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CONTRACTION SEMIGROUPS FOR DIFFUSION WITH DRIFT

BY

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ABSTRACT. Recently Dodziuk, Karp and Li, and Strichartz have given results on existence and uniqueness of contraction semigroups generated by the Laplacian Δ on a manifold M; earlier, Yau gave related results for $L = \Delta + V$ for a vector field V. The present paper considers $L = \Delta - V - c$, with c a real function, and gives conditions for (a) uniqueness of semigroups on the bounded continuous functions, (b) preservation of C_0 (functions vanishing at ∞) by the minimal semigroup, and (c) existence and uniqueness of contraction semigroups on $L^p(\mu)$, $1 \leqslant p < \infty$, for an arbitrary smooth density μ on M. The conditions concern $L\rho/\rho$, where ρ is a smooth function, $\rho \to \infty$ as $x \to \infty$. They variously extend, strengthen, and complement the previous results mentioned above.

Introduction. Consider an operator

$$Lu = \Delta u - Vu - cu, \tag{1}$$

where Δ is the Laplacian on a noncompact Riemannian manifold M, V is a vector field, and c a function. (All coefficients are assumed reasonably smooth.) L represents a diffusion with "drift" V and "local dissipation" c. The evolution in time of an initial distribution f is a solution of the Cauchy problem for the heat equation:

$$\begin{cases} \partial u/\partial t = Lu, & t > 0, \\ u = f \quad \text{when } t = 0, \\ u \text{ continuous for } t \geqslant 0. \end{cases} \tag{2}$$

The sign of V in (1) has a simple interpretation: if $V(x) \cdot du(x) > 0$ then the stuff brought toward x by V is at a lower temperature than the stuff being taken away, thus causing a drop in the temperature u at x.

If $c \geqslant 0$, equation (2) always has a "minimal solution" which can be obtained as follows (see Dodziuk **[D]** for more details). Represent M as the union of an increasing sequence of compact manifolds with boundary M_n. Let u_n be the solution of (2) on M_n with $u_n = 0$ on ∂M_n. This solution can be represented as

$$u_n(t, x) = \int_{M_n} p_n(t, x, y) f(y)\, dv_y,$$

where p_n is positive, $\int p_n\, dv_y \leqslant 1$, $\lim_{t \to 0^+} \int p_n\, dv_y = 1$, and

$$\partial p_n/\partial t = L_x p_n = L_y^* p_n.$$

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(L_x indicates L acting in the x variable, and L^* is the formal adjoint of L.) By the maximum principle and the monotone convergence theorem, the p_n converge up to a function p with $\int p\, dv_y \leqslant 1$, and (in the sense of distributions)

$$\partial p/\partial t = L_x p = L_y^* p, \qquad t > 0.$$

Since p satisfies the parabolic equation $2p_t = L_x p + L_y^* p$, it is smooth in all variables for $t > 0$. If f is a bounded continuous function on M ($f \in \mathrm{BC}(M)$), then the function

$$u(x, t) = \int p(t, x, y) f(y)\, dv_y \tag{3}$$

solves the Cauchy problem (2). The map $f \to u(\cdot, t)$ defines a contraction semigroup on the space $\mathrm{BC}(M)$, with infinitesimal generator L defined on an appropriate domain containing $C_c^2(M)$, the C^2 functions with compact support.

§1 considers uniqueness of the solution (3), and §2 concerns the preservation of C_0 (continuous functions vanishing at ∞). §3 considers the existence of contraction semigroups on L^p, $1 \leqslant p < \infty$, generalizing some results of Strichartz [**S**]. The interesting condition there relates the divergence of V to the local dissipation c. Specifically, Theorem 3 says: The operator L is dissipative on L^p if

$$(1/p)\operatorname{div} V \leqslant c.$$

As $p \to \infty$ this reduces to the familiar condition $c \geqslant 0$. The condition is also necessary when $p = 1$, or when $L = -V - c$ ($\Delta = 0$). In any case, if it is uniformly violated on a sufficiently large set, then L is not dissipative.

There are several results similar to those in §§1 and 2. A standard reference (particularly for counterexamples) is Azencott [**A**]. Generalizing results of Feller and Hille in the case $M \subset R^1$, he gives conditions based on integrals involving coefficients A, B, C in the expression

$$Lu(\rho) = Au''(\rho) + Bu'(\rho) + cu(\rho),$$

where $\rho \to \infty$ as $x \to \infty$. The conditions seem closely related to ours, yet not directly comparable.

