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CONTRIBUTIONS FROM CONJUGACY CLASSES OF REGULAR ELLIPTIC ELEMENTS IN Sp(n, Z) TO THE DIMENSION FORMULA

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ABSTRACT. The dimension of the space of cusp forms on the degree n Siegel upper half-space can be obtained from the Selberg trace formula; in this paper we compute the contribution from the conjugacy classes of regular elliptic elements in $\mathrm{Sp}(n, \mathbf{Z})$ using Weyl's character formula for representations of $\mathrm{GL}(n, \mathbf{C})$.

Introduction. Let H_n be the Siegel upper half-space; specifically,

$$H_n = \{ Z \in M_n(\mathbf{C}) \mid Z = {}^tZ, \operatorname{Im} Z > 0 \}.$$

The real symplectic group of degree $2n$, $\mathrm{Sp}(n, \mathbf{R})$, acts transitively on H_n as a group of holomorphic automorphisms by the action,

$$M(Z) = (AZ + B)(CZ + D)^{-1}, \qquad M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ in } \mathrm{Sp}(n, \mathbf{R}).$$

Let $\mathrm{Sp}(n, \mathbf{Z}) = \mathrm{Sp}(n, \mathbf{R}) \cap M_{2n}(\mathbf{Z})$ be the full modular subgroup of $\mathrm{Sp}(n, \mathbf{R})$.

Denote by $S(k; \mathrm{Sp}(n, \mathbf{Z}))$ the vector space of holomorphic cusp forms of weight k and degree n with respect to $\mathrm{Sp}(n, \mathbf{Z})$. (See [6] for the detail of definition.) If $k \geqslant 2n + 3$, k is even when n is odd and $n \geqslant 1$, the dimension of $S(k; \mathrm{Sp}(n, \mathbf{Z}))$ over $\mathbf{C}$ is given by Selberg's trace formulas as follows [3]:

$$\dim_{\mathbf{C}} S(k; \mathrm{Sp}(n, \mathbf{Z})) = C(k, n) \int_F \sum_M K_M(Z, \bar{Z})^{-k} (\det Y)^{k-n-1} \, dX \, dY,$$

where

(1)

$$C(k, n) = 2^{-n} (2\pi)^{-n(n+1)/2} \prod_{i=0}^{n-1} \Gamma\left(k - \frac{n-i-1}{2}\right) \left[\prod_{i=0}^{n-1} \Gamma\left(k - n + \frac{i}{2}\right) \right]^{-1},$$

(2) F is a fundamental domain on H_n for $\mathrm{Sp}(n, \mathbf{Z})$,

(3) in the summation, M ranges over all matrices $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ in $\mathrm{Sp}(n, \mathbf{Z})/\{\pm 1\}$ and

$$K_M(Z, \bar{Z}) = \det\left[(1/2i)\left(Z - \overline{M(Z)}\right)\right] \det \overline{(CZ + D)}.$$

An element M of $\mathrm{Sp}(n, \mathbf{Z})$ is regular elliptic if M has an isolated fixed point on H_n. Let M be a regular elliptic element of $\mathrm{Sp}(n, \mathbf{Z})$. Then the fixed point of M can be

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transformed into the point iE of H_n by an action of $\mathrm{Sp}(n,\mathbf{R})$. Here E is the identity of the matrix ring $M_n(\mathbf{C})$. Note that the isotropy group of iE in $\mathrm{Sp}(n,\mathbf{R})$ is $U(n)$, the $n \times n$ unitary group, via the identification

$$\begin{bmatrix} P & Q \\ -Q & P \end{bmatrix} \to P + Qi.$$

Hence M is conjugate in $\mathrm{Sp}(n,\mathbf{R})$ to an element $[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}]$ with $A + Bi$ in $U(n)$. Furthermore, we can assume $A + Bi$ is a diagonal matrix since every unitary matrix can be diagonalized by a conjugation of a unitary matrix. Since $\mathrm{Sp}(n,\mathbf{Z})$ is discrete and $U(n)$ is compact, it follows that M is an element of finite order. Thus the eigenvalues of $A + Bi$ are roots of unity. Also, the centralizer $C_{M,\mathbf{Z}}$ of M in $\mathrm{Sp}(n,\mathbf{Z})/\{\pm 1\}$ is a group of finite order.

