SOME APPLICATIONS OF THE TOPOLOGICAL CHARACTERIZATIONS OF THE SIGMA-COMPACT SPACES $l_2$ AND $\Sigma$

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ABSTRACT. We use a technique involving skeletoids in $\sigma$-compact metric ARs to obtain some new examples of spaces homeomorphic to the $\sigma$-compact linear spaces $l_2$ and $\Sigma$. For example, we show that (1) every $\aleph_0$-dimensional metric linear space is homeomorphic to $l_2$; (2) every $\sigma$-compact metric linear space which is an AR and which contains an infinite-dimensional compact convex subset is homeomorphic to $\Sigma$; and (3) every weak product of a sequence of $\sigma$-compact metric ARs which contain Hilbert cubes is homeomorphic to $\Sigma$.

1. Introduction. We consider the $\sigma$-compact pre-Hilbert spaces $l_2^\sigma = \{(x_i) \in l^2: x_i = 0$ for almost all $i\}$ and $\Sigma = \{(x_i) \in l^2: \sup |ix_i| < \infty\}$. For any $\sigma$-compact locally convex metric linear space $E$, with completion $\bar{E}$, the following results are known from work of Anderson, Bessaga and Pełczynski, and Torunczyk (see [3, Chapter VIII]):

(I) $(\bar{E}, E) \approx (l^2, l_2^\sigma)$ if $E$ is $\aleph_0$-dimensional;

(II) $(\bar{E}, E) \approx (l^2, \Sigma)$ if $E$ contains an infinite-dimensional compact convex subset.

In this paper we extend the above results to all $\sigma$-compact metric linear spaces $E$ for which the completion $\bar{E}$ is an AR. More generally, it is shown that if $C$ is a $\sigma$-compact convex subset of a metric linear space such that the closure $\bar{C}$ is nonlocally compact, then:

(I) $C \approx l_2^\sigma$ if $C$ is $\sigma$-fd-compact, (the countable union of finite-dimensional compacta);

(II) $C \approx \Sigma$ if $C$ is an AR and contains an infinite-dimensional locally compact convex subset;

(III) if the closure $\bar{C}$ in some complete metric linear space is nonlocally compact and an AR, then $(\bar{C}, C) \approx (l^2, l_2^\sigma)$ if $C$ is $\sigma$-fd-compact, and $(\bar{C}, C) \approx (l^2, \Sigma)$ if $C$ contains an infinite-dimensional locally compact convex subset.

The proof of (III) is based on the theory of skeletoids (cap sets and fd-cap sets) in $l^2$, and a result from [9]. However, it does involve many of the same constructions that appear in the proofs of (I) and (II), for which there is developed a method of skeletoids in $\sigma$-compact metric ARs based on the topological characterizations of $l_2^\sigma$ and $\Sigma$ given in [13].

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This method is also used to obtain results on weak products. Specifically, every weak product of a sequence of nondegenerate \( \sigma \)-fd-compact metric ARs is homeomorphic to \( l^2 \), and every weak product of \( \sigma \)-compact ARs which contain Hilbert cubes is homeomorphic to \( \Sigma \).

2. Strongly universal properties and skeletoids. We say that a metric space \( X \) is strongly universal for compacta (respectively, strongly universal for finite-dimensional compacta) if, for every map \( f: A \to X \) of a compactum (respectively, finite-dimensional compactum), for every closed subset \( B \) of \( A \) such that \( f \mid B \) is an imbedding, and for every \( \varepsilon > 0 \), there exists an imbedding \( h: A \to X \) such that \( h \mid B = f \mid B \) and \( d(h, f) < \varepsilon \).

2.1 Theorem [13]. A metric AR is homeomorphic to \( \Sigma \) (respectively, homeomorphic to \( l^2 \)) if and only if it is \( \sigma \)-compact and strongly universal for compacta (respectively, \( \sigma \)-fd-compact and strongly universal for finite-dimensional compacta).

In verifying the strongly universal properties for the spaces discussed in §1, we find it convenient to work with skeletal versions of these properties, formulated with respect to a tower of subsets \( X_1 \subset X_2 \subset \cdots \) in \( X \). We say that \( \{X_i\} \) is a strongly universal tower for compacta (respectively, strongly universal tower for finite-dimensional compacta) if, for every map \( f: A \to X \) of a compactum (respectively, finite-dimensional compactum), for every closed subset \( B \) of \( A \) such that \( f \mid B: B \to X_m \) is an imbedding into some \( X_m \), and for every \( \varepsilon > 0 \), there exists an imbedding \( h: A \to X_n \), for some \( n \geq m \), such that \( h \mid B = f \mid B \) and \( d(h, f) < \varepsilon \). We refer to \( \bigcup^\infty_1 X_i \subset X \) as a skeletoid for compacta (respectively, skeletoid for finite-dimensional compacta).

We also require the notion of a Z-set. A closed set \( F \) of a metric space \( X \) is a Z-set in \( X \) if all maps of compacta into \( X \) can be arbitrarily closely approximated by maps into \( X \setminus F \). When \( X \) is an ANR, it suffices to consider maps of the Hilbert cube into \( X \), or equivalently, maps of n-cells for all finite \( n \).

2.2 Proposition. Let \( X \) be a metric ANR such that every compact subset is a Z-set. Then if \( X \) contains a skeletoid for compacta (respectively, skeletoid for finite-dimensional compacta), \( X \) is strongly universal for compacta (respectively, strongly universal for finite-dimensional compacta).

