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ABSTRACT. In a previous paper we proved that for some special levels, in the space of elliptic modular forms with weight 3/2 the orthogonal complement of the subspace of cusp forms with respect to the Petersson inner product is generated by the Eisenstein series. In this paper we prove that this fact is true for any level.

1. Introduction. We prove the following theorem. The notations defined in [**1**] will be used throughout this paper.

THEOREM. *For any level N and any even character* ω, *the space* $\mathfrak{S}(N, \omega)$ *is generated by the Eisenstein series* $E(\psi, m)$ *with* $\psi^2 \neq \mathrm{id}$, $f_1(\mathrm{id}, 4D)$, $f_1(\mathrm{id}, 8D)$ *and their transforms under the elements of the group G.*

The dimension of $\mathfrak{S}(N, \omega)$ is known (see §5 of [**1**]). What we are going to do is to find a basis of $\mathfrak{S}(N, \omega)$, which is generated by the Eisenstein series. The first thing we do is to find a set of forms in $\mathfrak{S}(N, \omega)$ of the number $\dim \mathfrak{S}(N, \omega)$. The second thing we do is to prove its linear independence, using their values at cusps. Now we fix a positive integer N with $4 \mid N$ and an even character ω modulo N with conductor f. Let c be a positive integer and ψ a character with conductor m. Given an integer e, we let $e(p)$ denote the highest exponent of a prime p in e, i.e. $p^{e(p)} \| e$. We write $n(p)$ instead of $N(p)$. The pair (ψ, c) is called admissible if one of the following holds:

(1) c and m satisfy

$$c \mid N, \qquad m \mid (c, N/c) \mid N/f, \tag{1.1}$$

when $n(2) \geqslant 4$. Here $(c, N/c)$ denotes the greatest common divisor of c and N/c.

(2) c and m satisfy $c \mid N$, $m \mid (c/2^{c(2)}, N/c) \mid N/f$, and $c(2) = 0, 1, 3$ when $n(2) = 3$, $f(2) = 3$.

(3) c and m satisfy (1.1) and $c(2) = 0, 2, 3$ when $n(2) = 3, f(2) = 0, 2$.

(4) c and m satisfy (1.1) and $c(2) = 0, 2$ when $n(2) = 2$.

The number of all admissible pairs is

$$\sum_{c \mid N, (c, N/c) \mid N/f} \phi((c, N/c)) \qquad \text{if } n(2) \geqslant 4,$$

$$3 \sum_{c \mid N', (c, N/c) \mid N/f} \phi((c, N/c)) \qquad \text{if } n(2) = 3,$$

$$2 \sum_{c \mid N', (c, N/c) \mid N/f} \phi((c, N/c)) \qquad \text{if } n(2) = 2,$$

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where $N' = N/2^{n(2)}$ and ϕ is the Euler function. Suppose t is a positive integer, ψ^* is a totally even (in the sense of Serre and Stark [**2**]) primitive character with conductor r, and the pair (ψ^*, r) satisfies

$$(1.2) \qquad \text{(i) } 4r^2t \mid N, \qquad \text{(ii) } \omega = \psi^*\chi_t = \psi^*\left(\frac{t}{\,}\right).$$

We know that the number of all such pairs (ψ^*, t) is equal to the dimension of $\mathfrak{S}_{1/2}(N, \omega)$ (see [**2**]). For a given (ψ^*, t), we choose one character K such that $K^2 = \psi^*$ and the conductor $\tilde{m}$ of K is r (when $2 \nmid r$) or $2r$ (when $2 \mid r$). Moreover, if two such characters ψ^* and ψ_1^* have the same p-part for some p, then we choose K and K_1 such that they also have the same p-part for this p. Now we put

$$(1.3) \qquad \tilde{c} = 2^e rt \prod_{p \mid t,\, p \nmid r} p^{-1}$$

where

$$e = \begin{cases} 1, & r(2) \geqslant 3, \\ -1, & r(2) = 0, t(2) \geqslant 1, \\ 0, & r(2) = t(2) = 0. \end{cases}$$

In this paper p always denotes an odd prime unless an additional note is made. By (1.2) (i) we have $\tilde{c} \mid N$ and $2^{2-e} r \prod_{p \mid t,\, p \nmid r} p \tilde{c} \mid N$. By (1.2) (ii) we can verify that $f(2) \leqslant 2 - e + r(2)$, $f(p) \leqslant r(p)$ when $p \mid r$, or $f(p) \leqslant 1$ when $p \mid t$, $p \nmid r$. Hence we have $f\tilde{c} \mid N$. It is easy to see that $\tilde{m} \mid \tilde{c}$ and $\tilde{m} \mid N/\tilde{c}$. Therefore the pair $(K, \tilde{c})$ is admissible. We call it exceptional. Every pair (ψ^*, t) satisfying (1.2) corresponds to a unique exceptional pair in the above way, hence the number of all exceptional pairs is the dimension of $\mathfrak{S}_{1/2}(N, \omega)$. Therefore the number of all admissible but nonexceptional pairs is the dimension of $\mathfrak{S}(N, \omega)$. We can see that if $n(2) = 2, 3$, then $\tilde{c}(2) = 0$ for any exceptional pair.

For every positive divisor c of N, we put $g(c) = \phi((c, N/c))$. Suppose $d_1, d_2, \ldots, d_{g(c)}$ is a full set of representatives of $(\mathbf{Z}/(c, N/c)\mathbf{Z})^*$, where the d_i are chosen so that $(d_i, c) = 1$. Then the set of cusps

$$S(N) = \{d_1/c, \ldots, d_{g(c)}/c \mid c \mid N\}$$

is a full set of representatives of $\Gamma_0(N)$-equivalent classes of cusps. Now we introduce an order of cusps of $S(N)$. First we put all prime factors of N in a fixed order as $p_0 = 2$, $p_1, \ldots, p_n$. Then every c corresponds to an ordered series $(c(p_0), c(p_1), \ldots, c(p_n))$. Let c_1 and c_2 be two divisors of N. If there exists an integer j $(0 \leqslant j \leqslant n)$ such that $c_1(p_j) < c_2(p_j)$ and $c_1(p_i) = c_2(p_i)$ $(0 \leqslant i < j)$, we say that c_1 precedes c_2, and denote it by $c_1 \prec c_2$. Thus we have an order of all c. According to the order of denominators c we arrange the cusps of $S(N)$ in a line, putting those cusps with the same denominator in any order.

In the following two sections we shall choose a form $F(\psi, c) \in \mathfrak{S}(N, \omega)$ generated by the Eisenstein series for every admissible but nonexceptional pair (ψ, c) such that

$$(1.4) \qquad V(F(\psi, c), d/c) = \psi(d)\rho_1(\psi)\rho_2(d),$$

$$V(F(\psi, c), d/\beta) = 0 \qquad (c \prec \beta, \beta \mid N), \tag{1.5}$$

where the term $\rho_1(\psi)$ does not depend on d, $\rho_2(d)$ does not depend on ψ and $\rho_1(\psi)\rho_2(d) \neq 0$. Define a set of cusps

$$S_f(N) = \{d_1/c, \ldots, d_{g(c)}/c \mid c \text{ satisfies one of } (1), (2), (3), (4)\}.$$

Let us construct a matrix $A = (V(F(\psi, c), s))$, where every row of A corresponds to a function $F(\psi, c)$ while every column of A corresponds to a cusp s of $S_f(N)$. We can introduce an order among the functions $\{F(\psi, c)\}$ as we did for the cusps. The rows and columns of A are arranged according to the order of $F(\psi, c)$ and s, respectively. By (1.4), (1.5) and Lemma 7.1 of **[1]** we see that A is nonsingular, therefore the set of functions $\{F(\psi, c)\}$ is a basis of $\mathfrak{S}(N, \omega)$. If we can find such a basis, our theorem is proved. Sometimes there are some additional terms in the right-hand side of (1.4); however, we can show that they do not change the nonsingularity of A.

At the end of this section we introduce a Lemma which we shall use later.