On the other hand, let $M \in \mathrm{Sp}(n,\mathbf{Z})$ be conjugate in $\mathrm{Sp}(n,\mathbf{R})$ to $[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}]$ with $U = A + Bi$ in $U(n)$, and U have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$; $\lambda_i\lambda_j \neq 1$ for $1 \leqslant i \leqslant j \leqslant n$. Then the set of fixed points of M is isomorphic to

$$\Omega = \{ W \in D_n \mid UW{}^tU = W \}$$

under the Cayley transform which maps H_n biholomorphically onto the generalized disc D_n defined by D_n: $W = {}^tW \in M_n(\mathbf{C})$, $E - W{}^t\overline{W} > 0$. Note that Ω is isomorphic to

$$\Omega' = \{ W = [w_{ij}] \in D_n \mid (1 - \lambda_i\lambda_j) w_{ij} = 0 \text{ for all } i, j \}$$

which is a single point 0 because $1 - \lambda_i\lambda_j \neq 0$ for all i, j. Consequently, the set of fixed points of M is a single point and hence M is a regular elliptic element of $\mathrm{Sp}(n,\mathbf{Z})$. This proves that the following statements are equivalent:

(1) M is a regular elliptic element of $\mathrm{Sp}(n,\mathbf{Z})$.

(2) $M \in \mathrm{Sp}(n,\mathbf{Z})$ and has an isolated fixed point on H_n.

(3) $M \in \mathrm{Sp}(n,\mathbf{Z})$ and is conjugate in $\mathrm{Sp}(n,\mathbf{R})$ to $[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}]$ with $A + Bi = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n] \in U(n)$; λ_i $(i = 1, 2, \ldots, n)$ are roots of unity and $\lambda_i\lambda_j \neq 1$ for all i, j.

In this paper, we shall prove the following Theorem with which we can compute the contribution from conjugacy classes of regular elliptic elements in $\mathrm{Sp}(n,\mathbf{Z})$ for general n.

THEOREM. *Suppose $M \in \mathrm{Sp}(n,\mathbf{Z})$ is conjugate in $\mathrm{Sp}(n,\mathbf{R})$ to $[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}]$ with $U = A + Bi$ in $U(n)$ and U has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$; $\lambda_i\lambda_j \neq 1$ for $1 \leqslant i \leqslant j \leqslant n$. Then the contribution to $\dim_{\mathbf{C}} S(k; \mathrm{Sp}(n,\mathbf{Z}))$ ($k \geqslant 4$, k is even if n is odd) of elements in $\mathrm{Sp}(n,\mathbf{Z})$ which are conjugate in $\mathrm{Sp}(n,\mathbf{Z})/\{\pm 1\}$ to M is given by*

$$N_{\{M\}} = |C_{M,\mathbf{Z}}|^{-1} \prod_{i=1}^{n} \bar{\lambda}_i^k \prod_{1 \leqslant i \leqslant j \leqslant n} \left(1 - \bar{\lambda}_i\bar{\lambda}_j\right)^{-1},$$

where $C_{M,\mathbf{Z}}$ is the centralizer of M in $\mathrm{Sp}(n,\mathbf{Z})/\{\pm 1\}$ and $|C_{M,\mathbf{Z}}|$ is its order.

1. Convergence of the series.

LEMMA 1 [4]. *Suppose E is the $n \times n$ identity matrix and Z, W are $n \times n$ symmetric matrix variables satisfying $E - Z{}^t\overline{Z} \geqslant 0$ and $E - W{}^t\overline{W} \geqslant 0$. Then*

$$\det(E - Z{}^t\overline{Z})\det(E - W{}^t\overline{W}) \leqslant |\det(E - Z{}^t\overline{W})|^2.$$

The equality holds only when $Z = W$.

With this lemma, we are able to prove

PROPOSITION. *For $k > n$ and an element M of finite order in* $\mathrm{Sp}(n, \mathbf{Z})$ *such that M has an isolated fixed point on H_n, we have*:

(1) $\int_{H_n} |K_M(Z, \overline{Z})|^{-k}(\det Y)^{k-n-1}\, dZ$ *is convergent*,

(2) *the contribution of elements in* $\mathrm{Sp}(n, \mathbf{Z})$ *which are conjugate in* $\mathrm{Sp}(n, \mathbf{Z})/\{\pm 1\}$ *to M is given by*

$$N_{\{M\}} = C(k, n)|C_{M,\mathbf{Z}}|^{-1}\int_{H_n} K_M(Z, \overline{Z})^{-k}(\det Y)^{k-n-1}\, dZ.$$

