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ON THE REALIZATION OF INVARIANT SUBGROUPS OF $\pi_*(X)$

BY

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ABSTRACT. Let p be a prime and T: $X \to X$ a self map. Let A be a multiplicatively closed subset of the algebraic closure of F_p. Denote by $V_{T,A}$ the set of characteristic values of $\pi_*(T) \otimes F_p$ lying in A. It is proved that under certain conditions $V_{T,A}$ is realizable by a pair $\tilde{X}, \tilde{T}$: There exist a space $\tilde{X}$, maps $\tilde{T}$: $\tilde{X} \to \tilde{X}$ and f: $\tilde{X} \to X$ so that $f \circ \tilde{T} \sim T \circ f$, $\pi_*(F)$ is mod p injective and $\operatorname{im}(\pi_*(f) \otimes F_p) = V_{T,A}$. This theorem yields, among others, examples of spaces whose mod p cohomology rings are polynomial algebras.

Introduction. The technique of using self maps T: $X \to X$ to obtain mod p splittings, retracts and realizable subalgebras of the mod p cohomology of a given space has been well exploited by now: **[Freyd]** used idempotents to split spectra, **[Nishida]** used the ψ_λ maps defined on $\mathrm{BU}(n)$ by **[Sullivan]** to obtain a mod p splitting of $\mathrm{SU}(n)$, **[Wilkerson]**$_1$ produced mod p retracts of H and H_0 spaces and **[Cooke and Smith]** splitted co-H-spaces, all using self maps.

Self maps were used to obtain geometric realizations of subalgebras of the mod p cohomology of spaces, e.g., **[Clark and Ewing]**, **[Cooke]**, **[Stasheff]** and **[Zabrodsky]**$_1$.

The two main methods used can be described briefly as follows:

The direct limit. One constructs the infinite telescope of T: $X \to X$, i.e., $\mathrm{Tel}(T) = X \times I \times N/\sim$ (N the set of natural numbers) where $\sim$ is the equivalence relation generated by $(x, 1, n) \sim (T(x), 0, n+1)$, $(*, t, n) \sim (*, t', n')$. If $H_m(T, M)$ is an idempotent, i.e., $H_m(T^2, M) = H_m(T, M)$, then $H_*(\mathrm{Tel}(T), M) = \operatorname{im} H_*(T, M)$ and one obtains a realization of a submodule of $H_*(X, M)$ or $H^*(X, M)$.

The orbit space. If G is a finite group acting freely on a topological space X, the orbit space X/G has the property:

$$\begin{aligned} H^*(X/G, Z/mZ) &= H^*_G(X, Z/mZ) \\ &= \{x \in H^*(X, z/mZ) \mid g^*x = x \text{ for every } g \in G\}, \end{aligned}$$

provided m is prime to the order of G.

The two methods do not have obvious Eckmann-Hilton duals. A third method, described in **[Zabrodsky]**$_1$ does have such a dual and this dual construction is the main subject of this paper. Because **[Zabrodsky]**$_1$ has not been published yet, we

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bring here in a complete, self-contained form the part of self maps theory needed in our proofs.

Our main result (Theorem A) deals with realizations of subgroups of $\pi_*(X)$ corresponding to splittings of the characteristic polynomial of $\pi_*(T) \otimes F_p$ ($F_p = Z/pZ$), where T $X \to X$ is a self map.

This realization is formed "to the left" of X and is illustrated by a commutative diagram

$$\begin{array}{ccc} \hat{X} & \xrightarrow{f} & X \\ \hat{T}\downarrow & & \downarrow T \\ \hat{X} & \xrightarrow{f} & X \end{array}$$

where $\pi_*(f)$ is mod p injective and $\operatorname{im} \pi_*(f) \otimes F_p$ is a prescribed subgroup of $\pi_*(X) \otimes F_p$.

This realization yields geometric realizations of polynomial algebras; in particular, we reconstruct Stasheff's realization of the polynomial algebra

$$F_p[x_{2m}, x_{4m}, \ldots, x_{2km}], \; m|p-1$$

[**Stasheff**, pp. 146–147]. The paper is organized as follows: §0 contains our basic conventions and notation, statements of the main results and some examples attempting to show that none of the hypotheses of the main theorems can be relaxed.

§§1 and 2 cover the basic results from the theory of self maps to be used in our proofs. §3 gives the final versions of the main results and their proofs.

Some consequences, examples and applications are given in §4. For convenience, we list the standard notation and terminology of this paper at the end of §4.

0. Notation, conventions and summary of results. We fix a prime p and denote by F_p and Z_p the field of p elements and the integers localized at p, respectively.

A commutative diagram, unless otherwise specifically stated, means commutative up to homotopy. Quite often equality between functions means equality of homotopy classes.

In general, spaces will be assumed to be of the homotopy type of simply connected CW complexes of finite type or their p-localizations.

This is a matter of convenience and all major results are valid for nilpotent spaces.

All spaces, maps and homotopies are pointed. Consequently all standard homotopy theoretic constructions are of the reduced type: CX, ΣX, etc. Homology and cohomology are always assumed to be the reduced theories.

We use the following standard notation:

$PX = \{\varphi | \varphi: I \to X\}$—the free path space.

$LX = \{\varphi \in PX | \varphi(0) = *\}$—the contractible path space.

$\Omega X = \{\varphi \in PX | \varphi(0) = * = \varphi(1)\}$—the loop space.

Homotopies $f_0 \sim f_1$: $X \to Y$ are considered to be maps V: $X \to PY$, $V(x)[\varepsilon] = f_\varepsilon(x)$, $\varepsilon = 0, 1$. Given a map, f: $X \to Y$, we denote by Pf, Lf, Ωf the maps on the function spaces induced by f.

For a map $f\colon X \to Y$ we write

$$j_f\colon V_f \to X \quad \text{and} \quad \hat{j}_f\colon Y \to C_f$$

for the homotopy fiber and mapping cone of f, respectively:

$$V_f = \{(x, \varphi) \in X \times LY | \varphi(1) = f(x)\} \quad \text{with}\, j_f(x, \varphi) = x.$$

$C_f = X \times I \amalg Y/\sim$, $\amalg$—the disjoint union, $\sim$ spanned by $x, 0 \sim *$, t; $x, 1 \sim fx$ with $\hat{j}_f$—the composition $Y \subset X \times I \amalg Y \to C_f$.

Given $f_1\colon X_1 \to X_0$, $f_2\colon X_2 \to X_0$, the homotopy pull back of f_1, f_2 is a triple $(\hat{X}, r_1, r_2)$ where $\hat{X}$ is the space and $r_i\colon \hat{X} \to X_i$, $i = 1, 2$, are the maps given by

$$\hat{X} = \{(x_1, \varphi, x_2) \in X_1 \times PX_0 \times X_2 | f_1(x_1) = \varphi(0), \varphi(1) = f_2(x_2)\},$$

with $r_i(x_1, \varphi, x_2) = x_i$, $i = 1, 2$.

Given $T_i\colon X_i \to X_i'$, $i = 0, 1, 2$, $f_i\colon X_i \to X_0, f_i'\colon X_i' \to X_0'$, $i = 1, 2$, $V_i\colon f_i' \circ T_i \sim T_0 \circ f_i$ $(V_i\colon X_i \to PX_0)$ then T_i, V_i induce a map from the pull back of f_1, f_2 to that of f_1', f_2' in a natural way.

The [**Cooke and Smith**] splitting of a p-localization of a finite CW suspension X induced by $T\colon X \to X$ corresponds to the splitting $P = P_1 \cdot P_2 \cdots P_t$ of the characteristic polynomial P of $H_*(T, F_p)$, $(P_i, P_j) = 1$ if $i \neq j$. One obtains a homotopy equivalence $\bigvee_{i=1}^{t} X_i \approx X$ and $f_i\colon X_i \to X$ satisfies

$$\operatorname{im} H_*(f_i, F_p) = \ker P_i\big(H_*(T, F_p)\big).$$

One can prove an Eckmann-Hilton dual of this theorem for H-spaces (see 4.2.2) where $\pi_*(X) \otimes F_p$ replaces $H_*(X, F_p)$.

However, one cannot expect such a splitting to exist for arbitrary spaces: The first obstruction is the multiplicative structures of $\pi_*(X) \otimes F_p$ and $H^*(X, F_p)$ which are preserved by self maps.

Consider the following example: If $T\colon S^{2n} \to S^{2n}$ has degree λ, $\lambda \not\equiv 0, 1 \bmod p$ and p-odd, then by the E.H.P. sequence one can see that the characteristic polynomial of $\pi_m(T)$ is of the form $P(x) = (x - \lambda)^{r_m^{(1)}}(x - \lambda^2)^{r_m^{(2)}}$. One cannot hope to have a splitting $S^{2n} \approx_p X_1 \times X_2$, where X_i corresponds to the $(x - \lambda^i)^{r_m^{(i)}}$ factor of $P(x)$. One cannot even expect to obtain a realization $h_1\colon X_1 \to S^{2n}$ with

$$\operatorname{im} \pi_m(h_1) \otimes F_p = \bigcup_r \ker\big(\pi_m(T) \otimes F_p - \lambda\big)^{r_m^{(i)}}$$

for if

$$u \in \bigcup_r \ker\big(\pi_{2m}(T) \otimes F_p - \lambda\big)^r = \operatorname{im} \pi_{2m}(h_1) \otimes F_p$$

then the Whitehead product $[u, u] \neq 0$ must lie in $\operatorname{im} \pi_{4m-1}(h_1) \otimes F_p$ but obviously $[u, u] \in \ker(\pi_{4m-1}(T) - \lambda^2)^{r_{4m-1}^{(2)}}$. On the other hand, the mod p Hopf fibration $S^{4n-1} \to S^{2n}$ realizes $\bigcup_r \ker(\pi_m(T) \otimes F_p - \lambda^2)^r$. This realization is possible because the multiplicative closure of the roots of $(x - \lambda^2)^r \in F_p[x]$ contains no root of $(x - \lambda)^s$.

Thus, first one concludes that if one deals with a nontrivial ring $\pi_*(X) \otimes F_p$ one cannot expect to have a splitting of rings $\pi_*(X) \otimes F_p \approx \oplus A_i$; one can only expect a realization of a vector subspace $A \subset \pi_*(X) \otimes F_p$ corresponding to a factor P_1 of the characteristic polynomial of $\pi_*(T) \otimes F_p$ (at least if $\pi_m(X) = 0$ for $m > N$), provided the multiplicative closure of the roots of P_1 contains no root of its complement. Indeed a consequence of our main theorem (Theorem A) easily yields

0.1. PROPOSITION. *Suppose $\pi_*(X) = \oplus \pi_m(X)$ is a finite group of order a power of p (in particular, $\pi_n(X) = 0$ for $n > N$ for some N). Given T: $X \to X$, if the characteristic polynomial $P \in F_p[x]$ of $\pi_*(T) \otimes F_p$ splits as $P = P_1 \cdot P_2$ and if the multiplicative closure of roots of P_1 (in some extension field) contains no root of P_2 then one has a commutative diagram*

$$\begin{array}{ccc} \hat{X} & \xrightarrow{f} & X \\ \hat{T}\downarrow & & \downarrow T \\ \hat{X} & \xrightarrow{f} & X \end{array}$$

where $\pi_*(f)$ is injective and $\operatorname{im}[\pi_*(f) \otimes F_p] = \ker P_1(\pi_*(F) \otimes F_p)$.

The restriction $\pi_*(X) \otimes Q = 0$ in Proposition 0.1 cannot be removed without a proper substitute. This is due to the fact that if $\pi_*(X) \otimes Q \neq 0$ a geometric realization of a T-invariant subgroup of $\pi_*(X) \otimes F_p$ will yield a T-invariant subgroup of $\pi_*(X) \otimes Q$ with an obvious relation between the two. This may fail to exist for algebraic reasons as we demonstrate by the following example.

0.2. EXAMPLE. Let T_0: $K(Z \oplus Z, 2n) \to K(Z \oplus Z, 2n)$ be given by the matrix

$$H^{2n}(T_0, Z) = \begin{pmatrix} 0 & -p \\ 1 & 1 \end{pmatrix}$$

with respect to some basis $u_1, u_2 \in H^{2n}(K(Z \oplus Z, 2n), Z)$. Then for $w = pu_1^2 - u_1u_2 + u_2^2$ one has $H^{4n}(T_0, Z)w = pw$ and if X is the two stage Postnikov system with w as the k-invariant one obtains:

$$\begin{array}{ccc} K(z, 4n-1) & \xrightarrow{p1} & K(z, 4n-1) \\ \downarrow & & \downarrow \\ X & \xrightarrow{T} & X \\ \downarrow r & & \downarrow r \\ K(Z \oplus Z, 2n) & \xrightarrow{T_0} & K(Z \oplus Z, 2n) \end{array}$$

The characteristic polynomial of $\pi_{2n}(T) \otimes F_p$ is $x(x-1)$ and that of $\pi_{4n-1}(T) \otimes F_p$ is x. Thus, the characteristic polynomial of $\pi_*(T) \otimes F_p$ is $x^2(x-1)$ and $x - 1$ is a factor with a multiplicatively closed set of roots, containing no roots of its complement x^2. But $\ker[\pi_*(T) \otimes F_p - 1]$ (even after localizing at p) is not realizable: Such a realization has to be of the form f: $K(Z_p, 2n) \to X_p$ with $H^*(f, F_p)\rho_p\tilde{u}_i = \rho_p\tilde{u}$, where $\tilde{u} \in H^{2n}(K(Z_p, 2n), Z_p)$ is a generator, $\tilde{u}_i \in H^{2n}(K(Z_p \oplus Z_p, 2n), Z_p)$ are the images of u_i, and ρ_p: $H^*(\ , Z_p) \to H^*(\ , F_p)$ is the reduction. Hence, $H^{2n}(f, Z_p)$

is surjective, so is $H^{2n}(r \circ f, Z_p)$ but then $H^{4n}(r \circ f, Z_p)\tilde{w} \neq 0$ ($\tilde{w}$ the image of w) as $px^2 - xy + y^2$ is an irreducible quadratic form over Z_p. This is a contradiction.

The reason for our failure to realize $\ker(\pi_*(T) \otimes F_p - 1)$ is the fact that the factorization $x^2(x-1)$ of the characteristic polynomial of $\pi_*(T) \otimes F_p = \pi_*(T)/\text{torsion} \otimes F_p$ is not a mod p reduction of a factorization of the characteristic polynomial of $\pi_*(T) \otimes Q$. This should explain the hypothesis of our main theorem given in its first version as follows:

THEOREM A. *Let T: $X \to X$. Given sequences of polynomials $P_1^{(n)}, P_2^{(n)} \in Z[x]$ so that*:

(1) $P_i^{(n)} | P_i^{(n+1)}$, $i = 1, 2$.

