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# ON THE ARITHMETIC AND HOMOLOGY OF ALGEBRAS OF LINEAR TYPE

BY

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ABSTRACT. Three modifications of the symmetric algebra of a module are introduced and their arithmetical and homological properties studied. Emphasis is placed on converting syzygetic properties of the modules into ideal theoretic properties of the algebras, e.g. Cohen-Macaulayness, factoriality. The main tools are certain Fitting ideals of the module and an extension to modules of a complex of not necessarily free modules that we have used in studying blowing-up rings.

**0. Introduction.** The terminology algebras of linear type refers to symmetric algebras of modules and mild modifications thereof. These are broad enough to include various blowing-up rings. To introduce them let $R$ be a commutative Noetherian ring and $E$ a finitely generated $R$-module. Denote by $\mathrm{Sym}(E)$, or simply $S(E)$, the symmetric algebra of $E$ over $R$. We place particular emphasis on the graded structure of $S(E)$:

$$S(E) = \bigoplus_{t \geq 0} \mathrm{Sym}_t(E). \tag{1}$$

If $R$ is an integral domain, $S(E)$ is hardly ever an integral domain itself: It is so if and only if each of the symmetric powers $\mathrm{Sym}_t(E)$ is a torsion-free $R$-module. It follows easily, however, that if $\mathrm{Sym}_t(E)_0$ denotes the torsion part of $\mathrm{Sym}_t(E)$ then

$$B(E) = \bigoplus_{t \geq 0} (\mathrm{Sym}_t(E)/\mathrm{Sym}_t(E)_0) \tag{2}$$

is an integral domain. An important special case is that of $E = I$, an ideal of $R$. $B(I)$ is then the blowing-up ring or, Rees ring

$$B(I) = \bigoplus_{t \geq 0} I^t = R[IT], \quad \text{for some indeterminate } T.$$

The localization of $B(E)$ at $K = R_{(0)}$ yields the polynomial ring

$$L = K[T_1, \ldots, T_e] = \mathrm{Sym}(V),$$

where $e$ is the rank of $E$, that is, $E \otimes_R K = K^e = V$.

(3) $C(E)$ = Integral closure of $B(E)$ in $L$.

Note that in the construction of either $B(E)$ or $C(E)$ one may replace $E$ by $E/E_0$; we thus assume $E$ is a torsion-free $R$-module.

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For the next algebra suppose $R$ is also integrally closed. In this case, for each prime ideal $P$ of height 1 of $R$, $E_P$ is a free $R_P$-module and thus $\mathrm{Sym}_t(E_P) = (\mathrm{Sym}_t(E)_P)$ embeds in $\mathrm{Sym}_t(V)$; it then follows that $\mathrm{Sym}_t(E)^{**}$ ($= \mathrm{Hom}_R(\mathrm{Hom}_R(\mathrm{Sym}_t(E), R), R)$), the $R$-bidual of $\mathrm{Sym}_t(E)$, is a submodule of $\mathrm{Sym}_t(V)$. In particular,

$$D(E) = \bigoplus_{t \geqslant 0} \mathrm{Sym}_t(E)^{**} \qquad (= \text{graded bidual of } S(E)) \tag{4}$$

is a subring of $L$. It turns out that $C(E)$ itself is a subring of $D(E)$.

Each of the algebras $B(E)$, $C(E)$ and $D(E)$ displays properties that would be of interest to have in $S(E)$. Thus $B(E)$ is an integral domain, while $C(E)$ and $D(E)$ are Krull domains. Furthermore, $D(E)$ is always factorial along with $R$. On the other hand, even for geometric rings, $D(E)$ may show some pathology: e.g. $D(E)$ may not be Noetherian (cf. Example 2.3), although no such example is known for $R$ regular.

We now outline the contents, leaving further comments to the appropriate sections. We study some arithmetical properties of these four algebras and consider comparisons in the sequence of homomorphisms ($R$ = normal): $S(E) \to B(E) \to C(E) \to D(E)$. To this purpose we attempt several approaches at converting syzygetic properties of the module $E$—the fine details of a projective resolution of $E$—into ideal theoretic properties of $S(E)$. When $R$ is a Cohen-Macaulay domain, a first level of necessary conditions for various equalities in the sequence above is obtained by considering the heights of the Fitting ideals of a presentation

$$R^m \xrightarrow{\phi} R^n \to E \to 0.$$

A recurring requirement involves sliding estimates on the sizes of the ideals $I_t(\phi)$ generated by the $t$-sized minors of $\phi$ of the type

$$(\mathcal{F}_k) \qquad \mathrm{height}(I_t(\phi)) \geqslant \mathrm{rank}(\phi) - t + 1 + k, \qquad 1 \leqslant t \leqslant \mathrm{rank}(\phi).$$

Thus the equality $S(E) = B(E)$ requires $(\mathcal{F}_1)$, while $S(E) = D(E)$ needs $(\mathcal{F}_2)$. None of these conditions is, in general, sufficient.

Full comparisons between any two such algebras present varying degrees of difficulty. Surprisingly, the equality $C(E) = D(E)$, for $R$ = Cohen-Macaulay, can be essentially decided at the level of the ideals $I_t(\phi)$ (Theorem 2.1). Testing for the equality $S(E) = C(E)$ will be largely ignored here and we shall focus on the equalities $S(E) = B(E)$ and $S(E) = D(E)$.

After discussing some of the arithmetical properties of these algebras in §§1 and 2, we introduce the $Z$-complex, $Z(E)$, of the module $E$. It is but a simple extension of one of the so-called approximation complexes of **[12–14]**, and used there to study Rees rings and associated graded algebras.

The construction hinges on the Koszul homology modules $H_i(S_+; S(E))$ ($S_+ =$ irrelevant ideal of $S(E)$). The $i$th graded component of this module, $Z_i = H_i(S_+; S(E))_i$, is often accessible from other properties of $E$. (For instance, when $E$ is an ideal $I$, the $Z_i$ are the modules of cycles of an ordinary Koszul complex on $I$.)

These modules can be put together into a complex of graded $\tilde{S} = S(R^n)$-modules:

$$Z(E): 0 \to Z_l \otimes \tilde{S}[-l] \to Z_{l-1} \otimes \tilde{S}[-l+1] \to \cdots \to Z_1 \otimes \tilde{S}[-1] \to \tilde{S} \to 0$$

($l = n$-rank$(E)$). Since $H_0(Z(E)) = S(E)$, and the complex is relatively short, it often turns that properties of $S(E)$ can be read off $Z(E)$.

To indicate the range of applicability and flexibility of the $Z$-complex, in §4 we discuss broad classes of examples where it is possible to transfer properties from $E$ to $S(E)$. Since the case of ideals has been dealt with elsewhere [**12**, **13**, **14**, **29** and **30**], the emphasis is now on modules of higher rank giving rise to integral domains or Cohen-Macaulay algebras.

The sequential criterion of acyclicity for the case of ideals of [**13**] is extended in §5 to torsion-free modules. As a consequence one obtains the following description of modules $E$ for which $Z(E)$ is acyclic (Theorem 5.6): For $R$ = normal, $E$ must arise from a (Bourbaki-) sequence $0 \to F \to E \to I \to 0$, where $I$ is an ideal generated by a proper sequence (the known condition for acyclicity of $Z(I)$) and $F$ is a free module and its basis forms a regular sequence on $S(E)$.

An important technical device is obtained by studying a resolution of $S(E)$ as an $\tilde{S} = S(R^n)$-module whenever $Z(E)$ is acyclic. It allows us, in §6, to derive very precise information on $E$ if $S(E)$ is to be Cohen-Macaulay and the full determination of the CM type of $S(E)$.

The last section deals with a discussion of a conjecture to the effect that if $S(E)$ is factorial, then $E$ must have projective dimension at most 1 ($R$ = regular). There is a considerable body of evidence for it and connections to the existence of modules with rather 'strange' properties. These resemble some of the difficulties of building indecomposable vector bundles over $P_n$, for $n$ large.

The rings considered in this paper will be commutative Noetherian (except for one instance) with an identity. For notation, terminology and basic results—especially when dealing with Cohen-Macaulay rings—we shall use [**18** and **22**].

**1. Samuel's criterion.** In [**28**] Samuel proved some elementary but ultimately very interesting properties of graded factorial rings. It exploits the relationship between factoriality of a ring $A = \oplus_{t \geq 0} A_t$ and the $A_0$-module structure of the components $A_t$ typified in the following:

THEOREM 1.1. *Let $A = \oplus A_t$ be a factorial graded Noetherian domain. Then $A_0$ is factorial and each $A_t$ is a reflexive $A_0$-module.*

More generally, if $A$ is integrally closed (= normal) the condition that each $A_t$ be $A_0$-reflexive is equivalent to the condition that divisorial primes of $A$ (i.e. height 1 primes) contract to 0 or divisorial primes of $A_0$. This is usually denoted PDE, cf. [**9**], where it is discussed in detail.

When applied to $S(E)$ this gives the more precise [**9**, **28**]:

THEOREM 1.2. *Let $R$ be a normal Noetherian domain and let $E$ be a finitely generated $R$-module. Suppose each* $\mathrm{Sym}_t(E)$ *is a reflexive $R$-module. Then $S(E)$ is a Krull domain and* $\mathrm{Cl}(R) = \mathrm{Cl}(S(E))$ ($\mathrm{Cl}(-)$ *denotes the divisor class group.*).

Referring to the algebras $C(E)$ and $D(E)$ of §0:

PROPOSITION 1.3. *Let $R$ be a normal Noetherian domain and a finitely generated $R$-module. Then $C(E)$ and $D(E)$ are (graded) Krull domains.*

PROOF. For $C(E)$ it is clear since $B(E)$ is a Noetherian domain. As for $D(E)$, we shall define a family of divisorial prime ideals with the requisite finite character of a Krull domain. First, however, we recall **[9]**:

LEMMA 1.4. *Let $R$ be a normal Noetherian domain and $M$ a finitely generated torsion-free $R$-module. The following are equivalent*:

(a) *$M$ is reflexive.*

(b) *$M = \cap M_p$ over the height 1 primes of $R$.*

(c) *Every regular sequence of 2 elements on $R$ is a regular sequence on $M$.*

To complete the proof of 1.3, consider the following (prime) ideals of $D(E)$:

(i) $P = Q \cap D(E)$, where $Q$ is a height 1 prime of $L$.

(ii) $P = \oplus_{t \geq 0}(P_0 \mathrm{Sym}_t(E))^{**}$, where $P_0$ is a height 1 prime of $R$.