PROOF. Since M is conjugate in $\mathrm{Sp}(n, \mathbf{R})$ to an element $\left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right]$ with $U = A + iB = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$, $\lambda_i\lambda_j \neq 1$, we may assume $M = \left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right]$ in the computation of $K_M(Z, \overline{Z})$. Then under the Cayley transform $Z = i(E + W)(E - W)^{-1}$, the integral in (1) is transformed into

$$2^{n(n+1)}\int_{D_n} \frac{\det(E - W\overline{W})^{k-n-1}\, dW}{|\det(E - {}^t\overline{U}W\overline{U}W)|^k},$$

where D_n is the generalized disc defined by

$$D_n\colon W = {}^tW \in M_n(\mathbf{C}), \quad E - W{}^t\overline{W} > 0.$$

For $k > n$, we have

$$\int_{D_n} \det(E - W\overline{W})^{k-n-1}\, dW < \infty.$$

To prove the integral is convergent, it suffices to prove that $\det(E - {}^t\overline{U}W\overline{U}\overline{W}) \neq 0$ if $E - W{}^t\overline{W} \geqslant 0$. Suppose on the contrary that $\det(E - {}^t\overline{U}W\overline{U}\overline{W}) = 0$; then Lemma 1 with $Z = {}^t\overline{U}W\overline{U}$ implies $\det(E - W\overline{W}) = 0$ and ${}^t\overline{U}W\overline{U} = W$. Let $W = [w_{ij}]$; then ${}^t\overline{U}W\overline{U} = W$ implies $(1 - \overline{\lambda_i\lambda_j})w_{ij} = 0$. Since $1 - \overline{\lambda_i\lambda_j} \neq 0$ for all i, j; this forces $W = 0$ which contradicts $\det(E - W\overline{W}) = 0$. Hence $\det(E - {}^t\overline{U}W\overline{U}\overline{W}) \neq 0$ for all $E - W\overline{W} \geqslant 0$ and (1) follows.

To prove (2), we use the notation

$$K_\gamma(Z, \overline{Z}) = \det\left[\frac{1}{2i}\left(Z - \overline{\gamma(Z)}\right)\right]\det\overline{(C_1Z + D_1)} \quad \text{if } \gamma = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}.$$

According to the Selberg trace formula, the contribution is given by

$$N_{\{M\}} = C(k,n)\int_F \sum_{M_1 \in \bar{\Gamma}_n / C_{M,\mathbf{Z}}} K_{M_1 M M_1^{-1}}(Z,\bar{Z})^{-k} (\det Y)^{k-n-1}\, dZ$$

$$= C(k,n)\cdot |C_{M,\mathbf{Z}}|^{-1} \int_F \sum_{M_1 \in \bar{\Gamma}_n} K_{M_1 M M_1^{-1}}(Z,\bar{Z})^{-k} (\det Y)^{k-n-1}\, dZ.$$

Here $\bar{\Gamma}_n = \mathrm{Sp}(n,\mathbf{Z})/\{\pm 1\}$ is the projective full modular subgroup of $\mathrm{Sp}(n,\mathbf{R})$. Note that

$$\int_F \sum_{M_1 \in \bar{\Gamma}_n} |K_{M_1 M M_1^{-1}}(Z,\bar{Z})|^{-k} (\det Y)^{k-n-1}\, dZ$$

$$= \int_{H_n} |K_M(Z,\bar{Z})|^{-k} (\det Y)^{k-n-1}\, dZ < +\infty.$$

Thus, we can exchange the order of integration and summation, so our assertion in (2) follows.

2. Proof of the Theorem. To prove the Theorem, we need the following two lemmas.

LEMMA 2 (THEOREM 3-3 IN [7]). *Suppose that* $\{\psi_i(W)\}$, $i = 1,2,\ldots,n,\ldots,$ *is an orthonomal base of the Hilbert space* $H^2(k, D_n)$ *with* $k \geqslant 4$. *Then*

$$2^{n(n+1)} C(k,n) \det(E - Z\bar{W})^{-k} = \sum_{i=1}^{\infty} \psi_i(Z)\overline{\psi_i(W)}.$$