(2) *Let $\hat{P}$ denote the* mod *p reduction of a polynomial $P \in Z[x]$. Then* $\deg P_i^{(n)} = \deg \hat{P}_i^{(n)}$ *and the multiplicative closure of the roots of $\hat{P}_1^{(n)}$ contains no root of $\hat{P}_2^{(m)}$ for all m, n.*

(3) *For every n there exists $r_n > 0$ so that*

$$\left[P_1^{(n)} \cdot P_2^{(n)}\right]^{r_n}\left[\pi_n(T)\right] \otimes Z_p = 0.$$

Then

$$\left\{\bigcup_r \ker\left[\hat{P}_1^{(n)}\right]^r\left(\pi_n(T) \otimes F_p\right)\right\}_{n=2}^{\infty}$$

is realizable, i.e., there exsits a commutative diagram

$$\begin{array}{ccc} \hat{X} & \xrightarrow{f} & X \\ \hat{T}\downarrow & & \downarrow T \\ \hat{X} & \xrightarrow{f} & X \end{array}$$

so that $\pi_n(f)$ is mod p injective,

$$\operatorname{im} \pi_n(f) \otimes F_p = \bigcup_r \ker\left[\hat{P}_1^{(n)}\right]^r\left(\pi_n(T) \otimes F_p\right).$$

Moreover, given a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ T'\downarrow & & \downarrow T \\ X' & \xrightarrow{f'} & X \end{array}$$

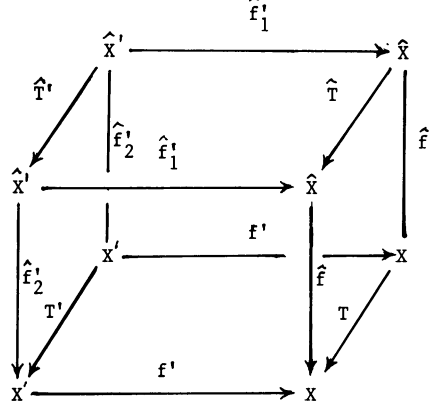
for which there exist polynomials $P'^{(n)} \in Z[x]$ so that:

(1) $P'^{(n)} | P'^{(n+1)}$.

(2) *The roots of $\hat{P}'^{(n)}$ are in the multiplicative closure of roots of $\hat{P}_1^{(n)}$.*

(3) *For every n there exists $r_n > 0$ so that* $[P'^{(n)}]^{r_n}(\pi_n(T') \otimes Z_p) = 0$.

Then there exists a commutative cube



so that $\hat{f}_2'$ *is a* mod p *equivalence.*

In case the spaces and maps are p-local, the conclusion of the second part could be simplified to state: f' *could be factored as* $f' = \hat{f} \circ \hat{f}', \hat{f}'\colon X' \to \hat{X}$ *and* $\hat{f}' \circ T' \sim \hat{T} \circ \hat{f}'$.

Theorem A follows from the following

THEOREM B. *Let* $T\colon X \to X$. *Given polynomials* P_1, $P_2 \in Z[x]$ *with the following properties*:

(1) *The leading coefficients of* P_i *are prime to p and the multiplicative closure of roots of* $\hat{P}_1$ *contains no root of* $\hat{P}_2$, *where* $\hat{P}_i$ *are the* mod p *reductions of* P_i, $i = 1, 2$.

(2) $$P_1\Big(\bigoplus_{m \leqslant n-1} H_m(T, Z_p)\Big) = 0 \quad \text{and} \quad P_1 \cdot P_2\big[H_n(T, Z_p)\big] = 0.$$

Then, if $T_n\colon X_n \to X_n$ *is the Postnikov approximation of* $T\colon X \to X$, *one has a commutative diagram*

$$\begin{array}{ccc} X_n & \xrightarrow{T_n} & X_n \\ \approx p \downarrow \hat{h} & & \approx p \downarrow \hat{h} \\ \hat{X}_n \times K(\hat{\pi}, n) & \xrightarrow{\hat{T}_n \times \tilde{T}} & \hat{X}_n \times K(\hat{\pi}, n) \end{array}$$

where $\hat{T}_n$, $\tilde{T}$ *satisfy* $P_1(\oplus_{m \leqslant n} H_m(\hat{T}_n, Z_p)) = 0$, $P_2[H_n(\tilde{T}, Z_p)] = 0$.

Theorems A and B are stated in their final versions and proved in §3. In §1 we introduce some notations and terminology which will somewhat simplify the statements of Theorems A and B.

In §4 some examples and applications of the main theorems are given. They include:

THEOREM A* (IN 4.1). *Let* $T\colon G \to G$ *be an endomorphism of a nilpotent group G. Suppose given polynomials* P_1, $P_2 \in Z[x]$ *with leading coefficients prime to p so that* *the multiplicative closure of the roots of the* mod *p reduction of* P_1 *contains no root of the* mod *p reduction of* P_2. *Suppose further that*

$$P_1 \cdot P_2\left[\bigoplus_i \Gamma_i(T)/\Gamma_{i+1}(T) \otimes Z_p\right] = 0$$

where $\Gamma_{i+1} \subset \Gamma_i$, $\Gamma_i(T)\colon \Gamma_i \to \Gamma_i$ *are the central series of G and T. Then there exists a T invariant subgroup* $\hat{G} \subset G$, *so that*

$$\Gamma_i(\hat{G})/\Gamma_{i+1}(\hat{G}) \otimes Z_p \to \Gamma_i(G)/\Gamma_{i+1}(G) \otimes Z_p$$

is injective and its image equals $\ker P_1(\Gamma_i(T)/\Gamma_{i+1}(T) \otimes Z_p)$.

Moreover, if $f'\colon G' \to G$ *is a homomorphism and* $T'\colon G' \to G'$ *satisfies* $f' \circ T' = T \circ f'$ *and* $P_1^r(H_1(T')) = 0$ *for some* $r > 0$, *then* $\operatorname{im} f' \subset \hat{G}$. (*This version slightly differs from the one in* 4.1. *They are obtained from one another by replacing the polynomials by their appropriate powers.*)

THEOREM B* (IN 4.1). *Given a central extension of a nilpotent group with endomorphisms*:

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "\sigma"] \arrow[d, "S"] & G \arrow[r, "\tau"] \arrow[d, "T"] & G_0 \arrow[r] \arrow[d, "T_0"] & 1 \\
0 \arrow[r] & C \arrow[r, "\sigma"] & G \arrow[r] & G_0 \arrow[r] & 1
\end{tikzcd}$$

Suppose $P_1, P_2 \in Z[x]$ *are as in Theorem* A*,

$$P_1\left[H_1(T_0) \otimes Z_p\right] = 0 \quad \textit{and} \quad P_1 \cdot P_2\left(S \otimes Z_p\right) = 0.$$

Then G splits mod *p as follows*:

$$\begin{tikzcd}
\tilde{C} \arrow[d, "\hat{S}"'] & G \arrow[l] \arrow[r] \arrow[d, "T"] \arrow[rr, bend left, "\tau"] & \hat{G}_0 \arrow[r] \arrow[d, "\hat{T}_0"] & G_0 \arrow[d, "T_0"] \\
\tilde{C} & G \arrow[l] \arrow[r] & \hat{G}_0 \arrow[r] & G_0
\end{tikzcd}$$

so that $G \to \tilde{C} \times \hat{G}_0$ *is a* mod *p isomorphism*, $P_1'(H_1(\hat{T}_0) \otimes Z_p) = 0$, $P_2(\hat{S} \otimes Z_p) = 0$, *where* $P_1' \in Z[x]$ *satisifies*: *The roots of* P_1' *are in the multiplicative closure of roots of* P_1.

For H-spaces one has the following consequences of Theorem A (4.2):

If $T\colon X \to X$ is a self map of a p-local H-space, $P_1^{(n)}$, $P_2^{(n)}$ are as in Theorem A, then $\hat{X}$ in the conclusion of Theorem A satisfies $H^*(\hat{X}, F_p)$ is isomorphic to the subalgebra of $H^*(X, F_p)$ generated by $\oplus_m\{\cup_r \ker[P_1^{(m)}]^r[QH^m(T, F_p)]\}$ and

$QH^*(f, F_p)$: $QH^*(X, F_p) \to QH^*(\hat{X}, F_p)$ corresponds to the projections

$$QH^m(X, F_p) \to QH^m(X, F_p)/A_m \approx QH^m(\hat{X}, F_p),$$

$A_m = \bigcup_r \ker[P_2^{(m)}]^r[QH^m(X, F_p)]$.

4.2.2. Let X be an H-space, T: $X \to X$, $\pi_m(X) = 0$ for $m > N$. If the characteristic polynomial P of $\pi_*(T) \otimes F_p$ factors as $P = P_1 \cdot P_2 \cdots P_n$, $(P_i, P_j) = 1$ for $i \neq j$, then $X \approx_p \prod X_i$, where

$$\pi_*(X_i) \otimes F_p = \ker P_i\left[\pi_*(T) \otimes F_p\right].$$

Finally, using 4.2 one can reconstruct the Quillen-Stasheff geometric realizations of the polynomial algebras $F_p[x_{2k}, x_{4k}, \ldots, x_{2rk}]$ for $k|p-1$ (see **[Quillen]**, **[Stasheff]**). The same method gives geometric realizations of some other polynomial algebras.

1. Annihilating polynomials of self maps. In this section we shall study some relations between a self map T: $X \to X$ and the linear algebra it induces on $H^*(X, M)$ and $\pi_*(X)$.

Let $P \in Z[x]$ be a polynomial with integral coefficients, $P(x) = \sum_{t=0}^r n_t x^t$. If T: $X \to X$ is a self map of either an H-space or a co-h-space, one can form $P(T)$:

$$P(T) = n_r T^r + n_{r-1} T^{r-1} + \cdots + n_0 1 \quad \text{where } T^t = \underbrace{T \circ T \circ \cdots \circ T}_{t},$$

$+$ represents the algebraic loop operation in $[X, X]$,

$$n_t T^t = \underbrace{T^t + \cdots + T^t}_{n_t}.$$

As $[X, X]$ is not necessarily associative, one chooses an arbitrary order of bracketing, e.g.,

$$P(T) = (\cdots \underbrace{(T^r + T^r) + \cdots + T^r)}_{n_r} + \underbrace{T^{r-1})}_{n_{r-1}} \cdots + \cdots + \cdots \underbrace{1) + 1) \cdots 1)}_{n_0}.$$

If X is an H-space then

$$\pi_k(P(T)) = P(\pi_k(T)) \quad \text{and} \quad Q_* H_k(P(T), F) = P[Q_* H_k(T, F)],$$

where Q_* is the submodule of primitives functor and F is a field. If X is a co-H-space then $H_k(P(t), M) = P(H_k(T, M))$.

1.1. Definitions. (A) Let R be an integral domain (usually we shall have $R = Q, Z, Z_p, F_p$). Let φ: $M \to M$ be an endomorphism of an R-module M. We say that a polynomial $P \in R[x]$ annihilates φ if for some $r \geqslant 1$, $P^r(\varphi) = 0$. Thus, φ is nilpotent if $P(x) = x$ annihilates φ.

(b) A polynomial of infinite degree $P_* \in R_*[x]$ is a sequence $\{P_n\}_{n=1}^\infty$, $P_n \in R[x]$ so that $P_n | P_{n+1}$. If $M_* = \{M_n\}_{n=1}^\infty$ is a graded R-module, $\varphi_* = \{\varphi_n\}_{n=1}^\infty$ a degree zero endomorphism of a graded R-module, we say that $P_* \in R_*[x]$ annihilates φ_* if for every $n \geqslant 1$ there exists $m \geqslant 1$ so that p_m annihilates φ_n.

Because one can consider any R-module M as a graded module, one says that $P_* \in R_*[x]$ annihilates $\varphi\colon M \to M$ if for some n, P_n annihilates φ.

The product in $R_*[x]$ is given by $(P_* \cdot \hat{P}_*)_n = P_n \cdot \hat{P}_n$.

(c) If $P_1, P_2 \in R[x]$ are polynomials of degree n_1 and n_2, respectively: $P_i(x) = \sum_{t=0}^{n_i} a_t^{(i)} x^t$, $i = 1, 2$, one can form the polynomial $P_1 \otimes P_2$ of degree $n_1 \cdot n_2$ as follows: If P_i are unitary, i.e., $a_{n_i}^{(i)} = 1$, $i = 1, 2$, consider P_i as a polynomial over $\hat{R}$, the algebraic closure of the field of fractions of R, $P_i(x)$ could be written as

$$P_i(x) = \prod_{j=1}^{n_i} \left(x - \lambda_j^{(i)}\right), \qquad \lambda_j^{(i)} \in \hat{R}.$$

Then

$$(P_1 \otimes P_2)(x) = \prod_{j=1}^{n_1} \left(\prod_{i=1}^{n_2} \left(x - \lambda_j^{(1)} \cdot \lambda_i^{(2)}\right) \right).$$

To see that $P_1 \otimes P_2 \in R[x]$ one can offer an alternative construction: Choose T_i: $R^{n_i} \to R^{n_i}$, $i = 1, 2$, to be R-endomorphisms of free R-modules so that the characteristic polynomial of T_i is $P_i \in R[x]$. Then the characteristic polynomial of $T_1 \otimes T_2$ is $P_1 \otimes P_2$.

If P_i are nonunitary, define

$$P_1 \otimes P_2 = \left(a_{n_1}^{(1)}\right)^{n_2} \left(a_{n_2}^{(2)}\right)^{n_1} \frac{P_1}{a_{n_1}^{(1)}} \otimes \frac{P_2}{a_{n_2}^{(2)}}.$$

Here the operation is performed in the field of fractions but the result is again in $R[x]$.

The following can be verified easily.

1.2. Lemma. (a) $(P_1 \cdot P_2) \otimes P = (P_1 \otimes P) \cdot (P_2 \otimes P)$ *and consequently if* $P_0 | P_1$ *then* $P_0 \otimes P | P_1 \otimes P$.

(b) *Suppose M_i are f.g. free R-modules.*

If $\varphi_i\colon M_i \to M_i$, $i = 1, 2$, *are annihilated by* P_i, $i = 1, 2$, *respectively, then* $P_1 \otimes P_2$ *annihilates* $\varphi_1 \otimes \varphi_2$.

