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ON CERTAIN ELEMENTARY EXTENSIONS OF MODELS OF SET THEORY

BY
ALI ENAYAT

ABSTRACT. In §1 we study two canonical methods of producing models of ZFC with no elementary end extensions. §2 is devoted to certain "completeness" theorems dealing with elementary extensions, e.g., using $\diamondsuit_{\omega_1}$ we show that for a consistent $T \supseteq$ ZFC the property "Every model $\mathfrak{A}$ of T has an elementary extension fixing $\omega^{\mathfrak{A}}$" is equivalent to $T \vdash$ "There exists an uncountable measurable cardinal". We also give characterizations of $T \vdash$ "κ is weakly compact" and $T \vdash$ "κ is measurable" in terms of elementary extensions.

Introduction. This paper deals with the study of elementary extensions of models of set theory, an area first systematically investigated by Keisler and Morley in **[KM]**, and later by Keisler and Silver in **[KS]**. Two recent contributions come from J. Hutchinson **[H1]** and M. Kaufmann **[Ka]**. Apart from its intrinsic interest as a chapter of (western) model theory, the study of elementary extensions of models of set theory has also benefitted general model theory, often by yielding soft proofs of known theorems, and sometimes, providing new results. The interested reader is referred to Hutchinson **[H2]** for a sample of such applications.

In §1 we discuss models of set theory with no elementary end extensions. Specifically, we provide two distinct methods which allow one to construct models of any consistent theory $T \supseteq$ ZFC with no elementary end extensions. This section is closely tied with the work of Kaufmann in **[Ka]**.

§2 deals with certain "completeness" theorems, and once more reminds us of the nice behavior of first-order logic. More precisely, we identify, in our "completeness" theorems, first-order equivalents of certain second-order properties of set theories. For example, in Theorem 2.17, we show that for a consistent theory $T \supseteq$ ZFC the second-order property "Every model $\mathfrak{A}$ of T has an elementary extension fixing $\omega^{\mathfrak{A}}$" is equivalent to

$$T \vdash \text{"There exists an uncountable measurable cardinal"}.$$

In contrast, in Theorem 2.6 we present an "incompleteness" phenomenon found by Kunen.

The somewhat spartan models first introduced by Keisler and Kunen in **[K3]**, and later generalized by Shelah in [S] and Rubin in **[RS]**, play a key role in §2. Because

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of their dependence on a result of Rubin and Shelah [**RS**], certain results in this section require $\diamondsuit_{\omega_1}$. We suspect that these results are true outright in ZFC. Some of the results of this paper were announced in [**E**].

0. Notation and conventions. (a) *Language*. Our language consists of $\{\in\}$ and possibly countably many constant symbols. All our results remain true if finitely many relation or function symbols are present, provided of course, they appear in the replacement scheme.

(b) *Structures*. Models of set theory are of the form $\mathfrak{A} = \langle A, E, \dots \rangle$ where E is a binary (usually non-well-founded) relation on A. Universes of structures are usually denoted by the corresponding Roman letters. Given a member c of $\mathfrak{A}$ we denote the extension of c in $\mathfrak{A}$ by c_E, hence $c_E = \{b \in A\colon b\ E\ c\}$. If α is an ordinal of $\mathfrak{A}$, $\mathfrak{A}_\alpha$ refers to what $\mathfrak{A}$ 'thinks' is $\langle R(\alpha), \in \rangle$. Any order-theoretic attribute of a model $\mathfrak{A}$ must be understood as the attribute of its linearly ordered set of ordinals $\langle \mathrm{Ord}(\mathfrak{A}), E\rangle$, hence $\mathrm{cf}(\mathfrak{A}) > \omega$ means $\mathrm{cf}(\langle \mathrm{Ord}(\mathfrak{A}), E\rangle) > \omega$. Given models $\mathfrak{A} = \langle A, E, \dots \rangle \subseteq \mathfrak{B} = \langle B, F, \dots \rangle$ we say that $\mathfrak{B}$ *fixes* an element c of $\mathfrak{A}$ if $c_E = c_F$, else $\mathfrak{B}$ is said to *enlarge* c. If $\mathfrak{B}$ fixes all elements of $\mathfrak{A}$ then we say that $\mathfrak{B}$ is an *end extension* of $\mathfrak{A}$, denoted $\mathfrak{A} \subset_e \mathfrak{B}$. $\mathfrak{B}$ is said to *rank extend* $\mathfrak{A}$, denoted $\mathfrak{A} \subset_r \mathfrak{B}$, if $\mathfrak{A} \subset_e \mathfrak{B}$ and every new element of $\mathfrak{B}$ is above $\mathfrak{A}$, i.e., if $b \in B \setminus A$ then $\mathrm{rank}(b) \in B \setminus A$. It is important to note that $\mathfrak{A} \prec_e \mathfrak{B}$ implies $\mathfrak{A} \prec_r \mathfrak{B}$. Lastly $\mathfrak{A} \prec_n \mathfrak{B}$ means that $\mathfrak{A}$ is a Σ_n-elementary submodel of $\mathfrak{B}$. We usually abbreviate the clause "elementary end extension" by "e.e.e."