Here D_n: $W = {}^tW \in M_n(\mathbf{C})$, $E - W\bar{W} > 0$, *and* $H^2(k, D_n)$ *is the space of holomorphic functions f defined on* D_n *satisfying*

$$\int_{D_n} \det(E - W\bar{W})^{k-n-1} |f(W)|^2\, dW < \infty.$$

LEMMA 3 (THEOREM 5 IN III OF [4]). *Let* $f = [f_1, f_2, \ldots, f_n]$ *be n-tuple integers such that* $f_1 \geqslant f_2 \geqslant \cdots \geqslant f_n \geqslant 0$ *and* $\xi = [\xi_1, \xi_2, \ldots, \xi_n]$ *with* $|\xi_i| \leqslant 1$. *Suppose that* $M_{2f}(\xi) = \det[\xi_i^{2f_j + n - j}]$ *and* $0 < t < 1$. *Then*

$$\sum_f M_{2f}(\xi) t^{f_1 + f_1 + \cdots + f_n} = \prod_{i<j} (\xi_i - \xi_j) \prod_{i \leqslant j} (1 - t\xi_i\xi_j)^{-1}.$$

PROOF OF THE THEOREM. We divide the proof into the following steps.

Step I. Since M has only isolated fixed points and $C_{M,\mathbf{Z}}$ is a discrete subgroup of the stabilizer of the fixed point, $C_{M,\mathbf{Z}}$ is a group of finite order. The contribution of the conjugacy class represented by M to the dimension formula is

$$N_{\{M\}} = C(k,n)|C_{M,\mathbf{Z}}|^{-1} \int_{H_n} K_M(Z,\bar{Z})^{-k} (\det Y)^{k-n-1}\, dX\, dY,$$

with

$$K_M(Z,\bar{Z}) = \det\left[\frac{1}{2i}\left(Z - \overline{(AZ+B)(-BZ+A)^{-1}}\right)\right] \det \overline{(-BZ+A)}.$$

Step II. With the Cayley transform $Z = i(E + W)(E - W)^{-1}$, the half-space H_n is transformed into the generalized disc D_n as defined in Lemma 2. Also we have

$$K_M(Z, \overline{Z}) = \det\left[\frac{-1}{2i}(ZB\overline{Z} - ZA + A\overline{Z} + B)\right]$$
$$= \det\left[(E - W)^{-1}(U - W\overline{U}\overline{W})(E - \overline{W})^{-1}\right]$$

and

$$(\det Y)^{-n-1}\,dX\,dY = 2^{n(n+1)}\det\left[(E - W)^{-1}(E - W\overline{W})(E - \overline{W})^{-1}\right]^{-n-1}dW.$$

Hence we can express $N_{\{M\}}$ as

$$N_{\{M\}} = C(k, n)2^{n(n+1)}|C_{M,\mathbf{Z}}|^{-1}\int_{D_n}\frac{\det(E - W\overline{W})^{k-n-1}dW}{\det(U - W\overline{U}\overline{W})^k}$$
$$= \frac{C(k,n)2^{n(n+1)}}{|C_{M,\mathbf{Z}}|(\det U)^k}\int_{D_n}\frac{\det(E - W\overline{W})^{k-n-1}dW}{\det(E - {}^t\overline{U}W\overline{U}\overline{W})^k}.$$

There exists a unitary matrix V such that ${}^t\overline{V}UV = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$. With the transformation $W \to VW{}^tV$, we get

$$N_{\{M\}} = \frac{C(k, n)2^{n(n+1)}}{|C_{M,\mathbf{Z}}|(\det U)^k}\int_{D_n}\frac{\det(E - W\overline{W})^{k-n-1}dW}{\det(E - {}^t\overline{U'}W\overline{U'}\overline{W})^k}, \qquad U' = {}^t\overline{V}UV.$$

Hence we can assume that U is a diagonal matrix in the beginning. Note that the operation of U on D_n is $W \to UW{}^tU = [\lambda_i\lambda_j w_{ij}]$.