LEMMA. *Let ψ and ω be primitive characters modulo m and f, respectively. Suppose $m \mid f$ and $\psi\omega$ is a primitive character modulo f. Then*

$$\sum_{a=1}^{m} \psi(a)\omega\left(1 + \frac{af}{m}\right) = \varepsilon m^{1/2} \quad \textit{where } |\varepsilon| = 1.$$

PROOF. It is sufficient to prove this Lemma for the case $m = p^r$, $f = p^s$ and $s \geqslant r$ (here p can be 2). If $s = r$ (in this case $p \neq 2$) we have

$$\begin{aligned}
&\sum_{a=1}^{p^r} \psi(a)\omega(1+a) \sum_{b=1}^{p^r} \bar{\psi}(b)\bar{\omega}(1+b) \\
&\quad = \sum_{a=1}^{p^r} \sum_{p\nmid b,\, b=1}^{p^r} \psi(ab^{-1})\omega(1+a)\bar{\omega}(1+b) \\
&\quad = \sum_{c=1}^{p^r} \psi(c) \sum_{p\nmid b,\, p\nmid 1+b,\, b=1}^{p^r} \omega\left(1+(c-1)b(1+b)^{-1}\right) \\
&\quad = \sum_{c=1}^{p^r} \psi(c) \left\{ \sum_{b=1}^{p^r} \omega(1+(c-1)b) - \sum_{b=1}^{p^{r-1}} \omega(1+(c-1)pb) \right. \\
&\qquad\qquad \left. - \sum_{b=1}^{p^{r-1}} \omega(1+(c-1)(1+pb)) \right\} \\
&\quad = p^r - p^{r-1} \sum_{a=1}^{p} \psi(1+ap^{r-1}) - p^{r-1} \sum_{a=1}^{p} \psi\omega(1+ap^{r-1}) \\
&\quad = p^r;
\end{aligned}$$

here we use Lemma 7.2 of [**1**]. If $s > r$ we have

$$\sum_{a=1}^{p^r} \psi(a)\omega(1+p^{s-r}a) \sum_{b=1}^{p^r} \bar{\psi}(b)\bar{\omega}(1+p^{s-r}b)$$
$$= \sum_{c=1}^{p^r} \psi(c) \sum_{p\nmid b,\, b=1}^{p^r} \omega\left(1+p^{s-r}(c-1)b(1+p^{s-r}b)^{-1}\right)$$
$$= \sum_{c=1}^{p^r} \psi(c)\left\{\sum_{b=1}^{p^r} \omega(1+p^{s-r}(c-1)b) - \sum_{b=1}^{p^{r-1}} \omega\left(1+p^{s-r+1}(c-1)b\right)\right\}$$
$$= p^r - p^{r-1}\sum_{a=1}^{p} \psi\left(1+ap^{r-1}\right) = p^r.$$

Now the Lemma is proved.

2. The case of $(\bar{\omega}\psi^2)^2 \neq$ **id.** For a given admissible pair (ψ, c) we define integers f_1, m_2, etc, as follows:

f_1 is the product of those $p^{f(p)}$ satisfying $c(p) < f(p)$;
m_2 is the product of those $p^{m(p)}$ satisfying $f(p) \leqslant m(p)$;
f_3 is the product of those $p^{f(p)}$ satisfying $0 < m(p) < f(p) \leqslant c(p)$;
f_4 is the product of those $p^{f(p)}$ satisfying $0 = m(p) < f(p) \leqslant c(p)$;
u_1 is the product of those p satisfying $p \mid c$, $p \nmid mf$, $2 \nmid c(p)$, $c(p) < n(p)$;
u_2 is the product of those p satisfying $p \mid c$, $p \nmid mf$, $2 \mid c(p)$, $c(p) < n(p)$;
v_1 is the product of those p satisfying $p \mid c$, $p \nmid mf$, $2 \nmid c(p)$, $c(p) = n(p)$;
v_2 is the product of those p satisfying $p \mid c$, $p \nmid mf$, $2 \mid c(p)$, $c(p) = n(p)$;
w is the product of those p satisfying $p \mid N$, $p \nmid cf$.

Put

$$f_0 = 2^{f(2)}, \qquad f_2 = \prod_{p \mid m_2} p^{f(p)},$$
$$m_0 = 2^{m(2)}, \qquad m_i = \prod_{p \mid f_i} p^{m(p)} \quad (i = 1, 3).$$

Hence $f = f_0 f_1 f_2 f_3 f_4$, $m = m_0 m_1 m_2 m_3$, $(f_i, f_j) = (m_i, m_j) = 1$ if $i \neq j$, and if p divides N then p divides some unique element of the set $\{f_1, m_2, f_3, f_4, u_1, v_1, u_2, v_2, w\}$.

We decompose ω and ψ into

$$\omega = \prod_{i=0}^{4} \omega_i, \qquad \psi = \prod_{i=0}^{3} \psi_i,$$

where the conductor of ω_i is f_i $(0 \leqslant i \leqslant 4)$, and the conductor of ψ_i is m_i $(0 \leqslant i \leqslant 3)$.

In this section we consider all admissible pairs (ψ, c) with the property $(\bar{\omega}\psi^2)^2 \neq$ id. For every such pair, we shall find a form $F(\psi, c) \in \mathfrak{S}(N, \omega)$ satisfying (1.4) and (1.5). We deal with this problem in the following cases. In every case, we shall first choose positive integers N_1, Q, η and characters ϕ_1, ϕ_2 such that ϕ_1 is a character modulo N_1, where $\phi_1^2 \neq \mathrm{id}$, Q satisfies $Q \mid N_1$, $(Q, N_1/Q) = 1$, then we shall define (see [**1**] for the definition of $E(\phi_1, N_1)$)

$$g = g(\psi, c)(z) = E(\phi_1, N_1) \mid W(Q), \tag{2.1}$$

$$h = h(\psi, c)(z) = \sum_{j=1}^{\sigma} \phi_2(j) g\left(\frac{z+j}{\sigma}\right), \tag{2.2}$$

$$q = q(\psi, c)(z) = h \mid V(\eta), \tag{2.3}$$

where $W(Q)$ belongs to the normalizer of $\Gamma_0(N)$ defined in §3 of [**1**], σ is the conductor of ϕ_2 and $V(\eta)$ is the shift operator. Our goal is to obtain $q \in \mathfrak{S}(N, \omega)$ and the desired property of its values at cusps. For a given integer s we define $c[s] = \prod_{p \mid s} p^{c(p)}$. Let

$$N_i' = f_1 m_2 f_3 f_4 u_1 u_2 v_1 w, \qquad Q' = f_1 m_2 u_1 u_2 w,$$
$$\eta' = c/m_1 m_2' f_3 f_4 u_1 v_1, \qquad \zeta = c[m_2] u_1, \quad \sigma' = m_1 m_2' m_3 u_1,$$
$$\phi_1' = \bar{\omega}_1 \omega_2 \omega_3 \omega_4 \bar{\psi}^2, \qquad \phi_2' = \bar{\omega}_2 \psi_1 \psi_2 \bar{\psi}_3 \chi_3'$$

where σ' is the conductor of ϕ_2', and $\chi_\zeta' = (\frac{\ }{\zeta})$ is primitive.

Case 1. $n(2) \geqslant 4$, $c(2) < f(2)$.

