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SLICE LINKS IN S^4

BY
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ABSTRACT. We produce necessary and sufficient conditions of a homotopy-theoretic nature for a link of 2-spheres in S^4 to be slice (i.e., cobordant to the unlink). We give algebraic conditions on the link *group* sufficient to guarantee sliceness, generalizing the known results for boundary links. The notion of a "stable link" is introduced and shown to be useful in constructing cobordisms in dimension 4.

1. Introduction. A *link in* S^4 is a smooth oriented submanifold $\mathscr{L}$ of S^4 where $\mathscr{L} = L_1 \cup L_2 \cup \cdots \cup L_n$ is the ordered disjoint union of 2-spheres. A link in S^4 is said to be *slice* (or *null-cobordant*) if the components of $\mathscr{L}$ bound disjoint, properly (and smoothly) embedded 3-balls in the 5-ball B^5. One of the difficult, outstanding problems in higher-dimensional knot theory is "Is every link of 2-spheres in S^4 a slice link?" [**24, 9**]. Kervaire has shown that all links of one component ($n = 1$) are slice [**23**]. For $n \geqslant 2$, very little is known. If the components of $\mathscr{L}$ bound disjoint, orientable 3-manifolds in S^4 (these are called *Seifert manifolds*), then there is known to exist a homomorphism $\phi\colon \pi_1(S^4 - \mathscr{L}) \to F_n$ taking a set of meridians of $\mathscr{L}$ to conjugates of generators of F_n (the free group of rank n). In fact these two conditions are equivalent, and any link satisfying them is called a *boundary link* [**17**]. It has been known for some time that boundary links are slice [**17, 9**] because, since the original technique of Kervaire involved the modification of a Seifert manifold, his techniques apply very well to this category of links. For a general link, the individual Seifert manifolds exist but will inevitably intersect each other.

There *are* elementary methods of constructing non-boundary slice links (for examples see §4), but they are ad hoc in the sense that they necessitate using the rigid geometric construction of the link to geometrically find the slice disks. Aside from Kervaire's techniques for boundary links, there is no general program for slicing a link. What is needed is a technique which can solve the slice problem for a general link.

In this paper we present algebraic criteria sufficient for the solution of the slice problem:

THEOREM 3.6. *For $\mathscr{L}$ to be slice it suffices that there exist a homomorphism ϕ: $\pi_1(S^4 - \mathscr{L}) \to P$ where P is an n-component higher-dimensional link group* (*see*

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definition in §2) *such that the normal closure of* image(ϕ) *is* P *and* $H_3(P; \mathbf{Z}_2) \cong H_4(P; \mathbf{Z}) \cong 0$.

Since any homology-boundary-link (see §4), for example, satisfies these conditions with $P = F_n$, this generalizes the known results. This theorem also provides hope that the slice question can be completely answered simply by knowledge of the link *group*. This is significant because higher-dimensional link groups have been "algebraically determined" [**23**] whereas, relatively speaking, we are still very ignorant of the geometry and homotopy of link exteriors. In §3 we derive *necessary* and *sufficient* conditions of a homotopy-theoretic nature for a link to be a slice link. This is the *only* dimension where such has been accomplished.

Even in higher dimensions it is an open question whether every link $(\coprod_{i=1}^{n} S^{2m}) \hookrightarrow S^{2m+2}$ is a slice link, although similar statements hold here for knots and boundary links ([**9**]; beware: Lemma 11 and hence Corollary 12 of [**18**] are incorrectly proven). Recently Roy Demeo, using techniques of Cappell and Shaneson, has announced a result entirely analagous to Theorem 3.6 for all even dimensions *greater* than four [**14**]. It is of interest to note that Cappell and Shaneson have shown in all *odd* dimensions that the cobordism of links *does not* reduce to the cobordism of individual components, indicating that the slice question in even dimensions may be complex [**9**]. (*Note*: In higher dimensions one works in the PL category.)

The primary *geometric* technique of researchers in this area has been the progressive modification of the Seifert manifolds for the link components. Unfortunately, one difficulty special to dimension four (caused by the failure of the Whitney trick) is that of getting 2-disks or 2-spheres *embedded* in order to perform these modifications. Many times this can be accomplished by "stabilizing", i.e., by connect-summing with copies of $S^2 \times S^2$. For this reason, in §5 we develop notions of links in $\#_r(S^2 \times S^2)$ and of "stably-slice". It is there shown how this device alleviates the embedding problems with *no loss of information*.

2. Notation. We will work in the category of smooth, oriented manifolds. The symbol $\langle G \rangle_P$ will denote the smallest normal subgroup of P containing G. The P will be omitted when it is clear from the context. A smooth, open tubular neighborhood of the submanifold A will be denoted $\mathscr{N}(A)$. The abbreviation $S^4 - A$ will sometimes be used for $S^4 - \mathscr{N}(A)$. The notation $A \hookrightarrow B$ means that A smoothly embeds in B. The terms knot group and link group (when not referring to the group of a specific link or knot) will mean a group isomorphic to the group of a higher-dimensional link group. We remind the reader that G is a higher-dimensional n-component link group if and only if G is finitely-presented, $H_1(G) \cong \mathbf{Z}^n$, $H_2(G) \cong 0$ and G has weight n [**23**].