More comparable are the results of Yau [**Y**]. He assumes $c = 0$ in (1) and assumes a function ρ such that $\rho(x) \to \infty$ as $x \to \infty$, while $L\rho \leqslant k$, $|d\rho| = o(\rho)$, and then constructs a kernel p with $\int p = 1$. (According to Theorem 1, this is the same as the minimal solution if ρ is C^2.) He shows further that if $V = 0$, then the solution preserves C_0 (continuous functions vanishing at ∞). His conditions on the coefficients of L are slight, and ρ may be just Hölder continuous, with $L\rho \leqslant k$ valid in the sense of distributions. He notes that the result applies when M is complete and Ric (the Ricci curvature) is bounded below.

Dod iuk [**D**] gives very similar results by more elementary methods. Karp and Li [**KL**] get nearly optimal results of this type for the case $L = \Delta$. If r denotes distance from a fixed point, they show that

$$\mathrm{Ric} \geqslant -k(r^2 + 1) \quad \Rightarrow \quad \mathrm{vol}\{r \leqslant R\} \leqslant e^{CR^2} \quad \Rightarrow \quad \text{uniqueness}$$

and

$$\mathrm{Ric} \geqslant -k(r^2 + 1) \quad \Rightarrow \quad C_0 \text{ is preserved}.$$

Our conditions are of the same type as Yau's. For uniqueness of solutions of (2), we require a positive C^2 function ρ such that $\rho(x) \to \infty$ as $x \to \infty$, with $L\rho \leqslant k\rho$ (Theorem 1). If ρ is the distance r_p from a fixed point P and M is complete, then the inequality need hold only within the cut locus of p. As Li mentions in a letter, this condition follows from Ric $\geqslant -C(1 + r_p^2)$; see the Appendix.

For preservation of C_0 we assume $\Delta\rho - V\rho \geqslant -k\rho$ and $|d\rho| \leqslant k\rho$ (which implies M is complete). This allows us to recover Karp's and Li's result involving the Ricci curvature, but only on a manifold M with a "pole", a point p where the exponential map is a diffeomorphism: $M_p \to M$.

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1. Uniqueness. First we consider whether (3) is the *only* solution of (2).

THEOREM 1. *Let $Lu = \Delta u - Vu - cu$ (where c has any sign) and suppose M carries a C^2 function $\rho > 0$ such that $\rho(x) \to \infty$ as $x \to \infty$ (in the one-point compactification) and $L\rho \leqslant k\rho$ for some constant k. Then*

$$u_t = Lu, \qquad u|_{t=0} = 0, \qquad |u| \leqslant e^{at} \quad \textit{for some } a$$

implies that $u \equiv 0$.

PROOF. Let $w = e^{-Kt}u\rho^{-1}$ with $K > \max(a, k)$. Then $w \to 0$ uniformly as $(x, t) \to \infty$ and

$$w_t = \Delta w - Vw + 2\frac{dw \cdot d\rho}{\rho} - \left(K - \frac{L\rho}{\rho}\right)w. \tag{1.1}$$

Since $K - (L\rho)\rho^{-1} > 0$, the maximum principle shows $w = 0$. In fact, if $w \neq 0$ then there is a positive maximum or a negative minimum. Suppose, say, $w(x_0, t_0)$ is a positive maximum. Then at (x_0, t_0), $0 = w_t = dw = Vw$, while $\Delta w \leqslant 0$ and $-(K - L\rho/\rho)w < 0$, contradicting (1.1).

REMARKS. In case $c \geqslant 0$, the existence of a ρ as in Theorem 1 guarantees that the minimal solution (3) is the only solution of (2).

In case $c = 0$, then $f \equiv 1$ admits the solution $u \equiv 1$. When Theorem 1 applies, this is the only bounded solution; hence

$$u(x, t) = \int p(t, x, y)\, dv_y = 1 \quad \text{for } t > 0.$$

This implies that the adjoint problem is conservative, i.e. solutions of $L^*u = u_t$ have $\int_M u$ independent of t. (For the adjoint problem, u is heat per unit volume.)

The condition $L\rho \leqslant k\rho$ cannot be replaced by $L\rho \leqslant k\rho^{1+\varepsilon}$ for any $\varepsilon > 0$. For example, with $M = (-1, 1)$, $u = u''$, and $\rho(x) = (1 - x^2)^{-2/\varepsilon}$, we have $L\rho \leqslant k\rho^{1+\varepsilon}$. But in this case there are two-well known distinct solutions of (2), one with $u(\pm 1, t) = 0$, and another with $u_x(\pm 1, t) = 0$.

Thinking in terms of heat, the condition $\Delta\rho - V\rho - c\rho \leqslant k\rho$ means that what happens at ∞ (the boundary of M) cannot diffuse to the finite part of M; note that the term $-V\rho$ gives the rate of drift in from ∞.