We choose

$$N_1 = [8, 2^{f(2)}] N_1', \quad Q = [8, 2^{f(2)}] Q', \quad \eta = \eta'/m_0,$$
$$\phi_1 = \bar{\omega}_1 \phi_1' \chi_{\eta Q}, \qquad \phi_2 = \psi_0 \phi_2',$$

where $[a, b]$ denotes the least commmon multiple of a and b. Then we have $\sigma = m_0 \sigma'$. It is easy to verify that ϕ_1 is a character modulo N_1. (The notations χ_t and $\chi_d'(2 \nmid d)$ always denote primitive characters.) Since $f(2) > c(2) \geqslant m(2)$, $f(p) > c(p) \geqslant m(p)$ $(p \mid f_1)$, the condition $(\bar{\omega}\psi^2)^2 \neq \mathrm{id}$ implies $\phi_1^2 \neq \mathrm{id}$. By Lemmas 3.1 and 3.2 of [**1**] and the property of the shift operator $V(\eta)$, we find that $g \in \mathfrak{S}(N_1, \omega\phi_2^2\chi_\eta)$, $h \in \mathfrak{S}(\sigma N_1, \omega\chi_\eta)$ and $q \in \mathfrak{S}([8, 2^{f(2)}] f_1 m_2 m_3 u_1 u_2 w c, \omega)$. Since $m(p) \leqslant \min(c(p), n(p) - c(p)) \leqslant n(p) - f(p)$ by (1.1), the condition $c(p) < f(p)$ (i.e. $p \mid f_1$) implies $c(p) = \min(c(p), n(p) - c(p)) \leqslant n(p) - f(p)$. This is also true for $p = 2$. If $f(2) \leqslant 2$, then $c(2) + 3 \leqslant 4 \leqslant n(2)$. Therefore we can show that $[8, 2^{f(2)}] f_1 m_2 m_3 u_1 u_2 w c \mid N$, noting the definition of u_1, u_2 and w. This means $q \in \mathfrak{S}(N, \omega)$.

By Lemmas 4.4 and 4.11 of [**1**], we know g vanishes at all cusps of $S(N_1)$ except $1/f_3 f_4 v_1$. Remember the fact that a cusp d/c is $\Gamma_0(N_1)$-equivalent to a cusp in $S(N_1)$ with (c, N_1) as its denominator. Let α_3, α_4, α_5 and α_6 be positive integers such that

$$f_i \mid \alpha_i \mid \prod_{p \mid f_i} p^{n(p)} \quad (i = 3, 4), \qquad v_1 \mid \alpha_5 \mid \prod_{p \mid v_1} p^{n(p)}, \qquad \alpha_6 \mid \prod_{p \mid v_2} p^{n(p)}.$$

We see that h cannot vanish only at the cusps $d/m_0 m_1 m_2' \alpha_3 \alpha_4 \alpha_5 \alpha_6 u_1$ in $S(N)$. Now we calculate the values of h at these cusps. The values at the cusps with $\alpha_6 \neq 1$ will not be used later, so we take $\alpha_6 = 1$. We have

$$(2.4)\quad h\left(z+\frac{d}{m_0m_1m_2'\alpha_3\alpha_4\alpha_5u_1}\right)=\sum_{j=1}^{\sigma}\phi_2(j)g\left(z+\frac{d+j\alpha_3\alpha_4\alpha_5m_3^{-1}}{m_0m_1m_2'\alpha_3\alpha_4\alpha_5u_1}\right)$$
$$=\sum_{j_1=1}^{\sigma/m_3}\sum_{j_2=1}^{m_3}\bar{\omega}_2\psi_0\psi_1\psi_2\chi_\zeta'(m_3j_1)\bar{\psi}_3(\sigma m_3^{-1}j_2)$$
$$\times g\left(z+\frac{d+j_1\alpha_3\alpha_4\alpha_5+j_2\sigma\alpha_3\alpha_4\alpha_5m_3^{-2}}{m_0m_1m_2'\alpha_3\alpha_4\alpha_5u_1}\right).$$

There is only one integer j_1^* satisfying

$$d+j_1^*\alpha_3\alpha_4\alpha_5=\lambda\sigma m_3^{-1},\qquad 1\leqslant j_1^*\leqslant\sigma m_3^{-1},$$

where λ is an integer. The fraction in the right side of (2.4) is $\Gamma_0(N_1)$-equivalent to $1/f_3f_4v_1$ only if $j_1=j_1^*$. Therefore

$$(2.5)\quad V(h, d/m_0m_1m_2'\alpha_3\alpha_4\alpha_5u_1)=\bar{\omega}_2\psi_0\psi_1\psi_2\chi_\zeta'(m_3j_1^*)(\sigma m_3^{-1})^{3/2}$$
$$\times\sum_{j=1}^{m_3}\bar{\psi}_3(\sigma m_3^{-1}j)V\left(g,\frac{\lambda+j\alpha_3\alpha_4\alpha_5m_3^{-1}}{\alpha_3\alpha_4\alpha_5}\right).$$

We have

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}1\\ f_3f_4v_1\end{pmatrix}=\begin{pmatrix}\lambda+j\alpha_3\alpha_4\alpha_5m_3^{-1}\\ \alpha_3\alpha_4\alpha_5\end{pmatrix},\qquad\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma_0(N_1),$$

hence

$$D\equiv\alpha_3\alpha_4\alpha_5/f_3f_4v_1\pmod{[8,2^{f(2)}]f_1m_2u_1},$$
$$A\equiv\lambda+j\alpha_3\alpha_4\alpha_5m_3^{-1}\pmod{f_3f_4v_1}.$$

Now we write c as $c[2f_1m_2u_1]c[f_3f_4]c[u_2v_1v_2]$ and ε_d as $\varepsilon(d)$. The law of quadratic reciprocity is being applied in the following form:

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right)=\varepsilon(a)\varepsilon(b)\varepsilon(ab)^{-1}\qquad(a,b\text{ odd integers}).$$

By Lemma (4.2) of [**1**] we obtain

$$(2.6)\quad V(g,(\lambda+j\alpha_3\alpha_4\alpha_5m_3^{-1})/\alpha_3\alpha_4\alpha_5)$$
$$=\bar{\omega}_3\psi_3^2(\lambda+j\alpha_3\alpha_4\alpha_5m_3^{-1})\bar{\omega}_4(\lambda)\omega_0\omega_1\bar{\omega}_2\psi_0^2\psi_1^2\psi_2^2\left(\frac{\alpha_3\alpha_4\alpha_5}{f_3f_4v_1}\right)$$
$$\times\varepsilon(c[f_3f_4]v_1)\varepsilon^{-1}(c[f_3f_4]f_3f_4\alpha_3\alpha_4\alpha_5)\left(\frac{d}{c[f_3f_4]f_3f_4\alpha_3\alpha_4\alpha_5}\right)$$
$$\times\left(\frac{c[2f_1m_2u_1]}{\alpha_3\alpha_4\alpha_5f_3f_4v_1}\right)\left(\frac{m_0m_1m_2'u_1}{c[f_3f_4]v_1}\right)V\left(g,\frac{1}{f_3f_4v_1}\right).$$

Now we consider the sum

$$\sum_{j=1}^{m_3} \bar{\psi}_3(\sigma m_3^{-1} j)\bar{\omega}_3\psi_3^2(\lambda + j\alpha_3\alpha_4\alpha_5 m_3^{-1})$$
$$= \bar{\omega}_3\psi_3(d)\psi_3(\alpha_4\alpha_5)\omega_3\bar{\psi}_3^2(\sigma m_3^{-1}) \sum_{j=1}^{m_3} \bar{\psi}_3(j)\bar{\omega}_3\psi_3^2\left(1 + \frac{j\alpha_3}{m_3}\right).$$

Take $n = m_3/(m_3, \alpha_3 f_3^{-1})$. Since $f_3 \mid \alpha_3/(m_3, \alpha_3 f_3^{-1})$, we have

$$\sum_{j=1}^{m_3} \bar{\psi}_3(j)\bar{\omega}_3\psi_3^2\left(1 + \frac{j\alpha_3}{m_3}\right)$$
$$= \sum_{a=1}^{n} \sum_{b=1}^{m_3/n} \bar{\psi}_3(a + bn)\bar{\omega}_3\psi_3^2\left(1 + a\alpha_3 m_3^{-1} + \frac{b\alpha_3}{(m_3, \alpha_3 f_3^{-1})}\right)$$
$$= \sum_{a=1}^{n} \bar{\omega}_3\psi_3^2(1 + a\alpha_3 m_3^{-1}) \sum_{b=1}^{m_3/n} \bar{\psi}_3(a + bn) = 0$$

if $n \neq m_3$. Furthermore, if $n = m_3$, then

$$\sum_{j=1}^{m_3} \bar{\psi}_3(j)\bar{\omega}_3\psi_3^2\left(1 + \frac{jf_3}{m_3}\right) \neq 0$$

by the Lemma in §1. Substituting (2.6) into (2.5) we obtain

$$V(h, d/m_0 m_1 m_2' f_3\alpha_4\alpha_5 u_1) = \bar{\omega}_2\bar{\omega}_3\bar{\omega}_4\psi(d)\chi_d(c[m_2 f_3 f_4]u_1 f_4\alpha_4\alpha_5)$$
$$\times \omega_0\omega_1\psi\chi'_\zeta(\alpha_4\alpha_5)\varepsilon(c[f_3 f_4]v_1)\varepsilon^{-1}(c[f_3 f_4]f_4\alpha_4\alpha_5)$$
$$\times \left(\frac{c[2f_1 m_2 u_1]}{f_4\alpha_4\alpha_5 v_1}\right)\left(\frac{m_0 m_1 m_2' u_1}{c[f_4]}\right)\rho,$$

where ρ is a constant independent of d, α_4, α_5.