1. Models with no rank extensions. Following the classical theorem of MacDowell and Specker on e.e.e.'s of models of Peano Arithmetic in [**MS**], Keisler and Morley proved that every model of ZF with countable cofinality has an e.e.e. Later [**KS**] dashed the hopes of a complete analogy with arithmetic by providing ad hoc examples of models of set theory with no elementary end extensions. In particular, they showed that the model $\langle R_\kappa, \varepsilon\rangle$, where κ is the first strongly inaccessible cardinal, cannot be elementarily end extended. This did not settle the question whether for some strengthening T of ZFC, every model of T has an e.e.e. Kaufmann recently answered the question in the negative in [**Ka**]. His proof utilized ω_1-like rather classless models (defined below). We later noticed that the property of being rather classless is sufficient to obstruct elementary end extensions (an idea crucial to the proof of Theorems 2.17 and 2.18 below) and that by using ω_1-like e.e.e.'s of the so-called D.O. models (also defined below) one may altogether dispense with the heavy artillery of rather classless models in Kaufmann's theorem. Indeed, our proofs show something stronger: *If* $\mathfrak{A}$ *is either rather classless or an* ω_1*-like e.e.e. of a D.O. model, then* $\mathfrak{A}$ *has no rank extension satisfying* ZFC. Note that although, as mentioned earlier, the model $\langle R_\kappa, \varepsilon\rangle$ (where κ is the first strongly inaccessible cardinal) has no e.e.e.'s, it may very well have a rank extension satisfying ZFC, e.g., if there are two strongly inaccessible cardinals.

We now give the definitions of rather classless and D.O. models and state their existence theorems.

DEFINITION 1.1. A subset $X \subseteq A$ is a *class* of $\mathfrak{A}$ if for all a in A, $X \cap a_E = b_E$ for some $b \in A$. $\mathfrak{A}$ is *rather classless* if all classes of $\mathfrak{A}$ are first-order definable (with parameters) in $\mathfrak{A}$.

The following is a specific instance of a theorem first proven by Keisler and Kunen using $\diamondsuit_{\omega_1}$ in **[K3]**. Later Shelah **[S]** eliminated $\diamondsuit_{\omega_1}$ by an absoluteness argument.

THEOREM 1.2 (KEISLER - KUNEN - SHELAH). *Every countable model of* ZF *has an* ω_1*-like rather classless elementary end extension.*

DEFINITION 1.3. A model $\mathfrak{A}$ of set theory is a D.O. model if every ordinal of $\mathfrak{A}$ is first-order definable in $\mathfrak{A}$.

D.O. models were first introduced by Paris in **[P]**. He gave the following existence and uniqueness theorem. The proof given in **[P]** of Theorem 1.4 can be considerably shortened by quoting, rather than reproving, the Henkin-Orey omitting types theorem.

THEOREM 1.4 (PARIS). *Every consistent extension T of* ZF *has a D.O. model. Moreover, if T is complete, T has a unique D.O. model iff* $T \vdash$ "V = HOD".

The main result of this section is

THEOREM 1.5. *A model* $\mathfrak{A} \models$ ZFC *has no rank extension satisfying* ZFC *if it satisfies one of the following conditions.*

(a) $\mathfrak{A}$ *is rather classless.*

(b) $\mathfrak{A}$ *is an* ω_1*-like e.e.e. of a D.O. model.*

PROOF OF (a). Assume on the contrary and suppose $\mathfrak{A}$ is rather classless and $\mathfrak{A} \underset{r}{\subset} \mathfrak{P} \models$ ZFC. Using the reflection theorem in $\mathfrak{P}$ let $\alpha \in \mathrm{Ord}(\mathfrak{P}) \setminus A$ such that $\mathfrak{P}_\alpha \underset{3}{\prec} \mathfrak{P}$ and pick some $c \in B_\alpha \setminus A$. Using choice in $\mathfrak{P}$ we may fix a well-ordering $\lhd$ of B_α and in $\mathfrak{P}$ define for each $\beta < \alpha$ the model $\mathfrak{C}_\beta < \mathfrak{P}_\alpha$ by defining C_β as the collection of all elements of R_α first-order definable in $\langle R_\alpha, \varepsilon, \lhd \rangle$, using parameters from $R_\beta \cup \{c\}$. Now from the outside, define $\mathfrak{C} = \bigcup_{\beta \in \mathrm{Ord}(\mathfrak{A})} \mathfrak{C}_\beta$. At this point we have

$$\mathfrak{A} \underset{e}{\subset} \mathfrak{C} \prec \mathfrak{P}_\alpha \underset{3}{\prec} \mathfrak{P}.$$

If $\mathrm{Ord}(\mathfrak{C}) \setminus A$ has a minimum element γ then we reach a contradiction by the following trick of Kaufmann: Since $\mathfrak{C} \underset{3}{\prec} \mathfrak{P}$, $\mathfrak{C}$ is strong enough to define the full satisfaction class of $\mathfrak{A} = \mathfrak{C}_\gamma$, i.e., the collection $\mathrm{Sat}(\mathfrak{A})$ defined as

$$\mathrm{Sat}(\mathfrak{A}) = \{\langle \varphi, \vec{a} \rangle : \mathfrak{C} \models (\langle R_\gamma, \varepsilon \rangle \models \varphi[\vec{a}])\}.$$

But $\mathrm{Sat}(\mathfrak{A})$ is easily seen to be a class of $\mathfrak{A}$ and hence definable in $\mathfrak{A}$, contradicting a version of Tarski's theorem on undefinability of truth.