It is clear that both (i) and (ii) are divisorial primes of $D(E)$ and that $D(E)_P =$ discrete valuation domain. Furthermore, using 1.4, it follows that the prime ideals obtained have the finite character property of a Krull domain **[18]**. □

In the special case that $R$ is, in addition, factorial, one has, from Nagata's lemma **[22]**:

COROLLARY 1.5. *Let $R$ be a factorial Noetherian domain and $E$ a finitely generated $R$-module. Then $D(E)$ is a factorial domain.*

**2. Finiteness.** We now compare the algebras $C(E)$ and $D(E)$ in terms of a free presentation of the module $E$. Let $R$ be a normal Noetherian domain and let $E$ be a module with a presentation $R^m \xrightarrow{\phi} R^n \to E \to 0$. Denote by $I_t(\phi)$ the ideal generated by the $t$-sized minors of a matrix representation $(a_{ij})$, $1 \leq i \leq m$, $1 \leq j \leq n$, of $\phi$. Although the $I_t(\phi)$ are not true invariants of $E$, note that the Fitting invariants $F_s(E)$ are just $F_s(E) = I_{n-s}(\phi)$. It is convenient, however, to work from a fixed presentation of $E$.

We consider the following condition on the ideals $I_t(\phi)$. Let $k \geq 0$ be an integer.

$$(\mathcal{F}_k) \qquad \mathrm{ht}(I_t(\phi)) \geq \mathrm{rank}(\phi) - t + 1 + k, \qquad 1 \leq t \leq \mathrm{rank}(\phi).$$

In terms of the Fitting ideals this can be written

$$(\mathcal{F}_k) \qquad \mathrm{ht}(F_s(E)) \geq s - \mathrm{rank}(E) + 1 + k, \qquad \mathrm{rank}(E) \leq s.$$

In turn, these global conditions can—by an immediate localization argument—be expressed in terms of the local number of generators of the module $E$:

$(\mathcal{F}_k)$ For each prime ideal $P$ of $R$, if $E_P$ is not a free $R_P$-module, then $v(E_P) \leq \mathrm{ht}(P) + \mathrm{rank}(E) - k$.

($v(-)$ = minimum number of generators of $(-)$.)

REMARKS. (i) These properties could also be defined for a nondomain, at least as long as the module $E$ admits a rank. This condition is equivalent to saying that the ideal generated by the largest sized minors contains regular elements.

(ii) When $R$ is a Cohen-Macaulay ring these conditions are important in the study of the Krull dimension of $S(E)$ (see also [**30**]). Thus, for instance, $(\mathcal{F}_0)$ means that, locally, $\dim S(E) = \dim R + \text{rank}(E)$. $(\mathcal{F}_k)$, $k \geqslant 1$, has the following interpretation. In the definition above, consider the case $t = \text{rank}(\phi)$ so that grade $I_t(\phi) \geqslant k + 1$. Let $P$ be a prime of $R$ and let $\mathbf{x} = \{x_1, \ldots, x_l\}$, $l \leqslant k$, be a regular sequence contained in $P$. Now reduce $E$ modulo $(\mathbf{x})$, $E' = E \otimes (R/(\mathbf{x}))$, $\phi' = \phi \otimes (R/(\mathbf{x}))$. Since $I_t(\phi')$ contains regular elements, $\text{rank}(E) = \text{rank}(E')$, and from the defintion of $(\mathcal{F}_k)$ we see that $E'$ satisfies $(\mathcal{F}_{k-l})$. Therefore $\dim S(E') = \dim R' + \text{rank}(E')$, thus proving that $(\mathbf{x})S(E)$ is an ideal of height $l$. This shows that for any prime ideal of $R$,

$$\text{ht}(PS(E)) \geqslant \inf\{\text{ht}(P), k\}.$$

It is clear that if, conversely, $\dim S(E) = \dim R + \text{rank}(E)$ and $S(E)$ satisfies this height condition, then $E$ must satisfy $(\mathcal{F}_k)$.

(iii) It follows that if $S(E)$ is an integral domain then $(E)$ satisfies $(\mathcal{F}_1)$, while if each $\text{Sym}_t(E)$ is reflexive then $E$ satisfies $(\mathcal{F}_2)$, since each regular sequence $\{a, b\}$ on $R$ is a regular sequence on each $\text{Sym}_t(E)$ and thus on $S(E)$. A point that shall be pursued later is that there are 'very few' modules satisfying $(\mathcal{F}_2)$ and therefore very few factorial domains that are symmetric algebras. This is one of the reasons for bringing up the algebra $D(E)$.

THEOREM 2.1. *Let $R$ be a normal, Cohen-Macaulay, universally Japanese domain.*
(a) *If $E$ satisfies $(\mathcal{F}_2)$, then $C(E) = D(E)$.*
(b) *Conversely, if $S(E)$ is a domain and $C(E) = D(E)$, then $E$ satisfies $(\mathcal{F}_2)$.*

PROOF. We may assume $R$ is a local ring. As noted, $(\mathcal{F}_2)$ implies $\dim S(E) = \dim R + \text{rank}(E)$ and thus $\dim S(E) = \dim B(E) = \dim C(E)$.

Let $f$ be a homogeneous element of $D(E)$. The set $I = \{r \in R \mid rf \in C(E)\}$ is an ideal of height at least 2. From Remark (ii) above, $\text{ht}(IS(E)) \geqslant 2$. If $I \neq R$ we shall find this to be impossible.

For simplicity we first argue the case $S(E) =$ domain, that is, $S(E) = B(E)$. Here we have

$$\text{ht}(IC(E)) = \text{ht}(IC(E) \cap B(E)) \geqslant \text{ht}(IB(E)) \geqslant 2,$$

the equality at the left following from [**23**, Theorem 34.8]. As $C(E)$ is a Krull domain and $f$ lies in its field of quotients, this is impossible.

If $S(E)$ is not a domain, $B(E) = S(E)/J$, where $J$ is a prime ideal of height 0. In this case, $\text{ht}((IS(E) + J/J))$ is still at least two, and the argument applies.

For the converse we check $(\mathcal{F}_2)$ in terms of the local number of generators. Let $\mathfrak{m}$ be the maximal ideal of $R$. We may assume $\text{ht}(\mathfrak{m}) \geqslant 2$ and $E$ is not free; we must show $v(E) \leqslant \text{ht}(\mathfrak{m}) + \text{rank}(E) - 2 = \dim S(E) - 2$.

Since a regular sequence $\{a, b\}$ on $R$ is regular on $D(E)$, we have $\text{ht}(\mathfrak{m}D(E)) \geqslant 2$. By the result of Nagata, $\text{ht}(\mathfrak{m}C(E)) = \text{ht}(\mathfrak{m}C(E) \cap S(E))$. As $\mathfrak{m}S(E)$ is a prime ideal of $S(E)$, $\mathfrak{m}S(E)C(E) \cap S(E) = \mathfrak{m}S(E)$, so $\text{ht}(\mathfrak{m}S(E)) \geqslant 2$. Therefore

$$v(E) = \dim S(E)/\mathfrak{m}S(E) = \dim S(E) - \text{ht}(\mathfrak{m}S(E)) \geqslant \dim S(E) - 2,$$

as desired. □

A simple context for this theorem is that of modules which are free on the punctured spectrum of a local ring. Let $E$ be such a module over a Cohen-Macaulay ring as above ($\dim R \geqslant 2$).

COROLLARY 2.2. *If* $v(E) \leqslant \dim R + \operatorname{rank}(E) - 2$, *then* $C(E) = D(E)$.

A construction of a family of modules over regular local rings with this property —thus yielding factorial $C(E)$'s—can be found in [**33**]; see also Example 4.4.

EXAMPLE 2.3. For modules of rank 1 the algebras $C(E)$ and $D(E)$ are almost never equal. Let $R$ be a normal domain and let $E = I$ be an ideal of $R$. As remarked, $B(I) = \oplus I^t$, the Rees algebra of $I$. $C(I) = \oplus \bar{I}^t$ where $\bar{J}$ denotes the integral closure of the ideal $J$. As for $D(I)$ one has two cases. If $\operatorname{ht}(I) \geqslant 2$, then $(I^t)^{**} = R$ and $D(I) = R[T]$. If $\operatorname{ht}(I) = 1$, to write $D(I)$ we may assume $I$ is unmixed, say with a primary decomposition $I = P_1^{(e_1)} \cap \cdots \cap P_n^{(e_n)}$, where $P^{(e)} = e$th symbolic power of the prime $P$. It follows easily from 1.4 that $\operatorname{Sym}_t(I)^{**} = I^{(t)}$ has a similar decomposition as $I$, with $e_i$ replaced by $te_i$.

Note that if $R$ is factorial then $D(I)$ will be isomorphic to $R[T]$ in all cases. For other rings, however, $D(I)$ may even fail to be Noetherian. Indeed, consider the following example. Let $A = C[x, y, z]$, $y^2z + yz^2 = x^3 - xz^2$ and $R = A_{\text{origin}}$; let $P = (x, y)$. It is easy to see that for this ideal, $\operatorname{Sym}_t(P) = P^t$. $(0, 0, 1)$ is, however, a nontorsion point of the elliptic curve [**31**] and therefore $P^{(t)}$ is nonprincipal for all $t \geqslant 1$. It follows from this that $D(P) = \oplus_{t \geqslant 0} P^{(t)}$ is non-Noetherian (cf. [**26**]).

REMARK. We know, however, of no example of a module $E$ over a regular local ring $R$ for which $D(E)$ is not Noetherian. For instance, let $E$ be a module given by the presentation (cf. [**28**]):

$$\phi = \begin{bmatrix} a & 0 \\ b & a \\ 0 & b \\ c & 0 \\ 0 & c \end{bmatrix},$$

$\{a, b, c\}$ a regular sequence in a regular local ring $R$. $E$ is a reflexive module and $S(E)$ is a domain ([**1**, **15**]; see also [**30**]) but does not satisfy $(\mathcal{F}_2)$. Samuel pointed out that $\operatorname{Sym}_2(E)$ is not reflexive. In fact, no $\operatorname{Sym}_t(E)$, $t \geqslant 2$, is reflexive (cf. §4). We do not know whether $D(E)$ is Noetherian.

**3. The *Z*-complex of a module.** Let $R$ be a Noetherian ring and $E$ a finitely generated $R$-module. We extend to $E$ the construction of a complex associated to ideals of $R$ (cf. [**12**, **13**, **14**, **29** and **30**]). Throughout, for convenience, we shall blur the distinction betwwen a complex and the complex augmented by its 0th homology.

Denote $S = S(E)$ and its irrelevant ideal by $S_+$.

DEFINITION 3.1. $Z(E) = M^*(S_+; S)$ is called the *approximation complex* of $E$, where $M^*(S_+; S)$ is the complex defined in [**14**] for the ideal $S_+$ of the ring $S$.