A *spin-structure* σ on a compact, smooth oriented manifold B^n ($n \geqslant 3$) can be defined as an equivalence class of trivializations of its oriented tangent bundle restricted to the 2-skeleton of B (see [**28**, p. 202]). Two trivializations are equivalent if their "difference" $d\colon B^2 \to \mathrm{SO}(n)$ is null-homotopic. The manifold B will possess a spin-structure if and only if $\omega_2(\tau(B)) = 0$ [**27, 28**].

We note that if A is a submanifold of B, then a spin-structure on B, together with a trivialization of $N(A, B)$ (the normal bundle of A) will induce a spin-structure on the submanifold A [**30**]. If σ is a spin-structure on an n-manifold X and η is a spin-structure on an $(n+1)$-manifold W then $(X, \sigma) \hookrightarrow (W, \eta)$ will mean that there is a smooth embedding $F\colon X \hookrightarrow W$ such that the spin-structure induced by η on $F(X)$ pulls back to σ. In particular, this defines the notion of (X, σ) being the *spin-boundary* of (W, η). A *spin-surgery* on a spin manifold (X, σ) is one in which there is a spin-structure on the trace of the surgery which restricts to σ on X.

For a CW-complex K, the bordism group $\Omega_4^{\mathrm{Spin}}(K)$ is defined to be the set of triples (X, σ, f) where (X, σ) is a closed spin 4-manifold and $f\colon X \to K$, subject to the equivalence relation that $(X, \sigma, f) \sim (Y, \omega, g)$ if there exists a compact (W^5, η, H) whose boundary, in the obvious sense, is $(X, \sigma, f) \amalg (Y, -\omega, g)$. Disjoint union or connected-sum makes this set into an abelian group (see [**13** or **10**]).

3. The slice conditions. Let $\mathscr{L}$ be a link of n components in S^4 and G be $\pi_1(S^4 - \mathscr{N}(\mathscr{L}))$, *the link group*. Let $(X_{\mathscr{L}}, \sigma)$ be the unique compact spin 4-manifold obtained by spin-surgeries on the components of $\mathscr{L}$. Note that $X_{\mathscr{L}}$ has the homology of $\#_n(S^1 \times S^3)$ (hence has index zero) and has fundamental group isomorphic to the link group G. The next proposition introduces our basic philosophy which is that the slice problem is a problem of homology-cobordism of the link complement (actually we use $X_{\mathscr{L}}$) with certain restrictions on how π_1 can change over the cobordism. Cappell and Shaneson used this approach in their study of higher-dimensional link cobordism and were able to call upon their earlier work with Γ-groups [**8, 9**].

PROPOSITION 3.1. *The n-component link $\mathscr{L}$ in S^4 is a slice link if and only if X is the boundary of a compact, orientable 5-manifold W such that*:

(1) *the image of $\pi_1(X)$ in $\pi_1(W)$ normally generates all of $\pi_1(W)$*,

(2) $H_1(W) \cong \oplus_n \mathbf{Z}$,

(3) $H_2(W) \cong 0$.

PROOF. ($\Rightarrow$) If $\mathscr{L}$ is slice, then, by taking W to be the exterior of the slice discs D_i in B^5, we can easily see that the requirements above are satisfied.

($\Leftarrow$) Assuming that such a W exists, we add the n 2-handles to $X = \partial W$ which exactly reverse the surgeries that *produced* X from S^4. This will yield $S^4 = \partial(\mathscr{B})$, and the hypotheses on W insure that $\mathscr{B}$ will be contractible. Since any contractible manifold whose boundary is diffeomorphic to S^4 is itself B^5, we have $S^4 = \partial B^5$ and the cocores of the 2-handles just added will form slice discs for the components of $\mathscr{L}$. □

With the aid of this proposition and some difficult but technical surgery results of [**11**], we are now able to give our necessry and sufficient conditions for $\mathscr{L}$ to be slice. Keep in mind that, geometrically speaking, the group P will be the fundamental group of the complement of the slicing discs in B^5.

THEOREM 3.2. *The n-component link $\mathscr{L} \hookrightarrow S^4$ is a slice link if and only if there is a homomorphism ϕ from $G = \pi_1(S^4 - \mathscr{L})$ to a finitely-presented group P such that*:

(1) $\langle \phi(G) \rangle = P$,

(2) $H_1(P) \cong \oplus_n \mathbf{Z}$,

(3) $H_2(P) \cong 0$,

(4) *the element* (X, σ, f) *vanishes in* $\Omega_4^{\mathrm{Spin}}(K(P,1))$ *where* $f\colon X \to K(P,1)$ *induces* ϕ *and* (X, σ) *is the result of spin-surgery on* $\mathscr{L}$ *in* S^4.

PROOF. We shall show that the hypotheses above are equivalent to the existence of the appropriate W of Proposition 3.1.