Furthermore, let $c' = ct$, where t is a positive integer divisible only by the primes dividing f_4. Assume (ψ, c') is admissible. Then the constant ρ for it remains unchanged.

In subsequent formulas, we let ρ denote a constant of the same nature, whose value may depend on each formula. Let τ and α be two integers such that

$$\tau \mid c, \quad (\tau, m_0 m_1 m_2' f_3 f_4 u_1 v_1) = 1, \quad \alpha \mid \prod_{p \mid f_4} p^{n(p)-c(p)}.$$

We have

$$V(q(\psi, c), d/c\alpha\tau^{-1})\tau^{-3/2}V(h, d\tau/m_0 m_1 m_2' f_3 f_4\alpha u_1 v_1)$$
$$= \bar{\omega}_2\bar{\omega}_3\bar{\omega}_4\psi(d\tau)\chi_{d\tau}(c[m_2 f_3 f_4]\alpha u_1 v_1)\omega_0\omega_1\psi\chi'_\zeta(\alpha)\varepsilon(c[f_3 f_4]v_1)$$
$$\times \varepsilon^{-1}(c[f_3 f_4]\alpha v_1)\left(\frac{c[2f_1 m_2 u_1]}{\alpha}\right)\left(\frac{m_0 m_1 m_2' u_1}{c[f_4]}\right)\rho\tau^{-3/2},$$

and q vanishes at all other cusps in $S(N)$. Now we define

$$F(\psi, c)(z) = \sum_{t|(f_4, N/c)} \mu(t)\omega_0\omega_1\psi\chi'_\zeta(t)\chi'_t(c[2f_1m_2]m_0m_1m'_2)$$
$$\times\varepsilon(c[f_3f_4]v_1)\varepsilon^{-1}(c[f_3f_4]v_1t)q(\psi, ct).$$

If t is a square-free positive integer satisfying $t|(f_4, N/c)$, then we can show $ct|N$ and $(ct, N/ct)|N/f$. The first divisibility is obvious. Now we consider the second one. Let p be a factor of f_4. If $n(p) - c(p) = \min(c(p), n(p) - c(p)) \leqslant n(p) - f(p)$, then, of course, $n(p) - c(p) - 1 < n(p) - f(p)$. If $c(p) < n(p) - c(p) \leqslant n(p) - f(p)$, then $c(p) + 1 \leqslant n(p) - f(p)$. This proves the second divisibility. We also have $m|(ct, N/ct)$. Hence (ψ, ct) is an admissible pair and $q(\psi, ct)$ can also be defined. Therefore $F(\psi, c)$ belongs to $\mathfrak{S}(N, \omega)$. Noting that $V(q(\psi, ct), d/c\alpha\tau^{-1}) = 0$ if $t \nmid \alpha$ and $ct[f_4] = c[f_4]t$, we have

$$V(F, d/c\alpha\tau^{-1}) = \bar{\omega}_2\bar{\omega}_3\bar{\omega}_4\psi(d\tau)\chi_{d\tau}(c[m_2f_3f_4]\alpha u_1v_1)$$
$$\times\omega_0\omega_1\psi\chi'_\zeta(\alpha)\tau^{-3/2}\varepsilon(c[f_3f_4]v_1)\varepsilon^{-1}(c[f_3f_4]\alpha v_1)$$
$$\times\left(\frac{c[2f_1m_2u_1]}{\alpha}\right)\left(\frac{m_0m_1m'_2u_1}{c[f_4]}\right)\rho\sum_{t|\alpha}\mu(t) = 0$$

if $\alpha \neq 1$. Finally, we obtain

$$V(F(\psi, c), d/c) = \rho\bar{\omega}_2\bar{\omega}_3\bar{\omega}_4\psi(d)\chi_d(c^*), \tag{2.7}$$

$$V(F(\psi, c), d/\beta) = 0 \qquad (\beta|N, c \prec \beta), \tag{2.8}$$

where ρ does not depend on d and $c^* = c/c[2f_1]$ does not depend on ψ.

Case 2. $n(2) \geqslant 4$, $m(2) \geqslant 3$, $f(2) \leqslant m(2)$ $(\leqslant c(2))$.

We choose

$$N_1 = 2^{m(2)}N'_1, \qquad Q = 2^{m(2)}Q', \qquad \eta = \eta'/m'_0,$$

$$\phi_1 = \begin{cases} \omega_0\phi'_1\chi_{\eta Q}, & f(2) \geqslant 4, \\ \bar{\omega}_0\phi'_1\chi_{\eta Q}, & f(2) \leqslant 3, \end{cases} \qquad \phi_2 = \begin{cases} \bar{\omega}_0\psi_0\phi'_2, & f(2) \geqslant 4, \\ \psi_0\phi'_2, & f(2) \leqslant 3, \end{cases}$$

where m'_0 is the conductor of $\bar{\omega}_0\psi_0$ when $f(2) \geqslant 4$, and $m'_0 = m_0$ when $f(2) \leqslant 3$. By the method of Case 1 we can find a form $F(\psi, c)$ in $\mathfrak{S}(N, \omega)$ satisfying (2.8) and

$$V(F(\psi, c), d/c) = \begin{cases} \rho\bar{\omega}_0\bar{\omega}_2\bar{\omega}_3\bar{\omega}_4\psi(d)\chi_d(c^*), & f(2) \geqslant 4, \\ \rho\bar{\omega}_2\bar{\omega}_3\bar{\omega}_4\psi(d)\chi_d(c^*), & f(2) \leqslant 3. \end{cases} \tag{2.9}$$

Case 3. (i) $n(2) \geqslant 4$, $f(2) \leqslant 3$, $m(2) \leqslant 2$, $f(2) \leqslant c(2) \leqslant n(2) - 3$;

(ii) $n(2) \geqslant 5$, $c(2) = n(2) - 2$, $m(2) = 2$, $\omega_0\chi_\nu = \mathrm{id}$ or χ_{-1} $(\nu = 2^{c(2)})$; or $n(2) = 4$, $c(2) = m(2) = 2$;

(iii) $n(2) = 3$, $c(2) = 0$ or 1;

(iv) $n(2) = 2$, $c(2) = 0$.

We choose

$$N_1 = aN'_1, \qquad Q = aQ', \qquad \eta = \eta'/m_0,$$
$$\phi_1 = \bar{\omega}_0\phi'_1\chi_{\eta Q}, \qquad \phi_2 = \psi_0\phi'_2,$$

where $a = (8, 2^{n(2)})$ in (i), (iii), (iv), and $a = 4$ in (ii). By the method of Case 1 we can find a form $F(\psi, c) \in \mathfrak{S}(N, \omega)$, satisfying (2.7) and (2.8).

Case 4. $n(2) \geqslant 4, f(2) \geqslant 4, m(2) < f(2) \leqslant c(2)$.