We point out that $M^*(S_+; S)$ is a complex of graded $\tilde{S}$-modules, where $\tilde{S}$ is a polynomial ring $R[e_1, \ldots, e_n]$ over $R$ in as many variables as a chosen set of generators of $S_+$. In degree $i$,

$$M_i^* = H_i(S_+; S)_i \otimes \tilde{S}[-i],$$

where $H_i(S_+; S)_i$ denotes the $i$th graded part of the Koszul homology $H_i(S_+; S)$ of $S$ with respect to a system $\mathbf{x} = \{x_1, \ldots, x_n\}$ of linear generators of $S_+$, or, what amounts to the same thing, of $E$. Further, we are using the notation for shifting the graded components of a module: $\tilde{S}[-i]_j = \tilde{S}_{j-i}$.

Let us briefly indicate how these complexes come about. Let $F = R^n \xrightarrow{\phi} E \to 0$ be a surjection. Consider the graded algebra—a double Koszul complex—$\mathfrak{L} = \{\wedge(F) \otimes S(F) \otimes S(E), \partial, \partial'\}$. In terms of ordinary Koszul complexes,

$$\{\mathfrak{L}, \partial\} = \mathcal{K}(\mathbf{x}; S(E)) \otimes S(F) \quad \text{and} \quad \{\mathfrak{L}, \partial'\} = \mathcal{K}(\mathbf{x}; S(F)) \otimes S(E).$$

From the commutativity of the differentials $\partial$ and $\partial'$, several complexes arise. In particular one obtains a complex $M(S_+; S)$, with $M_i = H_i(S_+; S) \otimes \tilde{S}[-i]$. While $H_i(S_+; S)$ may be cumbersome to deal with, its $i$th graded part, $H_i(S_+; S)_i$, is given simply as

$$\ker\left( \overset{i}{\wedge} F \xrightarrow{\partial} \overset{i-1}{\wedge} F \otimes E \right),$$

$$\partial(a_1 \wedge \cdots \wedge a_i) = \sum (-1)^j (a_1 \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_i) \otimes \phi(a_j).$$

Note that $H_1(S_+; S)_1 = \ker(\phi) =$ first syzygy module of $E$, which explains the notation $Z(E)$; see also Lemma 3.3.

If we write $H_i(S_+; S)_i = Z_i(E) = Z_i$,

$$Z(E): 0 \to Z_n \otimes \tilde{S}[-n] \to \cdots \to Z_1 \otimes \tilde{S}[-1] \to \tilde{S} \to H_0(Z(E)) = S(E) \to 0.$$

The differential of $Z(E)$ is that included by $\partial'$. An important point is the actual length of this complex; it will be shown later that if $E$ has a rank, say $\operatorname{rank}(E) = e$, then $Z_i = 0$ for $i > n - e$. Furthermore, its homology is independent of the chosen presentation $\phi$. Note also that in each degree, we have a complex of finitely generated $R$-modules

$$0 \to Z_n \otimes \tilde{S}_{t-n} \to \cdots \to Z_1 \otimes \tilde{S}_{t-1} \to \tilde{S}_t \to \operatorname{Sym}_t(E) \to 0;$$

in this form it is convenient for checking acyclicity (cf. [**12**, **30**]).

The advantage of considering $M^*(S_+; S)$ rather than the full $M(S_+; S)$ lies in its simplicity since the higher symmetric powers of $E$ do not get directly involved in its construction. Moreover, one has

LEMMA 3.2. *The following conditions are equivalent*:

(a) $M(S_+; S)$ *is acyclic.*

(b) $Z(E)$ *is acyclic.*

(c) $H_i(S_+; S)_j = 0$ *for* $j > i \geqslant 0$.

*Furthermore, if the equivalences hold, then* $M(S_+; S) = Z(E)$.

PROOF. (a) $\Leftrightarrow$ (c) $\Leftrightarrow$ (b) If $M(S_+; S)$ is acyclic, then $H_i(S_+; S)_j = 0$ for $j > i \geqslant 0$ and, conversely, by [**14**, (11.9)]; clearly in such a case $M(S_+; S) = Z(E)$.

(b) $\Leftrightarrow$ (a), (c) If $Z(E)$ is acyclic, it may be used to compute $H_i(S_+; S)$, since $\operatorname{Tor}_j^{\tilde{S}}(\tilde{S}/(\mathbf{e}), M_i^*) = 0$ for $j > 0$. One concludes that $H_i(S_+; S)_j = 0$ for $j > i \geqslant 0$, and thus the complexes $M(S_+; S)$ and $Z(E)$ coincide. □

We now compare this construction with the approximation complexes associated with ideals, cf. [**14**] for notations.

LEMMA 3.3. (a) *Let $E = I$ be an ideal; then $Z(E)$ is the Z-complex of $I$.*

(b) *Let $E = I/I^2$ be the conormal module of the ideal $I$; then there exists a natural inclusion $H_i^*(I; R) \to H_i^*(S_+; S(I/I^2))$. If the M-complex of $I$ is acyclic, then $M(I; R) = Z(I/I^2)$.*

PROOF. (a) is immediate. (b) The elements of $H_i^*(I; R)$ considered as elements of $\wedge^i(R/I)^n$ are obviously in the kernel of $\wedge^i(R/I)^n \to \wedge^{i-1}(R/I)^n \otimes (I/I^2)$. The rest of the proof proceeds in the same manner as 3.2. □

**4. Modified Koszul complexes.** We now compare the complex $Z(E)$ to the Koszul complex associated to a presentation

$$0 \to Z_1(E) = L \overset{\psi}{\to} R^n = F \to E \to 0.$$

It will provide a convenient vehicle for exchanging information between the syzygies of $E$ and depth properties of $S(E)$.

The Koszul complex associated to the map $L \overset{\psi}{\to} F$ is $\mathcal{K}(E) = \wedge(L) \otimes S(F)$ with differential

$$\partial((a_1 \wedge \cdots \wedge a_r) \otimes w) = \sum (-1)^j (a_1 \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_r) \otimes \psi(a_j) \cdot w.$$

Since $L = Z_1(E)$, the skew-commutative structure of $Z(E)$, and the fact that the various differentials are also derivations, gives rise to a chain map $\mathcal{K}(E) \to Z(E)$ arising out of the maps $\wedge^r L \to Z_r(E)$. Let $e$ be the rank of $E$, that is, assume $E \otimes K = K^e$, where $K$ is the total ring of fractions of $R$. Since the constructions above localize, it follows that $Z(E)$ is a complex of length $n - e = l = \operatorname{rank}(L)$. $\mathcal{K}(E)$ on the other hand may be much longer since $\wedge^r(L)$ may be nonzero for $r > l$. For the purpose of comparison, it is useful to cut down the length of $\mathcal{K}(E)$. This may be done in several ways, most simply by reducing $\mathcal{K}(E)$ modulo its $R$-torsion. We prefer, however, to carry out another modification.

Suppose, for instance, that $E$ is a torsion-free module; in this case each $Z_r(E)$ is a second syzygy module and thus reflexive. Replacing $\wedge^r L$ by its double dual $(\wedge^r L)^{**}$, we get a chain map $\mathcal{K}(E)^{**} \to Z(E)$.

These modifications are actual identifications in the following cases.

PROPOSITION 4.1. *Let $E$ be a finitely generated $R$-module.*

(a) *If $E$ is a torsion-free module free at the primes $P$ with* depth $R_P \leqslant 1$, *then* $\mathcal{K}(E)^{**} \cong Z(E)$.

(b) *Let $R$ be a Cohen-Macaulay ring; if $E$ has a resolution $0 \to R^m \to R^n \to E \to 0$ and satisfies $(\mathcal{F}_0)$, then $\mathcal{K}(E) \cong Z(E)$. In this case $Z(E)$ provides a projective $\tilde{S}$-resolution of $S(E)$. Furthermore, if $R$ is an integral domain, then $S(E)$ is a domain if and only if $E$ satisfies $(\mathcal{F}_1)$.*

PROOF. (a) Here both $(\wedge^r L)^{**}$ and $Z_r(E)$ are reflexive modules which agree in depth $\leqslant 1$ and thus coincide.

(b) Consider the exact sequence $0 \to \wedge^r R^m \to Z_r(E) \to C \to 0$. To show $C = 0$, it is enough to check the height one primes. But the $(\mathcal{F}_0)$-condition now means

$v(E_P) \leqslant \mathrm{ht}(P) + \mathrm{rank}(E)$, so $E_P$ admits a resolution $0 \to R_P \to R_P^{e+1} \to E_P \to 0$. The assertion follows from the underlying mapping cone construction of both complexes. As for the $\tilde{S}$-projective resolution of $S(E)$, see [**1** or **30**]. □

REMARK. (i) Note that in the above proposition, (b) means that the graded components of $Z(E)$ provide free resolutions of $\mathrm{Sym}_t(E)$; the complex $Z(E)$ is, for this case, an aggregate of some of the resolutions of [**20** and **34**].

(ii) When $I$ is a perfect ideal of height two, (a) and (b) imply that the modules of cycles of an ordinary Koszul complex of $I$ are all free. This in turn is easily seen to be equivalent to saying that the Koszul homology modules $H_i(I; R)$ are Cohen-Macaulay; see also [**2** and **16**].

Before we consider other examples, we point out some duality features of these complexes. First we recall a result of [**25**]; see also [**8**].

LEMMA 4.2. *Let $M$ be finitely generated of finite projective dimension. Assume $M_{\mathfrak{p}}$ is free for each prime with* depth $R_{\mathfrak{p}} \leqslant 1$. *If $r = \mathrm{rank}(M)$, then $\det(M) = (\wedge^r M)^{**}$ is an invertible ideal.*

We use this in the context of the complex $\mathcal{K}(E)^{**}$, where $E$ is a torsion-free module and $\mathrm{pd}\, E < \infty$. Thus for the presentation above, $L$ is a reflexive module and $(\wedge^l L)^{**}$ is an invertible ideal ($l = \mathrm{rank}(L)$). Furthermore, for any integer $r < l$, the canonical pairing

$$\overset{r}{\wedge} L \otimes \overset{l-r}{\wedge} L \to \overset{l}{\wedge} L \to \left(\overset{l}{\wedge} L\right)^{**} = \det(L)$$

yields a mapping $\wedge^r L \to \mathrm{Hom}_R(\wedge^{l-r} L, \det(L))$ which is an isomorphism in depth $\leqslant 1$.

