf{a}_0, \mathbf{a}_1 (\mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{0}' \to \exists i \leqslant 1\, \exists \mathbf{a}_{i0}, \mathbf{a}_{i1} \\ (\mathbf{a}_i = \mathbf{a}_{i0} \cup \mathbf{a}_{i1} \text{ and } \mathbf{a}_{i0} \text{ and } \mathbf{a}_{i1} \text{ are s.n.c.})).$$

Lachlan [**7**] and, independently, Shoenfield and Soare (unpublished) have shown that there are incomparable r.e. degrees $\mathbf{a}_0$ and $\mathbf{a}_1$ s.t. $\mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{0}'$ and $\mathbf{a}_0 \cap \mathbf{a}_1$ exists. Since such degrees $\mathbf{a}_0$ and $\mathbf{a}_1$ are not s.n.c., an application of (24) yields a pair of s.n.c. degrees whose supremum is not strongly noncappable.

Obviously not every nonzero r.e. degree can be split into two s.n.c. degrees. Examples of degrees which allow such a splitting are given in Theorem 2 and in (24). We can obtain more examples by combining the s.n.c. construction with the low nondiamond technique of [**2**]: If **a** is *low cuppable*, i.e. if there is a low **b** s.t. $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'$, then **a** can be split into two s.n.c. degrees.[2]

[2] Recently Ambos-Spies, Jockusch, Shore and Soare have shown that every n.c. degree is low cuppable. Hence an r.e. degree can be split into s.n.c. degrees iff it is noncappable.

All constructions mentioned above are finite (or no-) injury arguments and automatically yield low degrees. We do not know whether there are s.n.c. degrees which are not low.

We conclude this section with an application of the above results to ascending sequences of uniformly r.e. degrees. We need the following notions: A sequence $\langle \mathbf{c}_n : n < \omega \rangle$ of r.e. degrees is *ascending* if $\forall n(\mathbf{c}_n \leqslant \mathbf{c}_{n+1})$ and $\forall n \exists m(\mathbf{c}_m \nleqslant \mathbf{c}_n)$; it is *uniformly recursively enumerable* (*u.r.e.*) if there is a recursive function f s.t. $\forall n(W_{f(n)} \in \mathbf{c}_n)$. A pair of incomparable r.e. degrees $\mathbf{a}$ and $\mathbf{b}$ is an *exact pair* for $\langle \mathbf{c}_n : n < \omega \rangle$ if $\forall n(\mathbf{c}_n < \mathbf{a}, \mathbf{b})$ and $\forall \mathbf{d} < \mathbf{a}, \mathbf{b} \exists n(\mathbf{d} < \mathbf{c}_n)$. An exact pair $(\mathbf{a}, \mathbf{b})$ of $\langle \mathbf{c}_n : n < \omega \rangle$ is *minimal* if $\mathbf{a}$ and $\mathbf{b}$ are minimal upper bounds for $\langle \mathbf{c}_n : n < \omega \rangle$. An upper bound $\mathbf{a}$ of $\langle \mathbf{c}_n : n < \omega \rangle$ is *uniform* if there is an r.e. set $A \in \mathbf{a}$ and recursive functions f and g such that $\forall n(W_{f(n)} \in \mathbf{c}_n$ and $W_{f(n)} = \{g(n)\}^A)$.

Cooper [3] has shown that there are u.r.e. ascending sequences with minimal upper bounds, while Sacks (see, e.g., Soare [9, Theorem 3.1]) has shown that no u.r.e. ascending sequence has a least upper bound and uniform upper bounds are not minimal. Yates [11] constructed a u.r.e. ascending sequence with exact pair; this exact pair consists of uniform upper bounds, i.e. it is not minimal. In [2] we have shown there exist u.r.e. ascending sequences with minimal exact pairs, and in [1] that there exist such sequences without exact pairs.

Note that for an exact pair $(\mathbf{a}, \mathbf{b})$ for an ascending sequence, $\mathbf{a} \cap \mathbf{b}$ does not exist. Conversely any pair of r.e. degrees without infimum is an exact pair for an ascending sequence of r.e. degrees. That in certain cases this sequence can be chosen to be u.r.e. is an easy consequence of Yates [12, Theorem 8].

LEMMA 8. *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are* low_2 *and* $\mathbf{a} \cap \mathbf{b}$ *does not exist, then* $(\mathbf{a}, \mathbf{b})$ *is an exact pair for a u.r.e. ascending sequence.*

For a proof of Lemma 8, see [2, Corollary 5].

LEMMA 9. *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are* low_2, $\mathbf{a} | \mathbf{b}$ *and* $\mathbf{a}$ *is s.n.c., then there is a u.r.e. ascending sequence for which* $(\mathbf{a}, \mathbf{b})$ *is an exact pair and* $\mathbf{a}$ *a minimal upper bound.*

PROOF. By Lemma 8 and the definition of an s.n.c. degree. (Note that if $\mathbf{a}$ is s.n.c. and $\mathbf{a} | \mathbf{b}$, then $\mathbf{a}$ is a minimal upper bound for $\{\mathbf{c} : \mathbf{c} \leqslant \mathbf{a}, \mathbf{b}\}$.) □

COROLLARY 4. (a) (AMBOS-SPIES [2]) *There is a u.r.e. ascending sequence with a minimal exact pair.*

(b) (COOPER [3]) *There is a u.r.e. ascending sequence with a minimal upper bound.*

PROOF. (a) By Theorem 2 and Lemma 9. (b) By (a). □

In the next section we obtain extensions of Corollary 4.

2. Pairs of r.e. degrees without infimum. From the s.n.c. degree construction we can extract the construction of a pair of r.e. degrees $\mathbf{a}$ and $\mathbf{b}$ without infimum: Using Friedberg-Muchnik type requirements we construct Turing incomparable r.e. sets A and B, $\deg A = \mathbf{a}$ and $\deg B = \mathbf{b}$. In addition, for these sets we satisfy the requirements $R_{\langle e,i \rangle}$ and conditions (4) and (5) of the proof of Theorem 1, where W_{e_2} is replaced by B. Then $\mathbf{a}$ is a minimal upper bound of $\{\mathbf{x} : \mathbf{x} \leqslant \mathbf{a}, \mathbf{b}\}$. Since $\mathbf{a} | \mathbf{b}$ this implies $\mathbf{a} \cap \mathbf{b}$ does not exist. The strategy to meet $\mathrm{R}_{\langle e,i\rangle}$ is essentially that of the proof of Theorem 1; only condition (7.6) in the definition of requiring attention has to be replaced by

$$(7.6') \qquad x \geqslant \max(\{r(k,s): k \leqslant \langle e,i\rangle\} \cup \{R(k,s): k < \langle e,i\rangle\}),$$

where $r(k,s)$ is the restraint imposed by the kth Friedberg-Muchnik requirement at the end of stage s. Further, if x enters E_e at some stage, then x is put into B at the same stage.

We use this construction to show that every nonzero r.e. degree can be split into a pair of r.e. degrees without infimum and the pairs without infimum are dense in the r.e. degrees.

THEOREM 3. *Let* $\mathbf{a}$ *and* $\mathbf{c}$ *be nonzero r.e. degrees. Then there are low r.e. degrees* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *such that* $\mathbf{a} = \mathbf{a}_0 \cup \mathbf{a}_1$, $\mathbf{c} \nleqslant \mathbf{a}_0, \mathbf{a}_1$ *and* $\mathbf{a}_0 \cap \mathbf{a}_1$ *does not exist.*

PROOF. W.l.o.g. we may assume $\mathbf{c} \leqslant \mathbf{a}$. Otherwise $\mathbf{c} \nleqslant \mathbf{a}_0, \mathbf{a}_1$ follows from $\mathbf{a} = \mathbf{a}_0 \cup \mathbf{a}_1$, and for the proof we may replace $\mathbf{c}$ by $\mathbf{a}$.