Theorem 1 applies to the generalized Ornstein-Uhlenbeck process in R^n, where

$$L = \Delta - l(x) \cdot \nabla \tag{1.2}$$

with $l(x)$ linear in x. Here $V = l(x)$ could represent a rotation, expansion, dilation, shear, etc. Taking $\rho(x) = 1 + |x|^2$, Theorem 1 guarantees uniqueness of solutions, and conservation.

In geometric applications the function ρ in Theorem 1 is generally taken to be (essentially) the distance r_p from a fixed point p in M. If p has a "cut locus" there are difficulties which can be avoided by a device due to Calabi [**C**]; see also [**D**].

THEOREM 1a. *Suppose M is complete and, for some point p, $Lr_p \leqslant kr_p$ inside the cut locus of p and for $r_p \geqslant \delta$. Then the conclusion of Theorem 1 holds.*

PROOF. Let $\rho = r_p$ for $r_p \geqslant \delta$, with $\rho > 0$ and smooth except on the cut locus of p. Then $L\rho \leqslant k\rho$, perhaps with a new k. Let $w = e^{-Kt}u\rho^{-1}$ as before, with $K > k$. If $u \not\equiv 0$ then w has, say, a positive maximum $w(x_0, t_0)$. If x_0 is not on the cut locus then r_p is smooth at x_0, and (1.1) gives a contradiction as before. If the maximum is achieved only with x_0 on the cut locus, let γ be a minimal geodesic from p to x_0. Let q be on γ at a small distance $\varepsilon > 0$ from p. On the segment of γ between q and x_0, $r_p = r_q + \varepsilon$; and everywhere else $r_p \leqslant r_q + \varepsilon$ by the triangle inequality. So the function

$$w_q = e^{-Kt}u(r_q + \varepsilon)^{-1}$$

has a maximum at (x_0, t_0). At this point r_q is smooth, and when ε is small then

$$L(r_q + \varepsilon)/(r_q + \varepsilon) < K$$

in a neighborhood of x_0. So again (1.1) gives a contradiction, with $\rho = r_q + \varepsilon$.

2. Vanishing at ∞.

THEOREM 2. *Let $L = \Delta - V - c$ with $c \geqslant 0$. Then L is the generator of a contraction semigroup on C_0 if there is a Hölder continuous function ρ such that as $x \to \infty$ then $\rho(x) \to \infty$ and*

$$|d\rho| \leqslant k\rho \qquad (k \text{ constant}), \tag{2.1}$$

$$V\rho \leqslant \Delta\rho + k\rho \quad \text{in the sense of distributions}. \tag{2.2}$$

REMARKS. (2.1) implies M is complete, and (2.2) says the drift to ∞ is not too rapid. A simple application of Theorem 2 is the Ornstein-Uhlenbeck process (1.2).

The proof uses a version of the Lumer-Phillips Theorem [**Yo**]: A closed operator L on a Banach space B is the generator of a contraction semigroup e^{Lt} if and only if L is *dissipative* and L^* has no positive eigenvalue. Dissipative means that for every u in the domain of B there is a $\tilde{u}$ in B^* with

$$\|\tilde{u}\| = 1, \quad \langle u, \tilde{u}\rangle = \|u\|, \quad \text{and} \quad \langle Lu, \tilde{u}\rangle \leqslant 0.$$

This implies that the range of $\lambda I - L$ is closed for $\lambda > 0$, and the lack of positive eigenvalues for L^* then implies $\lambda I - L$ is surjective.

LEMMA 2.1. *L is dissipative on the domain $\{u$ in C_0: Lu in $C_0\}$ iff $c \geqslant 0$.*

This is more or less well known, but for completeness we prove the "if" part.

On C_0 the vector $\tilde{u}$ is a unit measure concentrated at a maximum point x_0 for u, or minus a unit measure concentrated at a minimum point. Suppose the first case. Since $Lu \in C_0$, the elliptic regularity theorem shows that u is $C^{2-\varepsilon}$ for every $\varepsilon > 0$; hence Vu and, with it, $\Delta u = Lu + Vu + cu$ are continuous. Since x_0 is a maximum point, $du(x_0) = 0$ and

$$\langle Lu, \tilde{u}\rangle = (Lu)(x_0) = \Delta u(x_0) - c(x_0)u(x_0) \leqslant \Delta u(x_0),$$

since we assume $c \geqslant 0$, and $u(x_0) \geqslant 0$ at the maximum of any function in C_0. By an argument as in Courant and Hilbert [**CH**, leading up to p. 286], any continuous function u such that Δu is also continuous satisfies

$$\Delta u(x_0) = \lim_{R\to 0} \frac{2n}{|S^{n-1}|} R^{-1-n} \int_{d(x,x_0)=R} [u(x) - u(x_0)]\, d\sigma,$$

where $|S^{n-1}|$ is the area of the $n-1$ sphere, $d(x, x_0)$ is geodesic distance, and $d\sigma$ is the induced measure on the sphere $\{d(x, x_0) = R\}$. Since $u(x_0)$ is a maximum, it follows that $\Delta u(x_0) \leqslant 0$, and $L = \Delta - V - c$ is dissipative, proving Lemma 2.1.