($\Rightarrow$) If $\mathscr{L}$ is slice then W exists as in 3.1. Letting ϕ be the inclusion $\pi_1(X) \to \pi_1(W)$, properties (1) and (2) follow immediately. Recalling the "Hopf" exact sequence for any space K,

$$\pi_2(K) \to H_2(K) \to H_2(\pi_1(K)) \to 0,$$

it follows a fortiori from $H_2(W)$ being trivial that $H_2(P)$ is also. Finally, the commutative diagram of groups

$$\begin{array}{ccc} \pi_1(W) & & \\ \phi\uparrow & \searrow \mathrm{id} & \\ \pi_1(X) & \xrightarrow[\phi]{} & P \end{array}$$

induces one of spaces

$$\begin{array}{ccc} W & & \\ \uparrow & \searrow \Psi & \\ X & \xrightarrow[f]{} & K(P,1) \end{array}$$

from which (4) follows.

($\Leftarrow$) Assuming (1)–(4) above, we shall construct W as required by 3.1. Hypothesis (4) guarantees the existence of a compact, spin-manifold A of which (X, σ) is the spin-boundary and such that the following commutes:

$$\begin{array}{ccc} A & & \\ j\uparrow & \searrow \Psi & \\ X & \xrightarrow[f]{} & K(P,1) \end{array}$$

We can cause Ψ to be an isomorphism on π_1 by first connect-summing with $S^1 \times S^4$ until Ψ_* is epic, then performing spin-surgery on embedded circles representing the kernel of Ψ. Since P is finitely presented the kernel of Ψ will be finitely *normally*-generated and thus we need only perform a *finite* number of such surgeries (see [**35**, p. 15]). Note that our sole remaining task is to kill $H_2(A)$, for then, setting $W = A$, we would be done. This can be accomplished by spin-surgeries on embedded 2-spheres (in $A - X$) representing elements of $H_2(A)$. For this we appeal to a surgery theorem of [**11, 10**].

THEOREM 3.3 (see Theorem 4.3 of [**11**] and remarks following the proof). *Let X be the spin-boundary of A and let $P = \pi_1(A)$. If $H_2(P) = 0$ then we can replace A by a surgered A which satisfies in addition that the map $H_2(X) \to H_2(A)$ be an epimorphism.*

Recalling that, in our present situation, $H_2(X) = 0$, it follows immediately that we kill $H_2(A)$. $\square$

COROLLARY 3.4. *Let $\mathscr{L}$, $\mathscr{L}'$ be links in S^4 and suppose that the exterior of $\mathscr{L}$ is homotopy-equivalent, relative its boundary, to the exterior of $\mathscr{L}'$. Then $\mathscr{L}$ is slice if and only if $\mathscr{L}'$ is slice.*

Note. S. Plotnick has constructed an infinite number of knots K, K' which satisfy the hypotheses of 3.4, but which have nondiffeomorphic exteriors [**31**]. Gordon has constructed inequivalent knots with diffeomorphic exteriors [**15**].

PROOF. The homotopy-equivalence can be extended to one $f\colon Y \to X$ where X, Y are the surgeries on $\mathscr{L}'$, $\mathscr{L}$, respectively. This is covered by a bundle map from ν, the tangent bundle of Y, to ξ, a trivial stable bundle over X, by defining $\xi = (f^{-1})^*(\nu)$ where f^{-1} is the homotopy-inverse of f. Since Y is spin and of index zero, it is stably-parallelizable so ν (and hence ξ) is trivial. Now assume $\mathscr{L}$ is slice. Then the conclusions of Theorem 3.2 hold for some $\phi\colon G \to P$, and some $\phi^{\#}\colon X \to K(P,1)$. Since $H_2(\pi_1(X); \mathbf{Z}_2) \cong 0$, the normal map f is normally-cobordant to a self-homotopy-equivalence $g\colon X \to X$ (see [**35**, Chapter 16]). It follows that (Y, f, σ_Y) is equivalent to (X, g, σ_X) in $\Omega_4^{\mathrm{Spin}}(X)$ for *some* spin-structures σ_Y, σ_X. Letting $\bar{g}$ stand for the homotopy-inverse of g, we have that $(Y, \phi^{\#} \circ \bar{g} \circ f, \sigma_Y) \sim (X, \phi^{\#} \circ \bar{g} \circ g, \sigma_X)$ in $\Omega_4^{\mathrm{Spin}}(K(P,1))$, and hence that $(Y, h, \sigma_Y) \sim (X, \phi^{\#}, \sigma_X)$ where $h = \phi^{\#} \circ \bar{g} \circ f$. Since ϕ is an isomorphism on $H^1(\ :\mathbf{Z}_2)$, it can be shown that the vanishing of $(X, \phi^{\#}, \sigma)$, as given by (4) of 3.2, is equivalent to the vanishing of $(X, \phi^{\#}, \sigma_X)$ for *any* spin-structure. We shall omit this verification to avoid bogging down in details. Therefore, $(Y, h, \sigma_Y) \sim 0$ in $\Omega_4^{\mathrm{Spin}}(K(P,1))$ for $h = \phi^{\#} \circ \bar{g} \circ f$. Applying Theorem 3.2, it is only necessary to verify that the normal closure of the image of the map $\phi \circ g_*^{-1} \circ f_*$ is P. Since f_* and g_* are isomorphisms, this is a triviality. □

In order to analyze the consequences of Theorem 3.2 it is necessary to better understand the structure of $\Omega_4^{\mathrm{Spin}}(K)$. The following lemma says that, to the first approximation, $\Omega_4^{\mathrm{Spin}}(K)$ is determined by the homology of the space K.