Step III. For each n-tuple $f = [f_1, f_2, \ldots, f_n]$ of integers $f_1 \geqslant f_2 \geqslant \cdots \geqslant f_n \geqslant 0$, we let A_{2f} denote the irreducible unitary representation of $\mathrm{GL}_n(\mathbf{C})$ on the Hilbert space H corresponding to the highest weight

$$\chi_{2f}(\mathrm{diag}[a_1, a_2, \ldots, a_n]) = a_1^{2f_1}a_2^{2f_2}\cdots a_n^{2f_n}.$$

Then H is a finite dimension vector space of dimension

$$N[2f] = \prod_{i<j}(2f_i - 2f_j + j - i).$$

Furthermore, H is spanned by homogeneous polynomial functions in $n(n+1)/2$ variables of degree $f_1 + f_2 + \cdots + f_n$. Let $\psi_{2f}^{(i)}(W)$, $i = 1, 2, \ldots, N[2f]$, be an orthonomal base of H, which consist of homogeneous polynomial functions. By Hua's argument (p. 234 of [4]), $\{\psi_{2f}^{(i)}\}$ is an orthonomal base of $H^2(k, D_n)$ when $2f$ ranges over all n-tuples of nonnegative even integers $2f_1 \geqslant 2f_2 \geqslant \cdots \geqslant 2f_n \geqslant 0$. Consequently, we have the decomposition of the kernel

$$C(k, n)2^{n(n+1)}\det(E - {}^t\overline{U}W\overline{U}\overline{W})^{-k} = \sum_f\sum_{i=1}^{N[2f]}\psi_{2f}^{(i)}({}^t\overline{U}W\overline{U})\overline{\psi_{2f}^{(i)}(W)},$$

by Lemma 2. When W is transformed into ${}^t\overline{U}W\overline{U}$, we have

$$\psi_{2f}^{(i)}({}^t\overline{U}W\overline{U}) = A_{2f}(\overline{U})\left(\psi_{2f}^{(i)}(W)\right) \quad \text{for } i = 1, 2, \ldots, N.$$

Let $N_{\{M\}}(tU)$ be the integral with U replaced by tU $(0 < t < 1)$. Then we have

$$N_{\{M\}} = \lim_{t \to 1} N_{\{M\}}(tU)$$

by continuity and $N_{\{M\}}(tU)$ can be expressed as the series

$$\begin{aligned} N_{\{M\}}(tU) &= |C_{M,\mathbf{Z}}|^{-1} \prod_{i=1}^{n} (t\lambda_i)^{-k} \int_{D_n} \det(E - W\overline{W})^{k-n-1} \\ &\quad \times \sum_f \sum_{i=1}^{N[2f]} A_{2f}(t\overline{U}) \big(\psi_{2f}^{(i)}(W)\big) \cdot \overline{\psi_{2f}^{(i)}(W)}\, dW \\ &= |C_{M,\mathbf{Z}}|^{-1} \prod_{i=1}^{n} (t\lambda_i)^{-k} \sum_f \operatorname{Trace} A_{2f}(t\overline{U}) \end{aligned}$$

which is convergent when $0 < t < 1$ we shall see soon. Here we use the identity

$$\int_{D_n} \det(E - W\overline{W})^{k-n-1} \sum_{i=1}^{N[2f]} A_{2f}(t\overline{U}) \big(\psi_{2f}^{(i)}(W)\big) \overline{\psi_{2f}^{(i)}(W)}\, dW = \operatorname{Trace} A_{2f}(t\overline{U})$$

which follows from the orthonormality of $\{\psi_{2f}^{(i)}(W)\}$, $i = 1, 2, \ldots, N[2f]$, in $H^2(k; D_n)$ for $k \geqslant 4$.

Now choose positive numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ so that $\prod_{i \neq j}(\varepsilon_i \bar{\lambda}_i - \varepsilon_j \bar{\lambda}_j) \neq 0$ and let $U_\varepsilon = \operatorname{diag}[\varepsilon_1\lambda_1, \varepsilon_2\lambda_2, \ldots, \varepsilon_n\lambda_n]$. The character formula of Hermann Weyl (Theorem 1.2, Chapter VI of [**1**]) tells us

$$\prod_{i<j} \big(t\varepsilon_i\bar{\lambda}_i - t\varepsilon_j\bar{\lambda}_j\big) \cdot \operatorname{Trace} A_{2f}\big(t\overline{U}_\varepsilon\big) = M_{2f}\big(\overline{U}_\varepsilon\big) t^{f_1+f_2+\cdots+f_n} t^{n(n-1)/2}.$$

By Lemma 3, we get

$$N_{\{M\}}(tU_\varepsilon) = |C_{M,\mathbf{Z}}|^{-1} \prod_{i=1}^{n} (t\varepsilon_i\lambda_i)^{-k} \prod_{i \leqslant j} \big(1 - t\varepsilon_i\bar{\lambda}_i\varepsilon_j\bar{\lambda}_j\big)^{-1}.$$

As t and ε_i $(i = 1, 2, \ldots, n)$ approach 1, we get

$$N_{\{M\}} = |C_{M,\mathbf{Z}}|^{-1} \prod_{i=1}^{n} \bar{\lambda}_i^k \prod_{1 \leqslant i \leqslant j \leqslant n} \big(1 - \bar{\lambda}_i\bar{\lambda}_j\big)^{-1}.$$

This proves our Theorem.