The construction of r.e. sets $A_0 \in \mathbf{a}_0$ and $A_1 \in \mathbf{a}_1$ and subsidiary sets E_e^j is essentially the one given in the proof of Theorem 2. It suffices to make the following changes.

Now A is an r.e. set of degree $\mathbf{a}$, and the requirement $\mathrm{R}_{\langle 2e+j,i\rangle}$, $e = \langle e_0, e_1, e_2\rangle$ and $j \leqslant 1$, says

$$\mathrm{R}_{\langle 2e+j,i\rangle}\colon \left(W_{e_0} = \{e_1\}^{A_j} \text{ and } A_j \nleqslant_T W_{e_0}\right) \to E_e^j \neq \{i\}^{W_{e_0}}$$

(it would suffice to meet $\mathrm{R}_{\langle 2e+j,i\rangle}$ for $e = \langle e_0, e_1, 0\rangle$ and $j = 0$). Instead of (20) read

$$(20') \qquad E_e^j \leqslant_T A_{1-j},$$

and in the definition of requiring attention replace (22.6) by

$$(22.6') \qquad g(s) \leqslant x.$$

To verify that the construction changed in such a manner has the desired properties, we first show

LEMMA 10. $\mathbf{a} = \mathbf{a}_0 \cup \mathbf{a}_1$.

PROOF. That $\mathbf{a} \leqslant \mathbf{a}_0 \cup \mathbf{a}_1$ follows from $A = A_0^{(0)} \cup A_1^{(0)}$ as in the proof of Theorem 2. Since at stage $s+1$, by (22.6′) and $f_e^j(x,s) \geqslant x$, only numbers greater than or equal to $g(s)$ can enter A_0 and A_1, we have

$$\forall x, s \forall j \leqslant 1 \left(A_s \restriction x = A \restriction x \to A_{j,s+1} \restriction x = A_j \restriction x\right),$$

where $A_s = \{g(0), \dots, g(s)\}$. This implies $\mathbf{a}_0 \cup \mathbf{a}_1 \leqslant \mathbf{a}$. □

Lemma 7 holds by the original proof. Hence $\mathbf{a}_0$ and $\mathbf{a}_1$ are low and $\mathbf{c} \nleqslant \mathbf{a}_0, \mathbf{a}_1$. Since by assumption $\mathbf{c} \leqslant \mathbf{a}$, the latter and Lemma 10 imply

$$(25) \qquad \mathbf{a}_0 \mid \mathbf{a}_1.$$

As in the proof of Theorem 2, from Lemma 7 we can deduce that the requirements $\mathrm{R}_{\langle 2e+j,i\rangle}$ are met: We must only replace the W_{e_2} of the original proof by A_{1-j}. Then the premise $W_{e_2} \nleqslant_T A_j$ in the original requirements is satisfied by (25). The assumption that $R_{\langle 2e+j,i\rangle}$ is not met and $R(\langle 2e+j,i\rangle)=0$ (see the proof of Lemma 5, Case 2) now implies $A \leqslant_T A_j$, contrary to Lemma 10 and (25).

That conditions (20′) and (21) hold is proven as Lemma 6. For the proof of (20′) note that if x is put into E_j^e at stage $s+1$, then $g(s)$ is put into A_{1-j}. Hence, by (22.6′),

$$\forall x, s\,\exists y \leqslant x\left(x \in E^j_{e,s+1} - E^j_{e,s} \rightarrow y \in A_{1-j,s+1} - A_{1-j,s}\right).$$

The requirements $R_{\langle 2e+j,i\rangle}$ and conditions (20′) and (21) imply $\mathbf{a}_0$ and $\mathbf{a}_1$ are both minimal upper bounds for the set $\{\mathbf{x}: \mathbf{x} \leqslant \mathbf{a}_0, \mathbf{a}_1\}$, i.e.

(26) $$\forall j \leqslant 1\,\forall \mathbf{d} < \mathbf{a}_j\,\exists \mathbf{c} \leqslant \mathbf{a}_0, \mathbf{a}_1\,(\mathbf{c} \nleqslant \mathbf{d}).$$

(25) and (26) imply $\mathbf{a}_0 \cap \mathbf{a}_1$ does not exist. ■

In the above proof we have actually shown that any nonzero r.e. degree $\mathbf{a}$ can be split into incomparable low r.e. degrees $\mathbf{a}_0$ and $\mathbf{a}_1$ such that $\mathbf{a}_0$and $\mathbf{a}_1$ are minimal upper bounds of $\{\mathbf{x}: \mathbf{x} \leqslant \mathbf{a}_0, \mathbf{a}_1\}$. Together with Lemma 8 this implies

COROLLARY 5. *Every nonzero r.e. degree can be split into a minimal exact pair for some u.r.e. ascending sequence.* □

Lachlan [7] has shown the correspondng result to Theorem 3 for pais of r.e. degrees with infimum, namely that every nonzero r.e. degree $\mathbf{a}$ can be split into incomparable r.e. degrees $\mathbf{a}_0$ and $\mathbf{a}_1$ such that $\mathbf{a}_0 \cap \mathbf{a}_1$ exists.

Using a technique of Robinson [8], the construction for Theorem 3 can be done above any low r.e. degree, i.e.

(27) $\forall \mathbf{d}$ low $\forall \mathbf{a} > \mathbf{d}\,\exists \mathbf{a}_0, \mathbf{a}_1$

$$(\mathbf{d} < \mathbf{a}_0, \mathbf{a}_1 < \mathbf{a},\ \mathbf{a} = \mathbf{a}_0 \cup \mathbf{a}_1 \text{ and } \mathbf{a}_0 \cap \mathbf{a}_1 \text{ does not exist}).$$

In (27) the assumption that $\mathbf{d}$ is low cannot be omitted: Lachlan [6] has shown that, in general, splitting and density cannot be combined. We can prove, however, that the pairs of r.e. degrees without infimum are dense in the r.e. degrees.

THEOREM 4. *Let r.e. degrees* $\mathbf{d} < \mathbf{c}$ *be given. Then there are* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *such that* $\mathbf{d} < \mathbf{a}_0, \mathbf{a}_1 < \mathbf{c}$, *and* $\mathbf{a}_0 \cap \mathbf{a}_1$ *does not exist.*

The proof of Theorem 4, which combines the construction of a pair of degrees without infimum with infinite injury priority arguments, requires some definitions from Soare [9]:

Given an r.e. set X and an effective enumeration $\langle X_s: s < \omega\rangle$ of X such that $\forall s(X_{s+1} - X_s \neq \varnothing)$, x_{s+1} denotes the least element of $X_{s+1} - X_s$ and $x_0 = 0$. A stage t is a *true* stage of X (with respect to $\langle X_s: s < \omega\rangle$) if $X_t \upharpoonright x_t = X \upharpoonright x_t$. A variant of the standard recursive approximation $\langle \{e\}_s^{X_s}: s < \omega\rangle$ to $\{e\}^X$ and the use function is defined by

$$\{\hat{e}\}_s^{X_s}(x) = \begin{cases} \{e\}_s^{X_s}(x) & \text{if } \{e\}_s^{X_s}(x) \text{ is defined and } u(e, X_s, x, s) \leqslant x_s, \\ \text{undefined} & \text{otherwise,} \end{cases}$$

and

$$\hat{u}(e, X_s, x, s) = \begin{cases} u(e, X_s, x, s) & \text{if } \{\hat{e}\}_s^{X_s}(x)\downarrow, \\ 0 & \text{otherwise,} \end{cases}$$

respectively. The crucial property of this modified approximation is that for true t, a convergent computation $\{\hat{e}\}_t^{X_t}(x)$ is correct, i.e. $\{\hat{e}\}_t^{X_t}(x) = \{e\}^X(x)$, $\hat{u}(e, X_t, x, t) = u(e, X, x)$ and $X_t \upharpoonright \hat{u}(e, X_t, x, t) = X \upharpoonright \hat{u}(e, X_t, x, t)$ (see Soare [**9**, p. 518]).

