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# A PARTIAL ORDER ON THE REGIONS OF $\mathbf{R}^n$ DISSECTED BY HYPERPLANES

BY

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ABSTRACT. We study a partial order on the regions of $\mathbf{R}^n$ dissected by hyperplanes. This includes a computation of the Möbius function and, in some cases, of the homotopy type. Applications are presented to zonotopes, the weak Bruhat order on Weyl groups and acyclic orientations of graphs.

**0. Introduction.** Let $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ be a set of hyperplanes in $\mathbf{R}^n$. Then the components of $\mathbf{R}^n - \bigcup_{H \in \mathcal{H}} H$ form a set $\mathfrak{R}$ of open $n$-cells we will call *regions*. Traditionally $\mathfrak{R}$ has been studied in terms of enumeration, for instance counting the number of regions and the number of intersections of various dimensions among the hyperplanes in $\mathcal{H}$. For a thorough discussion of this problem see Zaslavsky's monograph [**Z1**] and the references therein. See also [**OS** and **Te**].

In this paper we will study $\mathfrak{R}$ from a different perspective. Our concern will be the way in which one can move from one region to another by "walking through" one hyperplane at a time. To analyze this situation we define a partial order on $\mathfrak{R}$ relative to a base region where the cover relations correspond to "walking through" a hyperplane and moving further away from the base.

The paper is organized as follows: In §I we define the partial order on $\mathfrak{R}$ and begin to study its structure. This culminates in Theorem 1.11 where an explicit computation of its Möbius function is performed. In §II we examine the special case where all the hyperplanes go through the origin. In certain cases we are able to compute the homotopy type of these posets. The next three sections are devoted to applications. §III applies our work to a partial order on a zonotope. §IV deals with the weak Bruhat order of a Weyl group as a special case of our posets. As a corollary we show that the homotopy type of the weak Bruhat order is that of a sphere. Finally, §V applies our results to a partial order on the acyclic orientations of a graph. Some of the relationships used in §§III, IV and V appear in an enumerative context in the excellent article of Greene and Zaslavsky [**GZ**].

Most of our poset notation is standard and can be found in [**A**]. When more than one partial order is being used we will sometimes use subscripts to indicate where the relation is, i.e., $x \leqslant_P y$ means that $x$ is less than or equal to $y$ in the poset $P$. By $\Delta(P)$ we mean the simplicial complex of chains (totally ordered sets) in $P$. We will think of $\Delta(P)$ as its geometric realization. By $P^*$ we mean the order dual of $P$, i.e.,

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$x \leqslant y$ in $P^*$ if and only if $x \geqslant y$ in $P$. If a poset has a maximum or minimum element, it will be denoted $\hat{1}$ or $\hat{0}$, respectively. The *interval* $[x, y]$ in $P$ is the set $\{z \mid x \leqslant z \leqslant y\}$ with the induced partial order.

If $x$ and $y$ are points in $\mathbf{R}^n$ then by $\langle x, y\rangle$ we mean the standard inner product. If $B$ is an unbounded region in $\mathbf{R}^n$ then a vector $r$ is called a *ray of B* if for any $b \in B$, $b + \lambda r \in B$ for $\lambda \geqslant 0$. If $R$ is a subset of $\mathbf{R}^n$, then $\bar{R}$ is its topological closure. Finally, by $[n]$ we mean the set $\{1, 2, \ldots, n\}$. Given a set $S$, $S^c$ is the complement of $S$ in the appropriate universal set.

G. Purdy has independently investigated the same poset on regions in $\mathbf{R}^2$ in his work on a classical problem in geometry due to Erdös [**Pu**]. Since the first version of our paper, Cordovil [**Co**] has generalized some of our results to oriented matroids.

**I. Structure theorems and the Möbius function.** Let $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ be a set of hyperplanes in $\mathbf{R}^n$, where $H_i = \{x \in \mathbf{R}^n \mid \langle z_i, x\rangle = \alpha_i\}$ and $\{z_1, z_2, \ldots, z_k\}$ span $\mathbf{R}^n$. Then $\mathbf{R}^n - \bigcup_{i=1}^k H_i$, which we will denote $\mathbf{R}^n - \bigcup \mathcal{H}$, is divided into a set $\mathfrak{R}$ of open connected $n$-cells, called *regions*. Let $H_i^+$ be the closed half space $H_i^+ = \{x \in \mathbf{R}^n \mid \langle z_i, x\rangle \geqslant \alpha_i\}$ and similarly let $H_i^- = \{x \in \mathbf{R}^n \mid \langle z_i, x\rangle \leqslant \alpha_i\}$.

For $R \in \mathfrak{R}$, $\bar{R}$ is a polytope. The set of *boundary hyperplanes* of $R$, $\mathfrak{B}(R)$, is defined by $\mathfrak{B}(R) = \{H \in \mathcal{H} \mid H \cap \bar{R}$ is of dimension $n-1\}$. Each hyperplane $H \in \mathfrak{B}(R)$ defines a *facet* of $\bar{R}$, $F'_H$, by $F'_H = H \cap \bar{R}$. The set of facets of $\bar{R}$ is denoted $\mathfrak{F}'(R)$.

A *face* of $\bar{R}$ is any intersection of facets, and the set of faces of $\bar{R}$ will be denoted $\mathfrak{F}(R)$. We will partially order $\mathfrak{F}(R)$ by containment. Note that $\bar{R}$ itself is a face of $\bar{R}$, namely the empty intersection. We will call $F$ a *face of* $\mathcal{H}$ if $F$ is a face of some $\bar{R}$, $R \in \mathfrak{R}$.

For $F \in \mathfrak{F}(R)$, let $\mathcal{H}(F) = \{H \in \mathcal{H} \mid F \subseteq H\}$. Define

$$\varepsilon_i = \begin{cases} 1 & \text{if } F \subseteq H_i^+, \\ -1 & \text{if } F \subseteq H_i^-. \end{cases}$$

Then we can write $F$ explicitly as

$$F = \left\{x \in \mathbf{R}^n \mid \langle x, z_i\rangle = \alpha_i \text{ for } H_i \in \mathcal{H}(F) \text{ and } \langle x, \varepsilon_j z_j\rangle \geqslant \varepsilon_j \alpha_j \text{ for } H_j \notin \mathcal{H}(F)\right\}.$$

The *relative interior* of $F$, relint($F$), is given by

$$\operatorname{relint}(F) = \left\{x \in \mathbf{R}^n \mid \langle x, z_i\rangle = \alpha_i \text{ for } H_i \in \mathcal{H}(F) \right.$$
$$\left. \text{and } \langle x, \varepsilon_j z_j\rangle > \varepsilon_j \alpha_j \text{ for } H_j \notin \mathcal{H}(F)\right\}.$$

Let $B$ be a fixed region. Without loss of generality we assume

$$B = \left\{x \in \mathbf{R}^n \mid \langle x, z_i\rangle > \alpha_i \text{ for } 1 \leqslant i \leqslant k\right\}.$$

For any region $R \in \mathfrak{R}$ consider the set $S(R)$ defined by $S(R) = \{H_i \in \mathcal{H} \mid \langle x, z_i\rangle < \alpha_i$ for $x \in R\}$. That is, $S(R)$ is the set of hyperplanes which separate $R$ from $B$. The function $S$ partially orders $\mathfrak{R}$ by

$$R_1 \leqslant R_2 \quad \text{if and only if } S(R_1) \subseteq S(R_2).$$

Let $P(\mathcal{H}, B)$ be the set $\mathfrak{R}$ with this partial ordering. In Figure 1 we present an example. The hyperplanes are labeled $a, b, c \ldots$ The base region $B$ is shaded and the other regions are labeled by the set $S(R)$.