LEMMA 3.5 ([**13**]; SEE ALSO [**11**]). *There is a spectral-sequence*

$$H_p\big(K; \Omega_{4-p}^{\mathrm{Spin}}(\mathrm{pt.})\big) \Rightarrow \Omega_4^{\mathrm{Spin}}(K)$$

whose $E^2_{p,4-p}$ terms are $H_p(K; \Omega_{4-p}^{\mathrm{Spin}}(\mathrm{pt.}))$.

Since the groups $\Omega_n^{\mathrm{Spin}}(\mathrm{pt.}) \cong \Omega_n^{\mathrm{Spin}}$ are $\mathbf{Z}$, $\mathbf{Z}_2$, $\mathbf{Z}_2$, $\{e\}$ and $\mathbf{Z}$ for $n = 0, 1, 2, 3$ and 4 [**27**], we have the following

THEOREM 3.6. *The n-component link $\mathscr{L}$ will be slice if there is a homomorphism ϕ from G to an n-component higher-dimensional link group P such that*

(i) $\langle \phi(G) \rangle = P$,

(ii) $H_3(P; \mathbf{Z}_2) \cong H_4(P; \mathbf{Z}) \cong 0$.

PROOF. The E^2 terms of the spectral-sequence for $\Omega_4^{\mathrm{Spin}}(P)$ vanish except for $E^2_{0,4}$ which is the image of Ω_4^{Spin} itself. Since $\mathrm{index}(X) = \omega_2(X) = 0$, the pair (X, σ) is trivial in Ω_4^{Spin} [**36**], and hence (X, σ, f) will be zero in the larger group as required by Theorem 3.2. □

REMARKS. Roy DeMeo has proved the analagous theorem for all even-dimensional links in dimensions greater than four [**14**]. Notice that Theorem 3.6 proves immediately that boundary links are slice by setting $P = F_n$, the free group on n generators. If one is willing to talk of slice links in *homology* B^5's then one can eliminate all of the normal-closure conditions above (retaining an isomorphism on H_1) and prove all of the analogous theorems.

The next section will discuss new classes of links which can be shown to be slice via 3.2 and will attempt to give the reader a feeling for the types of groups P one might encounter.

We close this section with an interesting improvement of Corollary 3.4.

COROLLARY 3.7. *Suppose $\mathscr{L}_0$ is a link with group G_0 which can be sliced by discs such that the group P satisfies $H_3(P) \otimes \mathbf{Z}_2 \cong 0$. Suppose that $\mathscr{L}_1$ is any link (with group called G_1). Then for $\mathscr{L}_1$ to be slice it suffices that there exist any map $f\colon X_1 \to X_0$ such that $\langle f_*(G_1) \rangle = G_0$ (where X_1, X_0 are the surgeries on the links).*

PROOF. Since $H_3(P) \otimes \mathbf{Z}_2 \cong 0$, $\Omega_4^{\text{Spin}}(K(P,1))$ is equal to $\Omega_4^{\text{Spin}} \oplus N$ where N is a subgroup of $H_4(P; \mathbf{Z})$. The image of an element (X_0, σ, g) in N is well known to be given by $g_*([X_0])$ (see [**10**, §8]). Since $\mathscr{L}_0$ is slice "over" P we known $g_*([X_0]) = 0$, and it follows that $(X_1, \sigma_1, g \circ f) \sim 0$ in $\Omega_4^{\text{Spin}}(K(P,1))$. Theorem 3.2 then applies. □

4. New classes of slice links and some perspective. We shall exhibit three classes of *nonboundary* links which can be shown to be slice by Theorem 3.2. These are "homology-boundary links", links whose groups are "arc-groups", and certain "semi-fibered links". We should remark that, although these classes extend the known results, many examples constructed to fall in these classes could (with work) be shown to be slice by some more "geometric" arguments, since *most* constructions are very geometric. This points out a fundamental difficulty with this subject—namely that very few techniques are available for *constructing* links in high dimensions. Spinning and its modifications yield knots and links which have 3-dimensional characteristics and hence cannot hope to illustrate the added complexities of higher dimensions. With this in mind, the primary importance of our work must lie not in these new classes of slice links, but in the hope for a global solution via Theorems 3.2 and 3.6.

First we would like to give the reader some perspective on the relationship between the link group G and the group P of the disk complement. We accomplish this via some elementary but revealing examples culled from the literature.

EXAMPLE 4.1. There are links which are not boundary links even though their groups are *isomorphic* to F_n. The trick is that no isomorphism takes meridians to conjugates of generators [**2**, or **32**, p. 94]. Even such a simple link as this cannot be handled by Kervaire's method, although Theorem 3.6 handles it easily.

