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DIFFERENCE EQUATIONS, ISOPERIMETRIC INEQUALITY AND TRANSIENCE OF CERTAIN RANDOM WALKS

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ABSTRACT. The difference Laplacian on a square lattice in $\mathbf{R}^n$ has been studied by many authors. In this paper an analogous difference operator is studied for an arbitrary graph. It is shown that many properties of the Laplacian in the continuous setting (e.g. the maximum principle, the Harnack inequality, and Cheeger's bound for the lowest eigenvalue) hold for this difference operator. The difference Laplacian governs the random walk on a graph, just as the Laplace operator governs the Brownian motion. As an application of the theory of the difference Laplacian, it is shown that the random walk on a class of graphs is transient.

The random walks we consider are defined as follows. Let K be a connected graph (i.e. a one dimensional simplicial complex). For a vertex $x \in K$, let $m(x)$ denote the number of edges emanating from x. The probability that a particle moves from x to another vertex $y \in K$ is $1/m(x)$ if x and y are connected by an edge and it is zero otherwise. As observed by Courant, Friedrichs and Lewy [**CFL**] for the case of a square lattice in the plane this random walk is intimately related to the difference analog of the Laplacian

$$\Delta f(x) = \sum_{y \sim x} f(y) - m(x) f(x), \tag{0.1}$$

where $y \sim x$ means that x and y are connected by an edge. The operator Δ defined by (0.1) and its relation to random walks have been studied extensively in the case of a lattice in $\mathbf{R}^n$ (cf. e.g. [**CFL**, **Du**]). Replacing a lattice by a more general graph corresponds to considering curved manifolds instead of flat ones in the continuous setting. Our results are motivated by [**Do**], where transience of the Brownian motion on certain manifolds was proved.

In the first section of this paper we show that many familiar facts (the Harnack inequality, the maximum principle, Green's formula, positivity of the first eigenfunction) hold for the discrete Laplacian on an arbitrary graph. In the second section we restrict our attention to graphs which, in a sense, correspond to manifolds of bounded curvature satisfying certain isoperimetric inequality (cf. (2.1) and (2.2)). We show that for such graphs there exists a positive function f defined on vertices for which $\Delta f < 0$. Such functions are called superregular in [**KSK**] (we prefer to call them superharmonic) and their existence implies that the random walk under consideration is transient (cf. [**KSK**, Chapter 6, §1]), i.e. a particle starting from

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a vertex x escapes to infinity with probability one for every vertex $x \in K$. The proofs here are motivated by the proofs of corresponding facts in the smooth case. In order to carry out this analogy we prove a counterpart of Cheeger's inequality [C] for the discrete Laplacian. Finally in §3 we exhibit many graphs which satisfy the geometric conditions required in §2.

1. The difference Laplacian. In this section K is an arbitrary graph, i.e. a connected simplicial complex of one dimension. We denote by $C^0(K)$ the space of all real valued 0-cochains, i.e. functions on vertices of K. Similarly, $C^1(K)$ is the space of all functions φ defined on oriented edges of K and satisfying

$$\varphi([x,y]) = -\varphi([y,x]),$$

where $[x,y]$ denotes an oriented (directed) edge beginning at x and ending at y. For every edge of K we fix, once and for all, a direction. Nothing will depend on this choice, but it is convenient to make in order to write certain formulae below in an unambiguous way. In what follows an edge will be understood to be an edge with the chosen direction.