It remains to prove that L^* has no positive eigenvalue. Suppose μ is a finite measure such that $L^*\mu = \lambda\mu$, $\lambda > 0$. By regularity, $d\mu = u\,dv$, where dv is the Riemann volume element, and u is a C^2 function, in $L^1(dv)$ since μ is a finite measure, satisfying $L^*u = \lambda u$. Let ψ be a C^2 function on the line, $\psi \geqslant 0$, $\psi(\rho) = 1$ for $0 \leqslant \rho \leqslant 1$, $\psi(\rho) = 0$ for $\rho \geqslant 2$, $\psi' \leqslant 0$. Set

$$\phi_m(x) = \psi(\rho/m).$$

Since $L^*u = \lambda u$ with $\lambda > 0$,

$$\begin{aligned} 0 &\leqslant \lambda \int \phi_m u \operatorname{sgn} u = \int \phi_m (L^*u) \operatorname{sgn} u \\ &\leqslant \int \phi_m (\Delta u - V^*u) \operatorname{sgn} u \quad [\text{since } c \geqslant 0] \\ &\leqslant \int (\Delta\phi_m - V\phi_m)|u| \end{aligned}$$

by Kato's inequality [**K**]. Further,

$$\Delta\phi_m - V\phi_m = \frac{\rho^2}{m^2}\psi'' \frac{|d\rho|^2}{\rho^2} + \frac{\rho}{m}\psi'\left(\frac{\rho}{m}\right)\frac{\Delta\rho - V\rho}{\rho} \leqslant K$$

in view of (2.1) and (2.2), since $\psi' \leqslant 0$. The integrand tends to 0 and is dominated by a constant times $|u|$, so $\lambda \int |u| = 0$. Hence, $u = 0$ and Theorem 2 is proved.

3. The L^p case. It is fair to ask: L^p with respect to what measure? We allow a general C^1 measure on M which, in local coordinates, is $m\,dx$, with m in C^1. Our operator can be written

$$L = \frac{1}{m}\sum \frac{\partial}{\partial x_j} g^{jk} m \frac{\partial}{\partial x_k} - \sum b_j \frac{\partial}{\partial x_j} - c \tag{3.1}$$
$$= A - V_m - c,$$

where g^{jk}, b_j, c, are sufficiently smooth, and the matrix (g^{jk}) is positive definite, defining a metric on the cotangent bundle. Define inner products

$$\langle u, v\rangle = \int uvm\,dx$$

and, for 1-forms,

$$\langle u, v\rangle = \int (u, v) m\,dx,$$

where (u, v) is the inner product in the cotangent bundle. Define the "m-divergence" of the vector field

$$V = \sum b_j \frac{\partial}{\partial x_j}$$

as

$$\nabla_m V = \frac{1}{m}\sum \frac{\partial b_j m}{\partial x_j}.$$

Then for u, w in C_c^2,

$$\langle Au, w\rangle = -\langle du, dw\rangle = \langle u, Aw\rangle,$$
$$\langle Vu, w\rangle = -\langle u, Vw\rangle - \langle u, (\nabla_m V) w\rangle = \langle u, V^* w\rangle.$$

These formulas show that the representation of A, and the "m-divergence", are coordinate invariant.

THEOREM 3. *For* $1 \leqslant p \leqslant \infty$, *$L$ is dissipative on the domain* $C_c^2 \subset L^p$ *if*

$$(1/p)\nabla_m V_m \leqslant c. \tag{D_p}$$

Hence, L is also dissipative on the closure of this domain in graph norm.

PROOF. Let $u \in C_c^2$, $p > 1$. Then $u\,d|u| = |u|\,du$ so

$$d(|u|^{p-2}u) = (p-1)|u|^{p-2}\,du.$$

Take $\tilde{u} = |u|^{p-2}u$, and compute

$$\left\langle Au - \frac{1}{p}V_m u - cu, |u|^{p-2}u\right\rangle = -(p-1)\langle du, |u|^{p-2}\,du\rangle + \frac{1}{p}\langle u, V_m(u|u|^{p-2})\rangle$$
$$+ \frac{1}{p}\langle u, (\nabla_m V_m) u|u|^{p-2}\rangle - \langle cu, u|u|^{p-2}\rangle.$$

But

$$\frac{1}{p}\left\langle u, V_m\left(u|u|^{p-2}\right)\right\rangle = \frac{p-1}{p}\left\langle u, (Vu)|u|^{p-2}\right\rangle,$$

so canceling and transposing gives

$$\begin{aligned}&\left\langle Au - V_m u - cu, |u|^{p-2}u\right\rangle \\ &\qquad = -(p-1)\left\langle du, |u|^{p-2}\,du\right\rangle + \left\langle\left(\frac{1}{p}\nabla_m V_m - c\right)u, u|u|^{p-2}\right\rangle \\ &\qquad \leqslant 0 \quad \text{if } (\mathrm{D}_p) \text{ holds.}\end{aligned}$$

For $p = 1$, take the limit in the last equality as $p \to 1$. The first term remains $\leqslant 0$, and the limit of the second term is $\leqslant 0$ by (D_1).