EXAMPLE 4.2. More generally it is easy to construct nonboundary links using the following trivial observation. If $\{K, J\}$ is a 2-component boundary link then a lift of J to the infinite-cyclic cover of the exterior of K must be null-homologous. Thus, if K is a fibered knot with fiber a punctured $M\#_r(S^1 \times S^2)$ and J is some $(* \times S^2)$, then $\{K, J\}$ is not a boundary link. Note that the construction of such fibered links necessitates finding a diffeomorphism of $M\#_r(S^1 \times S^2)$ with certain algebraic properties. Some examples with $M \cong S^3$ are given in [**3**]. These might then be connect-summed to any "twist-spun" knot (see [**32**]) with fiber say M. For more complicated examples one could employ §3 of [**26**] which shows how to create understandable diffeomorphisms via paths in the Kirby calculus.

EXAMPLE 4.3. There are 2-component slice links for which P cannot be chosen to be $\mathbf{Z} * \mathbf{Z}$. These include many of the links in 4.2 and a more interesting one obtained by "spinning" the arcs in Figure 1 (for a definition of spinning see [**32**, p. 85]). This embedding is due to W. Jaco [**22**] who computes that

$$\pi_1(\mathbf{R}^3_+ - \text{arcs}) \cong G \cong \{a, b, c \mid a = [\bar{c}, a][c, b]\}.$$

This will also be the group of the resultant link in S^4, and can be recognized as one of Baumslag's nonfree *parafree* groups [**4**]. As such it possesses no epimorphism to $\mathbf{Z} * \mathbf{Z}$ [**33**]. (Recall that a group G is said to be parafree if $\bigcap_{n=1}^{\infty} G_n$ is trivial and $G/G_n \cong F/F_n$ for all n where F is a free group, $G_1 \equiv G$ and $G_n \equiv [G, G_{n-1}]$.) However, it is known that every spun link L is slice since (S^4, L) can be defined as the boundary of $(B^3, \text{arcs}) \times D^2$ (see [**32**, p. 96]). Thus Example 4.3 (and 4.1) is a *non*boundary slice link. In fact, 4.3 cannot be sliced with $P \cong \mathbf{Z} * \mathbf{Z}$.

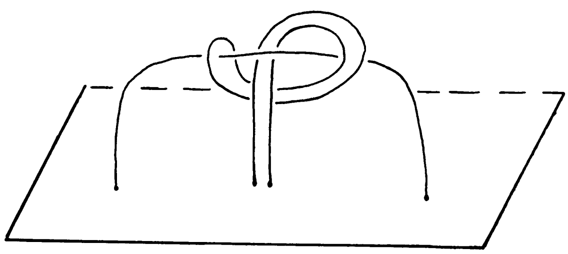

FIGURE 1

EXAMPLE 4.4. There are link groups with nonvanishing H_3 and H_4. The simplest example is the group of the 4-twist-spin of the trefoil knot, which is isomorphic to $\mathbf{Z} \times D$ where D is the binary icosahedral group [**16**]. Here $H_3 \cong H_4 \cong \mathbf{Z}_{120}$.

EXAMPLE 4.5. In contrast to 4.3 above, one might ask if there are links which cannot be sliced with $P \cong G$. Many *can*, including all spun-links, but, interestingly enough, for a *fibered knot* in S^4 to be slice "over" its own group the fiber must have *free* fundamental group [**12**]. Thus the Cappell-Shaneson-Akbulut-Kirby fibered knots with fiber $(S^1 \times S^1 \times S^1)^0$ cannot be sliced over their groups (see [**12**, §5; **1**; and 7]).

We now exhibit our three classes of links for which there existed no a priori argument for sliceness but which can be shown to be slice by our algebraic criteria.

DEFINITION. A link $\mathscr{L}$ of n components is said to be a *homology-boundary link* if there is a homomorphism ϕ: $G \to F_n$ (free group of rank n) whose image normally generates F_n. (This differs slightly from [**20**].)

DEFINITION. A group G is an *arc-group* if $G \cong \pi_1(B^3 - A)$ where A is the disjoint union of properly (and locally-flatly) embedded arcs in the 3-ball.

Examples of homology boundary links may be found among Examples 4.1 and 4.2 while any spun-link has group isomorphic to an arc-group.

COROLLARY 4.6. (a) *Homology boundary links are slice.*
(b) *Any link whose group is an arc-group is slice.*

PROOF. The proof of (b) requires a trivial lemma that the cohomological dimension of an arc-group is less than or equal to two. (Thus not all *knots* in S^4 fall into category (b) although they all fall into (a)).

Finally, we introduce a more complex class of links which are neither spun-links nor homology-boundary links (in general).

THEOREM 4.7. *Suppose $\mathscr{L} = \{K, J\}$ is a 2-component link in S^4 where K is fibered with fiber $V^0 = M^0\#_r(S^1 \times S^2)$ and glueing map ϕ. Suppose J is some $* \times S^2 \hookrightarrow V^0$ and that the composite*

$$\pi_1(M^0) \xrightarrow{i} \pi_1(V^0) \xrightarrow{\phi} \pi_1(V^0) \xrightarrow{P} \pi_1(\#_r(S^1 \times S^2))$$

is the zero map. Then $\mathscr{L}$ is a slice link.