Assume now that K is finite. For $f_1, f_2 \in C^0(K)$ and $\varphi_1, \varphi_2 \in C^1(K)$ we define inner products as follows:

$$(f_1, f_2) = \sum_x f_1(x) f_2(x), \quad (\varphi_1, \varphi_2) = \sum_\sigma \varphi_1(\sigma) \cdot \varphi_2(\sigma), \tag{1.1}$$

where x ranges over all vertices of K, and σ runs over the set of all edges of K. The coboundary operator

$$df([x,y]) = f(y) - f(x) \tag{1.2}$$

maps $C^0(K)$ into $C^1(K)$. We define for $f \in C^0(K)$,

$$\Delta f = -d^* df, \tag{1.3}$$

where d^* is the adjoint of d with respect to the inner products (1.1). A simple calculation using (1.1)–(1.3) yields

LEMMA 1.4. *For every $f \in C^0(K)$*

$$\Delta f(x) = \sum_{y \sim x} f(y) - m(x) f(x),$$

where $x \sim y$ indicates that x and y are connected by an edge and $m(x)$ is the number of edges emanating from x.

REMARK. If K is not finite we define Δ by the formula above.

DEFINITION 1.5. A cochain $f \in C^0(K)$ is called superharmonic at a vertex x if $\Delta f(x) \leq 0$, i.e. if the value $f(x)$ is greater than or equal to the arithmetic mean of the values at neighboring vertices.

The following is an obvious analog of the maximum (minimum) principle.

LEMMA 1.6. *Suppose $f \in C^0(K)$ is superharmonic at $x \in K$. If for every neighboring vertex $y \sim x f(y) \geq f(x)$, then $f(y) = f(x)$ for $y \sim x$.*

The proof is trivial.

LEMMA 1.7 (HARNACK INEQUALITY). *Suppose $f \in C^0(K)$ is superharmonic at x and y, where $x \sim y$, and $f > 0$. Then*

$$\frac{1}{m(y)} f(x) \leq f(y) \leq m(x) f(x).$$

PROOF. $0 \geq \Delta f(x) = \sum_{z \sim x} f(z) - m(x) f(x)$. Hence $m(x) f(x) \geq \sum_{z \sim x} f(z) \geq f(y)$ because $f \geq 0$. By symmetry $m(y) f(y) \geq f(x)$.

To state the analog of Green's formula we recall the notion of a relative cochain. For a subcomplex $L \subset K$, $C^0(K, L)$ consists of those cochains $f \in C^0(K)$ for which $f(x) = 0$ whenever $x \in L$. We define the Laplacian $\Delta_{K,L} f$ by the formula (0.1) if $x \notin L$ and $\Delta_{K,L} f(x) = 0$ for $x \in L$. Since $C^0(K, L) \subset C^0(K)$ the inner product and the coboundary are defined for cochains in $C^0(K, L)$.

LEMMA 1.8. *Let $f, g \in C^0(K, L)$. Then*

$$(\Delta_{K,L} f, g) = -(df, dg) = (f, \Delta_{K,L} g).$$

PROOF. This can be calculated directly or derived by observing that

$$(\Delta_{K,L} f, g) = (\Delta f, g) = -(df, df)$$

since at the vertices x, where $\Delta_{K,L} f(x) \neq \Delta f(x)$, $g(x) = 0$.

Thus the operator $-\Delta_{K,L}$ on $C^0(K, L)$ is selfadjoint and nonnegative. The following lemma is an analog of some familiar facts in the smooth case.

LEMMA 1.9.

(a) *the smallest eigenvalue λ of $-\Delta_{K,L}$ is given by $\lambda = \min((df, df)/(f, f))$, where the minimum is taken over $f \in C^0(K, L) \backslash \{0\}$. Moreover, if $(df, df)/(f, f) = \lambda$, then $\Delta_{K,L} f + \lambda f = 0$.*

(b) *Assume that the pair (K, L) is connected in the sense that every two vertices $x, y \notin L$ can be joined by a chain of edges $[x, y_1], [y_1, y_2], \ldots, [y_{n-1}, y_n], [y_n, y]$ so that $y_i \notin L$ for $i = 1, 2, 3, \ldots, n$. Then the multiplicity of λ is one and we can choose a positive eigencochain of $-\Delta_{K,L}$ belonging to λ.*