COROLLARY 4.3. (a) *If $l = 2$,*

$$L = \mathrm{Hom}_R(L, \det(L)) = L^* \otimes \det(L),$$

*the standard formula for rank 2 bundles.*

(b) *If* $\det(L) = R$ (*e.g. $R$ = factorial, or $L$ admits a finite free resolution*), *for any $r \leqslant l$, $(\wedge^r L)^{**} = (\wedge^{l-r} L)^*$, that is, $Z_r(E) \cong Z_{l-r}(E)^*$.*

EXAMPLE 4.4. Let us consider in some detail some of the modules of [**33**]. For a regular local ring or, more generally, for a Cohen-Macaulay local ring, it describes a matrix ($\dim R = n$) $R^{2n-3} \overset{\psi}{\to} R^n$ of rank $n-1$ which splits on the punctured spectrum of $R$. Denote $E = \mathrm{image}(\psi)$. A first remark is that such a module satisfies $(\mathcal{F}_2)$. In [**33**] it is proved that $\mathrm{pd}\, E \geqslant n-2$. In fact it is the case that $\mathrm{pd}\, E = n-1$. Indeed, otherwise $\mathrm{coker}(\psi)$ would be torsion-free, free on the punctured spectrum of $R$ so it would be isomorphic to a primary ideal generated by $n$ elements that is, to an ideal generated by a system of parameters. In this case the module of relations $E$ would have at least $\binom{n}{2}$ generators, but then $2n-3 \geqslant \binom{n}{2}$, which is possible only if $n \leqslant 3$.

Thus, at least for $n \geqslant 4$, $\mathrm{coker}(\psi)$ has nontrivial torsion concentrated in $\mathfrak{m}$ = maximal ideal. In particular, we obtain $\mathrm{Ext}^i_R(E, R) = 0$ for $i = 1, \ldots, n-3$.

Consider the case $n = 5$: $0 \to L \to R^7 \to E \to 0$. Here depth $L = 2$, while depth $L^* \geqslant 3$, since $\operatorname{Ext}^1_R(L, R) = 0$ implies $L^*$ is a third syzygy module. The $Z$-complex of $E$ is then

$$0 \to R \otimes \tilde{S}[-3] \to L^* \otimes \tilde{S}[-2] \to L \otimes \tilde{S}[-1] \to \tilde{S} \to S(E) \to 0.$$

Applying the acyclicity lemma of [**24**] to the graded components (cf. [**14**]), it will follow that the complex is exact and $\operatorname{Sym}_t(E)$ is a torsion-free $R$-module, i.e. $S(E)$ is an integral domain. It has Krull dimension 9 and the grade of the maximal homogeneous ideal is $\geqslant 8$. We shall see, however (cf. §6), that $S(E)$ is not a Cohen-Macaulay ring.

EXAMPLE 4.5. Let $R$ be a Cohen-Macaulay integral domain and $E$ a torsion-free $R$-module of projective dimension two satisfying $(\mathcal{F}_1)$ and admitting a resolution $0 \to L \to R^n \to E \to 0$ with $\operatorname{rank}(L) = 2$ (this is [**30**, (3.5)]).

The $Z$-complex of $E$ is then

$$0 \to R \otimes \tilde{S}[-2] \to L \otimes \tilde{S}[-1] \to \tilde{S} \to S(E) \to 0.$$

$Z(E)$ is again acyclic and $S(E)$ is a Cohen-Macaulay integral domain.

EXAMPLE 4.6. Let $R$ be an integral domain and $E$ a torsion-free $R$-module. We consider another case in projective dimension two. In general, to obtain $S(E) =$ domain, one needs high depths in the modules $Z_r(E)$; in dimension two this may often be done since the projective dimensions of the exterior powers of $L = Z_1(E)$ are easier to estimate.

Consider the case of a module satisfying $(\mathcal{F}_1)$. In addition, to use the $Z$-complex it is convenient that $\wedge^r L = (\wedge^r L)^{**}$ for each $r < \operatorname{rank}(L)$. One way to achieve it is to strengthen $(\mathcal{F}_1)$ in higher codimension by requiring

$$(*) \qquad v(E_P) \leqslant \tfrac{1}{2}(\operatorname{ht}(P) + 1) + \operatorname{rank}(E) \text{for each prime } P.$$

We claim $Z(E)$ is acyclic and $S(E)$ is a domain.

We may assume we are dealing with a minimal resolution of $E$ and $E$ is not free. Since, from $(\mathcal{F}_1)$, $v(E_p) \leqslant \operatorname{ht}(P) + \operatorname{rank}(E) - 1$,

$$v(E) - \operatorname{rank}(E) = \operatorname{rank}(L) \leqslant \operatorname{ht}(P) - 1 \qquad (P = \text{maximal ideal}).$$

To apply the acyclicity lemma as above and the underlying mapping cone property of the $Z$-complex (cf. [**12**, **30**]), $\wedge^r L$ must have depth at least $r + 1$ and be reflexive in the range $1 < r < \operatorname{rank}(L)$. Since the projective dimension of $\wedge^r L$ is $r$, we must have $\dim R - r \geqslant r + 1$, which is provided by $(*)$ above.

*Problem.* Determine necessary and sufficient conditions for a module of projective dimension two to admit an acyclic $Z$-complex.

Here is an example of a nice module of projective dimension two without an acyclic $Z$-complex. Let $R$ be a Cohen-Macaulay local ring and $\mathcal{K}(\mathbf{x}; R)$ the Koszul complex associated to a system of parameters $\mathbf{x} = \{x_1, \ldots, x_n\}$, $n \geqslant 3$. Let $E = Z_{n-3}$, that is, consider the tail of $\mathcal{K}(\mathbf{x}; R)$, $0 \to K_n \to K_{n-1} \to K_{n-2} \to E \to 0$. Alternatively, given the free module $R^n$, with basis $\{e_1, \ldots, e_n\}$, then $E$ is the cokernel of the mapping $R^n \to \wedge^2 R^n$ defined by multiplication by $\zeta = \sum x_i e_i$.

Counting ranks, it follows that $E$ satisfies $(\mathcal{F}_1)$. As $Z_{n-2}(E)$ has depth 2, it follows easily that $Z(E)$ is not acyclic if $n \geqslant 3$.

EXAMPLE 4.7. In order to show the exactness of the $Z$-complex of a module, we have been striving for high depth for the coefficients $Z_r(E)$. Normally this entails that, locally, depth $Z_r(E) \geqslant r$. We shall shortly take another approach—that of acyclic sequences. We shall look, however, at a last example where estimation of the depth of $Z_r(E)$ is possible.

Let $R$ be a Gorenstein local ring and $I$ an ideal of grade $g \geqslant 1$, minimally generated by $n$ elements. The $Z_r(I)$ are then the modules of cycles of the associated Koszul complex. For $n-1 \geqslant r \geqslant n-g+1$, $Z_r = B_r = r$-boundaries. Since, in that range, pd $B_r = n-(r+1)$, depth $Z_r = d-n+r+1$ ($d = \dim R$).

When $I$ is a Cohen-Macaulay ideal, the depth of one extra $Z_r$ may be determined. Indeed, let us show that depth $Z_{n-g} \geqslant d-g+2$ (we may assume $g \geqslant 2$).

Consider the sequence

$$0 \to B_{n-g} \to Z_{n-g} \to H_{n-g}(I; R) \to 0.$$

Note that depth $B_{n-g} = d-g+1$, while $H_{n-g}(I; R)$ is the canonical module of $R/I$ and thus has depth $d-g$. The exact sequence already says that depth $Z_{n-g} \geqslant d-g$. To determine depth $Z_{n-g}$ we test the vanishing of the modules $\operatorname{Ext}^i_R(Z_{n-g}, R)$ for $i = g, g-1$ (cf. [**21**]). From the exact sequence we have the homology sequence

$$\begin{aligned}\operatorname{Ext}^{g-1}(H_{n-g}, R) \to \operatorname{Ext}^{g-1}(Z_{n-g}, R) \to \operatorname{Ext}^{g-1}(B_{n-g}, R)\\ \to \operatorname{Ext}^{g}(H_{n-g}, R) \to \operatorname{Ext}(Z_{n-g}, R) \to \operatorname{Ext}^{g}(B_{n-g}, R).\end{aligned}$$

Here $\operatorname{Ext}^{g-1}(B_{n-g}, R) = R/I$, from the exactness of the corresponding tail of the Koszul complex. On the other hand, $\operatorname{Ext}^{g}(B_{n-g}, R) = \operatorname{Ext}^{g-1}(H_{n-g}, R) = 0$, while $\operatorname{Ext}^{g}(H_{n-g}, R) = R/I$ since $R$ is a Gorenstein ring. Thus we have the exact sequence

$$0 \to \operatorname{Ext}^{g-1}(Z_{n-g}, R) \to R/I \xrightarrow{\phi} R/I \to \operatorname{Ext}^{g}(Z_{n-g}, R) \to 0.$$

Localizing at primes of height $g$ and $g+1$, we get that $\phi$ is an isomorphism since $Z_{n-g}$ is a second syzygy module. Thus $\phi$ is an isomorphism and the desired assertion follows.

The first instance of interest to which this applies is when $n = g+2$. One obtains that depth $Z_2 = d-g+2$. Since depth $Z_1 = d-g+2$, we conclude that the $H_i(I; R)$ are Cohen-Macaulay modules. (See [**2**] for the original proof.)

For $n = g+3$, we get depth $Z_3 = d-g+2$, depth $Z_1 = d-g+2$ and depth $Z_2 \geqslant 2$. If the ideal $I$ satisfies ($\mathcal{F}_0$), it will follow that the complex $Z(I)$ is acyclic [**12**]. Thus, for the ideal generated by the $2 \times 2$ minors of a generic matrix,

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix},$$

$Z(I)$ is acyclic.

In case $n = g+3$ if depth $Z_2 \geqslant 3$ and $I$ satisfies ($\mathcal{F}_1$), we have that $S(I)$ and the Rees algebra $R(I) = R[IT]$ coincide [**12**]. In the above example, because of the Plücker relations, we must have depth $Z_2 = 2$.

Now suppose $I$ is the ideal generated by the $2 \times 2$ minors of the generic symmetric matrix

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{bmatrix}.$$

Here $n = g + 3$ and $I$ satisfies $(\mathcal{F}_1)$. We do not know whether depth $Z_3 \geqslant 3$.

**5. Acyclicity.** We begin by recalling the notions that play for the $Z$-complex the role of acyclic sequences.

DEFINITION 5.1. Suppose $\mathbf{x} = \{x_1, \ldots, x_n\}$ is a sequence of elements in a ring $R$. The sequence $\mathbf{x}$ is called a:

(a) *d-sequence* if:

($a_1$) $\mathbf{x}$ is a minimal generating set of the ideal $I = (\mathbf{x}) = (x_1, \ldots, x_n)$;

($a_2$) $(x_1, \ldots, x_i)\colon x_{i+1}x_k = (x_1, \ldots, x_i)\colon x_k$, for $i = 0, \ldots, n-1$ and $k \geqslant i+1$;

(b) *proper sequence* if $x_{i+1}H_j(x_1, \ldots, x_i; R) = 0$ for $i = 0, \ldots, n-1, j > 0$, where $H_j(x_1, \ldots, x_i; R)$ denotes the Koszul homology associated to the initial subsequence $\{x_1, \ldots, x_i\}$.