Proof. Given a map \( f: A \to X \) of a compactum, a closed subset \( B \) of \( A \) such that \( f \mid B \) is an imbedding, and \( \varepsilon > 0 \), we must construct an imbedding \( g: A \to X \) such that \( g \mid B = f \mid B \) and \( d(g, f) < \varepsilon \). Let \( \{X_i\} \) be a strongly universal tower for compacta, and let \( \{A_i\} \) be a tower of compacta such that \( \bigcup^\infty_1 A_i = A \setminus B \). We will inductively construct a sequence of maps \( \{g_n: A \to X\} \) such that:

(i) \( g_n(A_n) \subset X_{i(n)} \) for some \( i(n) \);
(ii) \( g_n \mid A_n \cup B \) is an imbedding;
(iii) \( g_n \mid A_{n-1} \cup B = g_{n-1} \mid A_{n-1} \cup B \) (set \( A_0 = \emptyset \) and \( g_0 = f \));
(iv) \( d(g_n, g_{n-1}) < \varepsilon/2^n \).

Then \( g = \lim_{n \to \infty} g_n \) is the required imbedding.

Suppose maps \( g_0, \ldots, g_{n-1} \) have been constructed. Since the compacta \( g_{n-1}(A_{n-1}) \) and \( g_{n-1}(B) \) are disjoint, there exists a neighborhood \( U \) of \( A_{n-1} \) in \( A \) such that \( \text{dist}(g_{n-1}(U), g_{n-1}(B)) > 0 \). Take

\[ \delta = \min\{\varepsilon/2^{n+1}, \text{dist}(g_{n-1}(U), g_{n-1}(B))\}. \]
Since $X$ is an ANR, there exists $\eta > 0$ such that every map $g': A \to X$ with $d(g', g_{n-1}) < \eta$ is $\delta$-homotopic to $g_{n-1}$. By the Z-set hypothesis, there exists a map $g': A \to X \setminus g_{n-1}(B)$ with $d(g', g_{n-1}) < \eta$. Let $h: A \times [0, 1] \to X$ be a $\delta$-homotopy between $g' = h_0$ and $g_{n-1} = h_1$, and let $\lambda: A \to [0, 1]$ be a Urysohn map with $\lambda(A_{n-1}) = 1$ and $\lambda(X \setminus U) = 0$. Define $\tilde{g}: A \to X$ by $\tilde{g}(a) = h(a, \lambda(a))$. Then $\tilde{g}(A) \cap g_{n-1}(B) = \emptyset$, $\tilde{g} | A_{n-1} = g_{n-1} | A_{n-1}$, and $\tilde{g}$ is $\varepsilon/2n+1$-homotopic to $g_{n-1}$.

Choose $0 < \mu < \text{dist}(\tilde{g}(A), g_{n-1}(B))$ such that every map $h: A \to X$ with $d(h, \tilde{g}) < \mu$ is $\varepsilon/2n+1$-homotopic to $\tilde{g}$. By the tower hypothesis, there exists an imbedding $h: A \to X_{i(n)}$ for some $i(n) > i(n-1)$, with $h | A_{n-1} = g_{n-1} | A_{n-1}$ and $d(h, \tilde{g}) < \mu$. Then $h(A) \cap g_{n-1}(B) = \emptyset$, and $h$ is $\varepsilon/2n$-homotopic to $g_{n-1}$. Using such a homotopy and a Urysohn map $\lambda: A \to [0, 1]$ with $\lambda(A_{n-1}) = 1$ and $\lambda(B) = 0$, we then construct the desired map $g_n$.

The identical construction works in the case that $\{X_i\}$ is strongly universal for finite-dimensional compacta and $A$ is finite-dimensional.

In general, the compact Z-set hypothesis in the above proposition is strictly necessary, and cannot be weakened to nowhere-local compactness. Consider the infinite-dimensional compact convex ellipsoid $M = \{(x_i) \in l^2: \sum_{i=1}^\infty i^2 x_i^2 \leq 1\}$, a topological Hilbert cube. Let $M_{\text{core}} = \{(x_i) \in l^2: \sum_{i=1}^\infty i^2 x_i^2 < 1\}$, and let $W$ be a wild (i.e., not a Z-set) Cantor set in $M$. Then $X = M_{\text{core}} \cup W$ is a $\sigma$-compact, nowhere-locally compact, convex subset of $l^2$ which contains a skeleton for compacta, but $X \not\subseteq \Sigma$, and is therefore not strongly universal for compacta, since $W$ is not a Z-set in $X$.

There also exists a $\sigma$-fd-compact counterexample. With $M$ and $W$ as above, let $M_f = M \cap l^2_f$, and consider $Y = M_f \cup W$. Then $Y$ is a $\sigma$-fd-compact, nowhere-locally compact AR which contains a skeleton for finite-dimensional compacta, but again $W$ is not a Z-set in $Y$. Thus $Y \not\subseteq l^2_f$ and is not strongly universal for finite-dimensional compacta. (Although $Y$ is nonconvex, it is easily seen that there exist maps $g: M \to Y$ arbitrarily close to the identity map such that $g | W = \text{id}$ and $g(M \setminus W) \subseteq M_f$. Thus $Y$ is arbitrarily finely dominated by $M$, and is therefore an AR.) It is an open question (see §4) whether the compact Z-set hypothesis is redundant when $X$ is an infinite-dimensional, $\sigma$-fd-compact, convex subset of a metric linear space.