If $P_* = \{P_n\}$, $\hat{P}_* = \{\hat{P}_n\}$ are in $R_*[x]$, one defines $P_* \otimes \hat{P}_*$ by $(P_* \otimes \hat{P}_*)_n = \prod_{m=1}^{n} P_m \otimes \hat{P}_{n-m}$, and by 1.2(a) $(P_* \otimes \hat{P}_*)_n | (P_* \otimes \hat{P}_*)_{n+1}$. If P_*, $\hat{P}_*$ annihilate $\varphi_*\colon M_* \to M_*$, $\hat{\varphi}_*\colon \hat{M}_* \to \hat{M}_*$, respectively, M_n, $\hat{M}_n$ f.g. free R-modules, then $P_* \otimes \hat{P}_*$ annihilates $\varphi_* \otimes \hat{\varphi}_*$.

We use the notation

$$\underbrace{P \otimes P \otimes \cdots \otimes P}_{n} = \bigotimes{}^n P$$

and $\prod_{m \leqslant n} \otimes^m P = \otimes^{\leqslant n} P$ for $P \in R[x]$. Define $\otimes P \in R_*[x]$ by $(\otimes P)_n = \otimes^{\leqslant n} P$. If P annihilates $\varphi\colon M \to M$ (M a f.g. free R-module), $\otimes P$ annihilates $\otimes \varphi$: $\otimes M \to \otimes M$. For $P_* \in R_*[x]$ one can form $\otimes^n P_*$ and $\otimes^{\leqslant n} P_*$ as follows:

$$(\otimes^n P_*)_m = \sum_{\Sigma r_t = m} P_{r_1} \otimes P_{r_2} \otimes \cdots \otimes P_{r_n},$$

$$(\otimes^{\leqslant n} P_*)_m = \prod_{\substack{t \leqslant n \\ \Sigma r_i = m}} P_{r_1} \otimes P_{r_2} \otimes \cdots \otimes P_{r_t}.$$

As we always have $r_i \geqslant 1$, one can define $(\otimes P_*)_m = \prod_{\Sigma r_i = m} P_{r_1} \otimes \cdots \otimes P_{r_t} = (\otimes^{\leqslant m} P_*)_m$. If M is a finitely generated abelian group, $\varphi\colon M \to M$, one can find a polynomial $P \in Z[x]$ annihilating φ: If $P^{(0)}$ is the characteristic polynomial of $\varphi \otimes Q$, $P^{(p)} \in Z[x]$ represents the characteristic polynomial of ${}^p\varphi \otimes F_p$ and $\mathbf{P}_1$, the set of torsion primes of M, then $P = P^{(0)}. (\prod_{p \in \mathbf{P}_1} P^{(p)})$ annihilates φ. $P^{(0)} \cdot P^{(p)}$ annihilates $\varphi \otimes Z_p$. If $\varphi_*\colon M_* \to M_*$ is an endomorphism of a graded abelian group of finite type, one can construct polynomials $P_* \in Z_*[x]$, $\hat{P}_* \in Z_{p*}[x]$ annihilating φ_* and $\varphi_* \otimes Z_p$: Using the above construction, one obtains polynomials $P_n \in Z[x]$, $\hat{P}_n \in Z_p[x]$ annihilating $\oplus_{m \leqslant n} \varphi_n$ and $\oplus_{m \leqslant n} \varphi_n \otimes Z_p$, respectively. The above procedure will yield $P_n | P_{n+1}$, $\hat{P}_n | \hat{P}_{n+1}$.

We shall study here relations between polynomials P annihilating $H_*(T, Z_p)$ or $\pi_*(T) \otimes Z_p$ for $T\colon X \to X$. If $P \in Z[x]$ we shall assume that the leading coefficient of P is prime to p, if $P \in Z_p[x]$ we shall assume it to be unitary (or equivalently, that its leading coefficient is a unit in Z_p). Given a map $T\colon X \to X$ let $T_n\colon X_n \to X_n$ be its Postnikov approximation in dim $\leqslant n$.

1.3. Lemma. *For $T\colon X \to X$ the following are equivalent*:

(a) *There exists $m > 0$ so that T^m factors through an n-connected space.*

(b) *There exists $m' > 0$ so that $H_k(T^{m'}, Z) = 0$ for $k \leqslant n$ (i.e., $H_k(T, Z)$ is nilpotent for $k \leqslant n$).*

(c) *There exists $m'' > 0$ so that $\pi_k(T^{m''}) = 0$ for $k \leqslant n$ (i.e., $\pi_k(T)$ is nilpotent for $k \leqslant n$).*

Proof (a) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c) are obvious.

(b) $\Rightarrow$ (a). For arbitrary M, one has a commutative diagram with exact rows

$$(T_1^* = \operatorname{Ext}(H_{k-1}(T, Z), M), \qquad T_2^* = \operatorname{Hom}(H_k(T, Z), M)):$$

$$\begin{array}{ccccccc}
0 \to & \operatorname{Ext}(H_{k-1}(X, Z), M) & \to & H^k(X, M) & \to & \operatorname{Hom}(H_k(X, Z), M) & \to 0 \\
& T_1^* \downarrow & & \downarrow H^k(T, M) & & \downarrow T_2^* & \\
0 \to & \operatorname{Ext}(H_{k-1}(X, Z), M) & \to & H^k(X, M) & \to & \operatorname{Hom}(H_k(X, Z), M) & \to 0
\end{array}$$

If $k \leqslant n$ then $(T_1^*)^{m'} = 0 = (T_2^*)^{m'}$ and consequently

$$[H^k(T, M)]^{2m'} = H^k(T^{2m'}, M) = 0.$$

Let $j^{(k)}\colon X^{(k)} \to X$ be the $k-1$ connective fibering of X. One has

$$X^{(k+1)} = V_{h_k} \xrightarrow{j_{h_k}} X^{(k)} \xrightarrow{h_k} K(\pi_k(X), k),$$

$j_{k+1} = j_k \circ j_{h_k}$. Suppose inductively (for $k \leqslant n$) that $T^{2(k-1)m'}$ factors up to homotopy as $T^{2(k-1)m'} \sim j^{(k)} \circ T_k$:

$$\begin{tikzcd}
X \arrow[rr, "T^{2m'}"] \arrow[dr, dashed, "T_{k+1}"'] & & X \arrow[rr, "T^{2(k-1)m'}"] \arrow[dr, "T_k"] & & X & \\
& X^{(k+1)} \arrow[rr, "j_{h_k}"] & & X^{(k)} \arrow[ur, "j^{(k)}"] \arrow[rr, "h_k"] & & K(\pi_k(X), k)
\end{tikzcd}$$

$[h_k \circ T_k] \in H^k(X, \pi_k(X))$ and as $H^k(T^{2m'}, \pi_k(X)) = 0$, $h_k \circ T_k \circ T^{2m'} \sim *$ and $T_k \circ T^{2m'}$ lifts to $T_{k+1}\colon X \to X^{(k+1)}$,

$$T_k \circ T^{2m'} \sim j_{h_k} \circ T_{k+1}.$$

As $j^{(k+1)} = j^{(k)} \circ j_{h_k}$ the inductive step is completed and $T^{2nm'}$ factors through $X^{(n+1)}$ which is n-connected.

(c) $\Rightarrow$ (a) is proved similarly using the inductive step given by the following commutative diagram:

$$\begin{tikzcd}
X \arrow[rr, "T^{(k-1)m''}"] \arrow[dr, "T'(k)"'] & & X \arrow[rr, "T^{m''}"] & & X \\
& X(k) \arrow[ur, "T''(k)"'] \arrow[rr, "r"] & & X(k+1)=C_{\hat{h}(k)} \arrow[ur, dashed, "T''(k+1)"'] & \\
\vee_i S^k_i \arrow[ur, "\hat{h}(k)"] & & & &
\end{tikzcd}$$

Here $X(k)$ is $k-1$ connected, $\pi_k(\hat{h}(k))$ surjective, as $\pi_k(T^{m''}) = 0$, $T^{m''} \circ T''(k) \circ \hat{h}(k) \sim *$ and $T^{m''} \circ T''(k)$ factors through the k-connected mapping cone $C_{\hat{h}(k)} = X(k+1)$ of $\hat{h}(k)$.

1.4. Proposition. *Let $T\colon X \to X$ be a self map, $T_n\colon X_n \to X_n$ its Postnikov approximation in* dim $\leqslant n$. *Let $F = F_p$ or Q. Then*:

(a) *If $P \in F[x]$ annihilates $H_m(T, F)$ for $m \leqslant n$, then $\otimes P$ annihilates $H_*(T_n, F)$ and $\pi_m(T) \otimes F$ for $m \leqslant n$.*

(b) *If P annihilates $\pi_m(T) \otimes F$ for $m \leqslant n$, then $\otimes P$ annihilates $H_*(T_n, F)$ and $H_m(T, F)$ for $m \leqslant n$.*

(c) *If X is an H-space, then $P_* \in F_*[x]$ annihilates $\pi_*(T) \otimes F$ if and only if it annihilates $Q_* H_*(T, F)$ where Q_* is the submodule of primitives functor.*

(d) *If X is an H-space, then $H_m(T, F_p)$ is nilpotent for $m \leqslant n$ if and only if for every $r > 0$ there exists $w_r\colon X_n \to X_n$ and $t_r > 0$ so that $[T_n^{t_r}] = [p^r 1] \circ [w_r] \in [X_n, X_n]$.*

Proof. The technical step in proving (a) and (b) is the inductive proof that $\otimes P$ annihilates $H_*(T_m, F)$, $m \leqslant n$: One has a ladder of fibrations which induces an exact sequence for $m \leqslant n$:

$$\begin{array}{ccccc}
K_m = K(\pi_m(X), m) & \overset{i_m}{\to} & X_m & \overset{k_{m,m-1}}{\to} & X_{m-1} \\
\hat{T}_m = K(\pi_m(T), m) \downarrow & & \downarrow T_m & & \downarrow T_{m-1} \\
K_m = K(\pi_m(X), m) & \overset{i_m}{\to} & X_m & \overset{k_{m,m-1}}{\to} & X_{m-1}
\end{array}$$

$$\begin{array}{ccccccc}
H_{m+1}(X_{m-1}, F) & \to & H_m(K_m, F) & \to & H_m(X_m, F) & \to & H_m(X_{m-1}, F) \\
\downarrow H_{m+1}(T_{m-1}, F) & & \downarrow H_m(\hat{T}_m, F) & & \downarrow H_m(T_m, F) & & \downarrow H_m(T_{m-1}, F) \\
H_{m+1}(X_{m-1}, F) & \to & H_m(K_m, F) & \to & H(X_m, F) & \to & H_m(X_{m-1}, F)
\end{array}$$

Suppose inductively that $\otimes P$ annihilates $H_*(T_{m-1}, F)$. If the hypothesis (a) holds, P annihilates $H_m(T, F) \approx H_m(T_m, F)$ and, by exactness, $\otimes P$ annihilates $H_m(\hat{T}_m, F) \approx \pi_m(T) \otimes F$. If the hypothesis (b) holds, P annihilates $\pi_m(T) \otimes F \approx H_m(\hat{T}_m, F)$ and, by exactness, as $\otimes P$ annihilates $H_m(T_{m-1}, F)$, it annihilates $H_m(T_m, F) \approx H_m(T, F)$.

Now, the structure of $H_*(K(\pi_m(X), m), F)$ (or more conveniently $H^*(K_m, F)$) is such that if $\hat{P}$ annihilates $H_m(\hat{T}_m, F)$, $\otimes \hat{P}$ annihilates $H_*(\hat{T}_m, F)$, ($QH^*(K_m, F)$ is generated over the algebra of cohomology operations by $H^m(K_m, F)$). If $\otimes P$ annihilates $H_*(T_{m-1}, F)$ and $H_m(\hat{T}_m, F)$, $\otimes(\otimes P) = \otimes P$ annihilates $H_*(\hat{T}, F)$ and using the Serre spectral sequence, the endomorphism

$$E^2(T) \approx H_*(T_{m-1}, F) \otimes H_*(\hat{T}, F): E_2 \to E_2$$

is annihilated by $(\otimes P \otimes (\otimes P)) = \otimes P$ and so is $E^\infty(T)$ and $H_*(T_m, F)$. This completes the inductive step of the proofs of (a) and (b).

(c) Suppose X is an H-space. If $P_* = \{P_n\}$ suppose that $\pi_m(T) \otimes F$, $m \leqslant n$ (resp. $Q_* H_m(T, F)$, $m \leqslant n$) is annihilated by $P = P_{r_n}$. Forming $P(T)$: $X \to X$, one has $\pi_m(P(T)) \otimes F = P[\pi_m(T) \otimes F]$ (resp. $Q_* H_*(P(T), F) = P(Q_* H_m(T, F)))$ and $\pi_m(P(T)) \otimes F$ is nilpotent (resp. $Q_* H_m(P(T), F))$ and consequently $H_m(P(T), F)$ are nilpotent.

Applying (a) and (b) with respect to the polynomials $P = x = \otimes P$, and letting $P(T)$ replace T, one has: $\pi_m(P(T)) \otimes F_p$ is nilpotent for $m \leqslant n$ if and only if $H_m(P(T)) \otimes F_p$ is nilpotnet for $m \leqslant n$ and as an endomorphism of a graded connected coalgebra is nilpotent through a given dimension if and only if its restriction to the submodule of primitives is nilpotent, (c) follows.

(d) As $\pi_m(p^r 1) \otimes F_p = 0$ if $[T_n^{t_r}] = [p^r 1] \circ [w_r]$ then $\pi_m(T_n^{t_r}) \otimes F_p = 0$ and $\pi_m(T) \otimes F_p$ is nilpotent for $m \leqslant n$. By (b) so is $H_m(T, F_p)$. Conversely, suppose $H_m(T, F_p)$ is nilpotent for $m \leqslant n$, then, so is $H^m(T, F_p)$. One can factor $p^r 1$ as follows:

$$X_n \approx Y_m \xrightarrow{h_m} \cdots \to Y_k \xrightarrow{h_k} Y_{k-1} \to \cdots \to Y_1 \xrightarrow{h_1} Y_0 = X_n,$$

where h_k: $Y_k \to Y_{k-1}$ is the homotopy fiber of a map g_{k-1}: $Y_{k-1} \to K(M_{k-1}, s_{k-1})$, M_{k-1} an F_p vector space.