PROOF. (a) follows trivially from Lemma 1.8. Note that $(f, f) = (|f|, |f|)$. However, for $f \in C^0(K, L)$,

$$\begin{aligned} (df, df) &= \sum_{\sigma = [x,y]} (f(x) - f(y))^2 \\ &\geq \sum_{\sigma = [x,y]} (|f(x)| - |f(y)|)^2 = (d|f|, d|f|). \end{aligned} \tag{1.10}$$

Thus (a) implies that if $f \in C^0(K, L)$ belongs to λ so does $g = |f|$. Hence $g \geq 0$ and satisfies $\Delta g = -\lambda g \leq 0$. By Lemma 1.6 and by connectedness $g(x) > 0$ whenever $x \notin L$. Suppose f takes both positive and negative values in K. By connectedness, there exists an edge $[x, y]$ such that $f(x) \cdot f(y) < 0$. In this case the inequality in (1.10) is strict, which contradicts Lemma 1.9(a). It follows that f cannot change sign. Now suppose that λ is not a simple eigenvalue. Let f_1, f_2 be two linearly independent eigenfunctions of $-\Delta_{K,L}$ belonging to λ. Choose a vertex $x \notin L$ and consider a linear map $A: \mathbf{R}^2 \to \mathbf{R}$ given by $A(a, b) = a f_1(x) + b f_2(x)$. This mapping has a nontrivial kernel, i.e. there exists a nonzero eigenfunction belonging to λ which vanishes at x. We saw above that this is impossible. Thus λ is a simple eigenvalue.

COROLLARY 1.11. *Suppose $L_1 \subset L_2 \subset K$. Let λ_i, $i = 1, 2$, be the smallest eigenvalue of $-\Delta_{K,L_i}$. Then $\lambda_1 \leq \lambda_2$.*

PROOF. This follows immediately from the characterization of the smallest eigenvalue in Lemma 1.9(a).

2. The analog of Cheeger's inequality and positive superharmonic functions. In this section M denotes an infinite, connected graph. The formalism of §1 can be applied to every finite subcomplex K of M. If K is such a graph, we define ∂K, the boundary of K, to consist of those vertices x of K for which at least one of the edges meeting at x is not in K, and of all edges of K spanned by such vertices.

We make two geometric assumptions on M. The first one corresponds to boundedness of the curvature in the Riemannian setting. Namely, we assume that there exists an integer $m > 0$ such that

$$m(x) \leq m \tag{2.1}$$

for all vertices $x \in M$.

The second assumption plays the role of an isoperimetric inequality (cf. [**Do**, §2]). We require that there exists a constant $\alpha > 0$ so that

$$\alpha V(K) \leq V(\partial K) \tag{2.2}$$

for all finite graphs $K \subseteq M$. Here $V(L)$ stands for the number of vertices of L.

Theorem 2.3 is an analog of Cheeger's inequality [**C**]:

THEOREM 2.3. *Suppose* (2.1) *and* (2.2) *are satisfied. If K is a finite subcomplex of M such that $(K, \partial K)$ is connected (cf. Lemma* 1.9(b)*), then the smallest eigenvalue λ of $-\Delta_{K,\partial K}$ satisfies $\lambda \geq \alpha^2/2m$.*

Before proving this theorem we will show that it implies that M carries a nonconstant positive superharmonic function (cochain).

THEOREM 2.4. *Suppose M satisfies* (2.1) *and* (2.2). *There exists a positive function $f \in C^0(M)$ and a real number $\lambda > 0$ so that $\Delta f + \lambda f = 0$. In particular f is positive, superharmonic and nonconstant. Consequently, the random walk on M is transient.*