The skeletonoids contained in the above counterexamples are proper subsets of the spaces. In the case that a $\sigma$-compact metric ANR is covered by a strongly universal tower, with each tower element $\sigma$-compact, compact subsets are automatically Z-sets. The following proposition will be used for weak products (§5).

2.3. PROPOSITION. Let $X$ be a metric ANR, and let $\{X_i\}$ be a strongly universal tower for compacta (respectively, strongly universal tower for finite-dimensional compacta), with each $X_i$ $\sigma$-compact (respectively, $\sigma$-fd-compact), and such that $\bigcup_1^\infty X_i = X$. Then every compact subset of $X$ is a Z-set.

PROOF. We first verify that every compact subset (respectively, finite-dimensional compact subset) of a tower element $X_i$ is a Z-set in $X$. Let $F$ be such a subset, let $f: K \to X$ be a map of a compactum, and let $\varepsilon > 0$. Consider the disjoint union $K \cup F$, and the map $\tilde{f}: K \cup F \to X$ defined by $\tilde{f} | K = f$ and $\tilde{f} | F = \text{id}$. Then $\tilde{f}$ can be approximated by an imbedding $h: K \cup F \to X_j$ for
some \( j > i \), with \( h \mid F = \tilde{f} \mid F = \text{id} \) and \( d(h, \tilde{f}) < \varepsilon \). Thus \( d(h \mid K, f) < \varepsilon \) and \( h(K) \cap F = \emptyset \).

Of course, since \( X \) is an ANR, it suffices to consider the case that \( K \) is an \( n \)-cell. Thus if \( \{X_i\} \) is strongly universal for finite-dimensional compacta, and \( F \subset X_i \) is finite-dimensional, the above procedure still works.

Since \( X = \bigcup_1^\infty X_i \), it follows that every compact subset of \( X \) is a \( \sigma Z \)-set (i.e., a countable union of \( Z \)-sets), and the proof of the proposition will be completed by the following.

\[ 2.4. \text{ LEMMA.} \quad \text{Every topologically complete closed} \ \sigma Z \text{-set in a metric ANR is a} \ Z \text{-set.} \]

\[ \text{PROOF.} \quad \text{Consider} \ F = \bigcup_1^\infty F_n, \text{with each} \ F_n \text{a} \ Z \text{-set in} \ X. \text{Choose a complete metric} \ d \text{for} \ F; \text{since} \ F \text{is closed in} \ X, \ d \text{can be extended to} \ X. \text{Let a map} \ f: K \to X \text{of a compactum and} \ \varepsilon > 0 \text{be given. Using the fact that each} \ F_n \text{is a} \ Z \text{-set, and the techniques in the second paragraph of the proof of 2.2, we may construct a sequence of maps} \ \{f_n: K \to X\} \text{and a sequence of positive constants} \ \{\varepsilon_n\} \text{such that:} \]

\begin{enumerate}
\item \( f_n(K) \cap F_n = \emptyset; \)
\item \( \varepsilon_n < \min\{\text{dist}(f_n(K), F_n), \varepsilon_{n-1}/2\}, \) with \( \varepsilon_0 = \varepsilon; \)
\item \( d(f_n, f_{n+1}) < \varepsilon_n/2, \) with \( f_0 = f; \)
\item \( f_{n+1} \mid K_n = f_n \mid K_n, \) where \( K_n = \{q \in K: \text{dist}(f_n(q), F) \geq 2^{-n}\}. \)
\end{enumerate}

The subsets \( K_n \) form a tower, and if \( \bigcup_1^\infty K_n = K, \tilde{f} = \lim_{n \to \infty} f_n \) is a well-defined map of \( K \) into \( X \setminus F \), with \( d(\tilde{f}, f) < \varepsilon \). Suppose there exists \( q \in K \setminus \bigcup_1^\infty K_n. \) Then for some sequence \( \{y_n\} \) in \( F \), \( d(f_n(q), y_n) < 2^{-n} \) for each \( n \). Since \( \{f_n(q)\} \) is Cauchy, so is \( \{y_n\}, \) and \( y_n \to y \in F. \) Hence \( f_n(q) \to y, \) and \( y \in F_m \) for some \( m. \) But since \( \text{dist}(f_m(K), F_m) > \varepsilon_m \) and \( d(f_m, f_n) < \varepsilon_m \) for each \( n > m, \) we cannot have \( f_n(q) \to y. \) Thus \( \bigcup_1^\infty K_n = K, \) and the proof is complete.

\[ 3. \text{Convex sets in metric linear spaces.} \quad \text{Throughout this section,} \ C \text{denotes a convex subset of a metric linear space} \ E. \text{We use a monotone invariant metric} \ d \text{on} \ E, \text{and the corresponding} \ F \text{-norm} \ | \cdot |: E \to [0, \infty), \text{defined by} \ |x| = d(x, \theta), \text{where} \ \theta \text{is the zero element. The monotone property means that} \ |tx| \leq |x| \text{if} |t| \leq 1. \]