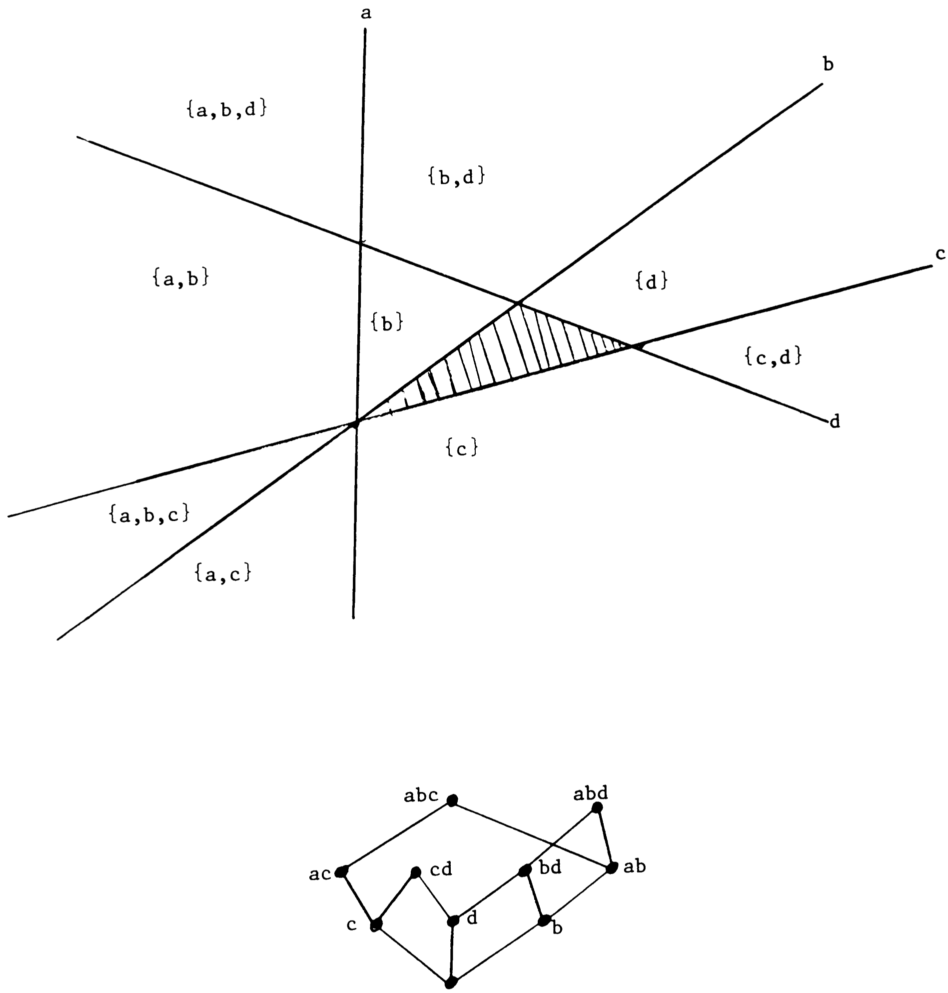



FIGURE 1

Unless noted otherwise the sets $\mathfrak{B}(B)$, $\mathfrak{F}(B)$ and $\mathfrak{F}'(B)$ will be denoted by $\mathfrak{B}$, $\mathfrak{F}$ and $\mathfrak{F}'$, respectively.

PROPOSITION 1.1. *$P(\mathcal{H}, B)$ is ranked by* $\mathrm{rk}(R) = |S(R)|$.

PROOF. This is geometrically clear based on the observation that if $S(R_1) \subset S(R_2)$ then there exists a hyperplane $H \in \mathcal{H}$ such that $H \in S(R_2) - S(R_1)$ and $H \in \mathfrak{B}(R_1)$. Hence there exists a region $R$ such that $S(R) = S(R_1) \cup \{H\}$ and so $R_1 < R \leqslant R_2$. We leave the details to the reader. $\square$

Our next goal is to explicitly compute the Möbius function $\mu$ for $P(\mathcal{H}, B)$. For the definition and the significance of the Möbius function see [**R** or **A**]. Since $P(\mathcal{H}, B)$ is not a lattice in general, the standard techniques for the computation of $\mu$ do not apply. We begin by identifying certain important regions.

LEMMA 1.2. *If $F \in \mathfrak{F}$ and $F \neq \varnothing$, then there exists a unique region $R(F) \in \mathfrak{R}$ such that $S(R(F)) = \mathcal{H}(F)$.*

PROOF. We will prove the existence of $R(F)$. Its uniqueness is clear. Let

$$\gamma \in \operatorname{relint}(F)$$

and $b \in B$. Define $y = b - \gamma$. Then we know that for all $l$, $1 \leqslant l \leqslant k$,

$$(*) \qquad \langle z_l, b\rangle = \langle z_l, \gamma + y\rangle = \langle z_l, \gamma\rangle + \langle z_l, y\rangle > \alpha_l.$$

For all $H_i \in \mathcal{H}(F)$ we have $\langle z_i, \gamma\rangle = \alpha_i$ and for $H_j \notin \mathcal{H}(F)$ we have $\langle z_j, \gamma\rangle > \alpha_j$. Hence, from $(*)$, for $H_i \in \mathcal{H}(F)$ we have $\alpha_i + \langle z_i, y\rangle > \alpha_i$ and so $\langle z_i, y\rangle > 0$ for $H_i \in \mathcal{H}(F)$. Consider $\langle z_i, \gamma - \varepsilon y\rangle$ for $\varepsilon > 0$. We see that

$$\langle z_i, \gamma - \varepsilon y\rangle = \langle z_i, \gamma\rangle - \varepsilon\langle z_i, y\rangle = \alpha_i - \varepsilon\langle z_i, y\rangle < \alpha_i$$

when $H_i \in \mathcal{H}(F)$. Choose $\varepsilon$ suitably small so that $\langle z_j, \gamma - \varepsilon y\rangle > \alpha_j$ for all $H_j \notin \mathcal{H}(F)$. Then the region $R$ containing $\gamma - \varepsilon y$ is the desired one. $\square$

LEMMA 1.3. *The mapping $F \to R(F)$ is an order-reversing injection into $P(\mathcal{H}, B)$ of the nonempty faces $F \in \mathfrak{F}$.*

PROOF. If $F_1 \leqslant F_2$ in $\mathfrak{F}$ then $\mathcal{H}(F_1) \supseteq \mathcal{H}(F_2)$ and since $S(R(F)) = \mathcal{H}(F)$ we have $R(F_1) \geqslant R(F_2)$. $\square$

A region $R$ of the form $R(F)$ where $F \in \mathfrak{F}$ will be called a *facial region*. Facial regions are critical since they are the only regions to have a nonzero value of the Möbius function.

The existence of a facial region $R(\varnothing)$ (such that $S(R(\varnothing)) = \mathcal{H}$) is dependent upon the choice of $B$. The only relevant fact for this paper is that if $B$ is bounded then $R(\varnothing)$ does not exist.