On the other hand, if $\mathrm{Ord}(\mathfrak{C}) \setminus A$ has no minimum element we argue as follows. First consider the set Σ defined as

$$\Sigma = \{\varphi(x, \vec{a}) : \vec{a} \in A \text{ and } \mathfrak{P} \models (\langle R_\alpha, \varepsilon, \lhd \rangle \models \varphi(c, \vec{a}))\}.$$

Here $\varphi(x, \bar{y})$ is a formula, in the sense of $\mathfrak{P}$, in the language $\{\varepsilon, \lhd\}$. So we may think of Σ as a one-type, i.e., the type of c in $\langle R_\alpha, \varepsilon, \lhd\rangle$ over A. Now the crucial idea is that Σ is definable in $\mathfrak{A}$ by virtue of being a class of $\mathfrak{A}$. This allows us to "talk about" $\mathfrak{C}$ within $\mathfrak{A}$. In particular, we may define the set Γ in $\mathfrak{A}$ as

$$\Gamma = \{t(x, \bar{a}) : \ulcorner t(x, \bar{a}) \text{ is an ordinal}\urcorner \in \Sigma \text{ and } \forall b \ulcorner t(x, \bar{a}) \neq b\urcorner \in \Sigma\}.$$

Here $t(x, \bar{y})$ is a (definable) term in the language $\{\varepsilon, \lhd\}$ in the sense of $\mathfrak{A}$, and $\ulcorner\varphi\urcorner$ codes φ. Hence Γ corresponds to $\mathrm{Ord}(\mathfrak{C})\setminus A$.

Invoking our assumption on the absence of any minimum element in $\mathrm{Ord}(\mathfrak{C})\setminus A$, we have

$$(*) \qquad \mathfrak{A} \models (\forall t \in \Gamma)(\exists t' \in \Gamma)(\ulcorner t' \in t\urcorner \in \Sigma).$$

But $\mathfrak{A}$ has a definable universal choice function since in $\mathfrak{P}$ we may fix any choice function g on some R_θ, where θ is in $\mathrm{Ord}(\mathfrak{P})\setminus A$, and then verify that $\{\langle a, a'\rangle \in A^2 : g(a) = a'\}$ is a class of $\mathfrak{A}$, and hence definable. Therefore $(*)$ implies

$$\mathfrak{A} \models \exists f \forall n < \omega \ulcorner f(n+1) \in f(n)\urcorner \in \Sigma$$

which, in turn, by absoluteness of ω between $\mathfrak{A}$ and $\mathfrak{P}$, implies

$$\mathfrak{P} \models \exists f \forall n < \omega(\langle R_\alpha, \varepsilon, \lhd\rangle \models f(n+1)(c) \in f(n)(c)),$$

contradicting the well-foundedness of R_α in $\mathfrak{P}$. □

REMARK 1.6. We remark in passing that the above *proof* shows that *no* model $\mathfrak{A} \models \mathrm{ZFC}$ has a *conservative* rank extension to another model $\mathfrak{P} \models \mathrm{ZFC}$. ($\mathfrak{P}$ is a conservative rank extension of $\mathfrak{A}$ if the intersection of any first-order definable (with parameters) subset of B with A is itself first-order definable with parameters.) This was also noticed by Kaufmann in "Added in Proof" of [**Ka**]. This is of interest since every model of *PA*, by the MacDowell-Specker theorem, has a conservative e.e.e., and in *PA* nonconservative extensions are harder to come by than conservative ones. See e.g. R. G. Phillips [**Ph**].

PROOF OF THEOREM 1.5(b). Suppose $\mathfrak{A}'$ is D.O. and $\mathfrak{A}' \prec_e \mathfrak{A}$ where $\mathfrak{A}$ is ω_1-like. Assume on the contrary that

$$\mathfrak{A} \subset_r \mathfrak{P} \models \mathrm{ZFC}.$$

At this point we quote a result of Kaufmann (Theorem 3.3 of [**Ka**]): Any ω_1-like model $\mathfrak{A}$ with an e.e.e. $\mathfrak{P}'$ has an e.e.e. $\mathfrak{P}$ such that $\mathrm{Ord}(\mathfrak{P})\setminus A$ has a minimum element. The proof given by Kaufmann proves the stronger result: If an ω_1-like model $\mathfrak{A}$ of ZFC has a rank extension $\mathfrak{P} \models \mathrm{ZFC}$ such that $\min(\mathrm{Ord}(\mathfrak{P})\setminus A)$ does not exist, then $\mathfrak{A}$ has an e.e.e. $\mathfrak{P}_1$ such that $\min(\mathrm{Ord}(\mathfrak{P}_1)\setminus A)$ exists ($\mathfrak{A}$ will also have an e.e.e. $\mathfrak{P}_2$ such that $\min(\mathrm{Ord}(\mathfrak{P}_2)\setminus A)$ does not exist). Hence without loss of generality, for some $\gamma \in \mathrm{Ord}(\mathfrak{P})$, $\mathfrak{A} = \mathfrak{P}_\gamma$. But then arguing in $\mathfrak{P}$, we note that every member of ω_1 is definable in $\langle R_\gamma, \varepsilon\rangle$, implying that ω_1 is countable! □