PROOF. Fix a vertex $x_0 \in M$. Define K_n to be the complex consisting of those vertices of M which can be joined to x_0 by a path consisting of at most n edges, together with all edges spanned by these vertices. Clearly $(K_n, \partial K_n)$ is connected for $n = 1, 2, \ldots$, and every finite subcomplex of M is contained in K_n for sufficiently large n. Let λ_n be the smallest eigenvalue of $\Delta_{K_n,\partial K_n}$, and let $f_n \in C^0(K_n, \partial K_n)$ be the corresponding eigencochain normalized so that

$$f_n(x_0) = 1. \tag{2.5}$$

By Corollary 1.11 and Theorem 2.3

$$\lambda_n \geq \lambda_{n+1} \geq \frac{\alpha^2}{2m},$$

i.e. $\lambda = \lim_{n\to\infty} \lambda_n > 0$. The functions f_n satisfy $\Delta f_n = -\lambda_n f_n < 0$ at interior vertices of K_n, i.e. are superharmonic. By the Harnack inequality the sequence

$(f_n(x))_{n=1}^{\infty}$ is bounded at every vertex x of M (strictly speaking $f_n(x)$ is defined only for sufficiently large n). Using the diagonal process we can find a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ exists for all $x \in M$. We see that $\Delta f + \lambda f = 0$. Moreover, by (2.5), $f(x_0) = 1$. Hence $f \geq 0$, $\Delta f \leq 0$ and f is not constant, since $\Delta f(x_0) = -\lambda f(x_0) = -\lambda < 0$. By the maximum principle or by repeated application of the Harnack inequality f is strictly positive everywhere. The transience of the random walk on M is a consequence of Proposition 6.3, Chapter 6 of [**KSK**].

PROOF OF THEOREM 2.3 Let $f \in C^0(K, \partial K)$ be the eigencochain of $-\Delta_{K,\partial K}$ belonging to the smallest eigenvalue λ. By Lemma (1.9)(a)

$$\lambda = (df, df)/(f, f). \tag{2.6}$$

Consider the expression $A = \sum_{\sigma=[x,y]} |f^2(x) - f^2(y)|$ where the summation is extended over all edges σ of K. Clearly

$$\begin{aligned} A &= \sum |f(x) + f(y)| \cdot |f(x) - f(y)| \\ &\leq (\sum |f(x) + f(y)|^2)^{1/2} \cdot (\sum |f(x) - f(y)|^2)^{1/2} \\ &\leq \sqrt{2}(\sum (f^2(x) + f^2(y)))^{1/2} \cdot (cf, df)^{1/2}. \end{aligned}$$

In $\sum(f^2(x)+f^2(y))$ every vertex contributes as many times as the number of edges emanating from it. Hence, by (2.1),

$$A \leq \sqrt{2m}(f, f)^{1/2} \cdot (df, df)^{1/2}. \tag{2.7}$$

On the other hand we can estimate A from below in terms of (f, f) as follows. Let $0 = \beta_0 < \beta_1 < \beta_2, < \cdots < \beta_N$ be the sequence of all values of f. Note that, since $\Delta f = -\lambda f < 0$, every interior vertex x of K has a neighbor y such that $f(y) < f(x)$. Define L_i, $i = 0, 1, \ldots, N$, as follows. A vertex x of K is in L_i if $f(x) \geq \beta_i$. An edge belongs to L_i if both endpoints are in L_i. Now

$$A = \sum_{i=1}^{N} \sum_{f(x)=\beta_i} \sum_{y\sim x, f(y)<\beta_i} (f^2(x) - f^2(y)).$$

If $f(x) = \beta_i$ and $f(y) = \beta_{i-k}$ for some $k \in \{1, 2, \ldots, i\}$, then, on the one hand, $f^2(x) - f^2(y) = \beta_i^2 - \beta_{i-k}^2 = (\beta_i^2 - \beta_{i-1}^2) + (\beta_{i-1}^2 - \beta_{i-2}^2) + \cdots + (\beta_{i-k+1}^2 - \beta_{i-k}^2)$ and, on the other hand, $x \in \partial L_i \cap \partial L_{i-1} \cap \cdots \cap \partial L_{i+1-k}$. It follows that

$$A = \sum_{i=1}^{N} \sum_{x\in\partial L_i} (\beta_i^2 - \beta_{i-1}^2) = \sum_{i=1}^{N} V(\partial L_i)(\beta_i^2 - \beta_{i-1}^2).$$