For $F \in \mathfrak{F}$ define $\mathfrak{B}(F)$ by $\mathfrak{B}(F) = \mathfrak{B} \cap \mathcal{H}(F)$.

LEMMA 1.4. *Let $F \in \mathfrak{F}$ and $F \neq \varnothing$. If $R \in \mathfrak{R}$ and $S(R) \supseteq \mathfrak{B}(F)$ then $R \geqslant R(F)$.*

PROOF. We will show that $S(R) \supseteq S(R(F))$. Recall that $S(R(F)) = \mathcal{H}(F)$. If $H_t \in \mathfrak{B}(F) \subseteq \mathcal{H}(F)$ then by hypothesis $H_t \in S(R)$.

Otherwise $H_t \in \mathcal{H}(F)$ and $H_t \notin \mathfrak{B}$. Since $H_t$ is not a boundary hyperplane of $\bar{B}$ we know that the inequalities

$$(*) \qquad \langle x, z_i\rangle > \alpha_i \quad \text{for } H_i \in \mathfrak{B}(F)$$

imply that $\langle x, z_t\rangle > \alpha_t$. Moreover, there exists a $\gamma$ such that $\langle z_i, \gamma\rangle = \alpha_i$ for $H_i \in \mathfrak{B}(F)$, and also $\langle z_t, \gamma\rangle = \alpha_t$, since the $H_i$ and $H_t$ all contain $F$. If $H_t \notin S(R)$ then for some $y$ in $R$ we have $\langle z_i, y\rangle < \alpha_i$ for $H_i \in \mathfrak{B}(F)$ and $\langle z_t, y\rangle > \alpha_t$. Then for all $H_i \in \mathfrak{B}(F)$, $\langle z_i, 2\gamma - y\rangle = 2\alpha_i - \langle z_i, y\rangle > \alpha_i$ and

$$\langle z_t, 2\gamma - y\rangle = 2\alpha_t - \langle z_t, y\rangle < \alpha_t,$$

which contradicts $(*)$. $\square$

COROLLARY 1.5. *Suppose $R \in \mathfrak{R}$ and $F \in \mathfrak{F}$, $F \neq \varnothing$. Then $R(F) \in [B, R]$ if and only if $\mathfrak{B}(F) \subseteq S(R) \cap \mathfrak{B}$.*

PROOF. If $\mathfrak{B}(F) \subseteq S(R) \cap \mathfrak{B}$ then $\mathfrak{B}(F) \subseteq S(R)$ since $\mathfrak{B}(F) \subseteq \mathfrak{B}$. By Lemma 1.4, $\mathfrak{B}(F) \subseteq S(R)$ implies that $R \geqslant R(F)$.

On the other hand, if $R(F) \leqslant R$ then $\mathcal{H}(F) = S(R(F)) \subseteq S(R)$ and so

$$\mathfrak{B}(F) = \mathcal{H}(F) \cap \mathfrak{B} \subseteq S(R) \cap \mathfrak{B}. \quad \square$$

An *atom* of a poset is an element of rank 1. If $R$ is an atom of $P(\mathcal{H}; B)$ then $S(R) = \{H\}$ for some $H \in \mathfrak{B}$, and conversely.

COROLLARY 1.6. *Suppose $F \in \mathfrak{F}$ and $F \neq \varnothing$. Then the set $A$ of all atoms $R(H)$ such that $H \in \mathfrak{B}(F)$ has $R(F)$ as its unique least upper bound.*

PROOF. It is clear that $R(F)$ is an upper bound for $A$. Lemma 1.4 shows that $R(F)$ is less than any other upper bound. □

A ranked poset $Q$ is called *Eulerian* if for all $p$ and $q$ in $Q$, $p \leqslant q$, we have $\mu_Q(p, q) = (-1)^{rk\, q - rk\, p}$. It is well known that $\mathfrak{F}$ is Eulerian, a fact which follows from Euler's formula. See [**Gru**, Chapter 8].

THEOREM 1.7 (STANLEY [**St**, PROPOSITION 2.2]). *Suppose $P$ is an Eulerian poset of rank $n$ and $Q$ is a subposet of $P$ containing $\hat{0}$ and $\hat{1}$. Set $\overline{Q} = (P \backslash Q) \cup \{\hat{0}, \hat{1}\}$. Then $\mu_Q(\hat{0}, \hat{1}) = (-1)^{n-1}\mu_{\overline{Q}}(\hat{0}, \hat{1})$.*

Recall that $\mathfrak{F}'$ is the set of facets of $\overline{B}$. For $F \in \mathfrak{F}$ define

$$\mathfrak{F}'(F) = \{F' \in \mathfrak{F}' \mid F \subseteq F'\}.$$

That is, $\mathfrak{F}'(F)$ is the set of all facets of $\overline{B}$ which contain $F$. Given a subset $\mathcal{C}$ of $\mathfrak{F}'$ let $\mathfrak{F}(\mathcal{C}) = \{F \in \mathfrak{F} \mid \mathfrak{F}'(F) \subseteq \mathcal{C}\}$. That is, $\mathfrak{F}(\mathcal{C})$ is the set of all faces $F$ of $\overline{B}$ such that every facet containing $F$ is in $\mathcal{C}$. Partially order $\mathfrak{F}(\mathcal{C})$ by containment. Let $\hat{\mathfrak{F}}(\mathcal{C}) = \mathfrak{F}(\mathcal{C}) \cup \hat{0}$.

THEOREM 1.8 (BRUGGESSER AND MANI [**BM**]). *Suppose $B$ is bounded. Let $R$ be a region and $\mathfrak{T} = S(R) \cap \mathfrak{B}$. Let $\mathcal{C} = \{F'_H \in \mathfrak{F}' \mid H \in \mathfrak{T}\}$. Then $Q = \mathfrak{F} - \mathfrak{F}(\mathcal{C})$ is the complex of an $(n-1)$-cell.*

(REMARK. In the notation of [**BM**], $\mathfrak{F} - \mathfrak{F}(\mathcal{C})$ is the complex $\mathfrak{U}(\overline{B}, r)$ where $r$ is some point in $R$.)

THEOREM 1.9. *In the situation of Theorem* 1.8, *with $\hat{Q} = Q \cup \hat{1}$, we have*

$$\mu_{\hat{Q}}(\hat{0}, \hat{1}) = 0.$$

PROOF. Since $Q$ is the complex of an $(n-1)$-cell, $\Delta(Q - \{\varnothing\})$ is the simplicial complex of the barycentric subdivision of $Q$. See for instance [**BB**, Example 1.3]. Thus $\chi(\Delta(Q - \{\varnothing\})) = 1$ where $\chi$ is the Euler characteristic. It is well known that $\mu(\hat{0}, \hat{1}) = \chi - 1$ [**R**, Proposition 6] and thus we conclude that $\mu_{\hat{Q}}(\hat{0}, \hat{1}) = 0$. □

LEMMA 1.10. *In the situation of Theorem* 1.8, $\mu(\hat{0}, \hat{1}) = 0$ *in* $\hat{\mathfrak{F}}(\mathcal{C})$.