3. Application. There are three conjugacy classes of regular elliptic elements in $\mathrm{PSL}_2(\mathbf{Z}) = \mathrm{SL}_2(\mathbf{Z})/\{\pm 1\}$, represented by

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

Their total contributions to the dimension formula with respect to $\mathrm{SL}_2(\mathbf{Z})$ is

$$\tfrac{1}{4}(-i)^k + \tfrac{1}{9}\big(\rho^k + \bar{\rho}^k - \rho^{k-2} - \bar{\rho}^{k-2}\big), \qquad \rho = e^{\pi i/3}.$$

Combining this with the contribution from identity and conjugacy classes represented by $\left[\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right]$, n a nonzero integer, we get that the dimension for the vector space of cusp forms of degree 1 and weight k (k even, $k \geqslant 4$) with respect to $\mathrm{SL}_2(\mathbf{Z})$ is

$$\dim_C S(k; \mathrm{SL}_2(\mathbf{Z})) = \frac{k-1}{12} - \frac{1}{2} + \frac{1}{4}(-i)^k + \frac{1}{9}\big(\rho^k + \bar{\rho}^k - \rho^{k-2} - \bar{\rho}^{k-2}\big).$$

It is more complicated for the case of degree two. By the computation in [**2**], there are 22 conjugacy classes of regular elliptic elements in $\mathrm{Sp}(2,\mathbf{Z})$ and the total contribution from these conjugacy classes is $N_1 + N_2$ with

$$N_1 = \begin{cases} 2^{-7}3^{-3} \times [1131, 229, -229, -1131, 427, -571, 123, \\ \qquad\qquad\qquad -203, 203, -123, 571, -427] \\ \qquad \text{if } k \equiv [0,1,2,3,4,5,6,7,8,9,10,11] \pmod{12}; \end{cases}$$

$$N_2 = \begin{cases} 5^{-1} & \text{if } k \equiv 0 \pmod 5, \\ -5^{-1} & \text{if } k \equiv 3 \pmod 5, \\ 0 & \text{otherwise.} \end{cases}$$

Here N_1 is the total contribution from regular elliptic conjugacy classes except conjugacy classes of order 5 and N_2 is the total contributions from conjugacy classes of order 5.

Let N_3 and N_4 be the contribution from conjugacy classes in $\mathrm{Sp}(2,\mathbf{Z})$, represented by elements of the form

$$\begin{bmatrix} a & 0 & b & * \\ * & 1 & * & * \\ c & 0 & d & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (c \neq 0) \quad \text{and} \quad \begin{bmatrix} U & S\,{}^tU^{-1} \\ 0 & {}^tU^{-1} \end{bmatrix},$$

respectively. Then by the computation in [**2**], we get

$$N_3 = \begin{cases} 2^{-5}3^{-3} \times [17k - 294, -25k + 325, -25k + 254, \\ \qquad\qquad 17k - 261, 17k - 86, -k + 53, -k - 42, \\ \qquad\qquad -7k + 91, -7k + 2, -k - 27, -k + 166, 17k - 181], \\ \text{if } \quad k \equiv [0,1,2,3,4,5,6,7,8,9,10,11] \pmod{12} \end{cases}$$

and

$$N_4 = \begin{cases} 2^{-7}3^{-3}5^{-1}(2k^3 + 96k^2 - 52k - 3231) & \text{if } k \text{ is even,} \\ 2^{-7}3^{-3}5^{-1}(2k^3 - 114k^2 + 2018k - 9051) & \text{if } k \text{ is odd.} \end{cases}$$

Consequently, we can write the dimension formula for the cusp forms of degree two and weight k ($k \geqslant 4$) as

$$\dim_{\mathbf{C}} S(k; \mathrm{Sp}(2,\mathbf{Z})) = N_1 + N_2 + N_3 + N_4.$$

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