In attempting to identify convex sets in metric linear spaces which may be homeomorphic to \( l^2_2 \) or \( \Sigma, \) we need first of all to determine whether compact subsets are \( Z \)-sets. As shown by the example following the proof of 2.2, it does not suffice to require only that the convex set be nowhere-locally compact. However, it is sufficient that the \emph{closure} of a convex set be nonlocally compact (note that a closed convex set which fails to be locally compact at some point is nowhere-locally compact).

\[ 3.1. \text{PROPOSITION.} \quad \text{If the convex set} \ C \text{has a nonlocally compact closure in} \ E, \text{then every compact subset of} \ C \text{is a} \ Z \text{-set in} \ C. \]

\[ \text{PROOF.} \quad \text{We may assume} \ \theta \in C. \text{Consider a compact subset} \ F \text{of} \ C, \text{a map} \ f: K \to C \text{of a compactum, and} \ \varepsilon > 0. \text{Choose} \ 0 < \delta < 1 \text{such that} \ |\delta f(q)| < \varepsilon/2 \text{for all} \ q \in K. \text{Set} \ D = \{(x - (1 - \delta)f(q))/\delta: x \in F \text{and} q \in K\}. \text{Then} \ D \subset E \text{is compact. Since there is no compact neighborhood of} \ \theta \text{in} \ C, \text{we must have} \ C \cap \{x \in E: |x| \leq \varepsilon/3\} \not\subset D, \text{and there exists} \ z \in C \setminus D \text{with} |z| < \varepsilon/2. \text{Define a map} \ g: K \to C \text{by the formula} \ g(q) = \delta z + (1 - \delta)f(q). \text{Since} \ z \notin D, \ g(q) \notin F \text{for} \]
any \( q \in K \), and
\[
|g(q) - f(q)| = |\delta z - \delta f(q)| \leq |z| + |\delta f(q)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]
Thus \( F \) is a \( Z \)-set in \( C \).

3.2. **Lemma.** Let \( \{C_i\} \) be a tower of convex sets such that \( \bigcup_{1}^{\infty} C_i \) is dense in the convex set \( C \), and suppose that \( C \) is an AR and each \( C_i \) is an AR. Then for every map \( f: A \to C \) of a compactum, for every closed subset \( B \) of \( A \) such that \( f(B) \subseteq C_m \) for some \( m \), and for every \( \varepsilon > 0 \), there exists a map \( g: A \to C_n \) for some \( n \geq m \), such that \( g | B = f | B \) and \( d(g, f) < \varepsilon \).

**Proof.** We will construct maps \( f_0, f_1: A \to C_n \), for some \( n \geq m \), such that \( f_0 | B = f | B \) and \( d(f_1, f) < \varepsilon/2 \). Then for any Urysohn map \( \lambda: A \to [0,1] \) such that \( \lambda(B) = 0 \) and \( \{a \in A: |f_0(a) - f(a)| \geq \varepsilon/2 \} \subseteq \lambda^{-1}(1) \), the required map \( g \) may be defined by the formula \( g(a) = (1 - \lambda(a))f_0(a) + \lambda(a)f_1(a) \).

The map \( f_0 \) is obtained as an extension of the map \( f | B \) into the AR space \( C_m \).

In constructing the map \( f_1 \), we may assume that \( A \) is a Hilbert cube, since \( C \) is an AR. Thus we may assume that \( A \) admits small self-maps into finite-dimensional subcompacta. (If \( A \) itself is finite-dimensional, the AR hypothesis on \( C \) is unnecessary.) Choose \( \delta > 0 \) such that \( |f(a) - f(a')| < \varepsilon/4 \) for all \( a, a' \in A \) with \( d(a, a') < \delta \). Choose a finite-dimensional subcompactum \( F \) of \( A \) for which there exists a map \( r: A \to F \) with \( d(r, id) < \delta \). Choose \( \eta > 0 \) such that \( |f(a) - f(a')| < \varepsilon/\delta(\dim F + 1) \) for all \( a, a' \in F \) with \( d(a, a') < \eta \). Let \( \mathcal{U} \) be a finite open cover of \( F \), with \( \dim \text{Nerve} \mathcal{U} \leq \dim F \) and \( \text{mesh} \mathcal{U} < \eta \). For each \( U \in \mathcal{U} \), choose \( \varphi(U) \subseteq \bigcup_{1}^{\infty} C_i \) such that for some \( a \in U \), \( |\varphi(U) - f(a)| < \varepsilon/\delta(\dim F + 1) \). This defines a partial realization of \( \text{Nerve} \mathcal{U} \) in some \( C_n \), \( n \geq m \), which may be extended linearly to a full realization \( \varphi \): \( \text{Nerve} \mathcal{U} \to C_n \). Let \( \alpha: F \to \text{Nerve} \mathcal{U} \) be any barycentric map. Then the composition \( \tilde{f} = \varphi \circ \alpha \) maps \( F \) into \( C_n \), and \( d(\tilde{f}, f | F) < \varepsilon/4 \). Finally, take \( f_1 = \tilde{f} \circ r: A \to C_n \). For each \( a \in A \), we have
\[
|f_1(a) - f(a)| \leq |\tilde{f}(r(a)) - f(r(a))| + |f(r(a)) - f(a)| \leq \varepsilon/4 + \varepsilon/4 = \varepsilon/2.
\]
This completes the proof of the lemma.

An infinite-dimensional compact convex set which can be affinely imbedded in \( l^2 \) is called a Keller cube. (For a discussion of such sets, including the fundamental theorem that all Keller cubes are homeomorphic to the Hilbert cube, we refer the reader to [3].)