REMARK 1.7. One cannot strengthen Theorem 1.5 by weakening "rank extension" to "end extension". This can be easily seen by a forcing argument. Since most expositions of forcing concentrate on generic extensions of standard models, we briefly elaborate. Given any model $\mathfrak{A}$ of set theory and any Boolean algebra **B**, complete in the sense of $\mathfrak{A}$, we denote the **B**-valued model, $V^{\mathbf{B}}$ of $\mathfrak{A}$, by $\mathfrak{A}^{\mathbf{B}}$ (see [**J**, §18]). If G is any ultrafilter over **B** we may then form the reduced two-valued model $\mathfrak{A}^{\mathbf{B}}/G$ in the usual way. This model, in general, is not an end extension of $\mathfrak{A}$, but if G is **B**-generic over $\mathfrak{A}$, then, in view of the equation

$$\|x \in y\| = \sum_{t \in \operatorname{dom}(y)} (\|x = t\| \cdot y(t)),$$

$\mathfrak{A}^{\mathbf{B}}/G$ will be an end extension of $\mathfrak{A}$. Also, given any such **B** in an ω_1-like model $\mathfrak{A}$ we may always find an ultrafilter which is **B** generic over $\mathfrak{A}$. Since, by Theorem 1.2, ω_1-like rather classless models exist and, by ω_1 applications of the Keisler-Morley theorem, every D.O. model has an ω_1-like e.e.e., we conclude that "rank extension" cannot be replaced by "end extension" in Theorem 1.5.

2. Large cardinal characterizations. In this section we provide characterizations of certain large cardinal properties in terms of elementary extensions. We start with some preliminaries. Note that in this section "definable" always means first order *with* parameters.

DEFINITION 2.1. A ranked tree is a structure $\langle T, <_T, O, <_0, r\rangle$ such that

(i) $\langle T, <_T\rangle$ is a tree, i.e. a partial order whose initial segments (determined by nodes) are linearly ordered;

(ii) $\langle O, <_0\rangle$ is a linear order with no last element; and

(iii) r is an order-preserving map from $\langle T, <_T\rangle$ onto $\langle O, <_0\rangle$ i.e. $s <_T t \to r(s) <_0 r(t)$.

A subset $B \subseteq T$ is a *branch* if r maps B onto O. Given a model $\mathfrak{A}$ and some ranked tree $T = \langle T, <_T, \dots\rangle$ definable in $\mathfrak{A}$ we say that T is *rather branchless in* $\mathfrak{A}$ if every branch of T is definable in $\mathfrak{A}$.

We heavily use the following result which, in the present form, is due to Shelah, [**S**, Theorem 12] who eliminated $\diamondsuit_{\omega_1}$ and generalized Theorem B of [**K3**].

THEOREM 2.2 (SHELAH). *Every countable model in a countable language has an elementary extension of power* $\aleph_1$ *in which every definable tree is rather branchless.*

As our first application we have

THEOREM 2.3. *Let* κ *be a distinguished constant of a theory* $T \supseteq$ ZFC. *The following are equivalent.*

(i) $T \vdash$ "*κ is weakly compact*".

(ii) $T \vdash$ "*κ is strongly inaccessible*" *and given any* $\mathfrak{A} \models T$ *and any* $X \in \mathfrak{A}_{\kappa+1}$ *the model* $\langle \mathfrak{A}_\kappa, X\rangle$ *has an e.e.e.*

PROOF. (i) $\Rightarrow$ (ii). Recall the Keisler characterization of weak compactness: κ is weakly compact iff κ is strongly inaccessible and for every $X \subseteq R_\kappa$ the model $\langle R_\kappa, X\rangle$ has an e.e.e. Hence this is the easy direction.

(ii) $\Rightarrow$ (i). Assume on the contrary. Thus $T \cup \{$"κ is not weakly compact"$\}$ is consistent and has a countable model $\mathfrak{A}$. Let $\mathfrak{B} = \langle B, F, \dots\rangle$ be any elementary extension of $\mathfrak{A}$ in which the binary (ranked) tree of height κ is rather branchless. *Note that this means that whenever* $X \subseteq \kappa_F$ *and* $X \cap \gamma \in B$ *for every* $\gamma \in \kappa_F$ *then*

$X \in B$. On the other hand, by the tree characterization of weakly compacts we know that for some $\tau \in \mathfrak{P}$

$$(*) \qquad \mathfrak{P} \models \text{“}\tau \text{ is a } \kappa\text{-Aronszajn (ranked) tree”}.$$

Now τ can be obviously coded in R_κ and we shall confuse τ with its code, so applying (ii) the model $\langle \mathfrak{P}_\kappa, \tau\rangle$ has an e.e.e. $\langle \mathfrak{C}, \tau'\rangle$. Note that τ' properly end extends τ, so if we pick some $c \in \tau' \setminus \tau$ then $\{b \in \tau' : b <_{\tau'} c\}$ is a branch of τ, hence belonging to $\mathfrak{P}$, contradicting $(*)$. □