Proof of Theorem 4. Let r.e. degrees $\mathbf{d} < \mathbf{c}$ be given. By density w.l.o.g. we may assume $\mathbf{d} \neq \mathbf{0}$. Choose r.e. sets $C \in \mathbf{c}$, $D \in \mathbf{d}$ and effective enumerations $\langle C_s: s < \omega \rangle$ and $\langle D_s: s < \omega \rangle$ of C and D, respectively, such that $D \subseteq \omega^{(0)}$, $C^{(0)} = D$, $D_0 = \varnothing$ and $\forall s (D_{s+1} - D_s \neq \varnothing)$. We construct r.e. sets A_0 and A_1 such that $\mathbf{a}_0 = \deg A_0$ and $\mathbf{a}_1 = \deg A_1$ have the desired properties.

To ensure $\mathbf{d} \leqslant \mathbf{a}_0, \mathbf{a}_1$ we set $A_0^{(0)} = A_1^{(0)} = D$. In general, we will have that $A_j^{(f)}$, $f < \omega, j \leqslant 1$, is recursive or recursive in D, and $A_0^{(f)} = A_1^{(f)}$, or one of them is empty, and we can decide which of them occurs. Hence for $A = A_0 \cup A_1$ and $\mathbf{a} = \deg A$, $\mathbf{a} = \mathbf{a}_0 \cup \mathbf{a}_1$. To guarantee $\mathbf{a} \leqslant \mathbf{c}$, we will make $A^{(f)} \leqslant_T C$ C-uniformly in f by strategies which depend on the type of requirement connected with $A^{(f)}$. For rows connected with the requirements which ensure that $\mathbf{a}_0 \cap \mathbf{a}_1$ does not exist, we use the (delayed) permitting method. The hereby imposed restraints on A interfere with our strategy to meet these requirements, and we succeed only if $C \nleqslant_T A$. Hence we have to meet the requirements

$$\mathrm{R}_{3e}: C \neq \{e\}^A$$

for all $e \geqslant 1$ (since for some $e \geqslant 1$, $\{e\}^A = \{0\}^A$, we may omit $e = 0$). To make $\mathbf{a}_0$ and $\mathbf{a}_1$ incomparable, we meet the requirements

$$\mathrm{R}_{3(2e+j)+1}: A_j \neq \{e\}^{A_{1-j}}$$

for all $e < \omega$ and $j \leqslant 1$. Then to ensure that $\mathbf{a}_0 \cap \mathbf{a}_1$ does not exist, it suffices to make, for some $j \leqslant 1$, $\mathbf{a}_j$ a minimal upper bound for $\{\mathbf{x}: \mathbf{x} \leqslant \mathbf{a}_0, \mathbf{a}_1\}$. For the sake of symmetry, we do this for both $\mathbf{a}_0$ and $\mathbf{a}_1$: we construct (not necessarily r.e.) sets E_e^j, $j \leqslant 1$, $e = \langle e_0, e_1 \rangle$, for which we meet the requirements

$$\mathrm{R}_{3\langle 2e+j, i\rangle+2}: \left(W_{e_0} = \{e_1\}^{A_j} \text{ and } A_j \nleqslant_T W_{e_0}\right) \to E_e^j \neq \{i\}^{W_{e_0}},$$

and for which we ensure

(28) $$E_e^j \leqslant_T A_{1-j},$$

(29) $$W_{e_0} = \{e_1\}^{A_j} \to E_e^j \leqslant_T A_j.$$

That this implies $\mathbf{a}_j, j \leqslant 1$, is a minimal upper bound for $\{\mathbf{x}: \mathbf{x} \leqslant \mathbf{a}_0, \mathbf{a}_1\}$ is shown as follows: For a contradiction assume there is an r.e. degree $\mathbf{v} < \mathbf{a}_j$ such that $\forall \mathbf{x} \leqslant \mathbf{a}_0, \mathbf{a}_1 (\mathbf{x} \leqslant \mathbf{v})$. Then we can choose $e = \langle e_0, e_1 \rangle$ such that $W_{e_0} \in \mathbf{v}$ and the premises of $\mathrm{R}_{3\langle 2e+j, i\rangle+2}$ and (29) hold. It follows that $\deg E_e^j \nleqslant \mathbf{v}$ and $\deg E_e^j \leqslant \mathbf{a}_0, \mathbf{a}_1$. Though $\deg E_e^j$ is not necessarily r.e. by Lachlan [**5**, Lemma 18], there is an r.e. degree $\mathbf{e}$ such that $\deg E_e^j \leqslant \mathbf{e} \leqslant \mathbf{a}_0, \mathbf{a}_1$, which gives the desired contradiction. In the construction of E_e^j a number put into E_e^j at some stage can be extracted from E_e^j at a later stage.

For a fixed number, however, this can happen only finitely often. Hence with $E^j_{e,s}$ denoting the numbers put into E^j_e at some stage $\leqslant s$ and not extracted from E^j_e at a later stage $\leqslant s$, $E^j_e = \lim_s E^j_{e,s}$.

We say a number $f \geqslant 1$ is of *type* i, $i \leqslant 2$, if $f = 3e + i$ for some e. A requirement is of type i if its index is. In the following l, k, m stand for numbers of type 0, 1, 2, respectively.

For meeting requirements of type 0 we use the preservation strategy, for that of type 1 the preservation and coding strategies of Sacks (see Soare [**9**, p. 525]). For this sake we need the following functions.

$$l(3e, s) = \max\{x: \forall y < x\,(C_s(y) = \{\hat{e}\}^{A_s}_s(y))\},$$
$$l(3(2e+j)+1, s) = \max\{x: \forall y < x\,(A_{j,s}(y) = \{\hat{e}\}^{A_{1-j,s}}_s(y))\}$$
$$(\textit{length function}),$$

$$m(3e, s) = \max\{x: \exists t \leqslant s\,(x \leqslant l(3e, t) \text{ and } \forall y < x\,(A_t \upharpoonright \hat{u}(e, A_t, y, t) = A_s \upharpoonright \hat{u}(e, A_t, y, t))\},$$
$$m(3(2e+j)+1, s) = \max\{x: \exists t \leqslant s\,(x \leqslant l(3(2e+j)+1, t) \text{ and } \forall y < x\,(A_{1-j,t} \upharpoonright \hat{u}(e, A_{1-j,t}, y, t) = A_{1-j,s} \upharpoonright \hat{u}(e, A_{1-j,t}, y, t)))\}$$
$$(\textit{modified length function}),$$

$$r(3e, s) = \max\{\hat{u}(e, A_s, x, s): x \leqslant m(3e, s)\},$$
$$r(3(2e+j)+1, s) = \max\{\hat{u}(e, A_{1-j,s}, x, s): x \leqslant m(3(2e+j)+1, s)\}$$
$$(\textit{restraint function}).$$

(We will define $r(3e+2, s)$ as a part of the construction below. $l(3e+2, s)$ and $m(3e+2, s)$ are undefined.)