We choose

$$N_1 = 2^{f(2)}N_1', \qquad Q = Q', \qquad \eta = \eta'/2^{f(2)},$$
$$\phi_1 = \omega_0\phi_1'\chi_{\eta Q}, \qquad \phi_2 = \psi_0\phi_2'.$$

Now $\sigma = m_0\sigma'$. Similarly to Case 1 we can show $g \in \mathfrak{S}(N_1, \omega\phi_2^3\chi_\eta)$, $h \in \mathfrak{S}(\sigma N_1, \omega\chi_\eta)$ and $q \in \mathfrak{S}(m_0 f_1 m_2 m_3 u_1 u_2 wc, \omega) \subset \mathfrak{S}(N, \omega)$. We can also obtain

$$V(h, d/\alpha_0 m_1 m_2' \alpha_3\alpha_4\alpha_5 u_1) = \bar{\omega}_2\psi_1\psi_2\chi_\zeta'(m_0 m_3 j^*)(m_1 m_2' u_1)^{3/2}$$
$$\times \sum_{j=1}^{m_0 m_3} \bar{\psi}_0\bar{\psi}_3(m_1 m_2' u_1 j) V\left(g, \frac{\lambda + j\alpha_0\alpha_3\alpha_4\alpha_5 m_0^{-1} m_3^{-1}}{\alpha_0\alpha_3\alpha_4\alpha_5}\right),$$

where α_0 satisfies $2^{f(2)} \mid \alpha_0 \mid 2^{n(2)}$, α_3, α_4, α_5 are defined as in Case 1 and j^* and λ satisfy

$$d + j^*\alpha_0\alpha_3\alpha_4\alpha_5 = \lambda m_1 m_2' u_1, \qquad \alpha \leqslant j^* \leqslant m_1 m_2' u_1.$$

We have

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 1 \\ 2^{f(2)} f_3 f_4 v_1 \end{pmatrix} = \begin{pmatrix} \lambda + j\alpha_0\alpha_3\alpha_4\alpha_5 m_0^{-1} m_3^{-1} \\ \alpha_0\alpha_3\alpha_4\alpha_5 \end{pmatrix}, \qquad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(N_1).$$

Hence,

$$D \equiv \alpha_0\alpha_3\alpha_4\alpha_5/2^{f(2)} f_3 f_4 v_1 \qquad (\bmod\ f_1 m_2 u_1),$$
$$A \equiv \lambda + j\alpha_0\alpha_3\alpha_4\alpha_5 m_0^{-1} m_3^{-1} \qquad (\bmod\ 2^{f(2)} f_3 f_4 v_1).$$

By Lemma (4.2) of [**1**] we obtain

$$V\big(g, (\lambda + j\alpha_0\alpha_3\alpha_4\alpha_5 m_0^{-1} m_3^{-1})/\alpha_0\alpha_3\alpha_4\alpha_5\big)$$
$$= \bar{\omega}_0\bar{\omega}_3\bar{\omega}_4\psi_0^2\psi_3^2(\lambda + j\alpha_0\alpha_3\alpha_4\alpha_5 m_0^{-1} m_3^{-1})$$
$$\times \omega_1\bar{\omega}_2\psi_1^2\psi_2^2(\alpha_0\alpha_3\alpha_4\alpha_5/2^{f(2)} f_3 f_4 v_1)$$
$$\times \varepsilon\big(dc[f_1 m_2 f_3 f_4] u_1 f_3 f_4 \alpha_3\alpha_4\alpha_5 + \alpha_0 m_0^{-1}\big)$$
$$\times \varepsilon^{-1}\big(c[f_1 m_2 f_3 f_4] m_1 m_2' f_3 f_4 \alpha_3\alpha_4\alpha_5\big)\left(\frac{2^{f(2)+c(2)}\alpha_0}{\lambda + j\alpha_0\alpha_3\alpha_4\alpha_5 m_0^{-1} m_3^{-1}}\right)$$
$$\times \left(\frac{d m_1 m_2' u_1}{c[f_3 f_4] f_3 f_4 \alpha_3\alpha_4\alpha_5}\right)\left(\frac{2^{f(2)} f_3 f_4 v_1 \alpha_0\alpha_3\alpha_4\alpha_5}{c[f_1 m_2] m_1 m_2'}\right)$$
$$\times V\big(g, 1/2^{f(2)} f_3 f_4 v_1\big).$$

Now consider the sum

$$\sum_{j=1}^{m_0 m_3} \bar{\psi}_0\bar{\psi}_3(j)\bar{\omega}_0\bar{\omega}_3\psi_0^2\psi_3^2\left(1 + \frac{j\alpha_0\alpha_3}{m_0 m_3}\right)\left(\frac{2^{f(2)+c(2)}\alpha_0}{1 + j\alpha_0\alpha_3/m_0 m_3}\right).$$

We can show, as in Case 1, that when $m_0 > 1$, the sum is not zero if and only if $\alpha_0 = 2^{f(2)}$ and $\alpha_3 = f_3$. When $m_0 = 1$ the sum is not zero if and only if $\alpha_3 = f_3$. Then we can find a form $F'(\psi, c) \in \mathfrak{S}(N, \omega)$, as in Case 1 (if $m_0 = 1$, we take $t\,|\,(2f_4, N/c)$ instead of $t\,|\,(f_4, N/c)$ when we define $F'(\psi, c)$) such that $F'(\psi, c)$ satisfies (2.8) and

$$(2.10) \qquad V(F'(\psi, c), d/c) = \rho\bar{\omega}_0\bar{\omega}_2\bar{\omega}_3\bar{\omega}_4\psi\chi_\nu(d)\chi_d(c^*)\varepsilon(dc^* + 2^{f(2)-m(2)})$$

where $\nu = 2^{c(2)}$. Let m' be the conductor of $\psi\chi_\nu$. We have $m'(2) < f(2)$, because $m(2) < f(2)$ and $f(2) \geqslant 4$. Hence, (2.8) and (2.10) hold for both $F'(\psi\chi_\nu, c)$ and $F'(\psi\chi_{-\nu}, c)$. Note that $\varepsilon_d = 2^{-1}\{1 + i + (1 - i)\chi_{-1}(d)\}$, so we define

$$F(\psi, c) = F'(\psi\chi_\nu, c) \pm i\chi_{-1}(c^*)F'(\psi\chi_{-\nu}, c)$$

(if $f(2) - m'(2) \geqslant 3$, we take $+$; if $f(2) - m'(2) = 1$ we take $-$). It is easy to verify that $F(\psi, c)$ satisfies (2.7) and (2.8).

Case 5. (i) $n(2) \geqslant 4, f(2) \leqslant 3, n(2) - 2 \leqslant c(2) \leqslant n(2), m(2) = 0$;

(ii) $n(2) \geqslant 5, c(2) = n(2) - 2, m(2) = 2, \omega_0\chi_\nu = \chi_2$ or χ_{-2} ($\nu = 2^{c(2)}$);

(iii) $n(2) = 3, c(2) = 2$ or 3;

(iv) $n(2) = 2, c(2) = 2$.

We choose

$$N_1 = aN_1', \qquad Q = Q', \qquad \eta = \eta'/a,$$
$$\phi_1 = \omega_0\phi_1'\chi_{\eta Q}, \qquad \phi_2 = \psi_0\phi_2',$$

where $a = (8, 2^{c(2)})$. Note that if $c(2) = 2, n(2) = 4$, then $f(2) \leqslant 2$ by (1.1). Similarly to Case 4 we can find a form $F(\psi, c) \in \mathfrak{S}(N, \omega)$ satisfying (2.8) and

$$(2.11) \qquad V(F(\psi, c), d/c) = \rho\bar{\omega}_0\bar{\omega}_2\bar{\omega}_3\bar{\omega}_4\psi\chi_\nu(d)\chi_d(c^*)\varepsilon(dc^* + a/m_0).$$

Cases 1–5 include all possible cases of admissible pairs with $(\bar{\omega}\psi^2)^2 \neq \mathrm{id}$.