Suppose inductively one has a commutative diagram:

$$\begin{tikzcd}
 & & Y_k \arrow[d, "h_k"] & \\
 & & Y_{k-1} \arrow[r, "g_{k-1}"] \arrow[d] & K(M_{k-1}, s_{k-1}) \\
X_n \arrow[r, "T_n^{\hat{t}}"] \arrow[uurr, dashed, "\hat{w}_k"] & X_n \arrow[r, "T_n^{t_{k-1}}"] \arrow[ur, "\hat{w}_{k-1}"] & X_n &
\end{tikzcd}$$

Now, by (a), for $P = x$, if $H_m(T, F_p)$ is nilpotent for $m \leqslant n$ then $H_m(T_n, F_p)$ (and, consequently $H^m(T_n, F_p)$) are nilpotent for all m. Say, $H^{s_{k-1}}(T_n^{\hat{t}}, F_p) = 0$.

$$[g_{k-1} \circ \hat{w}_{k-1}] \in H^{s_{k-1}}(X, M_{k-1}) = H^{s_{k-1}}(X, F_p) \otimes M_{k-1}$$

and

$$g_{k-1} \circ \hat{w}_{k-1} \circ T_n^{\hat{t}} \sim *;$$

hence, $\hat{w}_{k-1} \circ T_n^{\hat{t}}$ lifts to the homotopy fiber of g_k, $h_k \circ \hat{w}_k \sim \hat{w}_{k-1} \circ T_m^{\hat{t}}$. Put $\tilde{t}_{k-1} + \hat{t} = \hat{t}_k$ for the inductive step. Then $\tilde{t}_m = t_r$ satisfies

$$\begin{tikzcd}
& X_n \arrow[d, "\approx"'] \arrow[dd, bend left=60, "p^{r_1}"] \\
& Y_m \arrow[d] \\
X_n \arrow[ur, "\hat{w}_m"] \arrow[r, "T^{t_r}"'] & X_n
\end{tikzcd}$$

and one obtains w_r: $X_n \to X_n$ with $[p^r \circ 1] \circ w_r \sim T^{t_r}$.

1.4.1. REMARK. One can obtain a result similar to 1.4(d) for co-H-spaces to conclude that if P annihilates $H_*(T, F_p)$, it will annihilate $E_*(T)$ for every homology theory E_* with values in the category of F_p vector spaces.

1.4.2. REMARK. Theorem 1.4 holds for nilpotent spaces as well with essentially the same proof.

2. The lifting and extension obstructions of self maps. In this section we shall study some obstructions in self maps theory and their fundamental properties. Further aspects which are not needed for the proof of our main theorems can be found in **[Zabrodsky]$_1$**.

Our fundamental diagram is given by

(D.2.1)
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "T"'] \arrow[rr, bend left, "\ell : * \sim h \circ f"] \arrow[dr, phantom, "U"] & Y \arrow[r, "h"] \arrow[d, "S"] \arrow[dr, phantom, "W"] & B \arrow[d, "\tilde{S}"] \\
X \arrow[r, "f"'] & Y \arrow[r, "h"'] & B
\end{tikzcd}$$

where

U: $X \to PY$ is a homotopy $S \circ f \sim f \circ T$,

W: $Y \to PB$ is a homotopy $\hat{S} \circ h \sim h \circ S$,

l: $X \to LB$ is a homotopy $* \sim h \circ f$.

These yield maps

$$\begin{aligned}
&f_l\colon X \to V_h\colon && f_l(x) = f(x),\quad l(x) \in V_h,\\
&\hat{h}_l\colon C_f \to B\colon && \hat{h}_l(x,t) = l(x)[t],\quad \hat{h}_l(y) = h(y),\\
&\tilde{S}\colon V_h \to V_h\colon && \tilde{S}(y,\varphi) = S(y),\quad L\hat{S}\circ\varphi + W(y),\\
&\hat{\hat{S}}\colon C_f \to C_f\colon && \hat{\hat{S}}(x,t) = \begin{cases} T(x), 2t, & 0 \leqslant t \leqslant \frac{1}{2},\\ U(x)[2-2t], & \frac{1}{2} \leqslant t \leqslant 1,\end{cases}\\
& && \hat{\hat{S}}(y) = S(y).
\end{aligned}$$

One has

$$j_h \circ f_l = f,\quad \hat{h}_l \circ \hat{j}_f = h,\quad j_h \circ \tilde{S} = S \circ j_h,\quad \hat{\hat{S}} \circ \hat{j}_f = \hat{j}_f \circ S.$$

Consider the following problems. (One can refer to these problems as lifting and extension problems of self maps.)

(V) Is there a homotopy $\tilde{U}\colon \tilde{S} \circ f_l \sim f_l \circ T$ so that $Pj_h \circ \tilde{U} = U$?

(C) Is there a homotopy $\tilde{W}\colon \hat{h}_l \circ \hat{\hat{S}} \sim \hat{S} \circ \hat{h}_l$ so that $\tilde{W} \circ \hat{j}_f = W$?

2.1. Proposition. (V) *has a solution if and only if the map* $\alpha(l, W, U)\colon X \to \Omega B$ *given by* $\alpha(l, W, U) = L\hat{S} \circ l + W \circ f + Ph \circ V - l \circ T$ *is null homotoptic.*

(C) *has a solution if and only if the map* $\hat{\alpha}(l, W, U)\colon \Sigma X \to B$, *given by*

$$\hat{\alpha}(l,W,U)(x,t) = \begin{cases} \hat{S}(l(x)[4t]), & 0 \leqslant t \leqslant \frac{1}{4},\\ W(f(x))[4t-1], & \frac{1}{4} \leqslant t \leqslant \frac{1}{2},\\ h\{U(x)[4t-2]\}, & \frac{1}{2} \leqslant t \leqslant \frac{3}{4},\\ l(T(x))[4-4t], & \frac{3}{4} \leqslant t \leqslant 1,\end{cases}$$

is null homotopic.

Obviously α *and* $\hat{\alpha}$ *are adjoints.*

Proof.

$$\tilde{S} \circ f_l(x) = S \circ f(x), L\hat{S} \circ l(x) + W \circ f(x),$$
$$f_l \circ T(x) = f \circ T(x), l \circ T(x).$$

U induces a homotopy $\tilde{U}_1\colon \tilde{S} \circ f_l \sim f \circ T, L\hat{S} \circ l + W \circ f + Ph \circ U$ as maps $X \to V_h$ and the restriction of $\tilde{U}_1$ on the first factor is U (i.e.: $Pj_h \circ \tilde{U}_1 = U$). $\tilde{U}\colon \tilde{S} \circ f_l \sim f_l \circ T$, with $Pj_h \circ \tilde{U} = U$ exists if and only if $\alpha(l, W, U) = L\tilde{S} \circ l + W \circ f + Ph \circ U - l \circ T \sim *$ as maps $X \to \Omega B$. Similar arguments hold for problem (C) and the obstruction $\hat{\alpha}(l, W, U)$.

We need the following properties of $\alpha(l, W, U)$ and $\hat{\alpha}(l, W, U)$.

2.2. Lemma. (A) *If in* (D.2.1) *one of the following holds*:

(A_1) $(X, f, T, U) = (V_h, j_h, \tilde{S}, \textit{constant})$ *and* $l\colon X = V_h \to LB$ *is the projection.*

(A_2) $(B, h, \hat{S}, W) = (C_f, \hat{j}_f, \hat{\hat{S}}, \textit{constant})$ *and* l *is the adjoint of* $CX \to C_f$, *then* $\alpha(l, W, U) \sim * \sim \hat{\alpha}(l, W, U)$.

(B) *For* $w\colon X \to \Omega B$, *put* $l_w = w + l\colon * \sim h \circ f$, *then* $\alpha(l_w, W, U) \sim \Omega\hat{S} \circ w + \alpha(l, W, U) - w \circ T$.

(C) *Suppose* (D.2.1) *is extended to obtain*:

(D.2.2)
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\ell:\ *\sim h\circ f"] \arrow[d, "T"'] \arrow[dr, phantom, "U"] & Y \arrow[r, "h"] \arrow[d, "S"] \arrow[dr, phantom, "W"] & B \arrow[r, "k"] \arrow[d, "\hat{S}"] \arrow[dr, phantom, "W_0"] & B_0 \arrow[d, "\hat{S}_0"] \\
X \arrow[r, "f"] & Y \arrow[r, "h"] & B \arrow[r, "k"] & B_0
\end{tikzcd} \qquad W_0:\ \hat{S}_0\circ k\sim k\circ\hat{S}$$

Denote by $W_0 * W$: $\hat{S}_0 \circ k \circ h \sim k \circ h \circ S$ *the homotopy* $W_0 \circ h + Pk \circ W$. *Then*

$$\alpha(Lk \circ l, W_0 * W, U) \sim \Omega k \circ \alpha(l, W, U).$$

(D) *Consider the following cube related to* (D.2.2)

$$\begin{tikzcd}
& V_{k\circ h} \arrow[rr, "\tilde{h}"] \arrow[dl, "S_1"'] \arrow[dd, "j_{k\circ h}" near start] & & V_k \arrow[dl, "\hat{S}_1"'] \arrow[dd, "j_k"] \\
V_{k\circ h} \arrow[rr, "\tilde{h}" near start] \arrow[dd, "j_{k\circ h}"'] & & V_k \arrow[dd, "j_k" near start] & \\
& Y \arrow[rr, "h" near start] \arrow[dl, "S"'] & & B \arrow[dl, "\hat{S}"] \\
Y \arrow[rr, "h"] & & B &
\end{tikzcd}$$

where the vertical squares strictly commute. Then:

(D_1) *There exists a homotopy* W_1: $\hat{S}_1 \circ \tilde{h} \sim \tilde{h} \circ S_1$ *so that* $Pj_k \circ W_1 = W \circ j_{k \circ h}$.

(D_2) *There exists a fibration* r: $V_{\tilde{h}} \to V_h$ *with a cross section* $\chi: V_h \to V_{\tilde{h}}$, $r \circ \chi = 1_{V_h}$, $\chi \circ r \sim 1_{V\tilde{h}}$. *The maps* r, χ *have the following properties*: *There exists a homotopy* $\hat{l}$: $* \sim \tilde{h} \circ f_{Lk \circ l}$ ($f_{Lk \circ l}$: $X \to V_{k \circ h}$ *corresponds to* $Lk \circ l$: $* \sim k \circ h \circ f$), $Lj_k \circ \hat{l} = l$. *The lifting* $f_{\hat{l}}$: $X \to V_{\tilde{h}}$ *corresponding to* $\hat{l}$ *satisfies* $f_{\hat{l}} = \chi \circ f_l$. *Moreover, if* $\tilde{S}$: $V_h \to V_h$ *is induced by* $\hat{S}$, S, W *and* $\tilde{S}_1$: $V_{\tilde{h}} \to V_{\tilde{h}}$ *is induced by* $\hat{S}_1$, S_1, W_1 *then* $r \circ \tilde{S}_1 = \tilde{S}$.

(D_3) *If* $\alpha(Lk \circ l, W_0 * W, U) \sim *$ *and* U_1: $S_1 \circ f_{Lk \circ l} \sim f_{Lk \circ l} \circ T$ *satisfies* $Pj_{k \circ h} \circ U_1 = U$ *then* $\alpha(l, W, U) = \Omega j_k \circ \alpha(\hat{l}, W_1, U_1)$.

PROOF. (A) If (A_1) holds, one has $f_l = 1$. As $T = \tilde{S}$ in this case $\tilde{U}$, the constant homotopy, is a solution for V and $\alpha(l, W, U) \sim *$. If (A_2) holds, one has an obvious solution for (C) and $\hat{\alpha}(l, W, U) \sim *$. As α and $\hat{\alpha}$ are adjoints, (A) follows.

(B)

$$\begin{aligned}
\alpha(l_w, W, U) &= L\hat{S} \circ l_w + W \circ f + Ph \circ U - l_w \circ T \\
&= \Omega\hat{S} \circ w + L\hat{S} \circ l + W \circ f + Ph \circ U - l \circ T - w \circ T \\
&= \Omega\hat{S} \circ w + \alpha(l, W, U) - w \circ T.
\end{aligned}$$

(C)

$$\begin{aligned}
\alpha(Lk\circ l, W_0 * W, U) &= L\hat{S}_0\circ Lk\circ l + W_0 * W\circ f + Pk\circ Ph\circ U - Lk\circ l\circ T\\
&= L\hat{S}_0\circ Lk\circ l + W_0\circ h\circ f + Pk\circ W\circ f + Pk\circ Ph\circ U - Lk\circ l\circ T\\
&\sim L\hat{S}_0\circ Lk\circ l + W_0\circ h\circ f - Lk\circ L\hat{S}\circ l\\
&\quad + Lk\circ L\hat{S}\circ l + Pk\circ W\circ f + Pk\circ Ph\circ U - Lk\circ l\circ T\\
&= L\hat{S}_0\circ Lk\circ l + W_0\circ h\circ f - Lk\circ L\hat{S}\circ l + \Omega k\circ\alpha(l, W, 0).
\end{aligned}$$

Now, one can easily see that $x, s, t \to W_0[l(x)[t]][s]$ induces a null homotopy $L\hat{S}_0\circ Lk\circ l + W_0\circ h\circ f - Lk\circ LS\circ l \sim *$ as maps $X\to \Omega B$ and (C) follows.

(D)–(D_1) $\tilde{h}$ is given by $\tilde{h}(y,\varphi) = h(y), \varphi$ where $\varphi\in LB_0$, $\varphi(1) = k\circ h(y)$.

$$\hat{S}_1\circ\tilde{h}(y,\varphi) = \hat{S}\circ h(y), L\hat{S}_0\circ\varphi + W_0\circ h(y),$$
$$\tilde{h}\circ S_1(y,\varphi) = h\circ S(y), L\hat{S}_0\circ\varphi + W_0\circ h(y) + Pk\circ W(y),$$

and the homotopy W: $\hat{S}\circ h \sim h\circ S$ on the first factor could be obviously extended to W_1: $\hat{S}_1\circ\tilde{h}\sim\tilde{h}\circ S_1$.

(D_2) $V_{\tilde{h}}$ consists of triples (y,φ,Φ), $y\in Y$, $\varphi\in LB$,

$$\Phi\in L^2B_0 = \left\{\Phi\colon I^2\to B_0 \middle| \Phi(0,t) = \Phi(s,0) = *\right\}$$

satisfying $h(y) = \varphi(1)$, $\Phi(s,1) = k\circ\varphi(s)$.