Conditions (28) and (29) are satisfied using markers and the extended marker concept of the proof of Theorem 1, respectively. Let γ_j be the standard marker of A_j, and for $e = \langle e_0, e_1 \rangle$ and $x = \langle x_0, x_1 \rangle$ define

$$f^j_e(x, s) = \langle x_0, x_1, \max\{\bar{u}(e_1, A_{j,t}, y, t): y \leqslant \gamma_j(x, s) \text{ and } t \leqslant s\}\rangle,$$

where

$$\bar{u}(e_1, A_{j,s}, y, s) = \hat{u}(e_1, A_{j,s}, y, s) \quad \text{if } \{\hat{e}_1\}^{A_{j,s}}_s(y)\downarrow,$$
$$= s \quad \text{otherwise}.$$

Note that the function f^j_e has the following properties: It is nondecreasing in both arguments; $(f^j_e(x, s))_0 = (x)_0$; for any $x \neq y$, s and t, $f^j_e(x, s) \neq f^j_e(y, t)$; and if $W_{e_0} = \{e_1\}^{A_j}$ then $f^j_e(x) = \lim_s f^j_e(x, s)$ is a total A_j-recursive function. Hence, to satisfy (28) and (29) it suffices to ensure

(30) $$\forall x \forall s\,(A_{1-j,s} \upharpoonright \gamma_{1-j}(x) = A_{1-j} \upharpoonright \gamma_{1-j}(x) \to E^j_{e,s} \upharpoonright x = E^j_e \upharpoonright x)$$

and

(31) $$W_{e_0} = \{e_1\}^{A_j} \to \forall x, s\,(A_{j,s} \upharpoonright f^j_e(x) = A_j \upharpoonright f^j_e(x) \to E^j_{e,s} \upharpoonright x = E^j_e \upharpoonright x).$$

The basic strategy for satisfying the requirements of type 2 is similar to that used for like requirements in the preceding proofs. Roughly speaking, to meet R_m, where $m = 3\langle 2e + j, i\rangle + 2$, $e = \langle e_0, e_1\rangle$, $j \leq 1$, we wait for a number x and a stage s such that $E^j_{e,s}(x) = \{i\}^{W_{e_0,s}}_s(x)$, $W_{e_0,s}$ agrees with $\{\hat{e}_1\}^{A_{j,s}}_s$ on all numbers y used in the computation $\{i\}^{W_{e_0,s}}_s(x)$, and $f^j_e(x, s)$ is not used in a computation $\{\hat{e}_1\}^{A_{j,s}}_s(y)$ for such a y. Then we set $E^j_{e,s+1}(x) = 1 - E^j_{e,s}(x)$ and try to preserve $\{i\}^{W_{e_0,s}}_s(x)$ by imposing an appropriated restraint on A_j. In order to satisfy (30) and (31) we put a number $z \leq \gamma_{1-j}(x, s)$ into A_{1-j} and $f^j_e(x, s)$ into A_j. (In the actual construction we require $\gamma_{1-j}(x, s) \geq f^j_e(x, s)$; we then put $f^j_e(x, s)$ in both A_0 and A_1, thus ensuring $A_0^{(m)} = A_1^{(m)}$.) There are the following three obstacles to this procedure.

The first obstacle is that R_m has to obey the restraints imposed by higher priority requirements, and these restraints can be unbounded now. It is necessary for satisfying R_m that at certain stages these restraints drop back simultaneously, i.e.

$$\liminf_s \max\{r(f, s): f < m\} < \omega.$$

For type 0 and 1 requirements we have the following: Let T^f (T^f_j) be the set of true stages of $A^{(<f)}$ ($A^{(<f)}_j$) with respect to $\{(A_s)^{(<f)}: s < \omega\}$ ($\{(A_{j,s})^{(<f)}: s < \omega\}$). Then, for $l < m$, $\lim_{t\in T^m} r(l, t) < \omega$ and, for $k = 3(2e + j) + 1 < m$, $\lim_{t\in T^m_{1-j}} r(k, t) < \omega$ will exist. To obtain infinitely many simultaneous "windows", we require $T^m \cap T^m_0 \cap T^m_1$ to be infinite. For this sake we consider the sets

$$T^f_D = \{s: s \in T^f \text{ and } a^f_s \in D_s\} \qquad (f \geq 1),$$

where $a^f_0 = 0$ and $a^f_{s+1} = \mu x(x \in A^{(<f)}_{s+1} - A^{(<f)}_s)$. Since, by construction, $\forall s\,(D_s = A^{(0)}_s = A^{(0)}_{0,s} = A^{(0)}_{1,s})$, $T^f_D \subseteq T^f \cap T^f_0 \cap T^f_1$, and our assumption that D is nonrecursive will imply T^f_D is infinite. Finally we will see that, for any $m' < m$ of type 2, $\lim_{t\in T^m_D} r(m', t) < \omega$ will exist. So $\lim_{t\in T^m_D} \max\{r(f, t): f < m\} < \omega$ will exist.

The second obstacle is that higher priority requirements can act infinitely often and the given set D is a part of A. So there are infinitely many numbers enumerated in A which do not have to obey the restraints imposed by R_m. Such numbers can destroy our attempt to preserve a computation $\{i\}^{W_{e_0,s}}_s(x)$ by restraining certain numbers from A. We prevent this from constantly happening by allowing R_m to be attacked by the same number x more than once.

The third obstacle is that we are required to make A recursive in C. Hence if we attack R_m as described above and put z and $f^e_j(x, s)$ into A_{1-j} and A_j, respectively, then we want these numbers permitted by C. It can happen, however, that C never permits at a stage at which the higher priority restraints drop back. To overcome this difficulty we allow a delay in the permitting: If C permits a number y at a stage s then this permitting is valid until the next stage $t + 1$ such that $t \in T^m_D$. On the one hand this guarantees

$$\forall m \forall x \in \omega^{(m)} \forall t \in T^m_D\,(C_t \upharpoonright x = C \upharpoonright x \rightarrow A_{t+1}(x) = A(x)),$$

and thus $A \leq_T C$, since we will show $T^m_D \leq_T C$ uniformly in m (i.e. the delay is C-recursive). On the other hand this delay guarantees that all permittings by C happen at T^m_D-stages, i.e.—as outlined above—at stages with minimal restraints.

The above considerations lead to the following definition:

Requirement R_m, where $m = 3\langle 2e + j, i \rangle + 2$, $e = \langle e_0, e_1 \rangle$ and $j \leq 1$, *requires attention* at stage $s + 1$ if there is an $x \in \omega^{(m)}$ such that

(32.1) $$r(m, s) = 0,$$

(32.2) $E_e^j(x) = \{i\}_s^{W_{e_0,s}}(x)$,

(32.3) $\gamma_j(x, s) \geq u$, where $u = u\big(i, W_{e_0,s}, x, s\big)$,

(32.4) $W_{e_0,s} \restriction u = \{\hat{e}_1\}_s^{A_{j,s}} \restriction u$,

(32.5) $f_e^j(x, s) \geq \max\{r(f, s): f < m\}$,

(32.6) $\gamma_{1-j}(x, s) \geq f_e^j(x, s)$ and $f_e^j(x, s) \notin A_s$,

(32.7) there is a stage $t \leq s$ such that

(32.7.1) $\exists y < f_e^j(x, s)\,(y \in C_{t+1} - C_t)$ and

(32.7.2) $\forall v\big(t \leq v < s \to a_v^m \notin D_v$ or $A_v^{(<m)} \restriction a_v^m \neq A_s^{(<m)} \restriction a_v^m\big)$.