REMARK. The vector field V_m represents drift. The condition (D_p) says that the divergence of the drift must be compensated by the dissipation c. Examples below show that this condition is sharp in simple cases where $\nabla_m \cdot V_m$ and c are constant. When $p = 1$, condition (D_1) is also necessary, as the last equality in the proof shows.

Finally, we give conditions eliminating positive eigenvalues for L^*, thus guaranteeing a contraction semigroup on L^p.

THEOREM 4. *There are no nonzero solutions of $L^*u = \lambda u$ in L^q $(1/p + 1/q = 1)$ if*

$$\lambda \geqslant (1/p)\nabla_m V_m - c$$

and there is a function ρ on M such that

$$\rho(\infty) = \infty, \qquad |d\rho| = o(\rho), \qquad |V_m \rho| \leqslant k\rho.$$

Note. If $V_m = 0$ this is in Strichartz [S], who refers also to Yau.

PROOF. It suffices to take $\lambda = 0$ (replace c by $c - \lambda$), so the hypothesis is

$$(1/p)\nabla_m V_m - c \leqslant 0. \tag{3.2}$$

Let $H(u) = u|u|^{q-2}$. For ϕ in C_c^2 consider

$$\begin{aligned}I &= \left\langle \phi^2 H(u), Au - \frac{1}{q}V_m^* u - cu\right\rangle \\ &= -\left\langle d(\phi^2 H), du\right\rangle - \frac{1}{q}\left\langle V_m(\phi^2 H), u\right\rangle - \left\langle c\phi^2 H, u\right\rangle \\ &= -2\left\langle \phi H\,d\phi, du\right\rangle - \left\langle \phi^2 H'\,du, du\right\rangle - \frac{1}{q}\left\langle \phi^2 (V_m u)H', u\right\rangle - \frac{2}{q}\left\langle \phi H V_m \phi, u\right\rangle \\ &\quad - \frac{1}{q}\left\langle \phi^2(\nabla_m V_m)uH', u\right\rangle + \frac{1}{q}\left\langle \phi^2(\nabla_m V_m)uH', u\right\rangle - \left\langle \phi^2 cH, u\right\rangle.\end{aligned}$$

But $uH' = (q-1)H$ and $V_m^* u = -V_m u - \nabla_m V_m u$, so

$$\begin{aligned}I = &-2\left\langle \phi H\,d\phi, du\right\rangle - \left\langle \phi^2 H'\,du, du\right\rangle + \frac{1}{p}\left\langle \phi^2 H, V_m^* u\right\rangle \\ &- \frac{2}{q}\left\langle \phi H V_m \phi, u\right\rangle + \frac{1}{p}\left\langle \phi^2(\nabla_m V_m)H, u\right\rangle - \left\langle \phi^2 cH, u\right\rangle.\end{aligned}$$

Transpose $(1/p)\langle \phi^2 H, V_m^* u\rangle$, use $L^* u = 0$ and (3.2) to get

$$0 = \langle \phi^2 H, L^* u\rangle \leqslant -2\langle \phi H\, d\phi, du\rangle - \langle \phi^2 H'\, du, du\rangle - \frac{2}{q}\langle \phi H V_m \phi, u\rangle .$$

But $H' = (q-1)|u|^{q-2}$, so

$$(3.3) \qquad (q-1)\|\phi |u|^{q/2-1}\, du\|^2 \leqslant 2\sup|d\phi| \cdot \|\phi|u|^{q/2-1}\, du\| \cdot \|u^{q/2}\| - \frac{2}{q}\langle \phi H V_m \phi, u\rangle ,$$

where all norms are in L^2. Now let $\phi = \psi(\rho/j)$, with ψ as in the proof of Theorem 2. Then

$$u\phi H V_m \phi = |u|^q \psi\left(\frac{\rho}{j}\right)\frac{\rho}{j}\psi'\left(\frac{\rho}{j}\right)\left(\frac{V_m\rho}{\rho}\right).$$

By hypothesis, $|V_m \rho| \leqslant k\rho$ and $u \in L^q$, so by dominated convergence,

$$\langle \phi H V_m \phi, u\rangle \to 0 \quad \text{as } j \to \infty .$$

Also, as $j \to \infty$,

$$|d\phi| = \frac{\rho}{j}\psi'\left(\frac{\rho}{j}\right)\left(\frac{|d\rho|}{\rho}\right) \to 0$$

uniformly, since $|d\rho|/\rho \to 0$ as $\rho \to \infty$. Hence, from (3.3),

$$\|\phi|u|^{q/2-1}\, du\| \to 0.$$

It follows that u is constant on any open set where $u \neq 0$. Since u is continuous and in L^q, $u \equiv 0$. Q.E.D.