PROOF. By Lemma 1.9, $\mu(\hat{0}, \hat{1}) = 0$ in $\hat{Q}$. If $P = \mathfrak{F}$ in Theorem 1.7, then

$$\hat{\mathfrak{F}}(\mathcal{C}) = (P \backslash \hat{Q}) \cup \{\hat{0}, \hat{1}\}.$$

By Theorem 1.7, $\mu_{\hat{\mathfrak{F}}}(\hat{0}, \hat{1}) = (-1)^{n-1}\mu_{\hat{Q}}(\hat{0}, \hat{1}) = 0$. □

We can now state and prove the main theorem of this section.

THEOREM 1.11. *The Möbius function on $P(\mathcal{H}, B)$ is given by*

$$\mu(B, R) = \begin{cases} (-1)^{n-k} & \text{if } R = R(F) \text{ where } F \text{ is of dimension } k \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. The proof is in two parts, the first for bounded $B$ and the second for unbounded $B$.

*Part* I. Suppose $B$ is a bounded region. We proceed by induction on the rank of $R$. The theorem is certainly true for $\mathrm{rk}(R) = 0$. There are two cases to consider, when $R$ is facial and when it is not facial.

*Case* 1. Suppose $R$ is a facial region so that $R = R(F)$. Let

$$[B, R]_\mu = \{R' \in \mathfrak{R} \mid B \leqslant R' < R \text{ and } \mu(B, R') \neq 0\} \cup \{R\}.$$

By induction, if $R' \in [B, R]_\mu$ and $R' \neq R$ then $R' = R(F')$ for some $F'$, and by assumption $R = R(F)$. By Lemma 1.3, the map $\phi$ defined by $\phi(F) = R(F)$ is an anti-isomorphism between $[B, R]_\mu$ and the interval $[F, \hat{1}]$ in $\mathfrak{F}$.

Since $\mathfrak{F}$ is Eulerian and $\mu_P(B, R(F')) = \mu_{\mathfrak{F}}(F', \hat{1})$ by induction for $F' < F$, we conclude that $\mu(B, R) = \mu_{\mathfrak{F}}(F, \hat{1}) = (-1)^{n-k}$.

*Case* 2. Suppose $R$ is not a facial region. Define $\mathfrak{T}$ and $\mathcal{C}$ as in Theorem 1.8. From Corollary 1.5 we know that $R(F) \in [B, R]$ if and only if $\mathfrak{B}(F) \subseteq \mathfrak{T}$, which is true if and only if $\mathfrak{F}(F) \subseteq \mathcal{C}$, which happens if and only if $F \in \hat{\mathfrak{F}}(\mathcal{C})$. So the map $\phi$ from $[B, R]$ to $\hat{\mathfrak{F}}(\mathcal{C})$ defined by $\phi(B) = \hat{1}$, $\phi(R) = \hat{0}$ and $\phi(R(F)) = F$ is an anti-isomorphism. By induction $\mu(B, R(F)) = \mu_{\hat{\mathfrak{F}}(\mathcal{C})}(F, \hat{1})$ for all $F \in \hat{\mathfrak{F}}(\mathcal{C})$, $F \neq \varnothing$. So we conclude that $\mu(B, R) = \mu_{\hat{\mathfrak{F}}}(\hat{0}, \hat{1})$, which is 0 by Lemma 1.10.

Before proceeding to Part II we require

LEMMA 1.12. *Let $B$ be an unbounded region in $\mathbf{R}^n - \bigcup \mathcal{H}$. Then there exists a hyperplane $H$ and a bounded region $B'$ in $\mathbf{R}^n - \bigcup(\mathcal{H} \cup \{H\})$ such that $P(\mathcal{H}, B)$ is isomorphic to an order ideal of $P(\mathcal{H} \cup \{H\}, B')$.*

PROOF OF LEMMA 1.12. Define $z'$ to be the unit vector in the direction of $\sum_{H_i \in \mathfrak{B}} z_i$. Let $S$ be a sphere around the origin of radius $s$ so that $S$ contains every bounded region and some point from every region. Let $H' = \{x \mid \langle z', x\rangle = 2s\}$. Let $B'$ be the region $B' = \{x \mid \langle z_i, x\rangle > \alpha_i$ for $H_i \in \mathfrak{B}$ and $\langle z', x\rangle < 2s\}$. Notice that $B'$ is nonempty since every point in $S$ satisfies the second inequality and the points in $B \cap S$ satisfy the first set of inequalities.

We wish to show that $B'$ is bounded. If $r$ is a ray of $B'$ then it also is a ray of $B$. Hence $\langle z_i, r\rangle \geqslant 0$ for all $H_i \in \mathfrak{B}$ and $\langle z_j, r\rangle > 0$ for some $H_j \in \mathfrak{B}$. Thus for every point $x \in B'$ there is some $\lambda \geqslant 0$ such that $\langle z', x + \lambda r\rangle > 2s$, which contradicts that $r$ is a ray of $B'$. Hence $B'$ must be bounded.

Consider the map from $P(\mathcal{H}, B)$ to $P(\mathcal{H} \cup \{H\}, B')$ defined by $R \to R'$ if $S(R) = S(R')$. This map is well defined since every region $R$ is represented by a point $x$ in $S$ and hence $x$ satisfies the inequality $\langle z', x\rangle < 2s$. The map is an isomorphism of $P(\mathcal{H}, B)$ to an order ideal of $P(\mathcal{H} \cup \{H\}, B')$.

We now return to Part II of the proof of Theorem 1.11.

Construct $H$ as in Lemma 1.12. Then $P(\mathcal{H}, B)$ is an order ideal in

$$P' = P(\mathcal{H} \cup \{H\}, B').$$

For $R \in P$ let $R'$ be its image in $P'$. It is easy to see that $R$ is a facial region of $P$ if and only if $R'$ is facial in $P'$. Since $P$ is an order ideal in $P'$, $\mu_P(B, R) = \mu_{P'}(B', R')$. Applying Part I to the poset $P'$ finishes the proof. □

**II. Central arrangements.** For the most part, the rest of this paper will be concerned with *central arrangements*, i.e., for all $H \in \mathcal{H}$, $H$ contains the origin. Thus we have $H_i = \{x \in \mathbf{R}^n \mid \langle z_i, x \rangle = 0\}$ for all $H_i \in \mathcal{H}$.

PROPOSITION 2.1. *Let $\mathcal{H}$ be a central arrangement. Then $P(\mathcal{H}, B)$ is selfdual.*

PROOF. Suppose $R$ is a region in $\mathfrak{R}$. Then $-R = \{x \mid -x \in R\}$ is also in $\mathfrak{R}$. Moreover $S(-R) = S(R)^c$. So the map

$$\phi \colon R \to -R$$

is an isomorphism from $P(\mathcal{H}, B)$ to $P^*(\mathcal{H}, B)$. $\square$

Let $W_k$ be the number of elements at rank $k$ in $P(\mathcal{H}, B)$. The $W_k$ are called the Whitney numbers of the second kind.