Applying (2.2) we obtain

$$A \geq \alpha \sum_{i=1}^{N} V(L_i)(\beta_i^2 - \beta_{i-1}^2).$$

"Summation by parts" yields now

$$A \geq \alpha \left\{ V(L_N)\beta_N^2 + \sum_{i=1}^{N-1} \beta_i^2 (V(L_i) - V(L_{i+1})) \right\}.$$

A vertex $x \in L_i \backslash L_{i+1}$ if and only if $f(x) = \beta_i$. Therefore $A \geq \alpha(f,f)$. This inequality combined with (2.7) yields

$$\lambda = \frac{(df, df)}{(f,f)} \geq \frac{\alpha^2}{2m}$$

which proves the theorem.

REMARK. The proof above is patterned after the proof of Cheeger's inequality in [C]. The expression A plays the role of $|d(f^2)|$ and the estimate (2.7) is analogous to

$$|d(f^2)| \leq 2f|df| \leq 2(|f|^2)^{1/2} \cdot (|df|^2)^{1/2}.$$

Finally, the inequality $\alpha V(L) \leq V(\partial L)$ enters into the proof in the way the analogous isomperimetric inequality enters into Cheeger's proof.

3. An isoperimetric inequality. Suppose S is a simply connected Riemannian surface of curvature bounded from above by a negative constant. It is well known that for every open, relatively compact subset D of S with smooth boundary

$$\alpha A(D) \leq L(\partial D), \tag{3.1}$$

where α is a constant independent of D, A stands for area and L denotes length. We shall derive a combinatorial analog of (3.1) and use it to exhibit many graphs satisfying hypotheses of Theorem 2.4.

Let N be an oriented, triangulated, open surface without boundary. For a finite subcomplex L of N, let $V(L)$, $E(L)$, $F(L)$ denote the number of vertices, edges and faces, respectively, of L. For such a subcomplex L define δL to be the graph consisting of those edges and vertices of L which are on the boundary of a triangle not in L.

PROPOSITION 3.2. *Suppose N is as above and suppose that $m(x) \geq 7$ for every vertex $x \in N$. If N is planar $F(L) \leq 26E(\delta L)$ for every finite subcomplex L of N.*

REMARK. The condition $m(x) \geq 7$ is the combinatorial analog of negative curvature bounded away from zero. Planarity means that every cycle on N disconnects. Equivalently (cf. [**AhS**, Chapter III, §4]) N is homeomorphic to a subset of the plane.

PROOF OF PROPOSITION 3.2. We want to prove the inequality $F(L) \leq 26E(\delta L)$. If L has edges or vertices which do not lie on the boundary of a triangle in L, we can remove those edges and vertices and prove the inequality for the resulting complex. Therefore we assume, with no loss of generality, that if a vertex on an edge belongs to L, then one of the adjacent triangles does too. Since N is planar, we can assume that L is contained in the sphere S^2. The Euler characteristic $\chi(L)$ satisfies

$$\chi(L) = V(L) - E(L) + F(L) = 2 - n,$$

where n is the number of components of $S^2 \backslash L$. Clearly $n \leq E(\delta L)$ and

$$E(\delta L) \geq n - 2 = E(L) - F(L) - V(L). \tag{3.3}$$

On the other hand, since $m(x) \geq 7$ for all vertices $x \in N$,

$$3F(L) \geq 7V_i(L) + V(\delta L),$$

where $V_i(L)$ denotes the number of interior vertices of L. Since $V(L) = V_i(L) + V(\delta L)$, we obtain

$$F(L) \geq \frac{7}{3}V(L) - 2V(\delta L). \tag{3.4}$$

Clearly δL consists of circles (polygons to be exact) possibly touching at some vertices. It follows that $V(\delta L) \leq E(\delta L)$. Substituting this into (3.4) yields

$$V(L) \leq \frac{3}{7}F(L) + \frac{6}{7}E(\delta L). \tag{3.5}$$

Moreover, if $E_i(L)$ denotes the number of interior edges of L,

$$3F(L) = 2E_i(L) + E(\delta L) \leq 2E(L).$$

Together with (3.3) and (3.5) this gives

$$E(\delta L) \geq -F(L) - \frac{3}{7}F(L) - \frac{6}{7}E(\delta L) + \frac{3}{2}F(L),$$

i.e. $26E(\delta L) \geq F(L)$.