We can now state the *construction*.

Stage 0. For all m, $r(m, 0) = 0$.

Stage $s + 1$. The stage consists of four steps. Numbers are put into or extracted from the sets under construction only during the first three steps. At the end of Step 1 we define a temporary restraint function $\hat{r}(f, s)$ valid for Step 3. The final restraint function $r(f, s + 1)$ for stage $s + 1$ is defined in Step 4.

Step 1 (*Attacking type* 2 *requirements*). Choose the least m such that R_m requires attention, say $m = 3\langle 2e + j, i \rangle + 2$ and x is the least element of $\omega^{(m)}$ which satisfies (32.1)–(32.7). If x is in $E_{e,s}^j$ extract x from E_e^j; otherwise put x into E_e^j. In either case put $f_e^j(x, s)$ into A_0 and A_1 and define

$$\hat{r}(f, s) = \begin{cases} \max\big\{\hat{u}(e_1, A_{j,s}, y, s): y < u\big(i, W_{e_0,s}, x, s\big)\big\} & \text{if } f = m, \\ r(f, s) & \text{otherwise.} \end{cases}$$

If no requirement R_m requires attention, set $\hat{r}(f, s) = r(f, s)$ for all f.

Step 2 (*Coding D into A*). For all $x \in D_{s+1} - D_s$, put x into A_0 and A_1.

Step 3 (*Coding strategy for type* 1 *requirements*). For $l = 3(2e + j) + 1$, $j \leq 1$, $x < \omega$ and $t \leq s$: If

$$x \in C_s,\ \langle l, x, t \rangle \geq \max\{\hat{r}(f, s): f < l\}$$

and

$$\forall v\,(t \leq v \leq s \to x \leq m(l, v)),$$

put $\langle l, x, t \rangle$ into A_j.

Step 4 (*Preservation*). For all $m = 3\langle 2e + j, i \rangle + 2$, $j \leq 1$, define

$$r(m, s + 1) = \begin{cases} \hat{r}(m, s) & \text{if } A_{j,s+1} \restriction \hat{r}(m, s) = A_{j,s} \restriction \hat{r}(m, s), \\ 0 & \text{otherwise.} \end{cases}$$

Define $r(k, s + 1)$, $r(l, s + 1)$ and the other auxiliary functions as described above.

This completes the construction. To show that it is correct, we prove a series of lemmas. We start with some simple facts:

The construction is effective; so A_0 and A_1 are r.e. For any $e, x < \omega$ and $j \leq 1$, $\lim_s E^j_{e,s}(x)$ exists, since by (32.6) x can be inserted in or extracted from E^j_e at at most $\gamma_{1-j}(x)$ stages. Hence E^j_e is well defined. Moreover, whenever $E^j_{e,s}(x) \neq E^j_{e,s+1}(x)$, $f^j_e(x, s)$ is enumerated in A_0 and A_1 at stage $s + 1$. So, by (32.6), (30) and (31) hold, i.e. conditions (28) and (29) are satisfied by the constructed sets.

For the sake of requirement R_f only elements of $\omega^{(f)}$ are put into A_0 and A_1. If f is of type 0, R_f does not contribute any elements to A_0 or A_1; i.e. for any k, $A_0^{(k)} = A_1^{(k)} = \varnothing$. If $f = 3(2e + j) + 1, j \leq 1$, then $A_{1-j}^{(f)} = \varnothing$, and if f is of type 2, then $A_0^{(f)} = A_1^{(f)}$. Finally, by Step 2 of stage $s + 1$, $A_0^{(0)} = A_1^{(0)} = D$. From all this we conclude that $\mathbf{d} \leq \mathbf{a}_0, \mathbf{a}_1$ and $\mathbf{a}_0 \cup \mathbf{a}_1 = \mathbf{a}$.

Note that, by Step 2 of stage $s + 1$, $A_{j,s+1}^{(<f)} \neq A_{j,s}^{(<f)}$ for any $f \geq 1, s < \omega$ and $j \leq 1$. So the numbers $a^f_{j,s}$ (a^f_s), where $a^f_{j,0} = 0$ and $a^f_{j,s+1} = \mu x(x \in A_{j,s+1}^{(<f)} - A_{j,s}^{(<f)})$ ($a^f_0 = 0$ and $a^f_{s+1} = \mu x(x \in A_{s+1}^{(<f)} - A_s^{(<f)})$) are well defined. Recall that T^f_j (T^f) denotes the set of true stages of $A_j^{(<f)}$ ($A^{(<f)}$), and $T^f_D = \{s: s \in T^f \text{ and } a^f_s \in D_s\}$. Note that for $e \leq f$, $T^f_D \subseteq T^e_D$ and, since $D = A_0^{(0)} = A_1^{(0)}$, $T^f_D \subseteq T^f_0, T^f_1, T^f$.

LEMMA 11. *Let $m, e, i, f, t < \omega$ and $j \leq 1$ be given such that $m = 3\langle 2e + j, i\rangle + 2$, $f \geq m$, $t \in T^f_j$ and $r(m, t) > 0$ or $\hat{r}(m, t) > 0$. Then*

(i) $\forall s > t(\hat{r}(m, t) = r(m, s) = \hat{r}(m, s) > 0)$,

(ii) *$A^{(m)}$ is finite, and*

(iii) *R_m is met.*

PROOF. Since $t \in T^f_j$, $A_{j,t}^{(<f)} \restriction a^f_{j,t} = A_j^{(<f)} \restriction a^f_{j,t}$, and since $a_{j,t} \leq a^f_{j,t}$ and $m \leq f$, this implies

$$A_{j,t}^{(<m)} \restriction a_{j,t} = A_j^{(<m)} \restriction a_{j,t}. \tag{33}$$

If $r(m, t) > 0$, then by Step 4 of stage t, $A_{j,t} \restriction r(m, t) = A_{j,t-1} \restriction r(m, t)$, i.e. $r(m, t) \leq a_{j,t}$, and $\hat{r}(m, t) = r(m, t)$, since by (32.1) R_m does not require attention at stage $t + 1$. If $r(m, t) = 0$, then R_m is active at Step 1 of stage $t + 1$. Hence for some x,

$$\hat{r}(m, t) = \max\left\{\hat{u}\left(e_1, A_{j,s}, y, s\right): y < u\left(i, W_{e_0,s}, x, s\right)\right\} \leq a_{j,t}.$$

So in either case, $0 < \hat{r}(m, t) \leq a_{j,t}$, and therefore by (33),

$$A_{j,t}^{(<m)} \restriction \hat{r}(m, t) = A_j^{(<m)} \restriction \hat{r}(m, t).$$

Since for $\hat{r}(m, s) > 0$, $\hat{r}(m, s + 1) \neq \hat{r}(m, s)$ only if a number less than $\hat{r}(m, s)$ enters A_j at stage $s + 1$, and since no such number can enter $A_j^{(\geq m)}$ at stage $s + 1$, it follows by induction on $s \geq t$ that

$$\forall s > t\left(\hat{r}(m, s) = r(m, s) = \hat{r}(m, t) > 0 \text{ and } A_{j,s} \restriction \hat{r}(m, t) = A_{j,s-1} \restriction \hat{r}(m, t)\right). \tag{34}$$