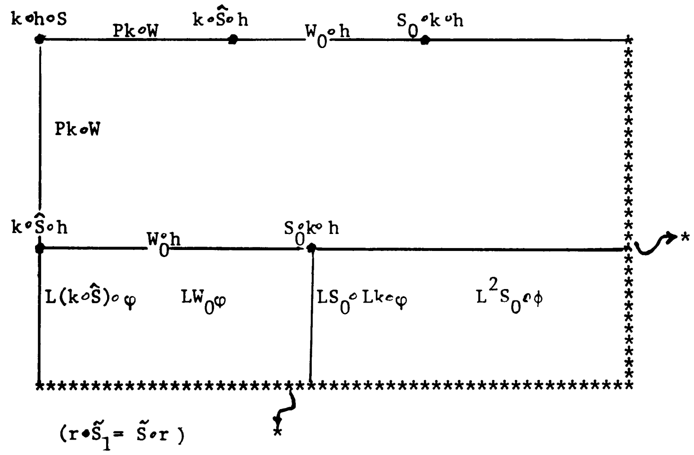
r: $V_{\tilde{h}}\to V_h$, given by $r(y,\varphi,\Phi) = y,\varphi$, is a fibration and χ: $V_h\to V_{\tilde{h}}$ is the cross-section given by

$$\chi(y,\varphi) = y,\varphi,\Phi_\varphi,$$

$$\Phi_\varphi(s,t) = \begin{cases} *, & s+t\leqslant 1,\\ k\circ\varphi(s+t-1), & s+t\geqslant 1,\end{cases}$$

$$h\circ f_{Lk\circ l}(x); = h\circ f(x),\qquad Lk\circ l(x)\in V_k,$$

and the homotopy l: $*\sim h\circ f$ on the first factor could be extended to a homotopy $\hat{l}$: $*\sim h\circ f_{Lk\circ l}$, $f_{\hat{l}}(x) = (f(x), l(x), \Phi_{l(x)}) = \chi\circ f_l(x)$. $\tilde{S}_1(y,\varphi,\Phi) = S(y)$, $L\hat{S}\circ\varphi + W(y)$, $\tilde{\Phi}$ where $\tilde{\Phi}$: $I^2\to B_0$ could be described by:



One has $r \circ \tilde{S}_1 = \tilde{S} \circ r$.

(D_3)

$$\begin{aligned}\Omega j_k \alpha(\hat{l}, W_1, U_1) &= Lj_k \circ L\hat{S}_1 \circ \hat{l} + Pj_k \circ W_1 \circ f_{Lk \circ l} + Pj_k \circ P\tilde{h} \circ U_1 - Lj_k \circ \hat{l} \circ T \\ &= L\hat{S} \circ Lj_k \circ \hat{l} + W \circ j_{k \circ h} \circ f_{Lk \circ l} + Ph \circ Pj_{k \circ h} \circ U_1 - Lj_k \circ \hat{l} \circ T \\ &= L\hat{S} \circ l + W \circ f + Ph \circ U - l \circ T = \alpha(l, W, U).\end{aligned}$$

2.3. EXAMPLE. Given $T: X \to X$. If $h_n: X \to X_n$ is the Postnikov approximation of X in dim $\leqslant n$, one has a commutative diagram:

$$\begin{array}{ccc} X & \overset{h_n}{\to} & X_n \\ T\downarrow & U_n & \downarrow T_n \\ X & \overset{h_n}{\to} & X_n \end{array}$$

One can construct X_n, h_n by an inductive procedure and obtain a sequence of fibrations $h_{n,n-1}: X_n \to X_{n-1}$ and liftings $h_n: X \to X_n$, $h_{n,n-1} \circ h_n = h_{n-1}$ as follows:

As we assume X to be simply connected, we have the "killing of homotopy groups" procedure to obtain $h_2: X \to X_2 = K(\pi_2(X), 2)$. If $h_{n-1}: X \to X_{n-1}$ is constructed, we proceed by forming $C_{h_{n-1}}$, and can see that $C_{h_{n-1}}$ is n-connected with its bottom Postnikov approximation given by $\hat{k}_{n-1}: C_{h_{n-1}} \to K(\pi, n+1)$. Here π turns out to be $\pi_*(X)$, $k_{n-1} = \hat{k}_{n-1} \circ \hat{j}_{h_{n-1}}: X_{n-1} \to C_{h_{n-1}} \to K(\pi_n(X), n+1)$ is the k-invariant and, using the natural null homotopy $l: * \sim \hat{j}_{h_{n-1}} \circ h_{n-1}$, the map h_n is then the lifting of h_{n-1} to $V_{k_{n-1}} = X_n$ induced by the null homotopy $L\hat{k}_{n-1} \circ l$: $* \sim k_{n-1} \circ h_{n-1}$.

Now one can incorporate the T-structure into the Postnikov system as follows: Complete the first stage to obtain

$$\begin{array}{cccc} X & \overset{h_2}{\to} & K(\pi_2(X), 2) = & X_2 \\ T\downarrow & U_2 & & \downarrow T_2 = K(\pi_2(T), 2) \\ X & \underset{h_2}{\to} & K(\pi_2(x), 2) = & X_2 \end{array}$$

Assume, given inductively

$$\begin{array}{ccccccc} X & \overset{h_{n-1}}{\to} & X_{n-1} & \overset{\hat{j}_{h_{n-1}}}{\to} & C_{h_{n-1}} & \overset{\hat{k}_{n-1}}{\to} & K(\pi_n(X), n+1) \\ T\downarrow & U_{n-1} & T_{n-1}\downarrow & W_{n-1} = \text{const.} & \downarrow \hat{T}_{n-1} & W_{0,n-1} & \downarrow K(\pi_n(T), n+1) \\ X & \underset{h_{n-1}}{\to} & X_{n-1} & \underset{\hat{j}_{h_{n-1}}}{\to} & C_{h_{n-1}} & \underset{\hat{k}_{n-1}}{\to} & K(\pi_n(X), n+1) \end{array}$$

$W_{0,n-1}$: $K(\pi_n(T), n+1) \circ \hat{k}_{n-1} \sim \hat{k}_{n-1} \circ \hat{T}_{n-1}$ exists and as $C_{h_{n-1}}$ is n-connected, $W_{0,n-1}$ is unique up to homotopy. Suppose inductively that T_{n-1} was induced by

T_{n-2}, $K(\pi_{n-1}(T), n)$, $W_{0,n-2}$ in the natural way, then by 2.2(A), (C)

$$\alpha(L\hat{k}_{n-1} \circ l, W_{0,n-1} * W_{n-1}, U_{n-1}) \sim *(W_{0,n-1} * W_{n-1} \sim W_{0,n-1})$$

and U_n can be constructed with $Ph_{n-1,n} \circ U_n = U_{n-1}$.

3. The main theorems. In this section we formulate and prove the main theorems:

3.1. THEOREM A. *Let T: $X \to X$. Suppose there exist polynomials $P_1, P_2 \in Z_*[x]$ ($P_{i,n}$ having leading coefficients prime to p) so that*:

(a) *The* mod *p reductions of $\otimes P_{1*}$ and P_{2*} are relatively prime.*

(b) *$P_{1*} \cdot P_{2*}$ annihilates $\pi_*(T) \otimes Z_p$.*

Then:

(i) *There exist a space $\hat{X}$, a self map $\hat{T}$: $\hat{X} \to \hat{X}$, and a map f: $\hat{X} \to X$ so that*:

(1) $f \circ \hat{T} \sim T \circ f$.

(2) *P_{1*} annihilates $\pi_*(\hat{T}) \otimes Z_p$.*

(3) *$\pi_*(f) \otimes Z_p$ is injective with* $\operatorname{im} \pi_*(f) \otimes Z_p = \bigcup_r \ker(P_{1*})^r(\pi_*(T) \otimes Z_p)$.

(ii) *Given X', T', f', T': $X' \to X'$, f': $X' \to X$ so that*:

(1) $f' \circ T' \sim T \circ f'$.

(2) *$\otimes P_{1*}$ annihilates $\pi_*(T') \otimes Z_p$.*

Then f', T' factors mod *p through f, $\hat{T}$ in the following sense: There exists a homotopy commutative cube*

$$\begin{tikzcd}[row sep=scriptsize, column sep=scriptsize]
& \hat X' \arrow[rr, "\hat f'_1"] \arrow[dl, "\hat T'"'] \arrow[dd, "\hat f'_2" near start] & & \hat X \arrow[dl, "\hat T"'] \arrow[dd, "\hat f"] \\
\hat X' \arrow[rr, "\hat f'_1" near start, crossing over] \arrow[dd, "\hat f'_2"'] & & \hat X \\
& X' \arrow[rr, "f'" near start] \arrow[dl, "T'"'] & & X \arrow[dl, "T"] \\
X' \arrow[rr, "f'"'] & & X \arrow[from=uu, crossing over, "\hat f" near end]
\end{tikzcd}$$

with $\hat{f}'_2$ — a mod *p equivalence. In particular, if X, $\hat{X}$, X', T, $\hat{T}$, T', f, f' are p-local, one may assume $\hat{f}'_2 = 1$, hence there exists $\hat{f}'$: $X' \to \hat{X}$ so that $\hat{f}' \circ T' \sim \hat{T} \circ \hat{f}'$ and $f' \sim f \circ \hat{f}'$.*

To prove Theorem A one proves

3.2. THEOREM B. *Let T: $X \to X$. Suppose there exist polynomials $P_1, P_2 \in Z[x]$ with leading coefficients prime to p, and suppose*

(1) *$\otimes \hat{P}_1$ and $\hat{P}_2$ are relatively prime where $\hat{P}_i$ are the* mod *p reductions of P_i.*

(2) *P_1 annihilates $H_m(T, Z_p)$, $m \leqslant n - 1$.*

(3) *$P_1 \cdot P_2$ annihilates $H_n(T, Z_p)$.*

Then there exists a homotopy commutative diagram

$$\begin{array}{ccc} X_n & \xrightarrow{T_n} & X_n \\ \hat{h}\downarrow \approx p & & \hat{h}\downarrow \approx p \\ \hat{X}_n \times K(\hat{\pi}, n) & \xrightarrow{\hat{T}_n \times \tilde{T}} & \hat{X}_n \times K(\hat{\pi}, n) \end{array}$$

where X_n, T_n *is the Postnikov approximation in* $\dim \leqslant n$; $\hat{T}_n: \hat{X}_n \to \hat{X}_n$, $\tilde{T}: K(\hat{\pi}, n) \to K(\hat{\pi}, n)$ *satisfy*: P_1 *annihilates* $H_m(\hat{T}_n, Z_p)$, $m \leqslant n$; P_2 *annihilates* $H_n(\tilde{T}, Z_p)$.

In particular, the Hurewicz homomorphism induces an isomorphism

$$\bigcup_r \ker P_2^r(\pi_n(T)) \otimes Z_p \to \bigcup_r \ker P_2^r\left[H_n(T, Z_p)\right].$$

First we prove the simple analogue for abelian groups:

3.2.1. Lemma. *Let* $\varphi: G \to G$ *be an endomorphism of a finitely generated abelian group. Suppose* $P_1, P_2 \in Z[x]$ *have leading coefficients prime to* p *and their* mod p *reductions are relatively prime. If* $P_1 \cdot P_2$ *annihilates* $\varphi \otimes Z_p$ *then one has a commutative diagram*

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G \\ \approx p \downarrow h & & \downarrow h \approx p \\ G_1 \oplus G_2 & \xrightarrow{\varphi_1 \oplus \varphi_2} & G_2 \oplus G_2 \end{array}$$

where P_i *annihilates* $\varphi_i \otimes Z_p$, $i = 1, 2$.

Proof. This is a minor variation of a standard linear algebra theorem:

Let $\hat{P}_i$ be the mod p reduction of P_i. If $\hat{P}_1$, $\hat{P}_2$ are relatively prime, so are P_1 and P_2. Let r be the smallest positive integer for which there exist polynomials Q_1, Q_2 so that $Q_1P_1 + Q_2P_2 = r$. We shall show now that r is prime to p. Suppose $p|r$, say $r = p \cdot r_1$, reducing mod p one has $\hat{Q}_1\hat{P}_1 + \hat{Q}_2\hat{P}_2 = 0$ and as $\hat{P}_1$, $\hat{P}_2$ are relatively prime one has $\hat{Q}_1 = \hat{Q} \cdot \hat{P}_2$, $\hat{Q}_2 = -\hat{Q}\hat{P}_1$. Let $Q \in Z[x]$ represent $\hat{Q}$ (with a leading coefficient prime to p); then $Q_1 - QP_2 = pQ_1'$, $Q_2 + QP_1 = pQ_2'$, $r = pr_1 = QP_2P_1 + pQ_1'P_1 - QP_1P_2 + pQ_2'P_1$ and $Q_1'P_1 + Q_2'P_2 = r_1 < r$, contradicting the minimality of r.

Replacing P_i by their suitable powers if necessary, one may assume

$$P_1 \cdot P_2(\varphi \otimes Z_p) = 0.$$

The homomorphism $\alpha: G \to \operatorname{im} P_1(\varphi) \oplus \operatorname{im} P_2(\varphi)$ given by $\alpha(x) = P_1(\varphi)x, P_2(\varphi)x$ is a mod p isomorphism: Indeed, if Q_1, Q_2, r are as above, $x_1, x_2 \in G$, arbitrary, put $z = P_1(\varphi)Q_1(\varphi)x_1 + P_2(\varphi)Q_2(\varphi)x_2$ then

$$P_1(\varphi)z = P_1(\varphi)rx_1 + P_1(\varphi)P_2(\varphi)Q_2(\varphi)[x_2 - x_1] = rP_1(\varphi)x_1 + y_1$$

with y_1 of finite order prime to p. Similarly, $P_2(\varphi)z = rP_2(\varphi)x_2 + y_2$, y_2 of order prime to p. Thus for some integer s prime to p, $sr(P_1(\varphi)x_1, P_2(\varphi)x_2) = \alpha sz$ and α is mod p surjective. Thus $\ker\alpha = \ker P_1(\varphi) \cap \ker P_2(\varphi)$. If $z \in \ker\alpha$, then $rz = Q_1(\varphi)P_1(\varphi)z + Q_2(\varphi)P_2(\varphi)z = 0$ and z is of order prime to p.

We need the following

3.2.2. LEMMA. *Let φ_1, φ_2: $M \to M$ be two commuting endomorphisms of an abelian group M. Suppose there exist polynomials $P_1, P_2 \in Z[x]$ with leading coefficients prime to p and with relatively prime* mod p *reductions so that P_i annihilates $\tilde\varphi_i = \varphi_i \otimes Z_p$. Then $\varphi_1 - \varphi_2$ is a* mod p *isomorphism.*

PROOF. One has to show that $\tilde\varphi_1 - \tilde\varphi_2$ is an isomorphism. With no loss of generality one may assume $P_i(\tilde\varphi_i) = 0$. Let Q_1, Q_2, r for $Q_i \in Z[x]$, r an integer prime to p, satisfy $P_1Q_1 + P_2Q_2 = r$. Now, for any polynomial $P \in Z[x]$, one has the following identity in $Z[x, y]$: $P(x) = (x - y)\hat P(x, y) + P(y)$. Consequently,

$$\begin{aligned} r &= Q_1P_1(\tilde\varphi_1) + Q_2P_2(\tilde\varphi_1) = Q_2P_2(\tilde\varphi_1) \\ &= (\tilde\varphi_1 - \tilde\varphi_2)\hat P(\tilde\varphi_1, \tilde\varphi_2) + Q_2P_2(\tilde\varphi_2) = (\tilde\varphi_1 - \tilde\varphi_2)\hat P(\tilde\varphi_1, \tilde\varphi_2). \end{aligned}$$

$\frac{1}{r}\hat P(\tilde\varphi_1, \tilde\varphi_2)$: $M \otimes Z_p \to M \otimes Z_p$ is the inverse of $\tilde\varphi_1 - \tilde\varphi_2$.