REMARKS. Note that $\mathscr{L}$ is *never* a boundary link. The hypotheses will always be satisfied if $\pi_1(M)$ is normally generated by a set of elements of finite order. The composite map above is always zero on the H_1-level. If $\pi_1(M)$ is not free then the group of $\mathscr{L}$ is *not* an arc-group because its cohomological dimension will be at least 3. These can be constructed as explained in Example 4.2.

PROOF. The surgered X_K is diffeomorphic to $S^1 \times_\phi V$. Choose a V away from where the glueing is done and note that it inherits a spin-structure σ for which $(V, \sigma) = M\#_r(S^1 \times S^2)$ is the spin-boundary of (W, η) where $W \cong \overline{W}\#_r(S^1 \times B^3)$ and $\overline{W}$ is 1-connected. For $i = 0, 1$ let W_i be copies of (W, η) and form the closed, spin 4-manifold

$$N = \left[W_0 \cup_{\partial W_0} (V \times [0, \tfrac{1}{2}])\right] \cup_\phi \left[W_1 \cup_{\partial W_1} (V_1 \times [\tfrac{1}{2}, 1])\right].$$

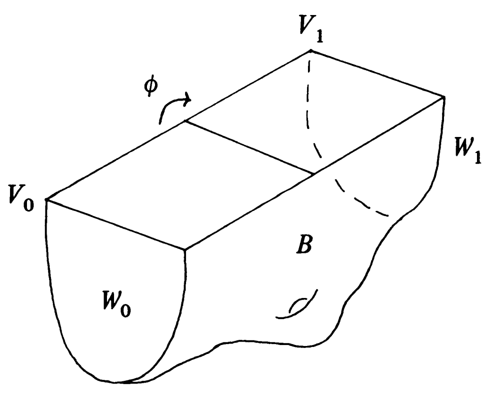

FIGURE 2

It is then an easy exercise to show that the hypotheses above imply

LEMMA 4.8. *There is a epimorphism* $\Psi: \pi_1(N) \to F_r$ *such that*

$$\pi_1(W_1) \xrightarrow{i} \pi_1(N) \xrightarrow{\Psi} F_r$$

is the identity map and

$$\pi_1(W_0) \xrightarrow{i} \pi_1(N) \xrightarrow{\Psi} F_r \equiv \pi_1(W_1)$$

is the composite $p \circ \phi \circ j: \pi_1(\#_r(S^1 \times B^3)) \to \pi_1(\#_r(S^1 \times B^3))$.

Then the element $[N, \sigma, \Psi]$ in $\Omega_4^{\mathrm{Spin}}(K(F_r, 1))$ is zero by Lemma 3.5, so $[N, \sigma, \Psi] = \partial[B, \xi, g]$ where we can assume that $g_*: \pi_1(B) \to F_r$ is the identity. Form the spin 5-manifold $A = B/W_0 \sim W_1$ so that $\partial A \cong X_K$. A Mayer-Vietoris argument produces the exact sequences:

$$\begin{array}{ccccccccc}
H_1(V) & \xrightarrow{\phi_* - \mathrm{id}_*} & H_1(V) & \to & H_1(X_K) & \xrightarrow{\cong} & \mathbf{Z} & \to & 0 \\
\downarrow j_* & & \downarrow j_* & & \downarrow j_* & & \downarrow & & \\
H_1(W) & \xrightarrow{i_0 - i_1} & H_1(B) & \to & H_1(A) & \to & \mathbf{Z} & \to & 0
\end{array}$$

Since $\phi_* - \mathrm{id}_*$ is an isomorphism, the map $i_0 - i_1$ is an epimorphism between free abelian groups of rank r, hence an isomorphism. Thus $H_1(X_K) \to H_1(A)$ is an isomorphism and furthermore $\pi_1(X_K) \twoheadrightarrow \pi_1(A)$ is easily seen to be epic since $\pi_1(N)$ maps onto $\pi_1(B)$. Since the inclusions i_0, i_1 of $\pi_1(W_i)$ into $\pi_1(B)$ can be shown to be monic, there is a corresponding exact sequence

$$H_2(\pi_1(B)) \to H_2(\pi_1(A)) \to H_1(\pi_1(W)) \underset{\cong}{\xrightarrow{i_0 - i_1}} H_1(\pi_1(B)).$$

Thus $H_2(\pi_1(A)) \cong H_2(\pi_1(B)) \cong 0$.

We can assume that the knot J lies on V_0 and bounds a 3-disk Δ in the appropriate $S^1 \times B^3$. Let $C = A - \mathcal{N}(\Delta)$ so $\partial C \cong X_{\mathscr{L}}$ as spin-manifolds. Clearly

$$\pi_1(C)/\langle m_J \rangle = \pi_1(A)$$

so the image $\pi_1(X_{\mathscr{L}}) \to \pi_1(C)$ normally generates $\pi_1(C)$. Since $i_0 - i_1$ (above) is an isomorphism, any element λ of $H_2(A)$ can be represented by an embedded 2-sphere in the interior of B. Thus $\lambda \cdot [\Delta] = 0$ so $H_1(C) \cong \mathbf{Z} \times H_1(A) \cong \mathbf{Z} \times \mathbf{Z}$, $H_2(C) \cong H_2(A)$ and hence $H_2(\pi_1(C)) \cong 0$. Applying Theorem 3.2 (using $\pi_1(X_{\mathscr{L}}) \to \pi_1(C)$), it follows that $\mathscr{L}$ is a slice link. □

We close this section with a discussion of avenues of future investigation as regards Theorem 3.6. Can every link be sliced in this way? Can the map $\phi: G \to P$ always be chosen to be an epimorphism? Is there a canonical subgroup N of G such that any link with group G is slice "over" G/N?