This implies (i). (ii) immediately follows from (i), since a number can enter $A^{(m)}$ only at stages $s + 1$ where R_m is active, i.e. where $0 = r(m, s) \neq \hat{r}(m, s)$. To prove (iii), we fix the least $s' \leqslant t$ s.t.

$$\forall s\,(s' \leqslant s \leqslant t \rightarrow \hat{r}(m,s) > 0).$$

Then, by induction on s with $s' \leqslant s \leqslant t$ and by (34),

$$\forall s > s'\,(\hat{r}(m,s) = r(m,s) = \hat{r}(m,s') > 0) \tag{35}$$

and

$$A_{j,s'} \upharpoonright \hat{r}(m,s') = A_j \upharpoonright \hat{r}(m,s'). \tag{36}$$

Moreover, R_m is active at stage $s'+1$; i.e., there is an $x \in \omega^{(m)}$ such that

$$E^j_{e,s'+1}(x) \neq \{i\}^{W_{e_0,s'}}(x),$$
$$W_{e_0,s'} \upharpoonright u(i, W_{e_0,s'}, x, s') = \{\hat{e}_1\}^{A_{j,s'}}_{s'} \upharpoonright u(i, W_{e_0,s'}, x, s'), \quad \text{and}$$
$$\hat{r}(m,s') = \max\left\{\hat{u}(e_1, A_{j,s'}, y, s') : y < u(i, W_{e_0,s'}, x, s')\right\}.$$

By (35), R_m does not require attention after stage $s'+1$ and therefore $E^j_e(x) = E^j_{e,s'+1}(x)$. By (36),

$$\{\hat{e}_1\}^{A_{j,s'}}_{s'} \upharpoonright u(i, W_{e_0,s'}, x, s') = \{e_1\}^{A_j} \upharpoonright u(i, W_{e_0,s'}, x, s'),$$

thus $W_{e_0} \neq \{e_1\}^{A_j}$ or $\{i\}^{W_{e_0,s'}}(x) = \{i\}^{W_{e_0}}(x)$. In either case R_m is met. □

LEMMA 12. *For all m,*

$$F_m = \left\{y \in A^{(m)} : \exists s\left(y \in A^{(m)}_{s+1} - A^{(m)}_s \text{ and } A^{(<m)}_s \upharpoonright y = A^{(<m)} \upharpoonright y\right)\right\}$$

is finite.

PROOF. Let $m = 3\langle 2e + j, i\rangle + 2$, $j \leqslant 1$, be given and assume $y \in F_m$. Then there is a stage s such that $y \in A^{(m)}_{s+1} - A^{(m)}_s$ and $A^{(<m)}_s \upharpoonright y = A^{(<m)} \upharpoonright y$. Hence R_m receives attention at stage $s+1$ via some x and $0 < \hat{r}(m,s) \leqslant f^j_e(x,s) = y$. As in the proof of Lemma 11 we can conclude that for all $s' > s$, $\hat{r}(m,s') = r(m,s') = \hat{r}(m,s) > 0$; i.e., R_m does not require attention after stage $s+1$, therefore $F_m \subseteq A^{(m)} = A^{(m)}_{s+1}$ is finite. □

LEMMA 13. *For all $f < \omega$ the following hold*:

(a) *If f is of type* 0 *or* 1, R_f *is met.*

(b) *If f is of type* 0 *or* 1, $A^{(f)}$ *is recursive. If $f = 0$ or f of type* 2, $A^{(f)} \leqslant_T D$.

(c) *The set T^{f+1}_D is infinite.*

(d) *For all* $1 \leqslant f' \leqslant f+1$:

$$\lim_{t \in T^{f+1}_D} r(f',t) = \lim_{t \in T^{f+1}_D} \hat{r}(f',t) < \omega \textit{ exists}; \tag{37}$$

$$\forall t \in T^{f+1}_D\, \forall s \geqslant t\,(r(f',s) \geqslant r(f',t) \textit{ and } \hat{r}(f',s) \geqslant \hat{r}(f',t)). \tag{38}$$

PROOF. By induction on f. For the inductive step fix f.

(a) We distinguish two cases.

Case 1: $f = 3e > 0$. For a contradiction assume R_f is not met, i.e. $C = \{e\}^A$. Then, by Soare [**9**, Lemma 2.1], $C \leqslant_T I_e$, where

$$I_e = \left\{x : \exists s\,(x \in A_{s+1} - A_s \text{ and } x < r(f,s)\right\}.$$

By construction, $I_e \subseteq A^{(<f)}$, therefore $I_e \leqslant_T A^{(<f)}$. Since by inductive hypothesis (b), $A^{(<f)} \leqslant_T D$, this contradicts $D <_T C$.

Case 2: $f = 3(2e + j) + 1$, $j \leqslant 1$. For a contradiction assume R_f is not met, i.e. $A_j = \{e\}^{A_{1-j}}$. Then $\lim_s m(f, s) = \omega$. Hence

$$M(x) = \mu s\,(\forall s' \geqslant s\,(m(f, s') \geqslant x))$$

is a total function. By construction, for any s such that $m(f, s) \geqslant x$, $s < M(x)$ iff $A^{(<f)}_{1-j,s} \restriction u \neq A^{(<f)}_{1-j} \restriction u$, where $u = \max\{\hat{u}(e, A_{1-j,s}, y, s): y < x\}$. Hence $M \leqslant_T A^{(<f)}_{1-j}$, thus, by inductive hypothesis, $M \leqslant_T D$. Note that for $s \geqslant M(x)$, $r(f, s) \geqslant m(f, s) \geqslant x$. Hence, by construction,

$$\forall x\left(A^{(\geqslant f)}_{1-j,M(x)} \restriction x = A^{(\geqslant f)}_{1-j} \restriction x\right)$$

(note that $A^{(f)}_{1-j} = \varnothing$). This implies $A^{(\geqslant f)}_{1-j} \leqslant_T D$, thus, by inductive hypothesis, A_{1-j} is recursive in D. Since, by assumption, $A_j \leqslant_T A_{1-j}$, this implies $A =_T A_0 \oplus A_1 \leqslant_T D$.

Now let $r = \liminf_s \max\{r(f', s): f' < f\}$. By inductive hypotheses (e) and (d), $r < \omega$. By construction, for $x \geqslant r$, $x \in C$ iff $\langle f, x, M(x)\rangle \in A_j$. Hence $C \leqslant_T M \oplus A_j \leqslant_T A_{1-j} \oplus A_j \leqslant_T D$, a contradiction.

(b) We distinguish four cases.

Case 1: *f is of type* 0. Then $A^{(f)} = \varnothing$.

Case 2: *f is of type* 1. Say $f = 3(2e + j) + 1, j \leqslant 1$. Then $A^{(f)}_{1-j} = \varnothing$, i.e. $A^{(f)} = A^{(f)}_j$. By (a) there is $p < \omega$ with $p = \liminf_s m(f, s)$. By inductive hypothesis, choose $r < \omega$ such that $r = \liminf_s \max\{r(f', s): f' < f\}$, and fix a stage s_0 such that $A_{j,s_0} \restriction p + 1 = A_j \restriction p + 1$ and $\forall s \geqslant s_0(\max\{r(f', s): f' < f\} \geqslant r)$. Now for given x and s, we can decide whether $\langle f, x, s\rangle \in A$ as follows: If $x > p$, find the least stage $s' \geqslant s$ such that $m(f, s') = p$. Then $\langle f, x, s\rangle \in A$ iff $\langle f, x, s\rangle \in A_{s'}$. If $x \leqslant p$, find the least stage $t \geqslant s, s_0$ s.t. $\max\{r(f', t): f' < f\} = r$. Then $\langle f, x, s\rangle \in A$ iff $\langle f, x, s\rangle \in A_t$.