THEOREM 2.4. *Let κ be a distinguished constant of a theory $T \supseteq$ ZFC where $T \vdash$ “$R_\kappa \models$ ZFC”. The following are equivalent.*

(i) $\forall n < \omega T \vdash$ “*R_κ has a Σ_n-e.e.e.*”

(ii) $\forall n < \omega$ *and for all $\mathfrak{A} \models T$ the model $\mathfrak{A}_\kappa$ has a Σ_n-e.e.e.*

PROOF. We prove the nontrivial direction. Given an arbitrary $n < \omega$ and any countable model $\mathfrak{A}_0$ of T we first let $\mathfrak{A} \succ \mathfrak{A}_0$ be a model in which the following ranked tree τ is rather branchless. The nodes of τ are ordered pairs (a, α) where $\alpha < \kappa$ and $\mathfrak{A} \models$“$a \subseteq R_\alpha$”. The ordering between nodes is defined in $\mathfrak{A}$ by

$$(a, \alpha) < (b, \beta) \quad \text{iff } \alpha < \beta \text{ and } b \cap R_\alpha = a.$$

The τ-rank of (a, α) is naturally α. It is easy to show that if T is rather branchless in $\mathfrak{A}$ then every class (see Definition 1.1) of $\mathfrak{A}_x$ belongs to $\mathfrak{A}$.

Using (ii), $\mathfrak{A}_\kappa$ has some Σ_{n+1}-e.e.e. $\mathfrak{P}$, hence by the (relativized) reflection theorem we may choose some $\gamma \in \mathrm{Ord}(\mathfrak{P}) \setminus A_\kappa$ such that $\mathfrak{P}_\gamma \prec_n \mathfrak{P}$ and then pick some $c \in B_\gamma \setminus A_\kappa$ and some $\lhd$ such that $\mathfrak{P} \models$“$\lhd$ well-orders R_γ”. Now consider

$$\Gamma = \{\varphi(x, \bar{a}) : \bar{a} \in A_\kappa \text{ and } \mathfrak{P} \models \text{“}\langle R_\gamma, \varepsilon, \lhd\rangle \models \varphi(c, \bar{a})\text{”}\}.$$

Here φ is a formula in the sense of $\mathfrak{P}$ in the language $\{\varepsilon, \lhd\}$. Γ is easily seen to be a class of $\mathfrak{A}_\kappa$. Hence by our choice of $\mathfrak{A}$, Γ belongs to $\mathfrak{A}$. The reader may verify that $\mathfrak{A} \models$“R_κ has a Σ_n-e.e.e.” Therefore $\mathfrak{A}_0$ says the same thing. In view of the completeness theorem the proof is complete. □

REMARK 2.5. One may guess that if for every model $\mathfrak{A}$ of a certain theory $T \supseteq$ ZFC the model $\mathfrak{A}_\kappa$ has an e.e.e. then $T \vdash$ “R_κ has an e.e.e.” This, however, need not be true since the theory $T_0 =$ ZFC $+ \{$“$R_\kappa \prec_n V$”$: n < \omega\} + \{$“R_κ has no e.e.e.”$\}$ $+ \{$“κ is inaccessible”$\}$ is consistent (relative to a weakly compact). This follows from Theorem 4.4 of **[Ka]**. Note that T_0 has no well-founded models since, by an observation of Kunen, if $\mathfrak{M} \prec_e \mathfrak{N} \models$ ZFC, $\mathfrak{N}$ is well founded and $\mathfrak{M} = \mathfrak{N}_\gamma$ then $\mathfrak{N} \models$“R_γ has an e.e.e.” For a sketch see the proof of Theorem 2.5 of **[Ka]**. Indeed, in contrast with other results of this section we have

THEOREM 2.6 (KUNEN). *Assume there exists a weakly compact cardinal. There is no consistent theory Φ such that for all consistent $T \supseteq$ ZFC, $T \vdash \Phi$ iff for every model $\mathfrak{A}$ of T, $\mathfrak{A}_\kappa$ has an e.e.e.*

PROOF. Assume to the contrary and let Φ be such a theory. We claim that ZFC $+ \Gamma \models \Phi$ where $\Gamma = \{$“R_κ has a Σ_n-e.e.e.”$: n < \omega\}$. If not, then for some $\varphi \in \Phi$ the theory $T_0 =$ ZFC $+ \Gamma + \neg\varphi$ is consistent. Since T_0 is a recursive theory there is some sentence $\neg\text{Con}(T_0)$ which expresses the inconsistency of T_0, hence $T_1 = T_0 + \neg\text{Con}(T_0)$ is a consistent theory with no ω-model. Therefore if $\mathfrak{M} \models T_1$ then by overspill there is an infinite integer H of $\mathfrak{m}$ such that

$$\mathfrak{M} \models \text{“}R_\kappa \text{ has a } \Sigma_H\text{-e.e.e.”}$$

In particular, for every model $\mathfrak{M}$ of T_1 the model $\mathfrak{M}_\kappa$ will have an e.e.e. Hence $T_1 \vdash \Phi$, which contradicts the fact that $\neg\varphi \in T_1$. At this point we know that every model of ZFC and Γ is also a model of Φ, so, by exhibiting a model $\mathfrak{A}$ of Γ such that $\mathfrak{A}_\kappa$ has no e.e.e., we would reach our final contradiction. Let κ be the first strongly inaccessible cardinal witnessing Γ in V (which exists since we are assuming there exists a weakly compact cardinal). By Theorem 1.2 of **[Ka]**, $\langle R_\kappa, \varepsilon\rangle$ has no e.e.e.'s which, in turn, implies that κ is smaller than the first weakly compact cardinal. In particular, $\kappa < \gamma$ for some strongly inaccessible γ, hence if $\mathfrak{A}$ is chosen to be $\langle R_\gamma, \varepsilon, \kappa\rangle$, our proof will be complete. □