When the conditions in Theorems 3 and 4 are met for two values $p_1 < p_2$, the resulting semigroups agree on $L^{p_1} \cap L^{p_2}$. The proof follows Strichartz [S], showing that the resolvent $(L-\lambda)^{-1}$ is the same in L^{p_1} as in L^{p_2} for large λ. If the resolvents are different, there is a u_1 in L^{p_1} and a u_2 in L^{p_2} with $u_1 \neq u_2$ and

$$(L-\lambda)u_1 = (L-\lambda)u_2$$

so $u_1 - u_2$ is an eigenvalue of L in $L^{p_1} + L^{p_2}$. We prove that the difference $u = u_1 - u_2 = 0$ if

$$(3.4) \qquad \frac{1}{p}\nabla_m V_m - c \leqslant \lambda, \quad \text{for } p = p_1 > 1, \text{ and } p = p_2 > p_1 .$$

The proof imitates Theorem 4 with an appropriate choice of H. Choose an increasing smooth function θ with

$$\theta(u) = \begin{cases} 1/p_2, & 0 \leqslant u \leqslant 1, \\ 1/p_1, & 2 \leqslant u. \end{cases}$$

Define an even function G with

$$G(u) = \frac{1}{u}\int_0^u \theta = \int_0^1 \theta(tu)\, dt, \qquad u > 0,$$
$$= \begin{cases} 1/p_2, & 0 \leqslant u \leqslant 1, \\ 1/p_1 - c_1/u, & 2 \leqslant u, \end{cases}$$

$G'(u) = \int_0^1 t\theta'(tu)\,dt \geqslant 0$, so $1/p_2 \leqslant G \leqslant 1/p_1$. Hence (3.4) gives

$$G(u)\nabla_m V_m - c \leqslant \lambda. \tag{3.4a}$$

Next define H odd with

$$H(u) = (p_2 Gu)^{-1}\exp\left(\int_1^u (Gs)^{-1}\,ds\right), \qquad u > 0,$$

$$= \begin{cases} u^{p_2-1}, & 0 \leqslant u \leqslant 1, \\ c_2(u - p_1 c_1)^{p_1-1}, & 2 \leqslant u. \end{cases}$$

Then

$$GH + (GH)'u = H \tag{3.4b}$$

so $H' = [1 - (Gu)']H/Gu = [1 - \theta]H/Gu > 0$. Hence further

$$\sqrt{H/u} \leqslant \text{const}\sqrt{H'}. \tag{3.4c}$$

Now suppose that $Lu = \lambda u$ with $u \in L^{p_1} + L^{p_2}$. Then $uH(u)$ is in L^1. Consider

$$I = \left\langle \phi^2 H(u), Au - G(u)V_m u - (c + \lambda)u \right\rangle$$

and calculate as before, using (3.4a, b), to get

$$\left\langle \phi^2 H'\,du, du \right\rangle \leqslant -2\left\langle \phi H\,d\phi, du \right\rangle + 2\left\langle GH\phi V_m \phi, u \right\rangle.$$

Thus

$$\|\phi\sqrt{H'}\,du\|^2 \leqslant 2\sup|d\phi| \cdot \|\phi\sqrt{H/u}\| \cdot \|\sqrt{Hu}\| + 2\left\langle GH\phi V_m\phi, u \right\rangle.$$

Applying (3.4c) and assuming a ρ as in Theorem 4, we find as before that $u \equiv 0$. This proves that the semigroups in L^{p_1} and in L^{p_2} agree on $L^{p_1} \cap L^{p_2}$. [Note: The above result is a revision in the galley proofs of a weaker result in the original article.]

We conclude with two examples. The first shows that the condition $(1/p)\nabla_m V_m - c \leqslant \lambda$ in Theorem 4 is essential, as is $c - \nabla_m V_m \geqslant 0$ in Theorem 3.

EXAMPLE 1. $Lu = u'' - bxu'$, where b is constant; take $m\,dx = dx$. (When $b < 0$, this is the Uhlenbeck process.)