Case 3: *f is of type* 2. By Lemma 12, fix z such that $\forall y \in F_t(y < z)$, and a stage s_0 s.t. $A^{(f)}_{s_0} \restriction z = A^{(f)} \restriction z$. Then

$$\forall x\, \forall s \geqslant s_0\left(A^{(<f)}_s \restriction x = A^{(<f)} \restriction x \rightarrow A^{(f)}_s \restriction x = A^{(f)} \restriction x\right).$$

Hence $A^{(f)} \leqslant_T A^{(<f)}$, therefore, by inductive hypothesis, $A^{(f)} \leqslant_T D$.

Case 4: $f = 0$. Then $A^{(f)} = A^{(0)} = D$.

(c) For a contradiction assume T^{f+1}_D is finite, i.e., there are only finitely many $t \in T^{f+1}$ with $a^{f+1}_t \in A^{(0)} = D$. Since for f' of type 0, $A^{(f')} = \varnothing$ and for $f' \leqslant f$ of type 2, by Lemma 12, there are only finitely many $t \in T^{f+1}$ with $a^{f+1}_t \in A^{(f')}$, it follows that there is an s_0 such that

$$\forall t > s_0\left(t \in T^{f+1} \rightarrow \exists l \leqslant f\left(l \text{ of type 1 and } a^{f+1}_t \in A^{(l)}\right)\right).$$

Hence, for all $x > a^{f+1}_{s_0}$ and $s \geqslant s_0$,

$$\bigcup\left\{A^{(l)}_s: l \leqslant f\right\} \restriction x = \bigcup\left\{A^{(l)}: l \leqslant f\right\} \restriction x \rightarrow A^{(<f+1)}_s \restriction x = A^{(<f+1)} \restriction x;$$

thus $A^{(<f+1)} \leqslant_T \bigcup\{A^{(l)}: l \leqslant f\}$. Since $D = A^{(0)} \leqslant_T A^{(<f+1)}$ and, by inductive hypothesis and (b), $\bigcup\{A^{(l)}: l \leqslant f\}$ is recursive, this implies D is recursive, contrary to our assumption.

(d) For $1 \leqslant f' < f+1$, (d) holds by inductive hypothesis, since for $f > 0$, $T_D^f \supseteq T_D^{f+1}$. Hence it suffices to consider $f' = f+1$.

Case 1: $f+1$ *is of type* 0. Then for $t \in T^{f+1}$, by induction on $s \geqslant t$,

$$\text{(39)} \qquad \forall s \geqslant t\big(m(f+1,t) \leqslant m(f+1,s),\ r(f+1,t) \leqslant r(f+1,s) = \hat{r}(f+1,s) \text{ and } A_t \upharpoonright r(f+1,t) = A_s \upharpoonright r(f+1,t)\big).$$

Since $T_D^{f+1} \subseteq T^{f+1}$, this implies (38). By (a) let $p = \mu x(C(x) \neq \{e\}^A(x))$ and choose $t \in T^{f+1}$ such that $C_t \upharpoonright p+1 = C \upharpoonright p+1$ and $m(f+1,t) \geqslant p$. Then

$$\forall s \geqslant t\big(s \in T^{f+1} \to m(f+1,s) \leqslant m(f+1,t)\big);$$

thus by definition of r and (39),

$$\forall s \geqslant t\big(s \in T^{f+1} \to \hat{r}(f+1,s) = r(f+1,s) = r(f+1,t)\big),$$

which implies (37).

Case 2: $f+1$ *is of type* 1. Say $f+1 = 3(2e+j)+1$, $j \leqslant 1$. Similar to Case 1 (consider stages in T_{1-j}^{f+1} and note that $T_D^{f+1} \subseteq T_{1-j}^{f+1}$).

Case 3: $f+1$ *is of type* 2. Then the claim follows from Lemma 11 and $T_D^{f+1} \subseteq T_j^{f+1}$, $j \leqslant 1$. □

Note that by Lemma 13(a), $C \not\leqslant_T A$ and $A_0 |_T A_1$.

LEMMA 14. *Let $m = 3\langle 2e+j, i\rangle + 2$, $j \leqslant 1$ and $e = \langle e_0, e_1\rangle$ be given and assume $W_{e_0} = \{e_1\}^{A_j}$. Then*

$$\exists x \in \omega^{(m)}\big(f_e^j(x) \in A\big) \to \mathrm{R}_m \textit{ is met.}$$

PROOF. Assume $f_e^j(x) \in A$, $x \in \omega^{(m)}$. Then there is a last stage s such that R_m receives attention via x at stage $s+1$; i.e.,

$$E_{e,s+1}^j(x) \neq \{i\}^{W_{e_0,s}}(x)\downarrow,$$
$$W_{e_0,s} \upharpoonright u\big(i, W_{e_0,s}, x, s\big) = \{\hat{e}_1\}_s^{A_{j,s}} \upharpoonright u\big(i, W_{e_0,s}, x, s\big),$$
$$f_e^j(x) = f_e^j(x,s) \geqslant v,$$

where $v = \max\{\hat{u}(e_1, A_{j,s}, y, s) \colon y < u(i, W_{e_0,s}, x, s)\}$, and $f_e^j(x,s)$ is put into A at stage $s+1$. To show that R_m is met, it suffices to show that $A_{j,s} \upharpoonright v = A_j \upharpoonright v$. If this is not the case, fix the least $t > s$ s.t. $A_{j,s} \upharpoonright v \neq A_{j,t} \upharpoonright v$. Then, by definition of f_e^j,

$$f_e^j(x,t) = \big\langle (x)_0, (x)_1, t\big\rangle > \big\langle (x)_0, (x)_1, s\big\rangle \geqslant f_e^j(x,s),$$

i.e. $f_e^j(x,s) \neq f_e^j(x)$, contrary to assumption. □

LEMMA 15. *For all m, R_m is met.*

PROOF. Fix $m = 3\langle 2e+j, i\rangle + 2$, $j \leqslant 1$, $e = \langle e_0, e_1\rangle$ and assume, for a contradiction, R_m is not met, i.e.

$$\text{(40)} \qquad W_{e_0} = \{e_1\}^{A_j},$$

$$\text{(41)} \qquad A_j \not\leqslant_T W_{e_0},$$

$$\text{(42)} \qquad E_e^j = \{i\}^{W_{e_0}}.$$

Since $T_D^m \subseteq T_0^m \cap T_1^m$, Lemma 11 implies every type 2 requirement $R_{m'}$, $m' \leqslant m$, requires attention at at most finitely many stages $t+1$ such that $t \in T_D^m$. So we can choose s_0 such that

(43) $$\forall t \geqslant s_0 \left(t \in T_D^m \to R_m \text{ does not require attention at stage } t+1 \right).$$

In the following we will refute (43), thus obtaining the desired contradiction.