PROOF OF THEOREM B. Consider the $(n - 1)$ Postnikov step of T:

$$\begin{array}{ccccc} X_n & \xrightarrow{h_{n,n-1}l} & X_{n-1} & \xrightarrow{k_{n-1}} & K(\pi_n(X), n+1) \\ T_n \downarrow & & \downarrow T_{n-1} & & \downarrow K(\pi_n(T), n+1) = \tilde T \\ X_n & \xrightarrow{h_{n,n-1}} & X_{n-1} & \xrightarrow{k_{n-1}} & K(\pi_n(X), n+1) \end{array}$$

As P_1 annihilates $H_m(T, Z_p)$, $m \leqslant n - 1$, it annihilates $H_m(T, F)$, $m \leqslant n - 1$, for $F = Q$ and $F = F_p$. By 1.4, $\otimes P_1$ annihilates $H_*(T_{n-1}, F)$ and consequently it annihilates $H_*(T_{n-1}, Z_p)$.

One has exactness of rows in the following commutative diagram:

$$\begin{array}{ccc} H_{n+1}(X_{n-1}, Z_p) \to & H_n(K(\pi_n(X), n+1), Z_p) = \pi_n(X) \otimes Z_p \to & H_n(X_n, Z_p) \\ H_{n+1}(T_{n-1}, Z_p)\downarrow & H_n(K(\pi_n(T), n+1), Z_p)\downarrow = \pi_n(T) \otimes Z_p & \downarrow H_n(T_n, Z_p) \\ H_{n+1}(X_{n-1}, Z_p) \to & H_n(K(\pi_n(X), n+1), Z_p) = \pi_n(X) \otimes Z_p \to & H_n(X_n, Z_p) \end{array}$$

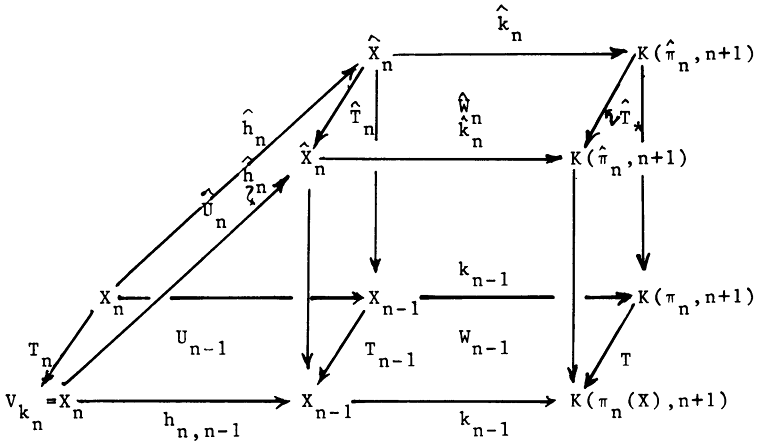
As $\otimes P_1$ annihilates $H_{n+1}(T_{n-1}, Z_p)$, and $P_2 \cdot P_1$ annihilates $H_n(T_n, Z_p) = H_n(T, Z_p)$, $(\otimes P_1) \cdot P_2$ annihilates $\pi_n(T) \otimes Z_p$. Put $P'_* = \otimes P_1$, $\hat P'_*$ the mod p reduction of P'_*. By hypothesis, $\hat P'_*$ and $\hat P_2$ are relatively prime and, by 3.2.1, one has an exact sequence, split mod p:

$$\begin{array}{ccccccccc} 0 & \to & \hat\pi_n & \xrightarrow{\hat u} & \pi_n(X) & \xrightarrow{u''} & \pi''_n & \to & 0 \\ & & \hat T \downarrow & & \downarrow \pi_n(T) & & \downarrow T''_n & & \\ 0 & \to & \hat\pi_n & \xrightarrow{\hat u} & \pi_n(X) & \xrightarrow{u''} & \pi''_n & \to & 0 \end{array}$$

$T_n'' \otimes Z_p$ is annihilated by P'_* and $\hat{T} \otimes Z_p$ is annihilated by P_2. Using 2.3 and 2.2(C) one obtains:

$$\begin{tikzcd}[column sep=large]
X_n \arrow[r, "h_{n,n-1}"] \arrow[d, "T_n"'] \arrow[dr, phantom, "U_{n-1}"] & X_{n-1} \arrow[r, "k_{n-1}"] \arrow[d, "T_{n-1}"] \arrow[dr, phantom, "W_{n-1}"] & K(\pi_n(X), n+1) \arrow[r, "u''_*"] \arrow[d, "\tilde{T}"] \arrow[dr, phantom, "W_0"] & K(\pi_n'', n+1) \arrow[d, "T''_{n*}"] \\
X_n \arrow[r, "h_{n,n-1}"] & X_{n-1} \arrow[r, "k_{n-1}"] & K(\pi_n(X), n+1) \arrow[r, "u''_*"] & K(\pi_1'', n+1)
\end{tikzcd}$$

$\alpha(l, W_{n-1}, U_{n-1}) \sim *$, $l\colon * \sim k_{n-1} \circ h_{n,n-1}$. Apply 2.2(D): $\hat{X}_n = V_{u''_* \circ k_{n-1}}$,



There exists $\hat{l}\colon * \sim \hat{k}_n \circ \hat{h}_n$ so that $\hat{h}_{n,\hat{l}}\colon X_n \to V_{\hat{k}_n}$ is the homotopy equivalence χ of 2.2(D_2).

By 2.2(D_3) one has $\Omega \hat{u}_* \circ \alpha(\hat{l}, \hat{W}_n, \hat{U}_n) \sim \alpha(l, W_{n-1}, U_{n-1}) \sim *$ and, as $\hat{u}$ is mod p split, $\alpha(\hat{l}, \hat{W}_n, \hat{U}_n)$ has order q_1 prime to p.

Now, P'_* annihilates $T_n'' \otimes Z_p = \pi_n(T''_{n*}) \otimes Z_p$ and as was stated above P'_* annihilates $H_*(T_{n-1}, Z_p)$, hence $\otimes P'_* = P'_*$ annihilates $\pi_*(T_{n-1}) \otimes Z_p$ and $\pi_*(\hat{T}_n) \otimes Z_p$ and consequently P'_* annihilates $H_*(\hat{T}_n, Z_p)$ and $H^*(\hat{T}_n, M \otimes Z_p)$ for all M. On the other hand, P_2 annihilates $\hat{T}_* \otimes Z_p$. Thus $[\hat{k}_n] \in H^{n+1}(\hat{X}_n, \hat{\pi}_n)$ is in $\ker(\hat{T}_* - \hat{T}_n^*)$, where $\hat{T}_*, \hat{T}_n^*\colon H^{n+1}(\hat{X}_n, \hat{\pi}_n) \to H^{n+1}(\hat{X}_n, \hat{\pi}_n)$ are the commuting endomorphisms induced by $\hat{T}_*, \hat{T}_n$, respectively. As $\hat{T}_* \otimes Z_p$ and $\hat{T}_n^* \otimes Z_p$ are annihilated by mod p relatively prime polynomials $\ker(\hat{T}_* \otimes Z_p - \hat{T}_n^* \otimes Z_p) = 0$ by 3.2.2.

It follows that $[\hat{k}_n]$ is of order q_2 prime to p. Consider:

$$\begin{tikzcd}[column sep=large]
\hat{X}_n \arrow[r, "\hat{k}_n"] \arrow[d, "\hat{T}_n"'] \arrow[rr, bend left, "\tilde{l}:\, * \sim q_2 \cdot 1 \circ k_n"] \arrow[dr, phantom, "\hat{W}_n"] & K(\hat{\pi}_n, n+1) \arrow[r, "q_2 \cdot 1"] \arrow[d, "\hat{T}_*"] \arrow[dr, phantom, "\hat{W}'_n"] & K(\hat{\pi}_n, n+1) \arrow[d, "\hat{T}_*"] \\
\hat{X}_n \arrow[r, "\hat{k}_n"] & K(\hat{\pi}_n, n+1) \arrow[r, "q_2 \cdot 1"] & K(\hat{\pi}_n, n+1)
\end{tikzcd}$$

Again, as $\hat{T}_n^*$, $\Omega\hat{T}_*$ are commuting endomorphisms of $H^n(\hat{X}_n, \hat{\pi}_n)$ and $\hat{T}_n^* \otimes Z_p$ and $\Omega\hat{T}_* \otimes Z_p$ are annihilated by polynomials with relatively prime mod p reductions, by 3.2.2. $\hat{T}_n^* - \Omega\hat{T}_*$ is a mod p surjection $H^n(\hat{X}_n, \hat{\pi}_n) \to H^n(\hat{X}_n, \hat{\pi}_n)$. Thus, for some prime to p integer r

$$r\alpha(\tilde{l}, \hat{W}_n', \hat{W}_n) \sim (\Omega\hat{T}_* - \hat{T}_n^*)\omega.$$

Put $g = q_1 \cdot q_2 \cdot r1\colon\ K(\hat{\pi}_n, n+1) \to K(\hat{\pi}_n, n+1)$, $\tilde{l}' = -q_1\omega + L(q_1 r1)\circ \tilde{l}\colon\ * \sim q_1 q_2 r \cdot 1 \circ \hat{k}_n = g \circ \hat{k}_n$, $\tilde{W}_n\colon\ \hat{T}_* \circ g \sim g \circ \hat{T}_*$ where $\tilde{W}_n = \hat{W}_n'' * \hat{W}_n'$. $\hat{W}_n'\colon\ \hat{T}_* \circ q_2 1 \sim q_2 1 \circ \hat{T}_*$ as above and $\hat{W}_n''\colon\ \hat{T}_* \circ q_1 r1 \sim q_1 r \circ \hat{T}_*$:

$$\begin{tikzcd}
X_n \arrow[r, "\hat{h}_n"] \arrow[d, "T_n"'] \arrow[dr, phantom, "\hat{U}_n"] \arrow[ddrr, phantom, bend right, "\ell:\ * \sim \hat{k}_n \circ \hat{h}_n"'] & \hat{X}_n \arrow[r, "\hat{k}_n"] \arrow[d, "\hat{T}_n"] \arrow[dr, phantom, "\hat{W}_n"] \arrow[rr, bend left, "\tilde{\ell}':\ * \sim g \circ \hat{k}_n"] & K(\hat{\pi}_n, n+1) \arrow[r, "g"] \arrow[d, "\hat{T}_*"] \arrow[dr, phantom, "\tilde{W}_n"] & K(\hat{\pi}_n, n+1) \arrow[d, "\hat{T}_*"] \\
X_n \arrow[r, "\hat{h}_n"] \arrow[rr, bend right, "\ell:\ * \sim \hat{k}_n \circ \hat{h}_n"'] & \hat{X}_n \arrow[r, "\hat{k}_n"] & K(\hat{\pi}_n, n+1) \arrow[r, "g"] & K(\hat{\pi}_n, n+1)
\end{tikzcd}$$

By 2.2(B), (C)

$$\alpha(\tilde{l}', \tilde{W}_n, \hat{W}_n) \sim -q_1(\Omega\hat{T}_* - \hat{T}_n^*)\omega + q_1 r\alpha(\tilde{l}, \hat{W}_n', \hat{W}_n) \sim *,$$

$$\alpha(Lg \circ \hat{l}, \hat{W}_n * \hat{W}_n, \hat{U}_n) = \Omega g \circ \alpha(\hat{l}, \hat{W}_n, \hat{U}_n) = rq_2q_1\alpha(\hat{l}, \hat{W}_n, \hat{U}_n) \sim *.$$

Hence

$$\begin{aligned}
* &\sim L\hat{T}_* \circ Lg \circ \hat{l} + \tilde{W}_n * \hat{W}_n \circ \hat{h}_n + P(g \circ \hat{k}_n) \circ \hat{U}_n - Lg \circ \hat{l} \circ T_n \\
&\sim L\hat{T}_* \circ Lg \circ \hat{l} - L\hat{T}_* \circ \tilde{l}' \circ \hat{h}_n + [L\hat{T}_* \circ \tilde{l}' \circ \hat{h}_n + \tilde{W}_n * \hat{W}_n \circ \hat{h}_n - \tilde{l}' \circ \hat{T}_n \circ \hat{h}_n] \\
&\quad + [\tilde{l}' \circ \hat{T}_n \circ \hat{h}_n + P(g \circ \hat{k}_n) \circ \hat{U}_n - \tilde{l}' \circ \hat{h}_n \circ T_n] + \hat{l}' \circ \hat{h}_n \circ T_n - Lg \circ \hat{l} \circ T_n.
\end{aligned}$$

The first brackets enclose $\alpha(\tilde{l}', \tilde{W}_n, \hat{W}_n) \circ \hat{h}_n \sim *$ and one can see directly that the second brackets enclose a null homotopic expression in $[X, \Omega K(\hat{\pi}, n+1)]$. Thus

$$* \sim \Omega\hat{T}_*(Lg \circ \hat{l} - \tilde{l}' \circ \hat{h}_n) - (Lg \circ \hat{l} - \tilde{l}' \circ \hat{h}_n) \circ T_n.$$

Now, by 2.2(D_3) $X_n \overset{\approx}{\to} V_{\hat{k}_n}$ and $V_{\hat{k}_n} \approx_p V_{g \circ \hat{k}_n} \approx \hat{X}_n \times \Omega K(\hat{\pi}_n, n+1)$ (as $g \circ \hat{k}_n \sim *$). One can easily see that the composition

$$\hat{h}\colon X_n \overset{\approx p}{\to} \hat{X}_n \times \Omega k(\hat{\pi}_n, n+1) \approx \hat{X}_n \times K(\hat{\pi}_n, n)$$

is given by

$$\hat{h}(x) = \hat{h}_n(x),\ Lg \circ \hat{l}(x) - \tilde{l}' \circ \hat{h}_n(x)$$

and as was shown above,

$$\begin{array}{ccc} X_n & \xrightarrow{p_2\hat{h}} & \Omega K(\hat{\pi}_n, n+1) \\ T_n \downarrow & & \downarrow \Omega\hat{T}_* = \tilde{T} \\ X_n & \xrightarrow{p_2 \circ \hat{h}} & \Omega K(\hat{\pi}_n, n+1) \end{array}$$

is homotopy commutative. As $\hat{h}_n \circ T_n \sim \hat{T}_n \circ \hat{h}_n$ Theorem B follows.