We now prove some results concerning measurable cardinals, but first

DEFINITION 2.7. $\mathfrak{B}$ is a κ-elementary end extension of $\mathfrak{A}$ (hereafter abbreviated as κ-e.e.e.) if $\kappa \in A$ and $\mathfrak{B}$ enlarges κ without enlarging any members of κ. Note that if $\mathfrak{A}$ has a κ-e.e.e. then $\mathfrak{A} \models$ “κ is regular”. The oldest theorem concerning κ-e.e.e.'s is the following result implicit in **[Sc]**.

THEOREM 2.8. *Let $\mathfrak{A} \models$ ‘κ is a measurable cardinal (ω included)’. Then $\mathfrak{A}$ has a κ-e.e.e.*

REMARK 2.9. When $\mathfrak{A} \models$ “$\kappa > \omega$” the usual (Scott ultrapower) proof of Theorem 2.8 yields a κ-e.e.e., $\mathfrak{B}$, of $\mathfrak{A}$ in which κ has a least new member. In view of Theorem 2.12 below it is worth mentioning that by iterating the ultrapower ω times “from outside” and within $\mathfrak{A}$, of course, we get a κ-e.e.e. of $\mathfrak{A}$ in which κ has no least new member.

κ-elementary end extensions are closely linked to ultrafilters; to make this connection exact we need

DEFINITION 2.10. Let $\mathfrak{A} \models$ ZFC and κ be a cardinal of $\mathfrak{A}$. $\mathfrak{U}$ is said to be a κ-complete ultrafilter over $\mathfrak{A}$ if

(1) $\mathfrak{U}$ is a nonprincipal ultrafilter over the Boolean algebra of $\mathfrak{A}$-subsets of κ;

(2) $\mathfrak{U}$ is κ-complete over $\mathfrak{A}$: whenever $\langle X_\alpha : \alpha < \lambda < \kappa\rangle \in A$ and $X_\alpha \in \mathfrak{U}$ then $\bigcap_{\alpha<\lambda} X_\alpha \in \mathfrak{U}$. Note that this is equivalent to: For every partition p of κ in $\mathfrak{A}$ of size less than κ, $p \cap \mathfrak{U}$ is nonempty.

THEOREM 2.11 **[K1]**. *The following are equivalent.*

(i) $\mathfrak{A}$ *has a κ-e.e.e.*

(ii) $\mathfrak{A}$ *has a Π_2-κ-e.e.e.*

(iii) *There exists a κ-complete over $\mathfrak{A}$ ultrafilter.*

Theorem 2.11 allows us to prove

THEOREM 2.12. *Let $\mathfrak{A} = \langle A, E, \ldots\rangle \models$ ZFC and κ be a regular cardinal of $\mathfrak{A}$ such that $(2^\kappa)_E$ is countable. Then $\mathfrak{A}$ has a κ-e.e.e., $\mathfrak{B}$. Moreover, if κ is uncountable in $\mathfrak{A}$ then $\mathfrak{B}$ may be required to either have a least new member of κ or have no least new member of κ.*

Theorem 2.12 generalizes a similar result of [**KM** and **H1**] where $\mathfrak{A}$ itself was assumed to be countable (cf. also Chapter 33 of [**K2**]).

PROOF (SKETCH). We outline an efficient generic ultrapower formulation. Consider the following partial orders in $\mathfrak{A}$:

$$\mathbf{P} = \{X \subseteq \kappa: X \text{ is stationary}\};$$
$$\mathbf{Q} = \{X \subseteq \kappa: |X| = \kappa\},$$

where the ordering in both cases is set inclusion. Consider the sets in $\mathfrak{A}$ of the form $D_f = \{X \subseteq \kappa: f\restriction X \text{ is constant}\}$ where f is a regressive function on κ in $\mathfrak{A}$. Each D_f is a dense subset of $\mathbf{P}$ by the Pressing down lemma. Hence, by our assumption of countability of 2^κ of $\mathfrak{A}$ there is an ultrafilter G_1 which meets all the D_f's. It is easy to check that the ultrapower $\mathfrak{B}_1 = \mathfrak{A}^\kappa/G_1$ is a κ-e.e.e. with a least new member of κ. On the other hand, we may look at sets in $\mathfrak{A}$ of the form

$$E_f = \{X \subseteq \kappa: \exists g: \kappa \to \kappa, g < f \text{ on } X \text{ and } g \text{ is one-to-one on } X\}$$

where

$$\mathfrak{A} \models \text{“}f: \kappa \to \kappa \quad \text{and} \quad \forall \alpha < \kappa \left|f^{-1}(\{\beta \in \kappa: \beta > \alpha\})\right| = \kappa\text{”}.$$