3. The case of $(\bar{\omega}\psi^2)^2 = \mathrm{id}$. In this section we deal with the admissible, but nonexceptional, pair (ψ, c) with $(\bar{\omega}\psi^2)^2 = \mathrm{id}$. The condition $(\bar{\omega}\psi^2)^2 = \mathrm{id}$ implies (i) $m(2) = f(2) + 1$ if $f(2) \geqslant 4$; $m(2) \leqslant 4$ if $f(2) \leqslant 3$. (ii) $c[f_1] = m_1 = 1$, f_1 and f_4 are square-free, and $\omega_1\prod_{p|f_1}\chi_p'$, $\omega_4 = \prod_{p\nmid f_4}\chi_p'$. (iii) $f_3 = m_3 = 1$. Now we define integers u_1', u_2', v_1', v_2' as follows:

u_1' is the product of u_1 and those p satisfying $p\,|\,f_4, c(p) < n(p), 2\,|\,c(p)$;

u_2' is the product of u_2 and those p satisfying $p\,|\,f_4, c(p) < n(p), 2\nmid c(p)$;

v_1' is the product of v_1 and those p satisfying $p\,|\,f_4, c(p) = n(p), 2\,|\,c(p)$;

v_2' is the product of v_2 and those p satisfying $p\,|\,f_4, c(p) = n(p), 2\nmid c(p)$.

Then we have

$$(3.1) \qquad \chi_d(u_1'v_1') = \omega_4(d)\chi_d(c[f_4]u_1v_1).$$

As in §2 we deal with our problem in the following cases. In every case we shall choose a character ϕ_2, an integer η and a form $g(z) = g(\psi, c)(z)$. Then we define $\phi_1 = \omega\phi_2^2\chi_\eta$ and $h(\psi, c)$, $q(\psi, c)$ according to (2.2) and (2.3), respectively. We will not check that fact that q belongs to $\mathfrak{S}(N, \omega)$ every time, but will leave it to the reader. Let

$$N_2 = \prod_{p|N,\, p\nmid v_2'} p, \qquad \zeta = c[m_2]u_1', \qquad \eta' = c/m_2'u_1'v_1',$$
$$\phi_2' = \bar{\omega}_2\psi_2\chi_\zeta', \qquad \sigma' = m_2'u_1'.$$

Here σ' is the conductor of ϕ_2'.

In [**1**] we found the following basis of $\mathfrak{S}(8D, \chi_l)$, where D is a square-free integer and $l \mid D$:

$$\{G(\chi_l, m, 8D) \mid m \mid D, m \neq 1\} \cup \{G(\chi_l, 4m, 8D) \mid m \mid D\} \cup \{G(\chi_l, 8m, 8D) \mid m \mid D\}.$$

We also found a basis of $\mathfrak{S}(8\mathrm{D}, \chi_{2l})$:

$$\{G(\chi_{2l}, m, 8D) \mid m \mid D, m \neq 1\} \cup \{G(\chi_{2l}, 2m, 8D) \mid m \mid D\} \cup \{G(\chi_{2l}, 8m, 8D) \mid m \mid D\},$$

and a basis of $\mathfrak{S}(4D, \chi_l)$:

$$\{G(\chi_l, m, 4D) \mid m \mid D, m \neq 1\} \cup \{G(\chi_l, 4m, 4D) \mid m \mid D\}.$$

These forms are all generated by the Eisenstein series. We shall use them in this section.

Case 1. (i) $n(2) \geqslant 4$, $n(2) - 2 \leqslant c(2) \leqslant n(2)$, $m(2) = 0$;
(ii) $n(2) \geqslant 5$, $c(2) = n(2) - 2$, $m(2) = 2$, $\omega_0 \chi_\nu = \chi_2$ or χ_{-2} ($\nu = 2^{c(2)}$);
(iii) $n(2) = 3$, $c(2) = 2$ or 3;
(iv) $n(2) = 2$, $c(2) = 2$.

We take $\phi_2 = \psi_0 \phi_2'$, $\eta = \eta'/a$ and $g = G(\phi_1, av_1', aN_2)$, where $a = (8, \nu)$. By the method of Case 5 of §2, we can find a form $F(\psi, c) \in \mathfrak{S}(N, \omega)$ satisfying (2.8) and (2.11).

Case 2. (i) $n(2) \geqslant 5$, $c(2) = n(2) - 2$, $m(2) = 2$, $\omega_0 \chi_\nu = \mathrm{id}$ or χ_{-1};
(ii) $n(2) = 4$, $c(2) = m(2) = 2$;
(iii) $n(2) = 3$, $c(2) = 1$.

In (i) and (iii) we take $\phi_2 = \psi_0 \phi_2'$, $\eta = \eta'/4$ and $g = G(\phi_1, v_1', 8N_2)$ if $v_1' \neq 1$, or $g = G(\phi_1, 4, 8N_2)$ if $v_1' = 1$. Note that ϕ_1 is χ_l ($l \mid N_2$) in (i) and (ii). In (iii) we take $\phi_2 = \phi_2'$, $\eta = \eta'/2$ and $g = G(\phi_1, 2v_1', 8N_2)$. We can show that $q(\psi, c)$ satisfies (2.7) and (2.8) in all three cases. Now we give the proof only for (iii). We know from §6 of [**1**] that g does not vanish only at cusps $1/2v_1'$ and 1 in $S(8N_2)$. Suppose integers j^* and λ satisfy

$$d + 2^{j^*} v_1' = \lambda \sigma', \qquad 1 \leqslant j^* \leqslant \sigma'.$$

Then we have

$$V(h, d/2\sigma' v_1') = \phi_2(j^*)(\sigma')^{3/2} V(g, \lambda/2v_1') = \rho \bar{\omega}_2 \omega_4 \psi(d) \chi_d(c^*),$$

and

$$V(h, d/\beta) = 0 \qquad (2\sigma' v_1' \prec \beta, \beta \mid 8\sigma' N_2).$$

Here we use Lemma 4.2 of [**1**] and the fact that $\phi_1 = \chi_{2l}$ with $l \mid N_2$, $(l, v_1') = 1$. Therefore q satisfies (2.7) and (2.8). We take $F(\psi, c) = q(\psi, c)$.

From now on we need only consider the case $c(2) \leqslant n(2) - 3$ if $n(2) \geqslant 4$, or $c(2) = 0$ if $n(2) = 2$ or 3. Put $b = (8, 2^{n(2)})$ and $\psi_0' = \psi_0$ if $f(2) \leqslant 3$, or $\psi_0' = \bar{\omega}_0 \psi_0$ if $f(2) \geqslant 4$. The conductor of ψ_0' is m_0.

Case 3. $c(2) \leqslant n(2) - 3$ if $n(2) \geqslant 4$, or $c(2) = 0$ if $n(2) = 2, 3$; $v_1' \neq 1$.

Take $\phi_2 = \psi_0' \phi_2'$, $\eta = \eta'/m_0$ and $g = G(\phi_1, v_1', bN_2)$. We can prove q satisfies (2.8) and (2.9). We take $F(\psi, c) = q(\psi, c)$.

Case 4. $c(2) \leqslant n(2) - 3$ if $n(2) \geqslant 4$, or $c(2) = 0$ if $n(2) = 2, 3$; $v_1' = 1, v_2' \neq 1$.

Let p be a prime factor of v_2'. Now we take $\phi_2 = \psi_0'\phi_2'$, $\eta = \eta'/m_0 p$ and $g = G(\phi_1, p, bN_2 p)$. ϕ_1 and χ_p' as its p-part. We can show that h vanishes at all cusps of $S(N)$ except $d/\sigma p^i$ ($\sigma = m_0\sigma'$, $i \geqslant 1$), and $V(h, d/\sigma p) = \rho\phi_2(d)$. Hence q satisfies (2.8) and (2.9). We take $F(\psi, c) = q(\psi, c)$.

Case 5. $c(2) \leqslant n(2) - 3$ if $n(2) \geqslant 4$ or $c(2) = 0$ if $n(2) = 2, 3$; $v_1'v_2' = 1$, $u_2' \neq 1$.