3.3. Proof of Theorem A. (i) One constructs $\hat{X}$ inductively, as follows: Suppose one has a commutative diagram

$$\begin{array}{ccc} X(n) & \xrightarrow{f(n)} & X \\ \downarrow T_n & & \downarrow T \\ X(n) & \xrightarrow{f(n)} & X \end{array}$$

so that:

(a)(n): $(\otimes P_{1*}) \cdot P_{2*}$ annihilates $\pi_*(T(n)) \otimes Z_p$.

(b)(n): $\otimes P_{1*}$ annihilates $\pi_m(T(n)) \otimes Z_p$ for $m \leqslant n-1$.

(c)(n): $\pi_*(f(n)) \otimes Z_p$ is an isomorphism for $m \geqslant n$ and a monomorphism for $m < n$, $\operatorname{im} \pi_n(f(n)) \otimes Z_p = \bigcup_r \ker P_{1*}^r(\pi_m(T) \otimes Z_p)$, $m < n$.

If $\{[X(n)]_m, [T(n)]_m\}$ are the Postnikov approximations of $X(n)$, $T(n)$ one has

$$\begin{array}{ccccc} K(\pi_n(X(n)), n) & \rightarrow & [X(n)]_n & \rightarrow & [X(n)]_{n-1} \\ K(\pi_n(T(n)), n) \downarrow & & \downarrow T(n)_n & & \downarrow T(n)_{n-1} \\ K(\pi_n(X(n)), n) & \rightarrow & [X(n)]_n & \rightarrow & [X(n)]_{n-1} \end{array}$$

As $\otimes P_{1*}$ annihilates $\pi_*(T(n)_{n-1}) \otimes Z_p$ it annihilates $H_*(T(n)_{n-1}, Z_p)$ and $H_m(T(n), Z_p)$, $m < n$. $(\otimes P_{1*})P_2$ annihilates $H_n(K(\pi_n(T(n)), n), Z_p) = \pi_n(T(n)) \otimes Z_p$ and consequently $(\otimes P_{1*}) \cdot P_{2*}$ annihilates $H_n(T(n)_n, Z_p) = H_n(T(n), Z_p)$ and one can apply Theorem B for $X(n)$, $T(n)$, P_{1*}, P_{2*} to obtain a commutative diagram.

$$\begin{array}{ccc} X(n)_n & \xrightarrow{T(n)_n} & X(n)_n \\ \approx p \downarrow & & \downarrow \approx p \\ \hat{X}(n)_n \times K(\hat{\pi}_n, n) & \xrightarrow{\hat{T} \times \tilde{T}} & \hat{X}(n)_n \times k(\hat{\pi}_n, n) \end{array}$$

$\hat{\pi} \otimes Z_p = \bigcup_r \ker(P_{2*}^r(\pi_n(T(n)) \otimes Z_p))$. In particular, one has a diagram

$$\begin{array}{ccc} X(n) & \xrightarrow{\hat{g}_n} & K(\hat{\pi}_n, n) \\ T(n) \downarrow & & \downarrow \tilde{T} \\ X(n) & \xrightarrow{\hat{g}_n} & K(\hat{\pi}_n, n) \end{array}$$

$X(n+1)$, $T(n+1)$—the fiber of $\hat{g}_n$—yield the next inductive step:

(a)($n+1$) follows from the fact that $(\otimes P_{1*}) \cdot P_{2*}$ annihilates $\pi_*(T(n))$ and $\pi_*(\tilde{T})$.

(b)–(c)($n+1$): $\pi_*(\hat g_n) \otimes Z_p$ is split surjective and is zero in dim $\neq n$. Thus, (c)(n) implies (b)($n+1$) and (c)($n+1$) in dim $\neq n$. $\pi_n(X(n+1)) \otimes Z_p = \pi_n(X(n)_n) \otimes Z_p$ which is isomorphic, by Theorem B, to $\bigcup_r \ker P_{1*}^r(\pi_n(T(n)) \otimes Z_p)$ and (b)($n+1$), (c)($n+1$) follow.

Passing to a limit, one obtains the desired $\hat X$, $\hat T$:

$$\begin{tikzcd}
\hat X = \varprojlim X(n) \arrow[r] \arrow[rrrrr, bend left, "f"] \arrow[d, "\hat T = \varprojlim T(n)"] & \cdots \arrow[r] & X(n) \arrow[r] \arrow[d, "T(n)"'] & X(n-1) \arrow[r] \arrow[d, "T(n-1)"'] & \cdots \arrow[r] & X(1) = X \arrow[d, "T(1)=T"'] \\
\hat X = \varprojlim X(n) \arrow[r] & \cdots \arrow[r] & X(n) \arrow[r] & X(n-1) \arrow[r] & \cdots \arrow[r] & X(1) = X
\end{tikzcd}$$

(ii) Given X', f', T' so that the following commutes:

$$\begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
T'\downarrow & & \downarrow T \\
X' & \xrightarrow{f'} & X
\end{array}$$

Form the pull back of f' and $f\colon \hat X \to X$ of (i) and the self maps induced to obtain:

$$\begin{tikzcd}
\tilde X' \arrow[rr, "\tilde f'_1"] \arrow[dr, "\tilde T'"'] \arrow[dd, "\tilde f'_2"'] & & \hat X \arrow[dr, "\hat T"'] \arrow[dd, "f"] & \\
& \tilde X' \arrow[rr, "\tilde f'_1"] \arrow[dd, "\tilde f'_2"'] & & \hat X \arrow[dd, "f"] \\
X' \arrow[rr, "f'"] \arrow[dr, "T'"'] & & X \arrow[dr, "T"] & \\
& X' \arrow[rr, "f'"] & & X
\end{tikzcd}$$

By the hypothesis $\otimes P_{1*}$ annihilates $\pi_*(T') \otimes Z_p$ and by (i) P_{1*} annihilates $\pi_*(\hat T) \otimes Z_p$, $P_{1*} \cdot P_{2*}$ annihilates $\pi_*(T) \otimes Z_p$.

By the Mayer-Vietoris exact sequence one has

$$\begin{array}{ccccccccc}
\cdots \to & \pi_{n+1}(X) & \xrightarrow{\delta} & \pi_n(X') & \xrightarrow{\sigma} & \pi_n(\hat X) \oplus \pi_n(\tilde X) & \xrightarrow{\tau} & \pi_n(X) & \xrightarrow{\delta} \cdots \\
& \pi_{n+1}(T)\downarrow & & \downarrow \pi_n(\tilde T') & & \downarrow \pi_n(\hat T) \oplus \pi_n(T') & & \downarrow \pi_n(T) & \\
\cdots \to & \pi_{n+1}(X) & \xrightarrow{\delta} & \pi_n(\tilde X') & \xrightarrow{\sigma} & \pi_n(\hat X) \oplus \pi_n(X') & \xrightarrow{\tau} & \pi_n(X) & \xrightarrow{\delta} \cdots
\end{array}$$

and $\otimes P_{1*}\cdot P_{2*}$ annihilates $\pi_*(\tilde{T}')\otimes Z_p$. Apply (i) of this theorem to $\tilde{X}', \tilde{T}'$, $(\otimes P_{1*})$, P_{2*} to obtain:

$$\begin{array}{ccc} \hat{X}' & \xrightarrow{h} & \tilde{X}' \\ \hat{T}'\downarrow & & \downarrow \tilde{T}' \\ \hat{X}' & \xrightarrow{h} & \tilde{X}' \end{array}$$

$\pi_*(h)\otimes Z_p$ injective onto $\bigcup_r \ker P_{1*}^r[\pi_*(\tilde{T}')\otimes Z_p]$. Theorem A(ii) will follow if one can prove that $\tilde{f}_2'\circ h$ is a mod p equivalence. Now, $(\otimes P_{1*})$ annihilates $\pi_n(\hat{T})\otimes Z_p \oplus \pi_n(T')\otimes Z_p$, hence $\operatorname{im}\tau\otimes Z_p \subset \bigcup_r \ker(\otimes P_{1*})^r(\pi_n(T)\otimes Z_p)$ and as P_{2*} and $\otimes P_{1*}$ are relatively prime mod p $\operatorname{im}\tau\otimes Z_p\subset \bigcup_r \ker P_{1*}^r(\pi_*(T)\otimes Z_p) = \operatorname{im}\pi_*(f)\otimes Z_p$. The inclusion in the other direction is obvious,

$$\operatorname{im}\tau\otimes Z_p = \operatorname{im}\pi_*(F)\otimes Z_p \quad \text{and} \quad \ker(\tau\otimes Z_p)\overset{\text{proj}}{\to}\pi_*(X')\otimes Z_p$$

is an isomorphism. Consequently, $\pi_*(\tilde{f}_2')\otimes Z_p$: $\pi_*(\tilde{X}')\otimes Z_p\to\pi_*(X')\otimes Z_p$ is surjective and its kernel is isomorphic to

$$\begin{aligned} \ker\sigma\otimes Z_p &= \operatorname{im}\delta\otimes Z_p = \operatorname{coker}\tau\otimes Z_p \\ &= \pi_*(X)\Big/\bigcup_r \ker\left(P_{1*}^r\pi_*(T)\otimes Z_p\right) \approx \bigcup_r \ker P_{2*}^r\left(\pi_*(T)\otimes Z_p\right). \end{aligned}$$

It follows that P_{2*} annihilates $\pi_*(\tilde{T}')\otimes Z_p/\operatorname{im}\delta\otimes Z_p$. Thus, the exact sequence $0\to\operatorname{im}\delta\otimes Z_p\to\pi_*(\tilde{X}')\otimes Z_p\to\pi_*(X')\otimes Z_p\to 0$ corresponds to the 3.2.1 splitting with respect to $\pi_*(\tilde{T}')$, $\otimes P_{1*}$ and P_{2*}, and

$$\pi_*(\hat{X}')\otimes Z_p \overset{\pi_* h\otimes Z_p}{\to} \pi_*(\tilde{X}')\otimes Z_p \overset{\pi_*(\tilde{f}_2')\otimes Z_p}{\to} \pi_*(X')\otimes Z_p$$

is an isomorphism.

3.4. Corollary. *$\hat{X}$, $\hat{T}$ of Theorem* A(i) *is unique up to* mod p *equivalence, i.e.: If X', T', f'; T': $X'\to X'$; f': $X'\to X$ satisfy: $f'\circ T'\sim T\circ f'$, $\pi_*(f')\otimes Z_p$ injective onto $\bigcup_r \ker P_{1*}^r(\pi_*(T)\otimes Z_p)$, then there exist $\hat{X}', \hat{T}', \hat{f}_1', \hat{f}_2'$; $\hat{T}'$: $\hat{X}'\to\hat{X}'$; $\hat{f}_1'$: $\hat{X}'\to\hat{X}$; $\hat{f}_2'$: $\hat{X}'\to X'$; $\hat{T}\circ\hat{f}_1'\sim\hat{f}_1'\circ\hat{T}'$; $T'\circ\hat{f}_2'\sim\hat{f}_2'\circ\hat{T}'$ with f_i'* mod p *equivalences.*

Proof. One applies Theorem A(ii) to obtain $\hat{X}', \hat{T}', \hat{f}_1', \hat{f}_2'$; $\hat{f}_2'$ a mod p equivalence, Now $\hat{f}_1'$ is a mod p equivalence as

$$\operatorname{im}\pi_*(f\circ\hat{f}_1')\otimes Z_p = \operatorname{im}\pi_*(f'\circ\hat{f}_2')\otimes Z_p = \operatorname{im}\pi_*(f')\otimes Z_p$$

and

$$\pi_*(f\circ\hat{f}_1')\otimes Z_p\colon \pi_*(\hat{X}')\otimes Z_p\to\operatorname{im}\pi_*(f')\otimes Z_p$$

and $\pi_*(f')\otimes Z_p$: $\pi_*(X')\otimes Z_p\to\operatorname{im}\pi_*(f')\otimes Z_p$ are isomorphisms.

3.5. Corollary. *Let X, T, P_1, P_2 be as in Theorem* A *and suppose X is p-local. If $T'\colon X \to X$ (homotopy) commutes with T, then T' induces a map $\hat{T}'\colon \hat{X} \to \hat{X}$ (where $\hat{X}$, $\hat{T}$, f are the constructed space and maps of Theorem* A(i)) *so that the following cube commutes*:

$$\begin{tikzcd}
& \hat{X} \arrow[rr, "f"] \arrow[dl, "\hat{T}"'] \arrow[dd, "\hat{T}'" near start] & & X \arrow[dl, "T"'] \arrow[dd, "T'"] \\
\hat{X} \arrow[rr, "f" near start, crossing over] \arrow[dd, "\hat{T}'"'] & & X & \\
& \hat{X} \arrow[rr, "f" near start] \arrow[dl, "\hat{T}"'] & & X \arrow[dl, "T"] \\
\hat{X} \arrow[rr, "f"] & & X \arrow[from=uu, "T'"' near start, crossing over] &
\end{tikzcd}$$

Proof. Apply the p-local version of Theorem A(ii) with $X', T', f' = \hat{X}, \hat{T}, T' \circ f$ to obtain a factorization $T' \circ f = f' = f \circ \hat{f}'$, $\hat{T} \circ \hat{f}' \sim \hat{f}' \circ T' = \hat{f}' \circ \hat{T}$. Now $\hat{f}' = \hat{T}'$ is the desired map.