REMARKS. (i) The relationship between (a) and (b) is, broadly, the following: Each $d$-sequence is a proper sequence and the linear forms in $S(I)$ corresponding to a proper sequence $\mathbf{x}$ generate a $d$-sequence relative to the ring $S(I)$ (cf. [**14**, (12.10)]).

(ii) In (b) it suffices to consider $j = 1$ [**19**].

(iii) In (a), ($a_2$) already embodies a measure of minimality. For instance, assume $I = (x_1, \ldots, x_n)$ is an ideal of grade $k$; then $x_1, \ldots, x_k$ is, as seen directly from ($a_2$), a regular sequence. Partly for this reason, we shall at times blur the definition by considering ($a_2$) alone.

For quick reference we quote the following criterion [**14**, (12.9), **19**].

PROPOSITION 5.2. *Let $R$ be a Noetherian local ring with infinite residue field and $E$ a finitely generated $R$-module. The following are equivalent*:

(a) *$Z(E)$ is acyclic.*

(b) *$S_+$ is generated by a $d$-sequence of linear forms of $S(E)$.*

Let us indicate how these conditions can be realized. For simplicity assume $R$ is a domain. A Bourbaki sequence (Bs) is an exact sequence of an $R$-module $0 \to F \to E \to I \to 0$, where $F$ is a free module and $I$ is an ideal.

Assume that in such a sequence $I$ is generated by a $d$-sequence. The symmetric algebra $S(I)$ in this case coincides with the Rees algebra, and therefore will be an integral domain [**17**, **32**]. Since $S(I) = S(E)/(F)$ ($(F) =$ ideal of $S(E)$ generated by the forms in $F$), $(F)$ will be a prime ideal of height $=$ rank$(F)$. It will follow from [**5**] that any basis of $F$ will generate a regular sequence in $S(E)$. Putting together a generating set of $E$ from a basis of $F$ and elements mapping to the $d$-sequence in $I$, we conclude that the irrelevant ideal of $S(E)$ is indeed generated by a $d$-sequence, whence $Z(E)$ will be acyclic. (The local and infinite residue field hypotheses are needed in the other direction only.)

In this construction, if $S(I)$ is integrally closed and $R$ is a Japanese ring, then $S(E)$ will also be integrally closed by Hironaka's lemma.

One of our purposes here is to prove a broad converse that expresses certain modules $E$ with $Z(E)$ acyclic in terms of Bourbaki sequences.

Assume from this point on that $R$ is a local ring with infinite residue field and let $E$ be a finitely generated $R$-module. The following lemmas provide the means to check whether a sequence generating $S_+$ is a $d$-sequence.

LEMMA 5.3. *Let $x_1,\ldots, x_n$ be 1-forms generating $S_+$. The following conditions are equivalent*:

(a) $\{x_1,\ldots, x_n\}$ *is a d-sequence.*

(b) $H_1(x_1,\ldots, x_i; S(E))_2 = 0$ *for* $i = 1,\ldots, n$.

(c) $H_1(x_1,\ldots, x_i; S(E))_k = 0$ *for* $i = 1,\ldots, n$, *and* $k \geqslant 2$.

PROOF. (a) $\Leftrightarrow$ (c) is [**14**, (12.7)].

(b) $\Rightarrow$ (c) We proceed by induction on $k$. Let $k > 2$ and suppose

$$H_1(x_1,\ldots, x_i; S(E))_{k-1} = 0 \quad \text{for } i = 1,\ldots, n.$$

From the long exact sequence of Koszul complexes (cf. [**22**]),

$$H_1(x_1,\ldots, x_i; S(E))_{k-1} \xrightarrow{x_{i+1}} H_1(x_1,\ldots, x_i; S(E))_k \to H_1(x_1,\ldots, x_{i+1}; S(E))_k,$$

and we inductively conclude that for each $i$,

$$H_1(x_1,\ldots, x_i; S(E))_k \to H_1(x_1,\ldots, x_n; S(E))_k$$

is injective. But this last module vanishes for $k > 1$ [**19**]. □

Let $E$ be a module with $\operatorname{rank}(E) = e$. In general, for the symmetric algebra $S(E)$, one has grade $S_+ \leqslant \operatorname{rank}(E)$.

LEMMA 5.4. *If $Z(E)$ is acyclic, then* grade $S_+ = \operatorname{rank}(E)$. *In particular, if $\{x_1,\ldots, x_n\}$ is a d-sequence of* 1-*forms generating $S_+$ then $\{x_1,\ldots, x_e\}$ is a regular sequence on $S(E)$.*

PROOF. Map a polynomial ring $\tilde{S} = R[e_1,\ldots, e_n]$ onto a generating set $y_1,\ldots, y_n$ of 1-forms of $S(E)$. The $Z$-complex of $E$ has length $n - e$ and, for each component, $\operatorname{Tor}_i^{\tilde{S}}(\tilde{S}/(\mathbf{e}), Z_r(E) \otimes S[-r]) = 0$ and $i > 0$. If $Z(E)$ is exact it can be used to read the grade of

$$S_+ = (\mathbf{e}) - \operatorname{depth} S(E) = n - \sup\left\{ i \middle| \operatorname{Tor}_i^{\tilde{S}}(\tilde{S}/(\mathbf{e}), S(E)) \neq 0\right\} = n - (n - e) = e.$$

The last assertion follows from an earlier remark. □

LEMMA 5.5. *Let $\{x_1,\ldots, x_n\}$ be a sequence of* 1-*forms in $S_+$. Suppose the first $r$ elements form a regular sequence. The following conditions are equivalent*:

(a) $\{x_1,\ldots, x_n\}$ *is a d-sequence.*

(b) $\{x^*_{r+1},\ldots, x^*_n\}$ *is a d-sequence on* $S^* = S(E)/(x_1,\ldots, x_r)$.

PROOF. The assertion follows from (5.3) and the isomorphism of graded modules $H_1(x^*_{r+1},\ldots, x^*_i; S^*) = H_1(x_1,\ldots, x_i; S(E))$, valid for all $i > r$. □

We can now prove the relationship between the acyclicity of $Z(E)$ and the special representations of $E$ in terms of Bourbaki sequences.

THEOREM 5.6. *Let $R$ be a local ring with infinite residue field. Suppose $E$ is a finitely generated torsion-free $R$-module and either* (i) pd $E < \infty$, *or* (ii) *$R$ is a normal domain. Then the following conditions are equivalent*:

(a) *$Z(E)$ is acyclic.*

(b) *$E$ admits a Bourbaki sequence $0 \to F \to E \to I \to 0$ such that*:

($b_1$) *$F$ is generated by elements which form a regular sequence of 1-forms on $S(E)$.*

($b_2$) *$I$ is generated by a proper sequence.*

PROOF. (b) $\Rightarrow$ (a) Note that a module $E$ as above has a well-defined rank, say rank$(E) = e$. Pick generators $\{x_1,\ldots, x_{e-1}\}$ of $F$ which form a regular sequence on $S(E)$ and pick elements $\{x_e,\ldots, x_n\}$ in $E$ whose images in $I$ form a proper sequence; $\{x_1,\ldots, x_n\}$ is a system of generators of $S_+$. By [**14**, (12.10)], the elements $\{x_3,\ldots, x_n\}$ form a $d$-sequence on $S(I)$; hence, by 5.5, $\{x_1,\ldots, x_n\}$ is a $d$-sequence on $S(E)$. The assertion now follows from 5.2.

(a) $\Rightarrow$ (b) We proceed by induction on $e$. If $e = 1$, then $E$ is isomorphic to an ideal and 5.2 applies.

Suppose then rank$(E) = e > 1$. Since $Z(E)$ is acyclic we have by 5.4 that grade $S_+ = e > 1$. Hence we can find $x \in E$, which is a nonzero divisor on $S(E)$. Our aim is to find an element $x$ with the additional property that $E/Rx$ is still torsion-free. We then apply the induction hypothesis to $E/Rx$ and the theorem will be proved.

Denote by $x^*$ the image of an element $x$ in $E/\mathfrak{m}E$ ($\mathfrak{m}$ = maximal ideal of $R$). To find $x$ which is regular on $S(E)$ amounts to finding $x$ such that $x^* \notin X$, where $X \subset E/\mathfrak{m}E$ is a finite union of proper linear subspaces which are determined by Ass$(S(E))$. The proof will be completed by the next lemma.

LEMMA 5.7. *Suppose $E$ satisfies the conditions of the theorem and* rank$(E) = e > 1$. *Let $X$ be a finite union of proper linear subspaces of $E/\mathfrak{m}E$. There exists $x \in E$ such that*

(i) *$E/Rx$ is torsion-free*,

(ii) *$x^* \notin X$.*

PROOF. (i) is satisifed if $x$ is $P$-basic for all primes $P$ of $R$ with depth $R_P \leqslant 1$. (ii) will be satisfied if $\lambda_i(x)$ is $\mathfrak{m}$-basic for $i = 1,\ldots, k$, where $\lambda_i$: $E \to L_i$ are epimorphisms and $X = -L_i$. Such an element exists according to [**4**, (2.4)]. □

We now relate the acyclicity of $Z(E)$ with the resolution of $S(E)$ as an $\tilde{S}$-module.

Let $\{\mathcal{G}, \partial\}$ be a minimal $\tilde{S}$-resolution of $S(E)$, i.e. $\mathcal{G}$ is a (graded) $\tilde{S}$-projective resolution of $S(E)$ and $\partial(\mathcal{G}) \subset (\mathfrak{m}\tilde{S} + \tilde{S}_+)\mathcal{G}$. We define the filtration $\mathcal{F}_{-i}\mathcal{G}$ on $\mathcal{G}$ by

$$(\mathcal{F}_{-i}\mathcal{G})_j = \bigoplus_{a_{jk} \leqslant i} \tilde{S}[-a_{jk}].$$

It is clear that for each $i$, $\mathcal{F}_{-i+1}\mathcal{G}$ is a subcomplex of $\mathcal{F}_{-i}\mathcal{G}$ and $\mathcal{F}_{-i}\mathcal{G}/\mathcal{F}_{-i+1}\mathcal{G} = \mathcal{L}_i \otimes \tilde{S}[-i]$, where $\mathcal{L}_i$ is a complex of $R$-modules.