3.3. **Lemma.** Let \( K \) be a Keller cube in a metric linear space \( E \). Then for every finite set \( \{x_1, \ldots, x_n\} \) in \( E \) the set \( L = \text{conv}\{K, x_1, \ldots, x_n\} \) is also a Keller cube. Furthermore, there exists \( z \in K \) with the property that, for every such \( L \), the subset \( \alpha_r L = \bigcup_{y \in L} [z, y] \) is a \( \sigma Z \)-set in \( L \).

**Proof.** Let \( \alpha: K \to l^2 \) be an affine imbedding. We may assume that \( \theta \in K \) and that \( \alpha(\theta) = (0,0,\ldots) \in l^2 \). For any \( L = \text{conv}\{K, x_1, \ldots, x_n\} \) \( \alpha \) can be extended to an affine imbedding of \( L \) as follows. If \( x_1 \in \text{span} K = \bigcup_{1}^{\infty} n(K - K) \), say \( x_1 = n(k_1 - k_2) \), set \( \alpha(x_1) = n(\alpha(k_1) - \alpha(k_2)) \). And if \( x_1 \notin \text{span} K \), choose \( \alpha(x_1) \in l^2 \setminus \text{span} \alpha(K) \). Then \( \alpha \) extends linearly to a homeomorphism between \( \text{conv}\{K, x_1\} \) and \( \text{conv}\{\alpha(K), \alpha(x_1)\} \). Repeating this procedure \( n \) times, we obtain the desired extension of \( \alpha \) over \( L \), with \( \alpha(L) = \text{conv}\{\alpha(K), \alpha(x_1), \ldots, \alpha(x_n)\} \).
By the foregoing, we may assume without loss of generality that $E = l^2$ and $(0,0,\ldots) \in K$. Choose an orthogonal sequence $\{u_i\}$ of nonzero vectors in the infinite-dimensional pre-Hilbert space span $K$. We may suppose that each $u_i \in K - K$; pick $v_i, w_i \in K$ such that $u_i = v_i - w_i$. Since $K$ is compact, the sequence $\{w_i\}$ is bounded. Consider $z = \sum_1^\infty 2^{-i}u_i$. We have $z \in K$, and $z + 2^{-i}u_i \in K$ for each $i$. It follows from Proposition 2.5 of [4] that for any compact convex set $L \supset K$, $\text{aur}_z L$ is a $\sigma Z$-set in $L$.

We are now ready to construct skeletaloids in convex sets.

3.4. PROPOSITION. Let $C$ be a separable infinite-dimensional convex set. Then $C$ contains a skeletaloid for finite-dimensional compacta, and if $C$ is an AR and contains a Keller cube, then $C$ contains a skeletaloid for compacta.

PROOF. Let $\{x_i\}$ be a dense sequence in $C$, and define $C_i = \text{conv}\{x_1, \ldots, x_i\}$, $i \geq 1$. We verify that $\{C_i\}$ is a strongly universal tower for finite-dimensional compacta. Given a map $f: A \to C$ of a finite-dimensional compactum, a closed subset $B$ of $A$ such that $f | B: B \to C_m$ is an imbedding into some $C_m$, and $\varepsilon > 0$, we must construct an imbedding $h: A \to C_r$ for some $r \geq m$, such that $h | B = f | B$ and $d(h, f) < \varepsilon$. By 3.2, $f$ may be approximated by a map $g: A \to C_n$ for some $n \geq m$, such that $g | B = f | B$ and $d(g, f) < \varepsilon/2$. (As noted in the proof of 3.2, the finite-dimensionality of $A$ makes the AR hypothesis on $C$ unnecessary.) Since $A$ is finite-dimensional, there exists a map $\varphi: A \to J$ into some finite-dimensional cell $J$, with $\varphi | B$ a constant map onto some boundary point $p$ of $J$, such that if $\varphi(a) = \varphi(a')$, then either $a = a'$ or $a, a' \in B$. Since $\{\dim C_i\}$ is unbounded, there exists an imbedding $e: C_m \times J \to C_r$, for some $r > n$, such that $e(x, p) = x$ and $|e(x, q) - x| < \varepsilon/2$ for all $x \in C_n$ and $q \in J$. Then the required imbedding $h: A \to C_r$ is defined by the formula $h(a) = e(g(a), \varphi(a))$.