THEOREM 2.2. *If $\mathcal{H}$ is central, then in $P(\mathcal{H}, B)$ we have $W_l = W_{n-l}$ for all $l$ and the numbers of maximal chains in $P(\mathcal{H}, B)$ is even.*

PROOF. That $W_l = W_{n-l}$ follows immediately from the selfduality of $P(\mathcal{H}, B)$. The map $\phi\colon P(\mathcal{H}, B) \to P^*(\mathcal{H}, B)$ from Proposition 2.1 has the property that $\phi(R)$ is not comparable to $R$; hence, no maximal chain gets mapped to itself. Since $\phi^2$ is the identity map the maximal chains are paired. Thus there are an even number of them. $\square$

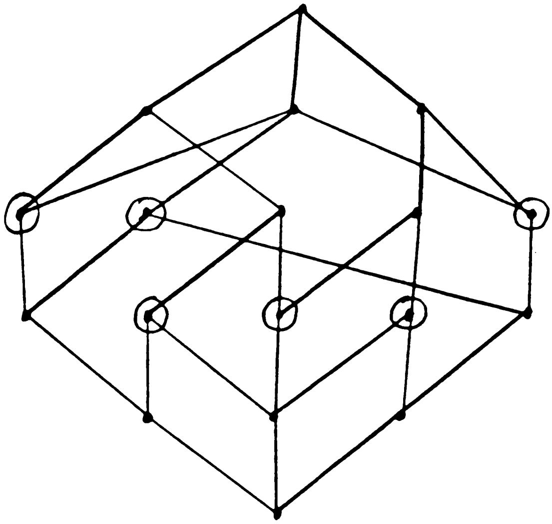
One might ask if $\{W_k\}$ is a unimodal sequence. This is not the case in general as the following example of J. Lawrence shows **[La]**: For $1 \leqslant i \leqslant 9$ let $z_i = (1, i, i^2)$ and let $z_{10} = (1, 2, 11/2)$. Then one can check that the Whitney numbers of $P(\mathcal{H}, B)$ are $\{1, 9, 9, 11, 11, 10, 11, 11, 9, 9, 1\}$. The poset $P(\mathcal{H}, B)$ need not be Sperner. In Figure 2 we show such an example.

In this section we will concern ourselves particularly with the situation where $B$ is bounded by a set of hyperplanes whose normals are linearly independent. Call such a region *basic*. If $B$ is basic then $\bar{B}$ is an infinite cone over an $(n-1)$-simplex. Our goal is to establish the homotopy type of $\Delta(P(\mathcal{H}, B) - \{B, -B\})$ in the case where $B$ is basic. If $B$ is not basic then we do not know what the homotopy type is.

LEMMA 2.3. *Suppose $B$ is a basic region. Then for each subset $A$ of the atoms of $P(\mathcal{H}, B)$ there exists a face $F \in \mathfrak{F}$ such that $R(F)$ is the unique least upper bound of $A$. Moreover, if $A$ is a proper subset of the atoms then the least upper bound is in $P(\mathcal{H}, B) - \{-B\}$.*

PROOF. If $B$ is basic then $\mathfrak{F}$ is isomorphic to the lattice of faces of a simplex. Hence every subset of the facets intersects in a face different for each set. Corollary 1.6 then implies the theorem. Since $\{0\}$ is the minimum nonempty face and $R(\{0\}) = -B$, $R(F) < -B$ for all $F \neq \{0\}$. $\square$

We recall the following definitions from **[Bj1]**. A subset $I$ of a poset $P$ is said to be *initial* if for every $p \in P$ there is an $i \in I$ such that $p \geqslant i$. Call $I$ *join-coherent* if whenever $T$ is a nonempty subset of $I$ which has an upper bound in $P$ then $T$ has a unique least upper bound in $P$. Define a simplicial complex $\Phi(P; I)$ on the set $I$ by taking as simplices those nonempty subsets of $I$ which have an upper bound in $P$.



$z_1 = (1,-1,0)$

$z_2 = (1,0,0)$ $\qquad B = \{x \mid \langle z_i, x\rangle > 0,\ i = 1,\dots,5\}$

$z_3 = (0,1,-1)$

$z_4 = (0,1,0)$

$z_5 = (0,0,1)$

The largest antichain is circled.

FIGURE 2

THEOREM 2.4 (BJÖRNER [**Bj1**, THEOREM 2.1]). *Let $I$ be a join-coherent initial subset of a poset $P$. Then $\Delta(P)$ and $\Phi(P; I)$ have the same homotopy type.*

THEOREM 2.5. *Let $\mathcal{H}$ be a central arrangement in $\mathbf{R}^n$ and $B$ a basic region. Then $\Delta(P(\mathcal{H}, B) - \{B, -B\})$ has the homotopy type of the $(n-2)$-sphere.*

PROOF. Let $\mathcal{A}$ be the set of atoms of $P(\mathcal{H}, B)$. By Lemma 2.3 they form a join-coherent initial subset in $P(\mathcal{H}, B) - \{B\}$. Moreover, since the upper bounds are all distinct, $\Phi(P - \{B, -B\}, \mathcal{A})$ is the simplicial complex of faces of an $(n-1)$-simplex and hence has the homotopy type of an $(n-2)$-sphere. $\square$

**III. Zonotopes.** Let $\{z_1, z_2, \dots, z_k\}$ be points in $\mathbf{R}^n$, which we will assume span $\mathbf{R}^n$. Suppose that none of the $z_i$'s is a multiple of any other. We define a *zonotope* $Z$ to be

$$Z = \left\{ x \in \mathbf{R}^n \middle| x = \sum_{i=1}^{k} \lambda_i z_i \text{ where } |\lambda_i| \leqslant 1 \text{ for all } i \right\}.$$

It is easy to check that $Z$ is a convex polytope. Our standard references on zonotopes are [**M** and **Sh**].

Let $\mathcal{H} = \{H_1, H_2, \ldots, H_k\}$ be the central arrangement of hyperplanes such that $H_i = \{x \in \mathbf{R}^n \mid \langle x, z_i \rangle = 0\}$. If $G$ is a face of $\mathcal{H}$, and $H_i \notin \mathcal{H}(G)$, define

$$\varepsilon_i = \begin{cases} 1 & \text{if } G \subseteq H_i^+, \\ -1 & \text{if } G \subseteq H_i^-. \end{cases}$$

Then let

$$\tau(G) = \left\{ x \in \mathbf{R}^n \middle| x = \sum_{H_i \in \mathcal{H}(G)} \lambda_i z_i + \sum_{H_j \notin \mathcal{H}(G)} \varepsilon_j z_j \text{ where all } |\lambda_i| \leqslant 1 \right\}.$$

LEMMA 3.1. *The map $\tau$ is an order-reversing bijection between the faces of $\mathcal{H}$ and the nonempty faces of $Z$.*

PROOF. We will show that for each face $G$ of the arrangement, $\tau(G)$ is a face of the zonotope. In [**M**, p. 92] it is shown that $\tau^{-1}$ exists, which completes the proof.