THEOREM 5.8. *The following conditions are equivalent*:

(a) *$Z(E)$ is acyclic.*

(b) *All the complexes $\mathcal{L}_i$ are acyclic.*

*If the equivalent conditions hold, then $\mathfrak{L}_i$ is a minimal R-free resolution of $H_i(S_+; S(E))_i = Z_i(E)$ shifted i steps to the left. In particular, one has the relation of Betti numbers*

$$\beta_i^{\tilde{S}}(S(E)) = \sum_j \beta_{i-j}^R(Z_j(E)).$$

PROOF. For any graded $\tilde{S}$-module $M$, put $M^* = M/\tilde{S}_+M$. We have the isomorphism of graded modules

$$H_i(S_+; S(E)) = \operatorname{Tor}_i^{\tilde{S}}(R, S(E)) = H_i(\mathcal{G}^*),$$

and

$$\mathcal{G}^* = \bigoplus_{i\geqslant 0} (\mathcal{F}_{-i}\mathcal{G}/\mathcal{F}_{-i+1}\mathcal{G})^* = \bigoplus_{i\geqslant 0} \mathfrak{L}_i,$$

where the $\mathfrak{L}_i$ are complexes of $R$-modules $L_{ik}$ which, considered as $\tilde{S}$-modules, are concentrated in degree $i$. It follows that

$$H_i(S_+; S(E))_j = H_i(\mathfrak{L}_j).$$

The equivalence of (a) and (b) now follows from 3.2. The additional assertions of the theorem follow trivially. □

REMARK. If $\beta_{ij}$ denotes the $(R-)\,j$th Betti number of $H_i(S_+; S(E))_i$, and $Z(E)$ is acyclic, then the $\tilde{S}$-resolution of $S(E)$ looks like

$$\cdots \bigoplus_{j=1}^{i} \tilde{S}[-j]^{\beta_{j,i-j}} \to \cdots \to \tilde{S}[-2]^{\beta_{2,0}} \oplus \tilde{S}[-1]^{\beta_{1,1}}$$
$$\to \tilde{S}[-1]^{\beta_{1,0}} \to \tilde{S} \to S(E) \to 0.$$

**6. Cohen-Macaulay and Gorenstein symmetric algebras.** Throughout this section we assume $R$ is a Cohen-Macaulay ring. We focus on obtaining symmetric algebras which are Cohen-Macaulay in the context of acyclic $Z$-complexes. The main point that emerges is strict depth conditions on the coefficient modules $Z_i(E)$ of $Z(E)$.

Since the $Z_i(E)$ are Koszul homology modules, several of the methods used to prove acyclicity of $Z(E)$ require high depth on such modules. In **[12]**, for instance, for an ideal $I$ we required that $H_i(I; R)$ be Cohen-Macaulay for all $i$ although the proofs themselves made different demands on the various $H_i(I; R)$. We now introduce a sliding condition that is closer to the needs of the $Z$-complex.

Let $I$ be an ideal of the Cohen-Macaulay local ring; say $\mathbf{a} = \{a_1, \ldots, a_n\}$ is a generating set for $I$. We shall look at conditions of the type

$(\mathcal{SD}_k)$ depth $H_i(\mathbf{a}; R) \geqslant d - n + i + k$ for all $i$.

Here $d = \dim R$, $k =$ fixed integer and, as usual, depth(0) $= \infty$. In case $I$ is a homogeneous ideal of a graded $R$-algebra $A$, we will restrict to the $i$th component of $H_i(I; A)$ the discussion below.

Let us remark on some elementary properties of this notion—generally denoted *sliding depth*.

(i) First we claim $(\mathcal{SD}_k)$ is independent of the generating set **a**. To see this it is enough to compare $(\mathcal{SD}_k)$ for two generating sets **a** and $\mathbf{a}' = \{\mathbf{a}, 0\}$. Suppose $(\mathcal{SD}_k)$ holds for **a**: Then, for each $i$, we have

$$(*) \qquad H_i(\mathbf{a}'; R) = H_i(\mathbf{a}; R) \oplus H_{i-1}(\mathbf{a}; R),$$

therefore depth $H_i(\mathbf{a}'; R) \geqslant d - n + (i-1) + k$, as required. Conversely, if $(\mathcal{SD}_k)$ holds for $\mathbf{a}'$ but not for **a**, let $i$ be highest so that the condition fails, that is, pick $i$ largest with depth $H_i(\mathbf{a}; R) < d - n + i - 1 + k$. But writing $(*)$ for $i+1$ instead of $i$, we would then get a contradiction.

(ii) If $R$ is a Cohen-Macaulay ring, the localizations of $I$ will have $(\mathcal{SD}_k)$ if it holds true for the maximal ideals. More generally, let $R$ be a Cohen-Macaulay local ring of dimension $d$ and let $G$ be a finitely generated $R$-module. Assume depth $G \geqslant d - r$. Then for any prime ideal $P$, depth $G_P \geqslant \mathrm{ht}(P) - r$. This is clear if $R$ is regular, since $d -$ depth $G$ is then the projective dimension of $G$ which does not increase under localization. But the underlying idea also works in the more general case: Assume first that $R$ admits a canonical module $\omega$. In this case

$$d - \operatorname{depth} G = \sup\{ j \mid \operatorname{Ext}_R^j(G, \omega) \neq 0\}.$$

Since the localization $\omega_P$ is a canonical module for $R_P$, the assertion again follows.

For general Cohen-Macaulay rings a simple argument with its $\mathfrak{m}$-adic completion yields the same conclusion.

In the case of an $R$-module $E$, we define $(\mathcal{SD}_k)$ on $S(E)$ as

$$\operatorname{depth} H_i(S_+; S(E))_i \geqslant d - n + i + k, \qquad i \geqslant 0.$$

The unqualified *sliding depth* condition will refer to the case $k = \mathrm{rank}(E)$.

(iii) Despite the appearance, there is no ambiguity regarding the sliding depth on the ideal $I$: whether one means $H_i(I; R)$ or $H_i(S_+; S(I))_i$. Indeed, as noted the latter is the corresponding module of cycles in the Koszul complex $\mathcal{K}(\mathbf{a}; R)$ associated to **a**. Thus, with $B_i$ and $Z_i$ denoting the boundaries and cycles of $\mathcal{K}(\mathbf{a}; R)$, to show the equivalence of the two conditions we simply chase depths in the sequences (cf. [**12**]) $0 \to Z_{i+1} \to K_{i+1} \to B_i \to 0$ and $0 \to B_i \to Z_i \to H_i(\mathbf{a}; R) \to 0$.

EXAMPLE 6.1. It is easy to see that if $I$ is generated by a proper sequence $\mathbf{x} = \{x_1, \ldots, x_n\}$ and satisfies sliding depth, then for any initial subsequence $\mathbf{x}' = \{x_1, \ldots, x_m\}$, $m < n$, $J = (\mathbf{x}')$ will satisfy the same sliding depth condition. This is obtained directly from the Koszul homology sequences associated to $\mathbf{x}'$ and $\{\mathbf{x}', x_{m+1}\}$ and the properness. As a consequence, for any of the ideals known to have to Cohen-Macaulay Koszul homology (or, *strongly Cohen-Macaulay* in the terminology of Huneke) and is generated by proper sequences, one obtains various other ideals with sliding depth and not always Cohen-Macaulay.

Another way to obtain ideals of height one satisfying sliding depth is the following. Let $I$ satisfy sliding depth and assume it is generated by a $d$-sequence. By Remark (iii) above it follows that the ideal $S_+$ of the Rees algebra $S(I)$ will satisfy sliding depth as well.

The naturality of $(\mathcal{SD}_e)$ is brought into relief by

THEOREM 6.2. *Let $R$ be a Cohen-Macaulay local ring and $E$ a finitely generated $R$-module of* $\operatorname{rank}(E) = e$. *The following conditions are equivalent*:

(a) *$Z(E)$ is acyclic and $S(E)$ is Cohen-Macaulay.*

(b) *$E$ satisfies sliding depth* $((\mathcal{SD}_e))$ *and* $(\mathcal{F}_0)$.

Note that $(\mathcal{F}_0)$ and $(\mathcal{SD}_e)$ together place rather strict bounds on the coefficient modules $Z_r(E)$ of the complex $Z(E)$. Indeed, $(\mathcal{F}_0)$ implies $d - n + e \geqslant 0$, $d = \dim R$, $n = v(E)$, while $(\mathcal{SD}_e)$ requires $\operatorname{depth} Z_r(E) \geqslant (d - n + e) + r$ for all $r$. We also recall that $(\mathcal{F}_0)$ just means $S(E)$ has its expected Krull dimension, that is, $\dim S(E) = \dim R + \operatorname{rank}(E)$.

PROOF. (b) $\Rightarrow$ (a) The exactness of the complex $Z(E)$ follows as in [**14**]. A simple depth-counting argument shows $\operatorname{depth} S(E) \geqslant d + \operatorname{rank}(E)$, so $S(E)$ is Cohen-Macaulay.

(a) $\Rightarrow$ (b) Since $S(E)$ is Cohen-Macaulay, we have $\dim S(E) = \dim R + \operatorname{ht}(S_+) = \dim R + \operatorname{rank}(E)$, as in 5.4.

We may assume $R$ admits a canonical module $\omega_R$. Note that $\omega_{\tilde{S}} = \omega_R \otimes \tilde{S}$. If $M$ is an $\tilde{S}$-module, we put $M^{\vee} = \operatorname{Hom}_{\tilde{S}}(M, \omega_{\tilde{S}})$. We also abbreviate the notations of 5.8 and write $\mathcal{F}_{-i}$ instead of $\mathcal{F}_{-i}\check{\mathcal{G}}$.

*Claim.* $H^j((\mathcal{F}_{-i}/\mathcal{F}_{-i+1})^{\vee}) = 0$ for $j > l = n - e$.

Let us first see what this claim entails. If $\mathcal{L}_i$ is a minimal $R$-free resolution of $H_i(S_+; S(E))_i = Z_i(E)$, then from 5.8,

$$\mathcal{F}_{-i}/\mathcal{F}_{-i+1} = \mathcal{L}_i[-i] \otimes \tilde{S}[-i].$$

Therefore

$$(\#)\quad H^j\big(\operatorname{Hom}_{\tilde{S}}(\mathcal{L}_i[-i]) \otimes_R \tilde{S}[-i], \omega_{\tilde{S}}\big) = H^j\big(\operatorname{Hom}_R(\mathcal{L}_i[-i], \omega_R)\big) \otimes_R \tilde{S}[i]$$
$$= H^{j-i}\big(\operatorname{Hom}_R(\mathcal{L}_i, \omega_R)\big) \otimes_R \tilde{S}[i] = \operatorname{Ext}_R^{j-i}\big(Z_i(E), \omega_R\big) \otimes \tilde{S}[i].$$

If the last module vanishes, then $\operatorname{depth} Z_i(E) \geqslant d - (j - i) + 1$; in particular, this holds for $j > n - e$, from which sliding depth follows.