E_f is dense in $\mathbf{Q}$ by a simple simultaneous induction argument. Also by the regularity of κ if $h: \kappa \to \kappa$ has bounded image then sets of the form $K_h = \{X \subseteq \kappa: h\restriction X \text{ is constant}\}$ are also dense in $\mathbf{Q}$. The reader may verify that if G_2 is any ultrafilter meeting all E_f's and K_h's, then $\mathfrak{B}_2 = \mathfrak{A}^\kappa/G_2$ is a κ-e.e.e. of $\mathfrak{A}$ with arbitrary small new members of κ. It is worth mentioning that both $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are minimal κ-e.e.e.'s of $\mathfrak{A}$ in the sense that if for some model $\mathfrak{C}$ we have $\mathfrak{A} \prec \mathfrak{C} \prec \mathfrak{B}$ then $\mathfrak{C} = \mathfrak{A}$ or $\mathfrak{C} = \mathfrak{B}$ where $\mathfrak{B} = \mathfrak{B}_1$ or $\mathfrak{B}_2$. □

REMARK 2.13. If $\mathfrak{A} \models$ “There exists an ω_1-Kurepa tree” and ω of $\mathfrak{A}$ is countable but ω_2 of $\mathfrak{A}$ is uncountable, then $\mathfrak{A}$ has no $\omega_1^{\mathfrak{A}}$-e.e.e.'s. Therefore, the condition of countability of 2^κ of $\mathfrak{A}$ in Theorem 2.12 cannot be weakened to countability of κ itself.

THEOREM 2.14. *Let* $T \supseteq$ ZFC *and* κ *be a distinguished constant of* T. *Then, assuming either* (a) $T \vdash 2^\kappa = \kappa^+$ *or* (b) $\diamondsuit_{\omega_1}$ (*in the real world*), *the following are equivalent.*

(i) $T \vdash$ “*κ is a measurable cardinal*”.

(ii) *Every model of T has a κ-e.e.e.*

PROOF. The proof using (a) is due to the author, and the one using (b) is due to M. Rubin. Rubin's proof is more powerful than ours and it led us to the proof of Theorems 2.17 and 2.18 below. We outline the proof of the nontrivial direction using (a). Assume on the contrary and let $\mathfrak{A}$ be a countable model of $T +$ “κ is not measurable”. Using Theorem 2.2 let $\mathfrak{B} \succ \mathfrak{A}$ in which the binary (ranked) trees of height κ and κ^+ are rather branchless. It is not hard to see that (ii) implies that $T \vdash$ ‘κ is strongly inaccessible’. Now if $\mathfrak{B}$ has a κ-e.e.e., $\mathfrak{C}$, then we may pick a new member c of κ and consider the set

$$\mathcal{U} = \{X \in B: \mathfrak{C} \models (c \in X \subseteq \kappa)\}.$$

To show that $\mathfrak{U} \in B$, it suffices (since $2^\kappa = \kappa^+$) to show that given any sequence $\langle X_\alpha: \alpha < \kappa \rangle$ of subsets of κ in $\mathfrak{P}$ the set $M = \{\alpha < \kappa: X_\alpha \in \mathfrak{U}\}$ is in $\mathfrak{P}$. But M is in $\mathfrak{P}$ essentially because of the fact that κ is strongly inaccessible and the binary tree of height κ is rather branchless in $\mathfrak{P}$. $\mathfrak{U}$ is obviously a κ-complete ultrafilter over $\mathfrak{P}$ and if $\mathfrak{U} \in B$ then κ is measurable in $\mathfrak{P}$. This is a contradiction. □

The reader should convince himself that the same method of proof shows that condition (a) can be relaxed to

(a′)
$$\text{For some } n < \omega,\ T \vdash \text{“}2^\kappa \leqslant \kappa^{\overbrace{+ + \cdots +}^{n\text{-times}}}\text{”}.$$

This was pointed out to us by Boban Velickovič.

To present the proof using (b), we first need to quote a theorem of Rubin (see [**RS**, Corollary 2.4 and Theorem 2.5]).

THEOREM 2.15 ($\diamondsuit_{\omega_1}$) (RUBIN). *Let $\mathfrak{A}$ be a countable model in a countable language. There exists $\mathfrak{P} \succ \mathfrak{A}$ such that:*

(I) *Every directed set definable in $\mathfrak{P}$ with no last element has a cofinal chain of length ω_1.*

(II) *If P is a definable partial order of $\mathfrak{P}$ and $X \subseteq P$ is maximal compatible and contains a cofinal chain (in X) of length ω_1 then X is definable (p and q are compatible if $\exists r > p, q$).*

REMARK 2.16. In Theorem 2.15 if $\mathfrak{A}$ is a model of ZF then (I) and (II) together imply that every tree in $\mathfrak{P}$ is rather branchless. In particular, $\mathfrak{P}$ would be rather classless.