Let $G$ be a face of the arrangement $\mathcal{H}$. Then $G$ has the form

$$G = \left\{ x \in \mathbf{R}^n \mid \langle x, z_i \rangle = 0 \text{ for } H_i \in \mathcal{H}(G) \text{ and } \langle x, \varepsilon_j z_j \rangle \geqslant 0 \text{ for } H_j \notin \mathcal{H}(G) \right\}.$$

We wish to show that $\tau(G)$ is a face of $Z$. Choose $r \in \operatorname{relint}(G)$ and $t \in \tau(G)$. Define $\alpha = \langle t, r \rangle = \sum_{H_j \notin \mathcal{H}(G)} \varepsilon_j \langle z_j, r \rangle > 0$. Notice that $\alpha$ is independent of the choice of $t$. Let $H_{r,\alpha} = \{x \in \mathbf{R}^n \mid \langle x, r \rangle = \alpha\}$. It is clear that $H_{r,\alpha} \supseteq \tau(G)$. Moreover, for any point $w \in Z$, $w = \sum_{H_k \in \mathcal{H}} \beta_k z_k$ where $|\beta_k| \leqslant 1$, we have

$$\langle w, r \rangle = \sum_{H_k \in \mathcal{H}} \beta_k \langle z_k, r \rangle = \sum_{H_j \notin \mathcal{H}(G)} \beta_j \langle z_j, r \rangle \leqslant \sum_{H_j \notin \mathcal{H}(G)} \varepsilon_j \langle z, r \rangle = \alpha$$

with strict inequality unless $\beta_j = \varepsilon_j$ for all $H_j \notin \mathcal{H}(G)$. Hence all of $Z$ is in $H_{r,\alpha}^-$ and $Z \cap H_{r,\alpha} = \tau(G)$. $\square$

COROLLARY 3.2. *The regions $\mathfrak{R}$ of the arrangement $\mathcal{H}$ are in one-to-one correspondence with the vertices of $Z$. The correspondence is given by $\hat{\tau}(R) = \tau(\bar{R}) = \sum_{i=1}^{k} \varepsilon_i z_j$ where*

$$\varepsilon_i = \begin{cases} 1 & \text{if } \bar{R} \subseteq H_i^+, \\ -1 & \text{if } \bar{R} \subseteq H_i^-. \end{cases}$$

Define $V(Z)$ to be the set of vertices of $Z$. Let $b$ be a fixed vertex in $V(Z)$. Without loss of generality assume that $b = \sum_{i=1}^{k} z_i$. If $v_1 = \sum_{i=1}^{k} \varepsilon_i z_i$ and $v_2 = \sum_{i=1}^{k} \varepsilon_i' z_i$ then we say that $v_1 \leqslant v_2$ when $\varepsilon_i \geqslant \varepsilon_i'$ for all $i$, $1 \leqslant i \leqslant k$. Call $V(Z)$ with this partial order $P(Z, b)$. It is easy to check that this is exactly the partial order induced by $P(\mathcal{H}, B)$ using the map $\hat{\tau}$. In fact the isomorphism can be more attractively given in the following way: For $S \subseteq [n]$ and $v \in V(Z)$, $v = -\sum_{i \in S} z_i + \sum_{j \notin S} z_j$, we let $R(v)$ be the region such that $S(R(v)) = \{H_i \mid i \in S\}$. Similarly if $R \in \mathfrak{R}$ define

$$v(R) = -\sum_{i \in S} z_i + \sum_{j \notin S} z_j$$

where $S = \{i \mid H_i \in S(R)\}$.

THEOREM 3.3. *In $P(Z, b)$, $v_1$ covers $v_2$ if and only if $v_1$ and $v_2$ lie on a one-dimensional face and $v_1 \geqslant v_2$.*

PROOF. In $P(Z, b)$, $v_1$ covers $v_2$ if and only if $R(v_1)$ covers $R(v_2)$ in $P(\mathcal{H}, B)$. So $\overline{R(v_1)} \cap \overline{R(v_2)}$ is a facet $F'_H$. Then $\tau(F'_H)$ is a one-dimensional face of $Z$ which contains $v_1$ and $v_2$. $\square$

We wish to find the phrasing of Theorem 1.11 in terms of zonotopes. Suppose $v = -\sum_{i \in S} z_i + \sum_{j \in S^c} z_j$ for some $S \subseteq [n]$. Define $C(b, v)$, the *cell of $v$ with respect to $b$*, to be the set $C(b, v) = \{x \mid x = \sum_{i \in S} \lambda_i z_i + \sum_{j \in S^c} z_j,\ |\lambda_i| \leq 1\}$. Notice that if $v' \in [b, v]_P$ then $v' \in C(b, v)$.

THEOREM 3.4. *In $P(Z, b)$ the Möbius function is given by*

$$\mu(b, v) = \begin{cases} (-1)^k & \text{if } C(b, v) \text{ is a face of } Z \text{ of dimension } k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. We will show that $C(b, v)$ is a face if and only if $R(v)$ is a facial region. Then the theorem follows immediately from the facts that $\mu(b, v) = \mu(B, R(v))$ and that the map $\tau$ takes a face of $\mathcal{H}$ to a face of complementary dimension of $Z$.

Suppose $C(b, v)$ is a face. Then Lemma 3.1 shows that

$$F = \{x \in \mathbf{R}^n \mid \langle x, z_i \rangle = 0 \text{ for } i \in S \text{ and } \langle x, z_j \rangle \geq 0 \text{ for } j \in S^c\}$$

is a face. Thus $S(R(F)) = \{H_i \mid i \in S\}$ and hence $R(F) = R(v)$.

Suppose $R(v)$ is a facial region, $R(v) = R(F)$, where

$$F = \{x \in \mathbf{R}^n \mid \langle x, z_i \rangle = 0 \text{ for } H_i \in \mathcal{H}(F) \text{ and } \langle x, z_i \rangle \geq 0 \text{ for } H_i \notin \mathcal{H}(F)\}.$$

Then $\tau(F)$ is a face of $Z$ and

$$\tau(F) = \left\{x \in \mathbf{R}^n \mid \sum_{H_i \in \mathcal{H}(F)} \lambda_i z_i + \sum_{H_j \notin \mathcal{H}(F)} \varepsilon_j z_j \text{ where all } |\lambda_i| \leq 1\right\}.$$

Let $S = \{i \mid H_i \in \mathcal{H}(F)\}$. Then we have $v = -\sum_{i \in S} z_i + \sum_{i \in S^c} z_i$ and so

$$\tau(F) = C(b, v). \quad \square$$

The importance of Theorem 3.4 is that the face structure of the zonotope can be analyzed by calculating the Möbius function of $P(Z)$. We give an example in Corollary 5.5.

**IV. The weak Bruhat order of a Weyl group.** In this section we present an application to a partial order on the Weyl group of a root system. This section will have much of the flavor of [**Z2**]. We will show that a natural group theoretic order on the Weyl group, the weak Bruhat order, is isomorphic to the partial order on the Weyl chambers gotten from our definition. As a corollary, a theorem of Björner [**Bj2**] about the homotopy type of the order is proven. For completeness we present the basic definitions. For more background and further details the reader might see [**H**].

A subset $\Phi$ of $\mathbf{R}^n$ is called a *root system* in $\mathbf{R}^n$ if the following axioms are satisfied:

(R1) $\Phi$ is finite, spans $\mathbf{R}^n$, and does not contain 0.

(R2) If $\alpha \in \Phi$, the only multiples of $\alpha$ in $\Phi$ are $\pm\alpha$.

(R3) If $\alpha \in \Phi$, $\sigma_\alpha$, the reflection with respect to $\alpha$, leaves $\Phi$ invariant.

(R4) If $\alpha, \beta \in \Phi$, then $2\langle \beta, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbf{Z}$.

Let $\mathfrak{W}$ be the subgroup of $\mathrm{GL}(\mathbf{R}^n)$ generated by the reflections $\sigma_\alpha$, $\alpha \in \Phi$. $\mathfrak{W}$ permutes the set $\Phi$ and is called the Weyl group of $\Phi$.