Let p be a prime factor of u_2'. Take $\phi_2 = \psi_0'\phi_2'$, $\eta = \eta'/m_0$ and $g = G(\phi_1, p, bN_2)$. We can show that g vanishes at all cusps of $S(N)$ except $d/\sigma p^i$ ($\sigma = m_0\sigma'$, $i \geqslant 0$), and

$$V(q, d/c) = V(h, d/c) = \phi_2(-d)\sigma^{3/2}V(g, 1) = -\phi_2(-d)\sigma^{3/2}p^{-1}\varepsilon_p\phi_1(p),$$

$$V(q, d/cp^i) = V(h, d/\sigma p^i) = \phi_2(-d)\chi_{d\sigma}(p^i)\overline{\phi_2}(p^i)\phi_1(p^{i-1})\varepsilon_p\varepsilon^{-1}(p^i)\sigma^{3/2} \qquad (i \geqslant 1).$$

The value of $V(g, 1)$ is from (5.9), (6.3) and (6.5) of [**1**]. Further, we define

$$g_1(\psi, c) = G(\phi_1\chi_p, p, bN_2),$$
$$h_1(\psi, c) = \sum_{j=1}^{\sigma} \phi_2(j) g_1\left(z + \frac{j}{\sigma}\right),$$
$$q_1(\psi, c) = h_1 \mid V(c/\sigma p).$$

It is easy to see that

$$V(q_1, d/cp^i) = \phi_2(-d)\chi_{d\sigma}(p^i)\bar{\phi}_2(p^{i+1})\phi_1(p^i)\varepsilon^{-1}(p^i)\sigma^{3/2}, \qquad (i \geqslant 0)$$

and q_1 vanishes at all other cusps of $S(N)$. Now we define

$$F(\psi, c) = q(\psi, c) - \varepsilon_p\phi_1\phi_2(p)q_1(\psi, c).$$

$F(\psi, C)$ satisfies (2.8) and (2.9).

Now let m_2^* denote the product of all prime factors of m_2. Put $\bar{\omega}_2\psi_2^2 = \chi_s'(s \mid m_2^*)$ and

$$\chi_d(sc[m_2]m_2 m_2^*) = \chi_d(y) \qquad (y \mid m_2^*, (d, m_2^*) = 1).$$

Case 6. $c(2) \leqslant n(2) - 3$ if $n(2) \geqslant 4$, or $c(2) = 0$ if $n(2) = 2, 3$; $u_2'v_1'v_2' = 1$. The conductor of $\psi_2\chi_y'$ is less than m_2 or $y \neq m_2^*$.

Let us take $\phi_2 = \psi_0'\overline{\psi_2}\chi_\xi'$, where $\xi = u_1'$, $\eta = c/\sigma$, $\sigma = m\xi$ and $g = G(\phi_1, m_2^*, bN_2)$. Using the same technique as above we can prove that $V(h, d/\beta) = 0$ ($\beta \mid N$, $\sigma \prec \beta$). Now we calculate $V(h, d/\sigma)$. Suppose integers j^* and λ satisfy $d + j^* m_2 = \lambda m_0\xi$, $1 \leqslant j^* \leqslant m_0\xi$. Then

(3.2)

$$V(h, d/\sigma) = \phi_2(-d)V(g, 1)\sigma^{3/2} + \psi_0'\chi_\xi'(j^*)(m_0'\xi)^{3/2} \sum_{a \mid m_2/m_2^*} a^{3/2} \sum_{\substack{x=1 \\ (x, m_2/a)=1}}^{m_2/a} \bar{\psi}_2(ax - \lambda)V\left(g, \frac{x}{m_2 a^{-1}}\right).$$

By Lemma 4.2 of [**1**] we have

$$V(g, x/(m_2a^{-1})) = \omega_0\omega_1(\psi_0')^2(m_2/am_2^*)\varepsilon(c[m_2]m_2m_2^*) \times \varepsilon^{-1}(c[m_2]a)\left(\frac{c[2]m_0(-1)^{(f_4-1)/2}}{m_2m_2^*a}\right)\left(\frac{x}{ac[m_2]s}\right).$$

Note that

$$(3.3)\quad \sum_{\substack{x=1\\(x,m_2/a)=1}}^{m_2/a} \bar{\psi}_2(ax-\lambda)\left(\frac{x}{ac[m_2]s}\right) = \sum_{\substack{x=1\\(x,m_2/a)=1}}^{m_2^*} \left(\frac{x}{ac[m_2]s}\right)\sum_{j=1}^{m_2/am_2^*} \bar{\psi}_2(ax+jam_2^*-\lambda) = 0,$$

if $a \neq m_2/m_2^*$, and

(3.4)

$$\sum_{\substack{x=1\\(x,m_2^*)=1}}^{m_2^*} \left(\frac{x}{y}\right)\bar{\psi}_2\left(-\lambda+\frac{xm_2}{m_2^*}\right) = \sum_{x=1}^{m_2^*}\left(\frac{x}{y}\right)\bar{\psi}_2\left(-\lambda+\frac{xm_2}{m_2^*}\right)\sum_{n|(x,m_2^*y^{-1})}\mu(n)$$

$$= \sum_{n|m_2^*y^{-1}}\mu(n)\left(\frac{n}{y}\right)\sum_{x=1}^{y}\left(\frac{x}{y}\right)\sum_{j=1}^{m_2^*/ny}\bar{\psi}_2\left(-\lambda+\frac{xnm_2}{m_2^*}+\frac{jynm_2}{m_2^*}\right)$$

$$= \bar{\psi}_2(-d)\chi_y'(-dm_0\xi m_2^*y^{-1})\psi_2(m_0\xi)\mu(m_2^*y^{-1}) \times \sum_{x=1}^{y}\left(\frac{x}{y}\right)\bar{\psi}_2\left(1+\frac{xm_2}{y}\right).$$

Substituting (3.3) and (3.4) into (3.2) we obtain

$$(3.5)\quad V(q(\psi,bc),d/c) = V(h,d/\sigma) = \bar{\omega}_2\omega_4\psi_0'\psi_2(d)\chi_d(c^*m_2m_2^*)\sigma^{3/2}(m_2^*)^{-3/2}y^{1/2}\rho_1 - \bar{\omega}_2\omega_4\psi_0'\psi_2\chi_y'(d)\chi_d(c^*m_2m_2^*)\sigma^{3/2}(m_2^*)^{-1/2}y^{-1/2}\rho_2,$$

where ρ_1, ρ_2 are constants with $|\rho_1|=|\rho_2|=1$. Here we use the Lemma in §1 and the fact that $\phi_1 = \chi_l$ or χ_{2l} with $l = m_2^*/y$. If the conductor of $\psi_2\chi_y'$ is less than m_2, the appearance of the second term in (3.5) does not influence the nonsingularity of the matrix A defined in §1, so we take $F(\psi, c) = q(\psi\chi_i', c)$ where $i = m_2m_2^*$. Note that the conductor of $\psi_2\chi_i'$ is still m_2. If the conductor of $\psi_2\chi_y'$ is equal to m_2, then

$$V(q(\psi\chi_y', c), d/c) = \bar{\omega}_2\omega_4\psi_0'\psi_2\chi_y'(d)\chi_d(c^*m_2m_2^*)\sigma^{3/2}(m_2^*)^{-3/2}y^{1/2}\rho_1 - \bar{\omega}_2\omega_4\psi_0'\psi_2(d)\chi_d(c^*m_2m_2^*)\sigma^{3/2}(m_2^*)^{-1/2}y^{-1/2}\rho_2,$$

and $V(q(\psi\chi_y', c), d/\beta) = 0$ $(\beta\,|\,N, c \prec \beta)$. If $y \neq m_2^*$ we can find a form $F(\psi, c)$ as a linear combination of $q(\psi\chi_i', c)$ and $q(\psi\chi_{yi}', c)$ such that $F(\psi, c)$ satisfies (2.8) and (2.9).