PROOF OF THE CLAIM. We show by induction on $r$ that

$$(\#\#)\qquad H^j\big((\mathcal{F}_{-l+r}/\mathcal{F}_{-l+r+1})^{\vee}\big) = H^j\big((\mathcal{F}_{-l+r})^{\vee}\big) = 0 \quad \text{for } j > l.$$

$r = 0$: Since $\mathcal{F}_{-l} = \mathcal{G}$ and $S$ is Cohen-Macaulay, we have

$$H^j\big((\mathcal{F}_{-l})^{\vee}\big) = \operatorname{Ext}^j_{\tilde{S}}(S, \omega_{\tilde{S}}) = 0 \quad \text{for } j > l.$$

The exact sequence $0 \to \mathcal{F}_{-l+1} \to \mathcal{F}_{-l} \to \mathcal{F}_{-l}/\mathcal{F}_{-l+1} \to 0$ gives rise to the homology exact sequence

$$H^{j-1}\big((\mathcal{F}_{-l+1})^{\vee}\big) \xrightarrow{\phi} H^j\big((\mathcal{F}_{-l}/\mathcal{F}_{-l+1})^{\vee}\big) \to H^j\big((\mathcal{F}_{-l})^{\vee}\big).$$

Suppose $j > l$; then $\phi$ is surjective. Now $H^j((\mathcal{F}_{-l}/\mathcal{F}_{-l+1})^{\vee})$ is generated by elements of degree $-l$, see (#), while $H^{j-1}((\mathcal{F}_{-l+1})^{\vee})$ is generated by elements of degree $\geqslant -l + 1$. It follows that $H^j((\mathcal{F}_{-l}/\mathcal{F}_{-l+1})^{\vee}) = 0$.

The proof of the induction step is similar. □

COROLLARY 6.3. *Let $R$ be a Cohen-Macaulay integral domain and $E$ a finitely generated $R$-module satisfying $(\mathcal{SD}_e)$ and $(\mathcal{F}_1)$. Then $S(E)$ is a Cohen-Macaulay integral domain.*

COROLLARY 6.4. *Let $R$ be a Cohen-Macaulay ring and $I$ an ideal generated by a d-sequence. The following conditions are equivalent:*

(a) *The Rees algebra $R(I) = \oplus_{t\geq 0} I^t$ is Cohen-Macaulay.*

(b) *The associated graded algebra $\mathrm{gr}_I(R) = \oplus_{t\geq 0} I^t/I^{t+1}$ is Cohen-Macaulay.*

(c) *$I$ satisfies sliding depth.*

With the notations of 6.2, we have

THEOREM 6.5. *Suppose $Z(E)$ is acyclic and $S = S(E)$ is Cohen-Macaulay. Then:*

(a) $\omega_S/S_+\omega_S = \oplus_{i=0}^{l} \mathrm{Ext}_R^{l-i}(Z_i(E), \omega_R)$.

(b) *The following conditions are equivalent:*

(b$_1$) *$S$ is Gorenstein.*

(b$_2$) $$\operatorname{depth} Z_i(E) \geq d - n + e + i + 1 \quad \textit{for } i = 0,\ldots, l-1,$$

*and* $\mathrm{Hom}_R(Z_l(E), \omega_R) = R$.

PROOF. (b) follows directly from (a). To prove (a), let $\omega_S = \mathrm{Ext}_{\tilde{S}}^{l}(S, \omega_{\tilde{S}})$ be the canonical module of $S$. We use the results of (6.2): By (# #) we have exact sequences

$$H^{l-1}\big((\mathcal{F}_{-i+1})\check{}\big) \xrightarrow{\phi_i} H^l\big((\mathcal{F}_{-i}/\mathcal{F}_{-i+1})\check{}\big) \xrightarrow{\psi_i} H^l\big((\mathcal{F}_{-i})\check{}\big) \to H^l\big((\mathcal{F}_{-i+1})\check{}\big) \to 0.$$

Denote again by '$*$' the reduction $S \to S/S_+$. We obtain the exact sequence of $R$-modules

$$\big(H^l\big((\mathcal{F}_{-i}/\mathcal{F}_{-i+1})\check{}\big)\big)^* \xrightarrow{\psi_i^*} \big(H^l\big((\mathcal{F}_{-i})\check{}\big)\big)^* \to \big(H^l\big((\mathcal{F}_{-i+1})\check{}\big)\big)^* \to 0.$$

Since $H^l((\mathcal{F}_{-i}/\mathcal{F}_{-i+1})\check{})$ is generated by elements of degree $-i$, cf. (#), $\psi_i^*$ equals the $(-i)$-graded part of $\psi_i$:

$$(\psi_i)_{-i}\colon H^l\big((\mathcal{F}_{-i}/\mathcal{F}_{-i+1})\check{}\big)_{-i} \to H^l\big((\mathcal{F}_{-i})\check{}\big)_{-i}.$$

Since, on the other hand,

$$\mathrm{Image}(\phi_i) \subset \bigoplus_{j\geq -i+1} H^l\big((\mathcal{F}_{-i}/\mathcal{F}_{-i+1})\check{}\big)_j,$$

it follows that $(\psi_i)_{-i} = \psi_i^*$ is injective. Hence we obtain the exact sequence

$$0 \to \big(H^l\big((\mathcal{F}_{-i}/\mathcal{F}_{-i+1})\check{}\big)\big)^* \xrightarrow{\psi_i^*} \big(H^l\big((\mathcal{F}_{-i})\check{}\big)\big)^* \to \big(H^l\big((\mathcal{F}_{-i+1})\check{}\big)\big)^* \to 0.$$

Arguing with degrees, we see that this exact sequence splits, which gives

$$(\omega_s)^* = \big(\mathrm{Ext}_{\tilde{S}}^{l}(S, \omega_{\tilde{S}})\big)^* = \big(H^l\big((\mathcal{F}_{-l})\check{}\big)\big)^* = \bigoplus_i \big(H^l\big((\mathcal{F}_{-i}/\mathcal{F}_{-i+1})\check{}\big)\big)^*.$$

The assertion now follows from (#) in the proof of 6.2. □

COROLLARY 6.6. *Let $R$ be a Cohen-Macaulay ring. If $I$ is a strongly Cohen-Macaulay ideal and $v(I_P) \leq \mathrm{ht}(P) + 1$ for all primes $P$, then $S = S(I)$ is Cohen-Macaulay. If $R$ admits a canonical module and $\mathrm{ht}(I) = g \geq 2$, then $\omega_S/S_+\omega_S = \omega_R \oplus (\omega_R/I\omega_R)^{g-2}$.*

PROOF. See [13] for the acyclicity of $Z(E)$. If $I$ is strongly Cohen-Macaulay and $Z$ denotes the cycles of a Koszul complex associated with $I$, then ($n = v(I), d = \dim R$)

$$\operatorname{depth} Z_i(E)\begin{cases} \geqslant d-g+2 & \text{for } i = 1,\ldots,n-g+1, \\ = d+i-(n-1) & \text{for } i = n-g+1,\ldots,n-1. \end{cases}$$

It follows that

$$\operatorname{Ext}_R^{l-i}(Z_i(E),\omega_R) = \operatorname{Ext}_R^{n-1-i}(Z_i(E),\omega_R) = 0 \qquad (l = n-1)$$

for $i = 1,\ldots,n-g$, since $(n-1-i)+(d-g+2) \geqslant d+1$ for $i$ in the first range. For $i \geqslant n-g+1$, we have the exact sequences

$$0 \to K_n \to K_{n-1} \to \cdots \to K_{i+1} \to Z_i(E) \to 0,$$

and therefore,

$$\operatorname{Ext}_R^{n-1-i}(Z_i(E),\omega_R) = \omega_R/I\omega_R.$$

The assertion now follows from 6.5(a). □

COROLLARY 6.7 [27]. *Let $R$ be a Cohen-Macaulay ring and $I$ an ideal containing regular elements. If $S(I)$ is Gorenstein then $R$ is Gorenstein and* $\operatorname{ht}(I) \leqslant 2$.

PROOF. By 6.4, $S(I)$ Gorenstein implies $\operatorname{Hom}_R(Z_l(I),\omega_R) = R$; since $Z_l(I) = R$, $R$ is Gorenstein. After localizing at a minimal prime of $I$, we may assume $I$ is primary relative to the maximal ideal. Since $S(I)$ is Cohen-Macaulay, we have $v(I) \leqslant \dim R + 1$ [13]. Thus we may apply 6.6 since $I$ is now strongly Cohen-Macaulay.

COROLLARY 6.8. *Let $R$ be a regular local ring and $E$ a finitely generated $R$-module. The following conditions are equivalent*:

(a) *$Z(E)$ is acyclic and $S(E)$ is Gorenstein.*

(b) *$E$ has projective dimension at most one and satisfies* ($\mathcal{F}_0$).

PROOF. (a) ⇒ (b) ($\mathcal{F}_0$) follows from the Cohen-Macaulayness of $S(E)$ [13]. As $Z(E)$ is acyclic, we have (cf. §5)

$$\mathcal{G}_i = \bigoplus_{j=1}^{i} \tilde{S}[-j]^{\beta_{j,i-j}}.$$

Since $S(E)$ is Gorenstein, $\mathcal{G}_l = \tilde{S}[-l]$ and $\mathcal{G}$ is self-dual. It follows that $\mathcal{G}_i = \tilde{S}[-i]^{\beta_{i,0}}$ and, hence, all the modules $Z_i(E)$ are free. In particular, $Z_1(E)$ is free and $\operatorname{pd} E \leqslant 1$.

(b) ⇒ (a) follows from 4.1. □

**7. Reflexive symmetric algebras.** We are now concerned with the symmetric algebra of a module $E$, $S(E) = \bigoplus \operatorname{Sym}_t(E)$, with each $\operatorname{Sym}_t(E)$ a reflexive module. When $R$ is a Cohen-Macaulay ring we have already identified a necessary condition, namely ($\mathcal{F}_2$). If $E$ is a module of projective dimension one, $0 \to R^m \to R^n \to E \to 0$, ($\mathcal{F}_2$) is also a sufficient condition [30]. The paucity of other classes of examples has led us to state

CONJECTURE 7.1. Let $R$ be a regular local and let $E$ be a finitely generated $R$-module. If $S(E)$ is reflexive, then $\operatorname{pd} E \leqslant 1$.

Stated otherwise it says that if $S(E)$ is factorial then $S(E)$ must be a complete intersection. In this section we shall describe some of the emerging evidence in support of this conjecture. We shall often depart from the hypothesis $R$ = regular, but the finiteness of the projective dimension will be kept.