A subset $\Delta$ of $\Phi$ is called a *base* if:

(B1) $\Delta$ is a basis of $\mathbf{R}^n$.

(B2) Each root $\beta$ can be written as $\beta = \Sigma_{\alpha\in\beta} k_\alpha \alpha$ with integral coefficients $k_\alpha$ all nonnegative or all nonpositive.

The roots in $\Delta$ are called *simple*. If $\beta = \Sigma k_\alpha \alpha$ and $k_\alpha \geqslant 0$ ($k_\alpha \leqslant 0$) then $\beta$ is called *positive* (*negative*). Let $\Phi^+$ ($\Phi^-$) be the set of positive (negative) roots.

Let $H_\alpha$ be the hyperplane $H_\alpha = \{x \mid \langle x, \alpha\rangle = 0\}$ for $\alpha \in \Phi$. Then a region $R$ of $\mathbf{R}^n - \bigcup_{\alpha\in\Phi} H_\alpha$ is called a *chamber*. Associated with each chamber is a base in the following way. For a fixed $R$ define

$$\Delta = \left\{\alpha \in \Phi \middle| H_\alpha \in \mathfrak{B}(R) \text{ and } R \subseteq H_\alpha^+ \right\}.$$

Then $\Delta$ is a base.

The Weyl group $\mathfrak{W}$ acts simply transitively on $\mathfrak{R}$, the set of chambers. That is, for any $R_1, R_2 \in \mathfrak{R}$ there exists a unique $\sigma \in \mathfrak{W}$ such that $\sigma(R_1) = R_2$. Moreover $\mathfrak{W}$ is generated by $\{\sigma_\alpha \mid \alpha \in \Delta\}$.

Let $\tau \in \mathfrak{W}$. Suppose $\tau = \sigma_{\alpha_1}\sigma_{\alpha_2} \cdots \sigma_{\alpha_k}$ where $\alpha_i \in \Delta$ and $k$ is minimal. Then we say the *length of* $\tau$ is $k$ or $l(\tau) = k$. For $\tau \in \mathfrak{W}$ define $n(\tau) = \{\alpha \in \Phi^+ \mid \tau(\alpha) \in \Phi^-\}$.

Fix a chamber $B \in \mathfrak{R}$. This also fixes a base $\Delta$. For each $R \in \mathfrak{R}$ we identify an element of $\mathfrak{W}$, $\sigma^R$ where $\sigma^R(B) = R$. Similarly given $\sigma \in \mathfrak{W}$, $R(\sigma) = \sigma(B)$.

LEMMA 4.1. *For each* $R \in \mathfrak{R}$,

$$S(R) = \left\{H_\alpha \middle| \alpha \in n\left((\sigma^R)^{-1}\right)\right\}.$$

PROOF. Let $b \in B$. Then

$$H_\alpha \in S(R) \quad \text{if and only if } \left\langle \sigma^R(b), \alpha\right\rangle < 0, \text{for } \alpha \in \Phi^+.$$

Multiplying both entries by $(\sigma^R)^{-1}$ we see that $H_\alpha \in S(R)$ if and only if $\langle b, (\sigma^R)^{-1}(\alpha)\rangle < 0$. This is equivalent to the statement of the lemma. $\square$

Define the *weak Bruhat order* on $\mathfrak{W}$ by $\tau \geqslant \sigma$ if and only if $l(\sigma^{-1}\tau) = l(\tau) - l(\sigma)$. This is equivalent to saying that $\tau$ can be written as $\tau = \sigma\sigma_{\alpha_1}\sigma_{\alpha_2} \cdots \sigma_{\alpha_k}$ where $\alpha_i \in \Delta$ and $l(\tau) = l(\sigma) + k$.

THEOREM 4.2. $\tau \geqslant \sigma$ *if and only if* $n(\tau^{-1}) \supseteq n(\sigma^{-1})$.

PROOF. This theorem is essentially in Bourbaki [**Bo**, Proposition 17, p. 157 and Corollary 2, p. 158]. For a more general version of this theorem, see [**Bj2**]. $\square$

COROLLARY 4.3. $\mathfrak{W}$ *under the weak Bruhat order is isomorphic to* $P(\mathfrak{H}; B)$ *where* $\mathfrak{H} = \{H_\alpha \mid \alpha \in \Phi\}$ *and* $B$ *is any region.*

PROOF. The corollary follows directly from Lemma 4.1 and Theorem 4.2. $\square$

Since each chamber of $\mathfrak{R}$ is bounded by $n$ hyperplanes we can apply Theorem 2.5 to reproduce a special case of Björner's result [**Bj2**] on Coxeter groups.

THEOREM 4.4. $\Delta(\mathfrak{W} - \{\hat{0}, \hat{1}\})$ *has the homotopy type of the sphere* $S^{n-2}$.

This application of Theorem 2.5 is equivalent to Björner's proof. One interesting aspect of the theorem is that for a slightly different partial order on $\mathfrak{W}$, the strong Bruhat order, the poset has the homotopy type of the sphere of dimension $(|\Phi|/2) - 2$. This was shown by Björner and Wachs [**BW**] with special cases done in [**E1**] and [**Pr**]. It should be remarked that the weak Bruhat order arises naturally when proving these results. See [**Bj3**, 4.15].

In the case of the symmetric group $\mathcal{G}_n$ the weak Bruhat order can be defined in more concrete terms. Consider a permutation to be a string of the numbers in $[n]$. Given two permutations $\tau$ and $\sigma$ we say that $\tau \geqslant \sigma$ if $\tau$ can be gotten from $\sigma$ by a sequence of adjacent transpositions that always moves the larger element to the left. For instance if $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ then $\tau$ covers $\sigma$ in this order if

$$\tau = \sigma_1\sigma_2 \cdots \sigma_{i-1}\sigma_{i+1}\sigma_i\sigma_{i+2} \cdots \sigma_n$$

and $\sigma_{i+1} > \sigma_i$.

This partial order was previously studied by Guilbaud and Rosenstiehl [**GR**]. The name *permutohedron* is used for the related zonotope. It is known that the vertex poset is a lattice. See Berge [**Ber**, p. 135]. An interesting open question is whether all the weak Bruhat orders are lattices.

**V. Acyclic orientations of graphs.** Let $G$ be a simple connected graph with vertex set $[n]$ and edge set $E = \{e_{ij} | i \text{ is adjacent to } j\}$. From this graph we construct an arrangement of hyperplanes $\mathcal{H} = \{H_{ij} | e_{ij} \in E\}$ where $H_{ij} = \{x | x_i - x_j = 0\}$. This arrangement was first used by Greene. See [**Gre** and **Z3**].

By an *acyclic orientation* (a.o.) we mean an orientation of the edges so that there are no directed cycles. The edge $e_{ij}$ will be denoted $(i, j)$ when the orientation is from $i$ to $j$. If $\alpha$ is an a.o. of $G$ let $\alpha^e$ be the orientation gotten from $\alpha$ by reversing the orientation on $e$. For each a.o. $\alpha$, denote the a.o. gotten from $\alpha$ by reversing all the edges by $\bar{\alpha}$. Finally, let $\mathcal{Q}$ be the set of all a.o.'s of $G$.