We are now ready to give a proof of Theorem 2.14 using (b). Assume on the contrary and let $\mathfrak{A}$ be a countable model of ‘κ is not measurable’+ T. Let $\mathfrak{P} \succ \mathfrak{A}$ be as in Theorem 2.15. By Theorem 2.11, and assumng (ii), there exists an ultrafilter $\mathfrak{U}$ which is κ-complete over $\mathfrak{P}$. To show that $\mathfrak{U} \in B$ we first use (I) to choose a cofinal sequence $\langle d_\alpha: \alpha < \omega_1 \rangle$ in the directed set of all finite partitions of κ in the sense of $\mathfrak{A}$ (directed by the refinement relation). Since $\mathfrak{U}$ is κ-complete over $\mathfrak{A}$ there exists a unique p_α in $\mathfrak{U} \cap d_\alpha$. It is easy to see that $p_\alpha \supseteq p_\beta$ whenever $\alpha < \beta$ and that the sequence $\langle p_\alpha: \alpha < \omega_1 \rangle$ is cofinal in $\mathfrak{U}$ (where $\mathfrak{U}$ is partially ordered by reverse inclusion). Thus, in view of (II) and the fact that $\mathfrak{U}$ is a maximal compatible subset of the partially ordered set of all nonempty subsets of κ (ordered by reverse inclusion), $\mathfrak{U}$ must belong to B. But then κ would be measurable in $\mathfrak{P}$. This is a contradiction. □

THEOREM 2.17 ($\diamondsuit_{\omega_1}$). *Let $T \supseteq$ ZFC. The following are equivalent.*

(i) $T \vdash$ “*There exists an uncountable measurable cardinal*”.

(ii) *Every model $\mathfrak{A}$ of T has an elementary extension which fixes $\omega^{\mathfrak{A}}$.*

PROOF. We prove the nontrivial direction. As usual we assume on the contrary that T + ‘There are no uncountable measurable cardinals’ is consistent and we let $\mathfrak{A}$ be a countable model of this theory. Now let $\mathfrak{P}$ be an elementary extension of $\mathfrak{A}$ as in Theorem 2.15. Using (ii), if some $\mathfrak{C} \succ \mathfrak{P}$ and $\mathfrak{C}$ fixes $\omega^{\mathfrak{P}}$, then, by Remark 2.16

and Theorem 1.5(a), $\mathfrak{C}$ must enlarge some $\lambda \in B$. So fix some new member c of λ and define the ultrafilter $\mathfrak{U}$ as $\{X \in B: \mathfrak{C} \models (c \in X \subseteq \lambda)\}$. But $\mathfrak{U}$ must be in $\mathfrak{P}$ by an argument similar to the proof of Theorem 2.14 using (b). Hence $\mathfrak{P} \models$ "$\mathfrak{U}$ is an ω_1-complete ultrafilter", therefore by a classical result of Ulam

$$\mathfrak{P} \models \exists \theta > \omega(\theta \text{ is measurable and } \mathfrak{U} \text{ is } \theta\text{-complete}).$$

This is a contradiction. □

With an argument similar to the previous proof we get the following result; the proof is omitted.

THEOREM 2.18 ($\diamondsuit_{\omega_1}$). *Let* $T \supseteq$ ZFC. *The following are equivalent.*

(i) $T \vdash$ "*There are arbitrarily large measurable cardinals*".

(ii) *For every model* $\mathfrak{A} \models T$ *there are arbitrarily large* κ*'s in* Ord($\mathfrak{A}$) *such that* $\mathfrak{A}$ *has a* κ*-e.e.e.*

(iii) *For every model* $\mathfrak{A} \models T$ *and every* $c \in A$ *there exists an elementary extension which fixes* c.

Lastly, we present two results concerning the problem of initial elementary submodels.

THEOREM 2.19. *Any extension of* ZF *has a model with no initial elementary submodels.*

PROOF. Use a D.O. model. □

THEOREM 2.20. *Suppose* $T \supseteq$ ZF *and for some formula* $\mathfrak{M}(x)$, *T proves* "$\mathfrak{M}$ *is transitive*" *and* $T \vdash \varphi^{\mathfrak{M}}$ *for each axiom* φ *of* ZF. *Assuming T is complete the following are equivalent.*

(i) *For every* $\mathfrak{A} \models T$, $\mathfrak{M}^{\mathfrak{A}}$ *has an initial elementary submodel.*

(ii) *There are formulas* $\psi(x)$, $\varphi_n(x)$, $n < \omega$, *such that T proves*:

(a) $\exists! x(\varphi_n(x)$ *and x is an ordinal*$)$, $n < \omega$;

(b) $\forall x \varphi_n(x) \to$ "$\mathfrak{M}_x \prec_n \mathfrak{M}$", $n < \omega$;

(c) $\forall x \forall y(\varphi_n(x) \wedge \varphi_m(y) \to x \in y)$, $n < m < \omega$;

(d) $\exists y \psi(y)$ *and* $\forall x \forall y(\varphi_n(x) \wedge \psi(y) \to x \in y)$, $n < \omega$.

PROOF. Use D.O. models and the reflection theorem in $\mathfrak{M}$ for the nontrivial direction. □

REMARK 2.21. Note that if $T_0 =$ ZF + "$O^\#$ exists", and $\mathfrak{M}(x) =$ "x is constructible", then T_0 satisfies (i) above. In light of §1 it is curious that by stretching Silver indiscernibles the constructible universe of every model of T_0 has an e.e.e.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706