In particular, as regards this last question, the candidate $N = G_\omega \cong \bigcap_{i=1}^{\infty} G_n$ is very promising. For knots, $G/G_\omega = \mathbf{Z}$ so all knots can be sliced in this way. Furthermore, a link is a homology-boundary link if and only if this quotient is free. In general G/G_ω is known to be *parafree* [33]. Since finitely-presented parafree groups resemble free groups in many ways, it is not unreasonable to hope that they have low homological dimension. In fact, all known finitely-presented parafree groups have cohomological dimension less than or equal to 2 [**5**, **4**]. Further, it has been conjectured that H_2 of a finitely-presented parafree group is trivial (a proof was even announced in [**19**]). The major obstacle to G/N in general is the question of its finite-presentability. Indeed

Question. Given a link group G, is there a "naturally" defined quotient G/N such that

(a) $H_2(G/N) = 0$ and

(b) G/N is finitely-presented?

The only such quotients I know are the trivial ones G and $\{e\}$.

5. Stable cobordism. We will now introduce a notion of a stable-link and of a stably-slice link. This will necessitate dealing with links of 2-spheres in an arbitrary connected-sum of $S^2 \times S^2$. Our major result is that "stabilizing" does not affect the "sliceness" of a link. This allows us to stabilize whenever we find it useful in modifying a given link exterior to a simpler one. We hope that other researchers in this field will make use of this simplification.

We must only consider links in $\#_k(S^2 \times S^2)$ which "look" as if they came from links in S^4.

DEFINITIONS. *A link* $\mathscr{L} = \{L_1, \ldots, L_n\}$ *in some* $\#_k(S^2 \times S^2)$ is a smooth submanifold diffeomorphic to an ordered disjoint union of 2-spheres subject to the restriction that the group of the link exterior be a (higher-dimensional) link group. Such a link is said to be *slice* if there is some $\#_k(S^2 \times B^3)$ in which the components bound disjoint, properly (and smoothly) embedded 3-balls.

Note. The restriction on the link group is essential. Without it, besides not correctly mirroring the S^4 case, it is easy to exhibit nonslice links [**10**, p. 61].

DEFINITIONS. The *stable n-link* $\vec{\mathscr{L}}$ induced by the n-link $\mathscr{L}$ in $\#_k(S^2 \times S^2)$ ($k \geqslant 0$) is a sequence of links $\mathscr{L}^i \hookrightarrow \#_{k+i}(S^2 \times S^2)$ ($i = 0, 1, \ldots$) where $\mathscr{L}^0 = \mathscr{L}$ and $\mathscr{L}^{i+1}$ is formed from $\mathscr{L}^i \hookrightarrow \#_{k+i}(S^2 \times S^2)$ buy connect-summing on another $S^2 \times S^2$ at some point far from $\mathscr{L}^i$. A *stable-link* $\vec{\mathscr{L}}$ is said to be *stably-slice* if some element in the sequence $\mathscr{L}^i$ is slice. Thus a *link* $\mathscr{L}$ is *stably-slice* if the stable-link $\vec{\mathscr{L}}$ which it induces is stably-slice.

The power of this approach is provided by the following

THEOREM 5.1. *A link* $\mathscr{L}$ *is slice if and only if it is stably-slice.*

PROOF. The "only if" is clear, so assume $\mathscr{L} \hookrightarrow \#_k(S^2 \times S^2)$ is stably-slice, i.e., $\mathscr{L} \hookrightarrow \#_k(S^2 \times S^2) \#_r(S^2 \times S^2)$ is slice in some $W = \#_{k+r}(S^2 \times B^3)$. Clearly $X_{\vec{\mathscr{L}}} = X \#_r(S^2 \times S^2)$. We could now prove that $\mathscr{L}$ is slice via a theorem analogous to Theorem 3.2 (in fact, this theorem holds word for word with S^4 replaced by $\#_j(S^2 \times S^2)$); but this would be unnecessarily repetitive. Instead, having noted this, we shall prove Theorem 5.1 by relating the question to one which we solved in earlier work [**11**].

LEMMA 5.2. *A link* $\mathscr{L}$ *in* $\#_k(S^2 \times S^2)$ *is slice if and only if* $X_{\mathscr{L}}$ *embeds in* S^5 *with one complementary component being* 1*-connected.*

LEMMA 5.3 (COROLLARY 7.2 OF [**11**]). *$X_{\mathscr{L}}$ embeds in S^5 with one complementary component* 1*-connected if and only if $X_{\mathscr{L}} \#_r (S^2 \times S^2)$ does also.*

The proof of Lemma 5.2 is like that of Proposition 3.1, the only new ingredient being the fact that if $\#_j(S^2 \times S^2)$ embeds in S^5 then each complementary component is diffeomorphic to $\#_j(S^2 \times B^3)$. This in turn follows from the 5-dimensional relative h-cobordism theorem [**29**]. This completes the proof of 5.1.