Now suppose $C$ contains a Keller cube $K$. Let $z \in K$ be a point with the property specified in 3.3. We may assume $z = \theta$. As before, let $\{x_i\}$ be a dense sequence in $C$, and define $L_i = \text{conv}\{K, x_1, \ldots, x_i\}$, $i \geq 1$. Then each Keller cube $L_i$ has the property that $\text{aur}_{L_i} ([0,1) \cdot L_i)$ is a $\sigma Z$-set in $L_i$. Equivalently, for each $0 < t < 1$ the Keller cube $tL_i$ is a $Z$-set in $L_i$. Let $\{t_i\}$ be a strictly increasing sequence of positive numbers such that $t_i \to 1$. For each $i$, set $C_i = t_iL_i$. Then $\{C_i\}$ is a tower of convex sets, with each $C_i \approx I^\infty$, and $\bigcup_1^\infty C_i$ is dense in $C$. Since the pair $(t_i+1L_{i+1}, t_iL_{i+1})$ is homeomorphic to the pair $(L_{i+1}, tL_{i+1})$ for some $0 < t < 1$, $t_iL_{i+1}$ is a $Z$-set in $t_i+1L_{i+1}$. Thus $C_i = t_iL_i \subset t_{i+1}L_{i+1}$ is a $Z$-set in $C_{i+1}$. By Anderson’s theorem on topological infinite deficiency [1], $(C_{i+1}, C_i) \approx (C_i \times I^\infty, C_i \times pt)$. We verify that $\{C_i\}$ is a strongly universal tower for compacta. Let a map $f: A \to C$, a closed subset $B$ of $A$, and $\varepsilon > 0$ be given as before (except that now we do not assume $A$ is finite-dimensional). Let $g: A \to C_n$ be the approximation given by 3.2, with $g | B = f | B$ and $d(g, f) < \varepsilon/2$. There exists a map $\varphi: A \to I^\infty$, with $\varphi | B$ a constant map onto a point $p$, such that if $\varphi(a) = \varphi(a')$, then either $a = a'$ or $a, a' \in B$. Let $e: C_n \times I^\infty \to C_{n+1}$ be an imbedding such that $e(x, p) = x$ and $|e(x, q) - x| < \varepsilon/2$ for all $x \in C_n$ and $q \in I^\infty$. Then as before, the formula $h(a) = e(g(a), \varphi(a))$ defines the required imbedding $h: A \to C_{n+1}$.

The hypothesis in 3.4 concerning the existence of Keller cubes in convex sets has an easier, but equivalent, formulation. We say that a convex set $C$ in a metric
linear space $E$ is **locally complete** at $x \in C$ if there exists a neighborhood of $x$ in $C$ which is complete with respect to an invariant metric on $E$.

### 3.5. PROPOSITION

**Every infinite-dimensional convex set which is somewhere locally complete contains a Keller cube.**

**PROOF.** We may assume $C$ is locally complete at $\theta \in C$, i.e., there exists $\varepsilon > 0$ such that every Cauchy sequence in $C \cap \{x \in E : |x| \leq \varepsilon\}$ converges in $C$. Let $\{x_i\}$ be a linearly independent sequence in $C$. We will construct a sequence of scalars $\{r_i\}$, with $0 < r_i \leq 2^{-i}$ for each $i$, such that the correspondence $(t_i) \rightarrow \sum_{i=1}^{\infty} t_i x_i$ defines an affine imbedding of the Keller cube $I^\infty = \prod_{i=1}^{\infty}[0, r_i] \subset I^2$ into $C$. Choose $0 < \tau_1 \leq 1/2$ such that $|\tau_1 x_1| < \varepsilon/2$. Suppose inductively that $\tau_1, \ldots, \tau_n$ have been chosen. For each $m$, $1 \leq m \leq n$, set

$$\delta_m = \inf \left\{ \left| \sum_{i=1}^{m} s_i x_i - \sum_{i=1}^{m} t_i x_i \right| : (s_i), (t_i) \in \prod_{i=1}^{m}[0, \tau_i], \right\}$$

and $|s_i - t_i| \geq 1/m$ for some $i$.

Since $(t_i) \rightarrow \sum_{i=1}^{m} t_i x_i$ is an imbedding of $\prod_{i=1}^{m}[0, \tau_i]$ into $C$, we have $\delta_m > 0$ for each $m$. Now choose $\tau_{n+1} > 0$ such that $|\tau_{n+1} x_{n+1}| < \varepsilon/2^{n+1}$ and $\tau_{n+1} < \min\{1/2^{n+1}, \delta_1/2^n, \ldots, \delta_n/2\}$. With the scalars $\{r_i\}$ so chosen, it is routine to verify that the correspondence $(t_i) \rightarrow \sum_{i=1}^{\infty} t_i x_i$ is an affine imbedding of $I^\infty$ into $C$.

Thus a convex set contains a Keller cube if and only if it contains an infinite-dimensional convex set which is somewhere locally complete. In particular, a convex set containing an infinite-dimensional locally compact convex set contains a Keller cube.

### 4. Convex sets homeomorphic to $I^2$ and $\Sigma$.

We can now prove the results stated in §1.

#### 4.1. THEOREM

**Let $C$ be a $\sigma$-compact subset of a metric linear space such that the closure $\overline{C}$ is nonlocally compact. If $C$ is $\sigma$-fd-compact, then $C \approx I^2$. If $C$ is an AR and contains an infinite-dimensional locally compact convex subset, then $C \approx \Sigma$.**

**PROOF.** Clearly, $C$ is infinite dimensional, and by 3.1 every compact subset of $C$ is a $Z$-set.

Suppose $C$ is $\sigma$-fd-compact. As a convex subset of a linear space, $C$ is contractible and locally contractible. Since every $\sigma$-fd-compact locally contractible metric space is an ANR [10], $C$ is an AR. By 3.4, $C$ contains a skeletoid for finite-dimensional compacta. Then 2.2 shows that $C$ is strongly universal for finite-dimensional compacta, and 2.1 gives $C \approx I^2$.

Now suppose that $C$ is an AR and contains an infinite-dimensional locally compact convex subset. Then $C$ contains a Keller cube by 3.5, and by 3.4 $C$ contains a skeletoid for compacta. Then $C$ is strongly universal for compacta, and $C \approx \Sigma$.