Now we suppose $u_2'v_1'v_2' = 1$, $y = m_2^*$ and the conductor of $\psi_2\chi_y'$ is equal to m_2. Then $\omega_1\omega_2\omega_4 = \psi_2^2\chi_\xi'$ where $\xi = cf_1u_1'/2^{c(2)}m_2$, and ψ_2 and ψ_2^2 have the same conductor. This means there exists an exceptional pair $(K_0K_2, \tilde{c})$ such that

$$\psi_2 = K_2\chi_n' \quad (n \mid m_2^*), \qquad c/2^{c(2)} = \tilde{c}/2^{\tilde{c}(2)}. \tag{3.6}$$

Take $r/r^{(2)} = m_2$, $t/t^{(2)} = \xi$ in (1.3), where the conductor of K_0 is a power of 2 and the conductor of K_2 is odd. By our choice of K in §1, K_2 is uniquely determined. In the following we fix $\tilde{c}/2^{\tilde{c}(2)}$ and K_2.

Case 7. $c(2) \leqslant n(2) - 3$ if $n(2) \geqslant 4$ or $c(2) = 0$ if $n(2) = 2, 3$;

$$\psi_2 = K_2\chi_n' \, (n \mid m_2^*, n \neq 1),$$

c satisfies (3.6).

We take $\phi_2 = \psi_0'\overline{K}_2\chi_\lambda'$ $(\lambda = m_2u_1')$, $\sigma = mu_1'$, $\eta = c/\sigma$ and $g = G(\phi_1, n, bN_2)$. Similarly to Case 6 we can show that

$$\begin{aligned} V(q(\psi, c), d/c) &= \overline{\omega}_2\omega_4\psi_0'\psi_2(d)\chi_d(c^*)\sigma^{3/2}n^{-1}\rho_1 \\ &\quad - \overline{\omega}_2\omega_4\psi_0'K_2(d)\chi_d(c^*)\sigma^{3/2}n^{-1}\rho_2, \end{aligned} \tag{3.7}$$

and $V(q(\psi, c), d/\beta) = 0$ $(\beta \mid N, c \prec \beta)$. Then we take $F(\psi, c) = q(\psi, c)$.

Up to this point we have completed the proof of our Theorem for the case $n(2) = 2,3$. If $n(2) \geqslant 4$ we need the following further consideration. In Cases 8 and 9, c always satisfies (3.6), $\phi_2 = \psi_0'\overline{K}_2\chi_\lambda'$ $(\lambda = m_2u_1')$, $\sigma = mu_1'$, $\eta = c/\sigma$. We can see that $\phi_1 = \chi_{f_1}$ or χ_{2f_1}.

Case 8. $n(2) \geqslant 4$, $c(2) \leqslant n(2) - 3$; $\psi_2 = K_2$, c satisfies (3.6), $\phi_1 = \chi_{2f_1}$ and $m(2) > 0$. Note that if $m(2) = 0$ in this case, (ψ, c) becomes an exceptional pair (take $r(2) = 0$, $t(2) = c(2) + 1$ in (1.3)). Now take $g = G(\chi_2, 2, 8N_2)$. Then we can show q satisfies (2.8) and

$$\begin{aligned} V(q, d/c) &= \sigma^{3/2}\phi_2(-d)V(g, 1) + (\sigma/2)^{3/2}\phi_2(-d + \sigma/2) \\ &= -2^{-1/2}\overline{\omega}_2\omega_4\psi_0'\psi_2(-d)\chi_{-d}(c^*)\sigma^{3/2}. \end{aligned}$$

Here we use $V(g, 1) = -2^{-3/2}$ from (6.5) of [**1**], and $\psi_0'(-d + \sigma/2) = -\psi_0'(-d)$. Therefore $F(\psi, c) = g(\psi, c)$ satisfies (2.8) and (2.9).

Case 9. $n(2) \geqslant 4$, $c(2) \leqslant n(2) - 3$; $\psi_2 = K_2$, c satisfies (3.6); $\phi_1 = \chi_{f_1}$, (ψ, c) is nonexceptional.

Note that if $m(2) = c(2) = 0$, then (ψ, c) is exceptional (take $r(2) = t(2) = 0$ in (1.3)). Suppose $m(2) > 0$ at first. The condition $\phi_1 = \chi_{f_1}$ implies $\omega_0 = \psi_0^2\chi_\lambda$ $(\lambda = 2^{c(2)-m(2)}\omega_0(-1))$. Taking $r(2) = m(2) - 1$, $t(2) = c(2) - m(2)$ in (1.3), we see there exists an exceptional pair (K_0K_2, c) such that $K_0^2 = \psi_0^2$. Since (ψ, c) is nonexceptional, we have $\psi_2 = K_0\chi_{-1}$, $K_0\chi_2$ or $K_0\chi_{-2}$. We take

$$\begin{aligned} g(K\chi_{-1}, c) &= g(K\chi_2, c) = G(\mathrm{id}, 4, 8N_2), \\ g(K\chi_{-2}, c) &= G(\mathrm{id}, 8, 8N_2). \end{aligned}$$

We can show that $q(K\chi_{-1}, c)$, $q(K\chi_2, c)$, $q(K\chi_{-2}, c)$ satisfy (2.8), and by a calculation we obtain

$$(3.8)\quad V(q(K\chi_{-1}, c), d/c) = -8^{-1}(1+i)\sigma^{3/2}\overline{\omega}_1'\overline{\omega}_2\omega_4(-d)\chi_{-d}(c^*) \times \left(K\chi_{-1}(-d) + iK(-d)\sum_{j=1}^{4}\left(\frac{-1}{j}\right)\phi_2\left(1+\frac{j\sigma}{4}\right)\right),$$

$$(3.9)\quad V(q(K\chi_2, c), d/c) = -8^{-1}(1+i)\sigma^{3/2}\overline{\omega}_1'\overline{\omega}_2\omega_4(-d)\chi_{-d}(c^*) \times \left(K\chi_2(-d) + iK\chi_{-2}(-d)\sum_{j=1}^{4}\left(\frac{-1}{j}\right)\phi_2\left(1+\frac{j\sigma}{4}\right)\right),$$

(3.10)

$$V(q(K\chi_{-2}, c), d/c) = -8^{-1}(1+i)\sigma^{3/2}\overline{\omega}_1'\overline{\omega}_2\omega_4(-d)\chi_{-d}(c^*) \times \left(K(-d) - 2^{-5/2}K\chi_2(-d)\sum_{j=1}^{8}\left(\frac{2}{j}\right)\phi_2\left(1+\frac{j\sigma}{8}\right) + 2^{-5/2}iK\chi_{-2}(-d)\sum_{j=1}^{8}\left(\frac{2}{j}\right)\phi_2\left(1+\frac{j\sigma}{8}\right)\right),$$

where $\overline{\omega}_1' = \overline{\omega}_1$ if $f(2) \geqslant 4$, $\overline{\omega}_1' = 1$ if $f(2) \leqslant 3$. Here we use

$$\varepsilon_k = 2^{-1}\{(1+i) + (1-i)\chi_{-1}(k)\}.$$

If $0 < m(2) \leqslant 3$, then $\psi_0 = \chi_{-1}$, χ_2 or χ_{-2}. We need only take id instead of K_0 above to get the desired form.

Now we suppose $m(2) = 0$, $c(2) > 0$. Take $\phi_2 = K_2\chi_\lambda'$ ($\lambda = m_2u_1'$), $\sigma = m_2u_1'$, $\eta = c/2\sigma$ and $g = G(\chi_2, 2, 8N_2)$. We can also prove that q satisfies (2.7) and (2.8). Finally we take $F(\psi, c) = q(\psi, c)$.

We have already found a form $F(\psi, c)$ for every admissible, but nonexceptional, pair (ψ, c). As mentioned in §1, we can prove that A is not singular by Lemma 7.1 of [**1**]. It is easy to see that the second terms in (3.5), where the conductor of $\psi_2\chi_\nu'$ is less than m_2, and in (3.7) do not influence the nonsingularity of the matrix A. When d runs over $d_1, d_2, \ldots, d_{g(c)}$, where $g(c) = \phi((c, N/c))$, the three vectors (3.8)–(3.10) are linearly independent. Then our theorem is proved. The proof here covers those of §7 in [**1**].

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