REMARK. The method of proof of the proposition above is borrowed from [**DB**] where an analogous inequality is proved in a somewhat different setting.

COROLLARY 3.6. *Let N be a triangulated planar surface with the property that $m \geq m(x) \geq 7$ at every vertex x of N, for an integer m independent of x. Then $V(\partial K) \geq V(K)/78m$ for every finite graph K contained in N.*

PROOF. Given K define L to be the smallest complex containing K and all the triangles of N all of whose edges are in K. According to the definitions of ∂K (cf. beginning of §2) and δL, the set of vertices of ∂K coincides with the set of vertices of δL. We can break L up into three parts, $L = L_0 \cup L_1 \cup L_2$, as follows. L_0 is the set of isolated vertices of L (i.e. of vertices x such that all the edges meeting at x are not in L, hence not in K). L_1 consists of all edges of K which are not on the boundary of any triangle of L, and L_2 is the remaining part of L.

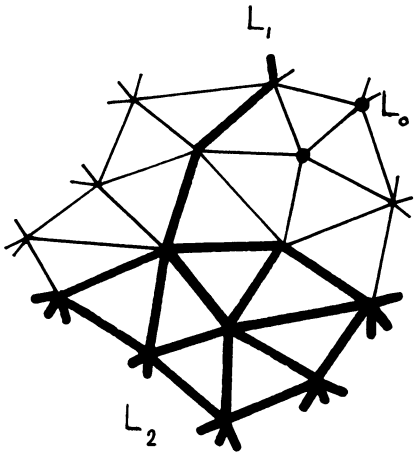

FIGURE 1

Clearly $V(L_2) \leq 3F(L_2)$. By Proposition 3.2 $F(L_2) \leq 26E(\delta L_2)$ so that

$$V(L_2) \leq 78E(\delta L_2). \tag{3.7}$$

Moreover $2E(\delta L_2) \leq mV(\delta L_2)$ and, therefore

$$V(L_2) \leq 39mV(\delta L_2). \tag{3.8}$$

Obviously $\partial K = L_0 \cup L_1 \cup \delta L_2$ and L_1 can have some vertices in common with δL_2. Therefore

$$V(\partial K) \geq \frac{1}{2}(V(L_0) + V(L_1) + V(\delta L_2)). \tag{3.9}$$

Similarly, since the sets of vertices of K and L are equal,

$$V(K) = V(L) \leq V(L_0) + V(L_1) + V(L_2). \tag{3.10}$$

(3.8), (3.9) and (3.10) yield $V(\partial K) \geq V(K)/78m$, which completes the proof.

REMARK. Triangulations satisfying assumptions of Corollary 3.6 (and, hence, satisfying (2.1) and (2.2)) occur naturally on planar surfaces which cover compact, oriented surfaces without boundary of genus $g > 1$. The simplest example is the plane (realized as the hyperbolic plane) with a tesselation into regular (in the sense of hyperbolic geometry) octagons. To manufacture a triangulation we subdivide every octagon as shown in Figure 2. The resulting triangulation has two classes of vertices with $m(x)$ equal to 8 and 16 respectively since eight octagons meet at every vertex of the original tesselation.

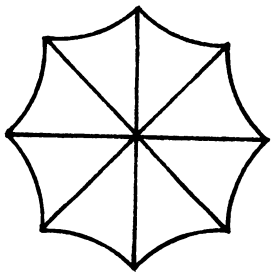

FIGURE 2

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