The Fourier transform with respect to x converts $Lu = u_t$ into the first order equation

$$-\xi^2\hat{u} + b\frac{\partial}{\partial\xi}(\xi\hat{u}) = \frac{\partial\hat{u}}{\partial t}.$$

The solution with $\hat{u}(0, \xi) = \hat{f}(\xi)$ is

$$\hat{u}(t, \xi) = \hat{f}(\xi e^{bt})\exp\left(bt - (\xi^2/2b)(e^{2bt} - 1)\right).$$

Taking a limit as $b \to 0$ gives the usual solution of $u'' = u_t$.

If this is a contraction on L^p, then the adjoint L^* has no positive eigenvalue in L^q. The eigenvalue equation $L^*u = \lambda u$ is solved by taking the Fourier transform:

$$\hat{u}(\xi) = |\xi|^{-\lambda/b}e^{-\xi^2/2b}.$$

If $b < 0$ there is no solution in L^q. If $b > 0$ we have

$$u(x) = c|x|^{-1+\lambda/b} * e^{-bx^2/2},$$

which is in L^q iff $q(-1 + \lambda/b) < -1$, that is iff $\lambda < b/p$, where $1/p + 1/q = 1$.

Combined with Theorems 3 and 4, this shows that $L - c$ (where c is constant) generates a contraction semigroup on L iff $c - b/p \geqslant 0$. Note that in this case $b = \nabla_m V_m$, so Theorems 3 and 4 are sharp.

EXAMPLE 2. $Lu = \Delta u - b(xu_y - yu_x)$; m is Lebesgue measure on R^2. Here $\nabla_m V_m = 0$, so there is a contraction semigroup on L^p for $1 < p < \infty$, and on C_0. We analyze the spectrum of L on the space L^2 by taking the Fourier transform:

$$\hat{L}v = -(\xi^2 + \eta^2)v + b\eta\frac{\partial v}{\partial \xi} - b\xi\frac{\partial v}{\partial \eta}.$$

The first order equation

$$\hat{L}\phi = \lambda\phi \tag{3.5}$$

gives the rotation vector field

$$\dot{\xi} = b\eta, \qquad \dot{\eta} = -b\xi,$$

where $\dot{\xi}$ is the derivative of ξ with respect to a parameter τ. The flow is

$$\xi = r\cos b\tau, \quad \eta = -r\sin b\tau, \qquad r \text{ constant}.$$

The solution of (3.5) is, with $b\tau = \theta$,

$$\phi(r\cos\theta, r\sin\theta) = e^{-(\lambda + r^2)\theta/b}\phi(r,0),$$

where we obviously need $e^{-(\lambda + r^2)2\pi/b} = 1$, or

$$\lambda + r^2 = ibk, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

The spectrum of L consists of the union of half-lines $\{\lambda\colon \lambda = ibk - r^2, k = 0, \pm 1, \ldots, r \geqslant 0\}$. This comes right down to $\operatorname{Im}\lambda = 0$, so $\|e^{tL}\| = 1$.

Appendix. We sketch a proof of some relations between Ricci curvature and Δr. The proof of Lemma 1 was given by C. L. Terng.

LEMMA 1. *Let r denote distance from a fixed point p. If* Ric $\geqslant -C(1 + r^2)$, *then* $(\Delta r)_r \leqslant -(\Delta r)^2/n + C(1 + r^2)$ *inside the cut locus of p.*

PROOF. Choose an orthonormal local frame $e_1, \ldots, e_n$ with $e_1 = \partial/\partial r$. Set $r_i = \nabla_{e_i} r$. Since $|\nabla r|^2 = 1$ and $r_{ij} = r_{ji}$,

$$0 = \Delta\left(\frac{1}{2}|\nabla r|^2\right) = \sum_i\left(\sum_j r_j r_{ji}\right)_i = \sum\left(r_{ji}^2 + r_j r_{jii}\right) = \sum\left(r_{ij}^2 + r_j r_{iji}\right). \tag{$*$}$$

The Ricci formula gives

$$r_{ijk} = r_{ikj} + \sum_l r_l R_{lijk},$$

so (now summing repeated indices)

$$r_{iji} = r_{iij} + r_l R_{liji} = r_{iij} + r_l R_{lj},$$

with R_{lj} the Ricci tensor. So $(*)$ gives

$$0 = \sum r_{ij}^2 + r_j\left(r_{iij} + r_l R_{lj}\right).$$

Set $f = \Delta r = r_{ii}$, and get

$$r_j f_j = r_j r_{iij} = -\sum r_{ij}^2 - r_j r_l R_{lj} \leqslant -\sum r_{ij}^2 + C(1 + r^2),$$

Since $r_1 = 1$, $r_2 = \cdots = r_n = 0$, $f_1 = \partial f/\partial r$, and

$$\sum r_{ij}^2 \geqslant \sum r_{ii}^2 \geqslant \frac{1}{n}\left(\sum r_{ii}\right)^2 = \frac{1}{n}(\Delta r)^2,$$

the lemma follows.