Since no infinite-dimensional metric linear space is locally compact, we have the following corollary.
4.2. COROLLARY. Every infinite-dimensional $\sigma$-fd-compact metric linear space (in particular, every $\aleph_0$-dimensional metric linear space) is homeomorphic to $l_2$. Every $\sigma$-compact metric linear space which is an AR and contains an infinite-dimensional locally compact convex subset is homeomorphic to $\Sigma$.

It was shown in [8] that every $\sigma$-compact locally convex metric linear space which contains a topological Hilbert cube is homeomorphic to $\Sigma$, and an example was given of such a space which contains no infinite-dimensional locally compact convex subsets. We do not know whether the hypothesis in the above corollary (or in the theorem) concerning the existence of an infinite-dimensional locally compact convex subset can be weakened by requiring only that $C$ contain a topological Hilbert cube.

As mentioned in §2, it is not known whether every infinite-dimensional $\sigma$-fd-compact convex subset of a metric linear space has the property that compact subsets are $Z$-sets. (Note that every such convex set must be locally infinite dimensional, and is therefore a first-category space. Thus in any case it is nowhere-locally compact). By 3.4, every such convex set $C$ contains a skeletoid for finite-dimensional compacta. Thus, the question of whether $C \approx l_2^n$ reduces to the question of whether every compact subset of $C$ is a $Z$-set. We do have the following partial answer.

4.3. COROLLARY. Let $C$ be an infinite-dimensional $\sigma$-fd-compact convex subset of a metric linear space $E$, and suppose that $E$ does not contain a Keller cube. Then $C \approx l_2^n$.

PROOF. $\overline{C}$ must be nonlocally compact, since otherwise it would contain a Keller cube, by 3.5. Thus the corollary follows from 4.1.

In particular, every infinite-dimensional $\sigma$-fd-compact symmetric convex subset $C$ of a metric linear space is homeomorphic to $l_2$, since in this case $\text{span } C = \bigcup_{i=1}^{\infty} nC$ is $\sigma$-fd-compact.

4.4. THEOREM. Let $C$ be a $\sigma$-compact convex subset of a complete metric linear space such that the closure $\overline{C}$ is nonlocally compact and an AR. Then $(\overline{C},C) \approx (l_2,l_2)$ if $C$ is $\sigma$-fd-compact, and $(\overline{C},C) \approx (l_2,\Sigma)$ if $C$ contains an infinite-dimensional locally compact convex subset.

PROOF. By [9], a closed convex subset of a complete metric linear space is homeomorphic to $l_2$ if it is separable, nonlocally compact, and an AR. Thus $\overline{C} \approx l_2$.

Applying 3.4 to $\overline{C}$, we obtain a strongly universal tower $\{C_i\}$ for finite-dimensional compacta, and the proof shows that the tower elements may be taken to be finite-dimensional cells in the dense convex subset $C$. Then, in the sense of Bessaga and Pelczynski [2], $\bigcup_{i=1}^{\infty} C_i$ is a skeletoid for the collection of finite-dimensional compacta in $\overline{C} \approx l_2$ (an fd-cap set for $\overline{C}$ in the sense of Anderson—see [5]). And if $C \supset \bigcup_{i=1}^{\infty} C_i$ is $\sigma$-fd-compact, then $C$ is also a skeletoid [14]. Since $l_2$ is a skeletoid for finite-dimensional compacta in $l_2$, and since all such skeletoids are equivalent under space homeomorphisms (see [3]), we have $(\overline{C},C) \approx (l_2,l_2)$.

On the other hand, if $C$ contains an infinite-dimensional locally compact convex subset, and therefore contains a Keller cube, 3.4 applied to $\overline{C}$ shows there exists a strongly universal tower $\{C_i\}$ for compacta in $\overline{C}$. Again, the construction may be done such that each $C_i$ is a compactum in $C$. Then $\bigcup_{i=1}^{\infty} C_i$ is a skeletoid for the collection of compacta in $\overline{C} \approx l_2$, and since $C \supset \bigcup_{i=1}^{\infty} C_i$, the $\sigma$-compact set $C$ is
also a skeletoid. Since $\Sigma$ is a skeletoid for compacta in $l^2$, we have by equivalence of skeletoids that $(\overline{C}, C) \approx (l^2, \Sigma)$.

5. **Weak products of $\sigma$-compact ARs.** For a sequence of pointed spaces $\{(X_i, p_i)\}$, the weak product $\Sigma(X_i, p_i)$ is defined by

$$
\Sigma(X_i, p_i) = \left\{ (x_i) \in \prod X_i : x_i = p_i \text{ for almost all } i \right\}.
$$

5.1. **Theorem.** If each $X_i$ is a nondegenerate $\sigma$-fd-compact metric AR, then $\Sigma(X_i, p_i) \approx l^2$. If each $X_i$ is a $\sigma$-compact metric AR containing a Hilbert cube, then $\Sigma(X_i, p_i) \approx \Sigma$.

**Proof.** For each $n = 1, 2, \ldots$, let

$$Z_n = \left\{ (x_i) \in \prod X_i : x_i = p_i \text{ for } i > n \right\}.$$

Since there exist arbitrarily small deformations of $\Sigma(X_i, p_i)$ into its AR subspaces $Z_n$ (use contractions of $X_i$ to $p_i$, for all large $i$), $\Sigma(X_i, p_i)$ is an AR [12].