THEOREM 5.1. *The regions of* $\mathbf{R}^n - \bigcup\mathcal{H}$ *are in one-to-one correspondence with the elements of* $\mathcal{Q}$. *The correspondence is given by*

$$R(\alpha) = \left\{x \in \mathbf{R}^n | x_i < x_j \text{ if } e_{ij} \text{ is oriented } (i, j) \text{ in } \alpha\right\}$$

*for each acyclic orientation* $\alpha$ *and inversely*

$$\alpha(R) = \left\{(i, j) | e_{ij} \in E \text{ and } x_i < x_j \text{ for } x \in R\right\}.$$

PROOF. This is a fundamental observation of Greene [**Gre**]. For the proof see [**GZ**]. □

Since every hyperplane $H_{ij} \in \mathcal{H}$ is orthogonal to the vector $(1, 1, \ldots, 1)$, the set of normal vectors to the $H_{ij}$'s spans an $(n - 1)$-dimensional subspace of $\mathbf{R}^n$. We will identify this subspace with $\mathbf{R}^{n-1}$ and consider the arrangement as being there.

COROLLARY 5.2. *For every acyclic orientation* $\alpha$ *of* $G$ *there is a unique minimal subset of the edges* $\mathrm{Bas}(\alpha)$, *called the basis of* $\alpha$, *such that every edge in* $E \setminus \mathrm{Bas}(\alpha)$ *is in the transitive closure of* $\mathrm{Bas}(\alpha)$.

PROOF. Let $R(\alpha)$ be the region assigned to $\alpha$ by Theorem 5.1. Define

$$\mathrm{Bas}(\alpha)=\{(i,j)\in\alpha\,|\,H_{ij}\in\mathfrak{B}(R(\alpha))\}.$$

Then $\mathrm{Bas}(\alpha)$ implies all the other edge orientations by transitive closure. □

If $G$ with orientation $\alpha$ is the comparability graph of a poset, then $\mathrm{Bas}(\alpha)$ corresponds to the edges of the Hasse diagram of the poset.

Fix an acyclic orientation $\beta\in\mathcal{Q}$. For each $\alpha\in\mathcal{Q}$ define

$$D(\alpha,\beta)=\{(i,j)\in\alpha\,|\,(j,i)\in\beta\}.$$

Now for each pair of a.o.'s $\alpha_1$ and $\alpha_2$ of $G$ we say $\alpha_1\leqslant\alpha_2$ if and only if $D(\alpha_1,\beta)\subseteq D(\alpha_2,\beta)$. It is easy to see that this partial ordering on $\mathcal{Q}$ is the same as the one induced by the partial order $P(\mathcal{H},B)$ where $B=R(\beta)$. We denote $\mathcal{Q}$ with this partial order by $\mathcal{Q}(\beta)$. In Figure 3 we have drawn two posets on the a.o.'s of a four-cycle. Notice that the poset of Figure 3(b) is a lattice whereas that of Figure 3(a) is not.

THEOREM 5.3. *Given any acyclic orientation $\alpha\in\mathcal{Q}$, there exists a linear ordering of the edges in $D(\alpha,\beta)$, $(e_1,e_2,\ldots,e_l)$, such that the orientations $\beta_0=\beta$, $\beta_i=\beta_{i-1}^{e_i}$ for $1\leqslant i\leqslant l-1$, and $\beta_l=\alpha$ are all acyclic.*

PROOF. This is a consequence of the fact that $\mathcal{Q}(\beta)$ is ranked. See Proposition 1.1. □

Suppose $\mathrm{Bas}(\beta)$ is a tree, when the orientation on the edges is ignored. Then $\beta$ will be called *tree-based*. This is equivalent to $R(\beta)$ being a basic region in the sense of §II. Thus we have, after applying Theorem 2.5,

THEOREM 5.4. *If $\beta$ is tree-based then $\Delta(\mathcal{Q}(\beta)-\{\beta,-\beta\})$ has the homotopy type of a sphere of dimension $n-3$.*

The zonotope $Z(G)$ associated with a graphic arrangement of hyperplanes (see §III) has been dubbed the *acyclotope* by Zaslavsky. By applying Theorem 3.3 we can make an observation about the face structure of the acyclotope. Let $\beta$ be tree-based with $\mathrm{Bas}(\beta)=T$, $b$ the corresponding vertex of $Z(G)$, and $a$ a vertex of $Z(G)$ with related acyclic orientation $\alpha$.

COROLLARY 5.5. *The cell $C(b,a)$ in $Z(G)$ is a face of $Z(G)$ if and only if the acyclic orientation $\alpha$ is gotten from $\beta$ by reversing the orientation on the edges of a subset $S\subseteq T$, and on the edges transitively implied by $S$.*

PROOF. By Theorem 3.4, $C(b,a)$ is a face of $Z(G)$ if and only if $\mu_Z(b,a)\neq0$, which is true if and only if $\mu(\beta,\alpha)\neq0$ in $\mathcal{Q}(\beta)$. Since $\beta$ is tree-based, $\mu(\beta,\alpha)\neq0$ if and only if $\alpha$ is the unique least upper bound of a set of atoms, by Lemma 2.3. Let $A$ be the set of atoms of $\mathcal{Q}(G)$ below $\alpha$. For $\sigma\in A$ we have $\sigma=\beta^e$ for some $e\in T$. Let $S=\{e\,|\,\beta^e\in A\}$. It is an easy exercise to see that the orientation $\alpha'$, gotten by reversing in $\beta$ all the edges in $S$ and those transitively implied by $S$, is acyclic. Moreover $\alpha'$ is the unique least upper bound of $A$ since the minimum set of reversals has been made. Thus $\alpha$ is the least upper bound of $A$ if and only if $\alpha=\alpha'$; this proves the corollary. □

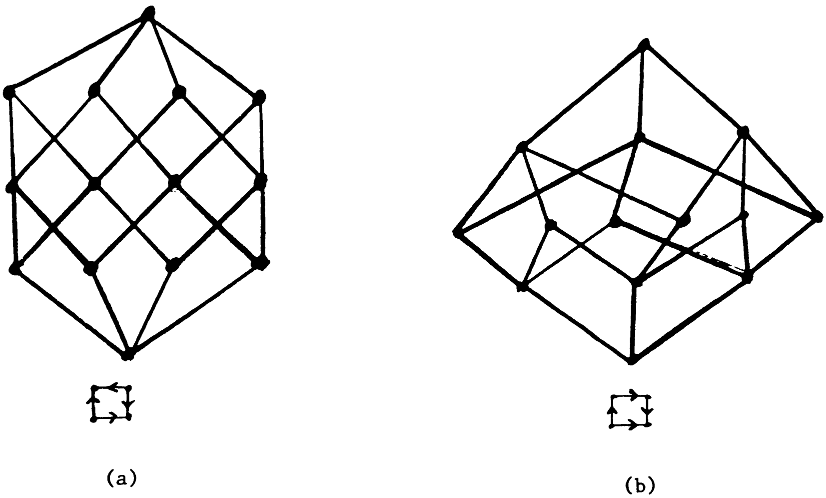


FIGURE 3

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NOTE ADDED IN PROOF. J. Walker and the author have established the homotopy type of any central arrangement of hyperplanes. A. Björner has shown that the weak Bruhat order of any Coxeter group is a lattice [**Bj2**].

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