PROPOSITION 7.2. *Let R be a Cohen-Macaulay ring and let E be a finitely generated R-module. If E satisfies* $(\mathcal{F}_2)$, *then* pd $E \neq 2$.

PROOF. Assume otherwise; pick $R$ local with lowest possible dimension, that is, we may assume $\mathrm{pd}_{R_P} E_P \leqslant 1$ for each prime $P \neq \mathfrak{m}$ = maximal ideal of $R$. Let

$$0 \to R^r \xrightarrow{\psi} R^m \xrightarrow{\phi} R^n \to E \to 0$$

be a minimal resolution of $E$. Since $E$ satisfies $(\mathcal{F}_2)$, we have

$$n = v(E) \leqslant \dim R + \mathrm{rank}(E) - 2,$$

that is

$$n - r = l = \mathrm{rank}(\phi) = n - \mathrm{rank}(E) \leqslant \dim R - 2.$$

Since $r \neq 0$, the ideal $I_r(\psi)$ is, by induction, $\mathfrak{m}$-primary. From [**6**], however, we have

$$\dim R = \mathrm{ht}(I_r(\psi)) \leqslant m - r + 1 = l - 1,$$

which is a contradiction. □

In the discussion of the next cases we shall bring in the $Z$-complex of $E$.

Let $E$ be a module of projective dimension 3:

$$0 \to R^r \xrightarrow{\theta} R^s \xrightarrow{\psi} R^m \xrightarrow{\phi} R^n \to E \to 0.$$

As above we assume $S(E)$ is reflexive and $R$ has lowest possible dimension. Further, assume $R$ contains a field.

Again from $(\mathcal{F}_2)$ we have $\mathrm{rank}(\phi) = l \leqslant d - 2$. Since $L = \mathrm{image}(\phi)$ has depth $d - 2$, by the induction hypothesis and the main result of [7], we may, in fact, assume $l = d - 2$. Consider the Koszul complex of the embedding $L \to R^n$ (cf. §4).

$$(*) \qquad 0 \to \bigwedge^{t} L \to \bigwedge^{t-1} L \otimes \tilde{S}_1 \to \cdots \to L \otimes \tilde{S}_{t-1} \to \tilde{S}_t \to \mathrm{Sym}_t(E) \to 0.$$

If each $\wedge^i L$, $1 \leqslant i \leqslant t$, has depth exceeding $i$, $(*)$ agrees with the corresponding subcomplex of $Z(E)$, cf. 4.1. Let us estimate the depth of these exterior powers. The free complex $\mathcal{C}_i$ of [**20** and **34**] has length $\lambda_i = \inf\{i + r, 2i\}$. If $\lambda_i - 2 + i \leqslant d - 2$, $\mathcal{C}_i$ is a minimal free resolution of $\wedge^i L$, which is, besides, an $i$th syzygy module.

Suppose there exists $t$ such that

$$(**) \qquad \lambda_t + t \leqslant d \leqslant \lambda_t + t + 1.$$

(Note that the right-hand side may not be the same as $\lambda_{t+1} + (t + 1)$; also, in this range the $\wedge^i L$ will have depth $\geqslant i$.)

Chasing depths in the complex $(*)$—taking into account that $\mathrm{Sym}_t(E)$ is reflexive and thus a second syzygy module—it follows that depth $\wedge^t L \geqslant t + 1$ if $d - \lambda_t = t$ and depth $\wedge^t L \geqslant t + 2$ if $d - \lambda_t = t + 1$, which contradicts the existence of $t$.

Unfortunately this does not occur for many values of $d$ and $r$. The first unsettled case in $d = 8$, $r = 3$. Nevertheless we have

PROPOSITION 7.3. *Let $E$ be a module of projective dimension* 3. *If the third Betti number $\beta_R^3(E) = 1$ or* 2, *then $S(E)$ is not factorial.*

Let us now outline a construction which may lead to a counterexample to 7.1. It involves the $Z$-complex more intimately.

Suppose $I$ is an ideal of the regular local ring $R$ with the following properties:

(i) $\mathrm{ht}(I) = 2$;

(ii) $I$ is generated by a $d$-sequence;

(iii) the powers $I^t$, $t \geqslant 1$, are unmixed.

For the construction, consider the module $\mathrm{Ext}^1_R(I, R)$: At each associated prime $\mathfrak{p}$ of $I$, $I_\mathfrak{p}$ is generated by a regular sequence of 2 elements from (i) and (ii); therefore we can find an extension

$$\xi\colon 0 \to R \to E \to 0$$

which generates $\mathrm{Ext}^1_R(I, R)$ at each such prime.

PROPOSITION 7.4. *$S(E)$ is factorial.*

PROOF. We show that each $\mathrm{Sym}_t(E)$ is reflexive. For $t = 1$ we use the choice of $\xi$ and Serre's lemma: Localizing at a prime $\mathfrak{p}$ of height 2 it follows that $E_\mathfrak{p}$ is free. If $\mathrm{ht}(\mathfrak{p}) \geqslant 3$ the unmixedness of $I$ guarantees depth $E \geqslant 2$. Thus $E$ is a second syzygy module.

For higher $t$ consider the exact sequence induced by $\xi$:

$$0 \to \mathrm{Sym}_{t-1}(E) \otimes R \to \mathrm{Sym}_t(E) \to \mathrm{Sym}_t(I) = I^t \to 0,$$

where (ii) is again used. From (iii) it follows that $\mathrm{Sym}_t(E)$ is reflexive along with $\mathrm{Sym}_{t-1}(E)$. □

The conjecture asserts that if $R$ is regular then such ideals are perfect. Note that the complex $Z(E)$ will, by 5.6, be acyclic. If $S(E)$ is Cohen-Macaulay, we have by 6.7, that pd $E \leqslant 1$.

Let us argue that the conditions of 7.4 are highly unlikely by considering how the properties of the complex $M(I; R)$ **[14]** place constraints on the homological properties of an ideal $I$ generated by a $d$-sequence.

PROPOSITION 7.5. *Let $R$ be a Gorenstein ring and $I$ an ideal of height* 2 *generated by a d-sequence.*

(a) *If $I$ is unmixed, then* pd $I \neq 2$.

(b) *If $I$ and $I^2$ are unmixed and $I$ is perfect at primes of height* 5, *then* pd $I \neq 3$ *as well.*

(c) *If the powers $I^t$, $t \geqslant 1$, are unmixed and $I$ satisfies sliding depth, then $I$ is a Cohen-Macaulay ideal.*

PROOF. For a presentation $0 \to L = Z_1(E) \to R^n \to I \to 0$, we shall prove $L$ is a Cohen-Macaulay module.

Since $I$ is generated by a $d$-sequence, the subcomplexes of $M(I;\, R)$,

$$0 \to H_t \to H_{t-1} \otimes \tilde{S}_1 \to \cdots \to H_1 \otimes \tilde{S}_{t-1} \to H_0 \otimes \tilde{S}_t$$
$$\to \mathrm{Sym}_t(I/I^2) = I^t/I^{t+1} \to 0,$$

are exact. Here $H_t$ denotes the homology of the Koszul complex $\mathcal{K}(\mathbf{a};\, R)$ associated to the $n$ generators in the presentation above. (Note that in this case, cf. 3.3, $Z(I/I^2) = M(I;\, R)$.)

As $I$ is unmixed and $\dim R = d \geqslant 4$, making $t = 1$ we have that $H_1$ is a submodule of $(R/I)^n$, and thus depth $H_1 \geqslant 1$ in case (a) and depth $H_1 \geqslant 2$ in case (b) (we have now used the fact that $d \geqslant 6$ for (b)). Using the complex for higher $t$'s, we similarly get depth $H_t \geqslant 1, 2$ in the respective cases (a), (b).

We use these values to estimate the depths of the modules of cycles of $\mathcal{K}(\mathbf{a};\, R)$. Denote by $B_i$ and $Z_i$ the boundaries and cycles of $\mathcal{K}(\mathbf{a};\, R)$. From the sequences $0 \to B_i \to Z_i \to H_i \to 0$ and $0 \to Z_{i+1} \to K_{i+1} \to B_i \to 0$ we get: In case (a), depth $Z_i \geqslant 3$, and in case (b), depth $Z_i \geqslant 4$, for all $i$.

Recall now (cf. 4.3) that $Z_{n-2} = Z_1^* = L^*$. We thus have

$$\text{depth } L + \text{depth } L^* \geqslant d - \text{pd } I + 1 + \{3(\text{a}) \text{ or } 4(\text{b})\} \geqslant d + 2.$$

By the duality theorem of [**10**], $L$ is a Cohen-Macaulay module.

To prove (c), note that the sliding depth condition implies

$$\text{depth } Z_1 + \text{depth } Z_{n-2} \geqslant (d - n + 2) + (d - n + (n - 1)) = d + 1 + (d - n),$$

and all that remains—to use the duality theorem—is to show $d > n$. For that, consider the algebra $\mathrm{gr}_I(R)$. Since depth $I^t/I^{t+1} \geqslant 1$, we may, by [**3**], find an element $x \in \mathfrak{m}$ = maximal ideal of $R$, regular on $\mathrm{gr}_I(R)$. It follows that the analytic spread, $l(I)$, is at most $d - 1$. But for ideals with $S(I/I^2) = \mathrm{gr}_I(R)$, $v(I) = l(I)$. □

REMARKS. (i) Note that for $R$ = regular, (c) is already taken care by the construction of 7.4 and the criterion 6.8.

(ii) The earliest dimension for which 7.1 may fail is $d = 5$. By $(\mathcal{F}_2)$, for such a module $E$ we must have $v(E) \leqslant 5 + \mathrm{rank}(E) - 2$, so we are in a situation similar to Example 4.4, i.e. $\mathrm{rank}(L) = 3$: Therefore the complex $Z(E)$ is acyclic. If $S(E)$ is known to be Cohen-Macaulay, the question will again be settled by 6.8.

(iii) There is a partial converse to the construction 7.4. Indeed, if $E$ is a module of rank 2 for which $Z(E)$ is acyclic and $S(E)$ is factorial, we would have: From 5.6 there exists a Bourbaki sequence $0 \to R \overset{\psi}{\to} E \to I \to 0$, where $I$ is generated by a proper sequence and is unmixed. Since $S(E)$ is factorial it is clear that the element $e = \psi(1)$ is prime, and therefore $S(I)$ is a domain. From [**13**] it follows that $I$ is generated by a $d$-sequence ($R$ = local, infinite residue field). Furthermore, the exact sequence

$$0 \to E \otimes R \to \mathrm{Sym}_2(E) \to \mathrm{Sym}_2(I) = I^2 \to 0$$

implies that, if depth $E \geqslant 3$, then $I^2$ will also be unmixed. This may also be the case for the higher powers of $I$.

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