LEMMA 2. *If*

(1) $$f'(r) < -a^2 f^2 + b^2 + c^2 r^2, \qquad r_0 > r > 0,$$

then

(2) $$f \leqslant (2/a)(1/ar + b + cr), \qquad r_0 > r > 0.$$

If $r_0 = \infty$ *then*

(3) $$f \geqslant -(\rho/a)\sqrt{b^2 + c^2 r^2}$$

as well, where ρ *is a constant* > 1 *satisfying*

(4) $$c\rho \leqslant ab^2(\rho^2 - 1).$$

PROOF OF (2).

Case 1. Suppose that for small r,

(5) $$\sqrt{b^2 + c^2 r^2} < af/\sqrt{2}\,.$$

Then $f'(r) \leqslant -a^2 f^2/2$, so $(1/f)' \geqslant a^2/2$, hence, by (5), $1/f \geqslant a^2 r/2$, or $f \leqslant 2/a^2 r$. Further, $f'(r) \leqslant 0$ as long as $f^2 \geqslant (b^2 + c^2 r^2)/a^2$. So f must lie below the dashed line in Figure 1; it cannot cross the graph of $(1/a)\,(b^2 + c^2 r^2)^{1/2}$ from below, since (1) implies $f' \leqslant 0$ at such a crossing. So, being generous,

$$f \leqslant \frac{2}{a^2 r} + \frac{\sqrt{2}}{a}(b^2 + c^2 r^2)^{1/2} \leqslant \frac{2}{a}\left(\frac{1}{ar} + b + cr\right).$$

Case 2. This is even easier; f starts out below the graph of $(\sqrt{2}/a)(b^2 + c^2 r^2)^{1/2}$.

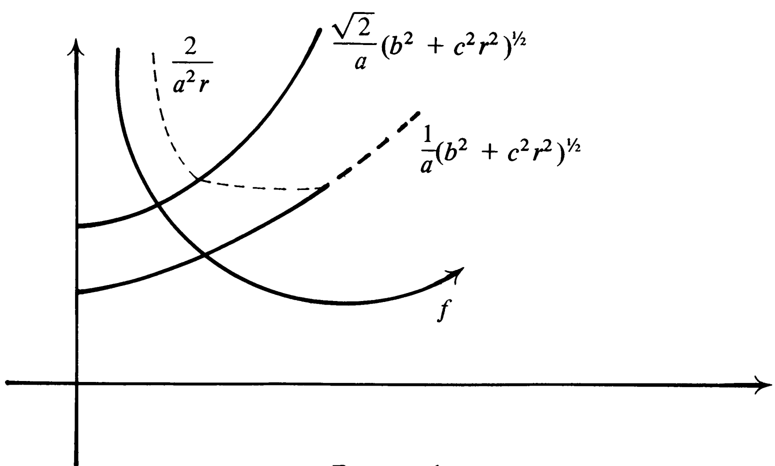

FIGURE 1

PROOF OF (3). Compare f to the function $g(r) = -(\rho/a)(b^2 + c^2r^2)^{1/2}$, with $\rho > 1$ satisfying (4). From (1), $f \leqslant g \Rightarrow f' < g'$, so f cannot cross the graph of g from below; if $f(r_1) \leqslant g(r_1)$, then $f(r) < g(r)$ for all $r > r_1$. But $f \leqslant g$ implies

$$f' \leqslant -a^2f^2 + b^2 + c^2r^2 \leqslant -a^2f^2 + a^2f^2/\rho^2 = -\alpha f^2, \qquad \alpha > 0.$$

So $(1/f)' \geqslant \alpha > 0$, and this implies that $1/f > 0$ eventually, contradicting $f \leqslant g < 0$.

Combining the lemmas with $f = \Delta r$, we find from (2) that

$$\operatorname{Ric} \geqslant -C\left(1 + r_p^2\right) \Rightarrow \Delta r_p \leqslant k\left(1/r_p + r_p\right)$$

inside the cut locus of p. (This also follows from the Laplacian Comparison Theorem of Greene and Wu (Lecture Notes in Math., vol. 699, Springer-Verlag, Berlin and New York) taking as model the space R^n with metric $dr^2 + \exp(r^2)\, d\theta^2$ in polar coordinates.)

If exp: $M_p \to M$ is a diffeomorphism, then (3) gives

$$\operatorname{Ric} \geqslant -C\left(1 + r_p^2\right) \Rightarrow \Delta r_p \geqslant -kr_p$$

since $\Delta r \to +\infty$ as $r \to 0$. (This inequality does not seem to follow from the theorem of Greene and Wu.) Hence Theorem 2 applies: C_0 is preserved.

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