We verify that $\{Z_n\}$ is a strongly universal tower for finite-dimensional compacta in $\Sigma(X_i, p_i)$. Let $f: A \rightarrow \Sigma(X_i, p_i)$ be a map of a finite-dimensional compactum, and $B$ a closed subset of $A$ such that $f \upharpoonright B: B \rightarrow Z_m$ is an imbedding into some $Z_m$. For each $i$, let $f_i$ denote the $i$'th-coordinate projection of $f$. Then $f$ can be arbitrarily closely approximated by a truncated map $\tilde{f}: A \rightarrow Z_n$, where

$$\tilde{f}(a) = (f_1(a), \ldots, f_n(a), p_{n+1}, \ldots).$$

And assuming $n > m$, $\tilde{f} \upharpoonright B = f \upharpoonright B$. Since $A$ is finite dimensional, and each $X_i$ is nondegenerate and path-connected, there exists a map $e: A \rightarrow \bigtimes_{i \in N} X_i$ into some finite product, with $e(B) = (p_{n+1}, \ldots, p_r)$, such that if $e(a) = e(a')$, then either $a = a'$ or $a, a' \in B$. Then the map $h: A \rightarrow Z_r$, defined by

$$h(a) = (f_1(a), \ldots, f_n(a), e_{n+1}(a), \ldots, e_r(a), p_{r+1}, \ldots),$$

is an imbedding which approximates $f$, and $h \upharpoonright B = f \upharpoonright B$.

If each $X_i$ is $\sigma$-fd-compact, then so is each $Z_n$, and since $\bigcup_{n=1}^{\infty} Z_n = \Sigma(X_i, p_i)$, it follows from (2.1), (2.2) and (2.3) that $\Sigma(X_i, p_i) \approx l^2$. (This result, in the case that each $X_i$ is finite-dimensional, was observed without proof in [11].)

The same type of argument as above shows that $\{Z_n\}$ is a strongly universal tower for compacta, provided that each $X_i$ contains a Hilbert cube containing the base point $p_i$. In such cases, then, $\Sigma(X_i, p_i) \approx \Sigma$.

Let $N = N_1 \cup N_2 \cup \cdots$ be a partition of the positive integers into infinite subsets. For each $k = 1, 2, \ldots$, let $W_k$ denote the weak product of the pointed spaces $(X_i, p_i)$ which are indexed by $N_k$, and let $q_k \in W_k$ denote the base point $(p_i : i \in N_k)$. Clearly, $\Sigma(X_i, p_i) \approx \Sigma(W_k, q_k)$. Thus to complete the proof, in the general case that each $X_i$ contains a Hilbert cube, we need to show that each weak product space $W_k$ contains a Hilbert cube containing the base point $q_k$. In other words, it suffices to show that $\Sigma(X_i, p_i)$ contains a Hilbert cube containing $(p_i)$.

Let $Q$ denote the Hilbert cube. Pick $q_0 \in Q$, and let $d$ be a metric on $Q$ such that $d(q, q_0) \leq 1$ for all $q$. For each $i \geq 1$, set $M_i = \{q \in Q : 2^{-i-1} \leq d(q, q_0) \leq 2^{-i+1}\}$ and $T_i = \{q \in Q : q = q_0 \text{ or } d(q, q_0) \geq 2^{-i+2}\}$. Since each $X_i$ is an AR and contains a Hilbert cube, there exist maps $g_i: Q \rightarrow X_i$ such that $g_i \upharpoonright M_i$ is an imbedding of $M_i$ into $X_i \setminus \{p_i\}$ and $g_i(T_i) = p_i$. It is easily verified that the formula $g(q) = (g_i(q))$
defines an imbedding \( g: Q \to \Sigma(X_i, p_i) \), with \( g(q_0) = (p_i) \). This completes the proof of the theorem.

If each \( X_i \) as above has an AR compactification \( K_i \), then the weak product \( \Sigma(X_i, p_i) \) is densely imbedded as a \( \sigma Z \)-set in the product space \( \prod_{i=1}^{\infty} K_i \), which is a Hilbert cube (see [6]). It is shown in [7] that, under the hypotheses of (5.1), \( \Sigma(X_i, p_i) \) is an \( \text{fd-cap} \) set, or a cap set, in \( \prod_{i=1}^{\infty} K_i \) if and only if each \( X_i \) is map-dense in \( K_i \), i.e., the identity map on \( K_i \) can be approximated by maps into \( X_i \).

In particular, let \( S \) be any dense \( \sigma \)-compact 1-dimensional AR in the 2-cell \( I^2 \), and pick \( p \in S \). Then \( \Sigma(S, p) \subseteq \prod_{i=1}^{\infty} I^2 = I^\infty \) is a dense \( \sigma Z \)-set and is homeomorphic to an \( \text{fd-cap} \) set in \( I^\infty \), but is not itself an \( \text{fd-cap} \) set, since the 1-dimensional space \( S \) cannot be map-dense in \( I^2 \).

ADDED IN PROOF. It has very recently been discovered that the characterization 2.1 requires the additional hypothesis that every compact subset \( F \) of \( X \) is a strong \( Z \)-set, i.e., for every open cover \( \mathcal{U} \) of \( X \) there exists a map \( f: X \to X \) limited by \( \mathcal{U} \) such that \( f(X) \cap F = \emptyset \). In all the applications of this paper, the strong \( Z \)-set hypothesis is satisfied.

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