By Lemma 13, let

$$r = \sup_{t \in T_D^m} \max\{ r(f,t) : f < m \}$$

and define $Z = \{ x \in \omega^{(m)} : x > r \}$. We will first show there is an A_j-recursive function $s(x)$ such that

(44) $$\forall x \in Z \forall s \geqslant s(x) \left(s \in T_D^m \to (32.1), (32.2), (32.4), (32.5) \text{ and the second part of } (32.6) \text{ hold} \right).$$

Since $f_e^j(x,s) \geqslant x$ for any x, s,

(45) $$\forall x \in Z \forall s \in T_D^m \left(f_e^j(x,s) > \max\{ r(f,s) : f < m \} \right).$$

By (40) and (42), $E_e^j \leqslant_T W_{e_0} \leqslant_T A_j$ and

$$\forall x \left(E_e^j(x) = \{i\}^{W_{e_0}}(x) \text{ and } W_{e_0} \upharpoonright u(i, W_{e_0}, x) = \{e_1\}^{A_j} \upharpoonright u(i, W_{e_0}, x) \right).$$

So we can A_j-recursively compute a stage $s(x)$ such that

$$E^j_{e,s(x)}(x) = \{i\}^{W_{e_0,s(x)}}_{s(x)}(x)$$

and

$$W_{e_0,s(x)} \upharpoonright u\left(i, W_{e_0,s(x)}, x, s(x)\right) = \{e_1\}^{A_{j,s(x)}}_{s(x)} \upharpoonright u\left(i, W_{e_0,s(x)}, x, s(x)\right)$$

via correct computations; i.e.,

(46) $$\forall x \forall s \geqslant s(x) \left(\{i\}^{W_{e_0,s}}_s(x) = \{i\}^{W_{e_0,s(x)}}_{s(x)}(x) \text{ and } u\left(i, W_{e_0,s}, x, s\right) = u\left(i, W_{e_0,s(x)}, x, s(x)\right) \right)$$

and

(47) $$\forall x \forall s \geqslant s(x) \left(W_{e_0,s} \upharpoonright u\left(i, W_{e_0,s}, x, s\right) = \{e_1\}^{A_{j,s}}_s \upharpoonright u\left(i, W_{e_0,s}, x, s\right) \text{ and } \forall y < u\left(i, W_{e_0,s}, x, s\right) \left(u\left(e_1, A_{j,s}, y, s\right) = u\left(e_1, A_{j,s(x)}, y, s(x)\right)\right)\right).$$

Since, by (42), $f_e^j(x) = \lim_s f_e^j(x,s)$ is a total A_j-recursive function and since $f_e^j(x,s)$ is nondecreasing in s, we can A_j-recursively find a stage s_x such that $f_e^j(x,s) = f_e^j(x)$ for $s \geqslant s_x$. W.l.o.g. we may assume $s_x \leqslant s(x)$, i.e.

(48) $$\forall x \forall s \geqslant s(x) \left(f_e^j(x,s) = f_e^j(x) \right).$$

This and Lemma 14 imply

(49) $$\forall x \in Z \forall s \geqslant s(x) \left(f_e^j(x,s) \notin A \right).$$

Since $E^j_{e,s+1}(x) \neq E^j_{e,s}(x)$ implies $f_e^j(x,s)$ is in A_{s+1}, we can conclude that

$$\forall x \in Z \forall s \geqslant s(x) \left(E^j_{e,s}(x) = E^j_e(x) \right)$$

and thus, by (42) and (46),

$$(50) \qquad \forall x \in Z \forall s \geqslant s(x)\left(E^j_{e,s}(x) = \{i\}^{W_{e_0,s}}_s(x)\right).$$

Finally, since R_m is not met, Lemma 11 implies

$$(51) \qquad \forall t \in T^m_D(r(m,t) = 0).$$

Facts (51), (50), (47), (45) and (49) imply (44) holds.

In the next step we define an infinite A-recursive subset Z'' of Z, and an A-recursive function $s''(x)$ such that

$$(52) \qquad \forall x \in Z'' \forall s \geqslant s''(x)\left(s \in T^m_D \to (32.1)\text{–}(32.6) \text{ hold}\right).$$

We first note that, by (42), $u(i, W_{e_0}, x)$ is a total W_{e_0}-recursive function. Hence, by (41) and Lemma 1, the set

$$Z' = \left\{x \in Z: \gamma_j(x) > u(i, W_{e_0}, x)\right\}$$

is infinite. Moreover, $Z' \leqslant_T A_j \oplus W_{e_0} \leqslant_T A_j$, and we can define an A_j-recursive function $s'(x)$ such that $\forall x(s'(x) \geqslant s(x))$ and

$$(53) \qquad \forall x \in Z' \forall s \geqslant s'(x)\left(\gamma_j(x,s) > u(i, W_{e_0,s}, x, s)\right).$$

Now, since Z' is an infinite A_j-recursive set, since $f^j_e(x)$ is A_j-recursive, and since, by Lemma 13(a), $A_{1-j} \nleqslant_T A_j$, a second application of Lemma 1 shows that the set

$$Z'' = \left\{x \in Z': \gamma_{1-j}(x) > f^j_e(x)\right\}$$

is infinite. $Z'' \leqslant_T A_0 \oplus A_1 =_T A$ and there is an A-recursive function s'' such that

$$(54) \qquad \forall x(s''(x) \geqslant s'(x) \geqslant s(x)) \quad \text{and}$$

$$(55) \qquad \forall x \in Z'' \forall s \geqslant s''(x)\left(\gamma_{1-j}(x,s) > f^j_e(x,s)\right).$$

Now (52) follows from (54), (44), (53) and (55).

Since Z'' is infinite and $Z'', s'' \leqslant_T A$,

$$\forall x \in Z'' \forall s > \max\{s''(x), s_0\}\ (C_s \upharpoonright x = C \upharpoonright x)$$

implies $C \leqslant_T A$, contrary to Lemma 13(a). Hence we can fix x, y, t such that $x \in Z''$, $y < x$, $y \in C_{t+1} - C_t$ and $t > \max\{s''(x), s_0\}$. Now if s is the least stage $\geqslant t$ in T^m_D then (32.7.1) and (32.7.2) hold (note that $x \leqslant f^j_e(x,s)$). So, by (52), R_m requires attention at stage $s+1$ contrary to (43). □

LEMMA 16. $A \leqslant_T C$.

PROOF. Since $A^{(0)} = D \leqslant_T C$, it suffices to show that for $f > 0$, $A^{(f)}$ is recursive in C, C-uniformly in f (i.e. there is a C-recursive function g s.t. $A^{(f)} = \{g(f)\}^C$). This is done by an effective induction. For the inductive step fix f. By inductive hypothesis we may use $C \oplus A^{(<f)}$ as oracle, and thus $C \oplus T^f_D$, since $T^f_D \leqslant_T A^{(<f)}$ uniformly in f.

Case 1: *f is of type* 0. Then $A^{(f)} = \varnothing$.

Case 2: *f is of type* 1. Say $f = 3(2e+j)+1$, $j \leqslant 1$. To $C \oplus T^f_D$-recursively decide whether a given $\langle f, x, t\rangle$ is in A, first check if $x \in C$. If not, $\langle f, x, t\rangle \notin A$. If so, choose s with $x \in C_s$ and the least element $s' > s, t$ of T^f_D. Then, by (38), $\langle f, x, t\rangle \in A$ iff $\langle f, x, t\rangle \in A_{s'}$.

Case 3: *f is of type* 2. To decide whether a given $\langle f, x\rangle$ is in A, find the least stage $t \in T_D^f$ with $C_t \upharpoonright \langle f, x\rangle = C \upharpoonright \langle f, x\rangle$. Then by (32.7), $\langle f, x\rangle \in A$ iff $\langle f, x\rangle \in A_{t+1}$. □

This completes the proof of Theorem 4. ■

Note that for $\text{low}_2\, \mathbf{c}$, the above constructed degrees $\mathbf{a}_0$ and $\mathbf{a}_1$ form a minimal exact pair for some u.r.e. ascending sequence of r.e. degrees.

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