One use of the stability theorem is in realizing "algebraic cobordisms" by actual ones. Let $\mathscr{L}$ be a link of n components in $\#_k(S^2 \times S^2)$ (in S^4 if $k = 0$), with group G.

DEFINITION. A *link cobordism* from the link $\mathscr{L}$ to another link $\mathscr{L}'$ is a proper submanifold W of $(\#_k S^2 \times S^2) \times I$ which is diffeomorphic to $\amalg_{i=1}^{n}(S^2 \times I)$ and such that $\partial_+ W = \mathscr{L}$ and $\partial_- W = \mathscr{L}'$.

DEFINITION. An *algebraic cobordism* of $\mathscr{L}$ is a homomorphism from G to an n-component link group P such that the image of ϕ normally generates P.

We say that an algebraic cobordism of L is *realized* by a link cobordism if $\vec{L}$ is cobordant to L' where the group of L' is isomorphic to P with a meridian set being given by the images under ϕ of a meridian set of G.

THEOREM 5.4. *Any algebraic cobordism ϕ: $G \to P$ of $\mathscr{L}$ can be realized by a link cobordism from an element in the stable-link $\vec{\mathscr{L}}$ to another link $\mathscr{L}'$.*

PROOF. Since P is finitely-generated, the map ϕ will factor as $G \xrightarrow{i} G * F_\mu \xrightarrow{\Psi} P$ where F_μ is a free group, Ψ is an epimorphism and $\Psi(F_\mu) \subset [P, P]$. Since P is finitely-presented, the kernel of Ψ will be normally generated by a finite number of elements $\{\gamma_1, \ldots, \gamma_m\}$ in $G * F_\mu$. Now form a 5-dimensional cobordism C by starting with $(\#_k(S^2 \times S^2) - \mathscr{L}) \times I$ and adding on μ one-handles and m two-handles (to $\partial_- C$) realizing the obvious algebra. Choose the framing on the two-handles so as to extend the original spin-structure over C. The "new boundary" $\partial_+ C$ will be a 4-manifold with boundary which has $\pi_1 \cong P$ and looks *homologically* like $\#_{k+m}(S^2 \times S^2) - \mathscr{L}'$ where $\mathscr{L}'$ is a link of n components. In order to make C into a homology-cobordism, we need to kill the classes looking like the $\#_m(S^2 \times S^2)$'s by adding 3-handles. Since $H_2(P) \cong 0$, the necessary classes are spherical but may not be representable in $\partial_+ C$ as embedded 2-spheres. However, after stabilizing the entire C (add on $\#_j(S^2 \times S^2) \times I$), we stabilize $\partial_+ C$ by adding $\#_j(S^2 \times S^2)$, and a theorem of Cappell and Shaneson assures us that the desired classes *are* now representable [**6**]. Furthermore, their theorem guarantees that the 2-spheres may be chosen in a π_1-negligible fashion. Thus, after adding these m 3-handles along $\partial_+ C$, we get a new cobordism C' and boundary $\partial_+ C'$, and C' will be a homology-cobordism between the stable-link exterior $E_- \cong \partial_- C' = (\#_k(S^2 \times S^2 - \mathscr{L}) \#_j S^2 \times S^2$ and $E_+ = \partial_+ C'$. The manifold E_+ will have n copies of $S^2 \times S^1$ as its boundary and thus is a link complement of some sort. It follows from the π_1-negligibility that $\pi_1(E_+)$ and $\pi_1(C')$ will be isomorphic to P in the desired manner. When $\mathscr{L} \times I$ is glued into C' it becomes an h-cobordism and hence is diffeomorphic, after stabilization, to a product [**6**]. Thus we have constructed a link cobordism from some element in the stable-link $\vec{\mathscr{L}}$ to a link $\mathscr{L}'$ with group P. □

REMARK. Of course, we have control of the meridians of the new link as they relate to ϕ and $\mathscr{L}$.

Another stumbling point in 4-dimensional knot theory is the attempted construction of a "minimal" Seifert manifold for a given knot or link. Recall that a Seifert manifold for a knot K is an orientable 3-dimensional submanifold of S^4 whose boundary is the knot K. There are various possible notions of minimality, but they all refer (intuitively) to some measure of complexity. If one could continually decrease the complexity of the Seifert manifolds (for the link components) via link cobordisms until they were all 3-disks, then the link would be slice. Given a Seifert manifold, the only reasonable way to reduce its "complexity" is to look for elements of its fundamental group which die in the link group, and try to perform an ambient surgical modification along a disk. Of course, such an embedded disc may not exist, but it does exist after connect-summing with $\#_j(S^2 \times S^2)$. Therefore, for questions of cobordism, we hope to use this approach of modifying the Seifert manifolds by (stable) cobordisms to reduce the slice problem and hopefully solve it completely.

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