4. Applications and examples.

4.1. *Nilpotent groups.* Although we have restricted our considerations to simply connected spaces, the main theorems hold for nilpotent spaces as well. Applying them to $K(G,1)$, G a finitely generated nilpotent group, one obtains some purely group theoretic observations (which could be proved by purely algebraic considerations):

Theorem A*. *Let $T\colon G \to G$ be an endomorphism of a finitely generated nilpotent group G. Given polynomials $P_1, P_2 \in Z[x]$ with leading coefficients prime to p and suppose the* mod p *reductions of $\otimes P_1$ and P_2 are relatively prime and that $(\otimes P_1) \cdot P_2$ annihilates*

$$\bigoplus_i \left[(\Gamma_i(T)/\Gamma_{i+1}(T)) \otimes Z_p \right]$$

($\Gamma_{i+1} \subset \Gamma_i$ and $\Gamma_i(T)\colon \Gamma_i \to \Gamma_i$ are the central series of G and T). Then there exists a T-invariant subgroup $\hat{G}$ of G so that

$$\left[\Gamma_i(\hat{G})/\Gamma_{i+1}(\hat{G})\right] \otimes Z_p \to \left[\Gamma_i(G)/\Gamma_{i+1}(G)\right] \otimes Z_p$$

is injective and its image is

$$\bigcup_r \ker P_1^r\left(\Gamma_i/\Gamma_{i+1}(T) \otimes Z_p\right).$$

Moreover, given a homomorphism $f'\colon G' \to G$, an endomorphism $T'\colon G' \to G'$ so that $f' \circ T' = T \circ f'$ and $\otimes P_1$ annihilates $H_1(T') \otimes Z_p$, then $f'(G') \subset \hat{G}$. (G' is not required to be nilpotent as one can replace it by $\operatorname{im} f' \subset G$ *and the hypothesis remains valid.)*

One has a simple finite procedure to obtain $\hat{G}$, as follows: Let $\hat{G} \subset H_1(G)$ represent $\bigcup_r \ker(\otimes P_1)^r(H_1(T) \otimes Z_p) \subset H_1(G) \otimes Z_p$ and $G_1 = r_0^{-1}(\hat{G}_1)$ (r_0: $G \to H_1(G)$). Inductively, if T-invariant subgroups $G_t \subset G_{t-1} \subset \cdots \subset G_0 = G$ are constructed, let $\hat{G}_{t+1} \subset H_1(G_t)$ represent $\bigcup_r \ker(\otimes P_1)^r(H_1(T|G_t) \otimes Z_p) \subset H_1(G_t) \otimes Z_p$ and let $G_{t+1} = r_t^{-1}\hat{G}_{t+1}$ (r_t: $G_t \to H_1(G_t)$). One can see that if G is nilpotent of order N, then $G_{N+t} = G_N$ for $t \geqslant 0$ and $G_N = \hat{G}$ is the desired subgroup.

THEOREM B*. *Given a central extension of a nilpotent group G_0 with endomorphisms*:

$$\begin{array}{ccccccccc} 0 & \to & C & \xrightarrow{\sigma} & G & \xrightarrow{\tau} & G_0 & \to & 1 \\ & & \downarrow S & & \downarrow T & & \downarrow T_0 & & \\ 0 & \to & C & \xrightarrow{\sigma} & G & \xrightarrow{\tau} & G_0 & \to & 1 \end{array}$$

Suppose $P_1, P_2 \in Z[x]$ are as in Theorem A*: $\otimes P_1$ *annihilates $H_1(T_0) \otimes Z_p$, $(\otimes P_1) \cdot P_2$ annihilates $S \otimes Z_p$. Then there exists a commutative diagram*

$$\begin{tikzcd} \tilde{C} \arrow[d, "\hat{S}"'] & G \arrow[l] \arrow[r] \arrow[d, "T"] \arrow[rr, bend left, "\tau"] & \hat{G}_0 \arrow[r] \arrow[d, "\hat{T}_0"] & G_0 \arrow[d, "T_0"] \\ \tilde{C} & G \arrow[l] \arrow[r] & \hat{G}_0 \arrow[r] & G_0 \end{tikzcd}$$

so that $G \to \hat{G}_0 \times \hat{C}$ is a mod p *isomorphism*, $\otimes P_1$ *annihilates $H_1(\hat{T}_0) \otimes Z_p$ P_2 annihilates $\hat{S} \otimes Z_p$.*

4.2. *H-spaces.*

4.2.1. THEOREM A_H. *Suppose in Theorem* A *one has the added assumption that X is an H-space. Then for the "indecomposables" functor Q,*

$$QH^*(f, F_p) \colon QH^*(X, F_p) \to QH^*(\hat{X}, F_p)$$

corresponds to the (*split*) *projection*

$$QH^*(X, F_p) \to QH^*(X, F_p) \Big/ \bigcup_r \ker P_{2*}^r QH^*(T, F_p).$$

Moreover, $H^*(\hat{X}, F_p)$ is isomorphic to the subalgebra of $H^*(X, F_p)$ generated by $\bigcup_r \ker P_{1*}^r[QH^*(T, F_p)] \subset QH^*(X, F_p)$.

PROOF. Because the claim is a mod p claim, and it suffices to prove it to be valid only up to an arbitrary but finite dimension, one may assume that X is p-local, $\pi_n(x) = 0$ for $n > N$ and $P_{1*} = P_1$, $P_{2*} = P_2 \in Z[x]$.

One can form $T' = P_1(T)$: $X \to X$. $\pi_*(T')$ is nilpotent on $\bigcup_r \ker P_1^r(\pi_*(T))$, and as P_2, P_1 are relatively prime mod p, $\pi_*(T')$ is an isomorphism on $\bigcup_r \ker P_2^r(\pi_*(T))$. It follows that for some $\hat{P} \in Z[x]$, with $\hat{P}(0)$ prime to p, $x\hat{P}(x)$ annihilates $\pi_*(T')$.

Here $\otimes x$ and $\otimes \hat{P}$ are relatively prime mod p and one can apply Theorem A with both assignments $\{\otimes x, \otimes \hat{P}\} = \{P_1, P_2\}$. One thus obtains

$$\begin{array}{ccc} \hat{X}_0 & \xrightarrow{\hat{f}_0} & X \\ \hat{T}_0 \downarrow & & \downarrow T' \\ \hat{X}_0 & \xrightarrow{\hat{f}_0} & X \end{array} \qquad \begin{array}{ccc} \hat{X}_1 & \xrightarrow{\hat{f}_1} & X \\ \hat{T}_1 \downarrow & & \downarrow T' \\ X_1 & \xrightarrow{\hat{f}_1} & X \end{array}$$

$$\hat{f}_0 \text{ realizing } \bigcup_r \ker \pi_*(T')^r, \qquad \hat{f}_1 \text{ realizing } \bigcup_r \ker \hat{P}^r(\pi_*(T')).$$

Now T' and T commute and one can apply 3.4 and 3.5 to conclude that $\hat{X}_0, \hat{T}_0, \hat{f}_0 \approx \hat{X}, \hat{T}, f$ where $\hat{X}, \hat{T}, f$ are the Theorem A realization for P_1, P_2. As $\pi_*(X) \approx \bigcup_r \ker \pi_*(T')^r \oplus \bigcup_r \ker \hat{P}^r(\pi_*(T'))$ the map

$$\hat{X}_0 \times \hat{X}_1 \xrightarrow{\hat{f}_0 \times \hat{f}_1} X \times X \xrightarrow{\mu_X} X$$

is a homotopy equivalence and $QH^*(\mu_X \circ \hat{f}_0 \times \hat{f}_1, F_p)$ corresponds to the P_1, P_2 splitting of $QH^*(X, F_p)$ and Theorem $\mathrm{A_H}$ easily follows. We only add the remark that if T is an H-map one obtains a commutative diagram:

$$\begin{array}{ccc} \hat{X}_0 \times \hat{X}_1 & \underset{\approx}{\xrightarrow{\mu_X(\hat{f}_0 \times \hat{f}_1)}} & X \\ \hat{T}_0 \times T_1 \downarrow & & \downarrow T' \\ \hat{X}_0 \times \hat{X}_1 & \xrightarrow{\mu_X(\hat{f}_0 \times \hat{f}_1)} & X \end{array}$$

4.2.2. *The Eckmann-Hilton dual of* **[Cooke and Smith]**. Let X be an H-space, $\pi_n(X) = 0$ for $n > N$ and let $T\colon X \to X$. Then X admits a mod p splitting $\prod X_i \xrightarrow{\approx p} X$ corresponding to a splitting of the characteristic polynomial $P \in F_p[x]$ of $\pi_*(T) \otimes F_p$ into relatively prime factors, i.e: If $P = \prod_{i=1}^r P_i$ then $f_i\colon X_i \to X$, $i = 1, \ldots, r$, satisfy $\operatorname{im} \pi_*(f_i) \otimes F_p = \ker P_i(\pi_*(T) \otimes F_p)$.

Proof. It suffices to prove 4.2.2 for $r = 2$. Suppose $P = P_1 \cdot P_2$, $(P_1, P_2) = 1$. We shall show that T can be replaced by $T'\colon X \to X$, $\pi_*(T') \otimes F_p = \pi_*(T) \otimes F_p$ and $\pi_*(T')$ is annihilated by $\tilde{P}_1 \cdot \tilde{P}_2 \in Z[x]$, where $\tilde{P}_i \in Z[x]$, represent P_i, $i = 1, 2$.

Once this is proved one can apply Theorem A and 4.2.1 and its proof for $\tilde{P}_1(T')\colon X \to X$ and the realization $X_1 \to X$ of $\bigcup_r \ker[\pi_*(\tilde{P}_1(T')) \otimes Z_p]^r$ represents $\ker P_1(\pi_*(T) \otimes F_p)$. To construct T', note first the following algebraic consideration:

4.2.2.1. Given an endomorphism $T\colon G \to G$ of a finitely generated abelian group, if the characteristic polynomial of $T \otimes F_p$ is $P_1 \cdot P_2$, $(P_1, P_2) = 1$, then for every integer t there exists $T^{(t)}\colon G \to G$ satisfying:

(a) For some $q \equiv 1 \bmod p$, $qT - T^{(t)} = t\phi$, $\phi \in \operatorname{End}(G)$.

(b) There exist integral representations $\tilde{P}_i \in Z[x]$ of P_i, $i = 1, 2$, so that $\tilde{P}_1 \cdot \tilde{P}_2$ annihilates $T^{(t)}$.

Proof of 4.2.2.1. One can replace t by any multiple of it; thus, one may assume $t = p^s q$, $q \equiv 1 \bmod p$, and q is a multiple of the exponent of the $\neq p$ torsion subgroup of G. Let $G_0, T_0 = G/\text{torsion}, T/\text{torsion}$, respectively, ρ: $G \to G_0$ the projection, χ: $G_0 \to G$ an arbitrary right inverse of ρ. $P_1 \cdot P_2$ is a product of the characteristic polynomials $P^{(0)}$, $P^{(p)}$ of $T_0 \otimes F_p$ and ${}^pT \otimes F_p$, respectively; thus, $P_i = P_i^{(0)} \cdot P_i^{(p)}$, $i = 1, 2$, $P_1^{(0)} \cdot P_2^{(0)} = P^{(0)}$ and $P_1^{(p)} \cdot P_2^{(p)} = P^{(p)}$. Let $\hat{P}_i^{(0)} \in Z[x]$ represent $P_i^{(0)}$. $\hat{P}_1^{(0)} \cdot \hat{P}_2^{(0)}$ annihilates $T_0 \otimes Z/p^sZ$ and, as $\hat{P}_1^{(0)}, \hat{P}_2^{(0)}$ are relatively prime mod p, $G_0 \otimes Z/p^sZ = \hat{G}_0^{(1)} \oplus \hat{G}_0^{(2)}$, $\hat{G}_0^{(i)} = \bigcup_r \ker[\hat{P}_i^{(0)}]^r(T_0 \otimes Z/p^sZ)$. This splitting could be lifted to $G_0 \approx G_0^{(1)} \oplus G_0^{(2)}$, $G_0^{(i)} \otimes Z/p^sZ = \hat{G}_0^{(i)}$. While $\hat{G}_0^{(i)}$ are $T_0 \otimes Z/p^sZ$-invariants, $G_0^{(i)}$ are not necessarily T_0-invariants; but one can define $\hat{T}_0$: $G_0 \to G_0$,

$$\hat{T}_0 = \hat{T}_0^{(1)} \oplus \hat{T}_0^{(2)}, \quad \hat{T}_0^{(i)}: G_0^{(i)} \to G_0^{(i)}, \quad \hat{T}_0^{(i)} \otimes Z/p^sZ = T_0 \otimes Z/p^sZ|\hat{G}_0^{(i)}.$$

It follows that $\hat{T}_0 - T_0 = p^s\hat{\phi}$, $\hat{\phi} \in \operatorname{End} G$. If the characteristic polynomial of $\hat{T}_0^{(i)}$ is $\tilde{P}_i^{(0)}$ then $\tilde{P}_1^{(0)} \cdot \tilde{P}_2^{(0)}$ annihilates $\hat{T}_0$ and $\tilde{P}_i^{(0)}$ reduces to $P_i^{(0)} \bmod p$. Put $\hat{T} = T + \chi \circ \hat{T}_0 \circ \rho - \chi \circ T_0 \circ \rho$, then $\hat{T} - T = p^s\chi\hat{\phi} = p^s\phi$, $q\hat{T} - qT = t\phi$. $q\hat{T}$ is annihilated by $(q\tilde{P}_1^{(p)} \cdot \tilde{P}_1^{(0)}) \cdot (\tilde{P}_2^{(p)} \cdot \tilde{P}_2^{(0)})$ where $\tilde{P}_i^{(p)} \in Z[x]$ represent $P_i^{(p)}$. $T^{(t)} = q\hat{T}$, $\tilde{P}_1 = q\tilde{P}_1^{(p)} \cdot \tilde{P}_1^{(0)}$, $\tilde{P}_2 = \tilde{P}_2^{(p)} \cdot \tilde{P}_2^{(0)}$ are the desired endomorphisms and polynomials.

Apply 4.2.2.1 to $\pi_*(T)$: $\pi_*(X) \to \pi_*(X)$ to obtain $\tilde{T}_*$: $\pi_*(X) \to \pi_*(X)$, $\pi_*(qT) - \pi_*(\tilde{T}) = t\phi$. Following [**Zabrodsky$_2$**, Propositions 1.7, 1.8] for an appropriate t, $\tilde{T}_*$ is realizable by T': $X \to X$ with the desired properties.

4.3. *Realizations of polynomial algebras*. Let X be a CW complex, $H^*(X, F_p)$ a polynomial algebra on even-dimensional generators. Suppose T: $X \to X$ satisfies $QH^r(T, Q) = \lambda_r 1$, $\lambda_r \in Z$. As $P(x) = \prod(x - \lambda_r)$ annihilates $QH^*(T, Q)$ and as $H^*(X, Z)$ has no p-torsion and $H^*(X, F_p)$ is a free algebra, P annihilates $QH^*(T, Z_p)$. The same argument shows P annihilates $QH^*(\Omega T, Z_p)$.

By 1.4(C), P annihilates $\pi_*(\Omega T) \otimes Z_p$ and $\pi_*(T) \otimes Z_p$.

If $\lambda \in Z$, $\lambda^q \equiv 1(p)$ and $q|p-1$, one splits the set $\{\lambda_r\}$ into $\Lambda_0 = \{\lambda_r | \lambda_r \equiv \lambda^i \bmod p\}$, $\Lambda_1 = \{\lambda_r | \lambda_r \not\equiv \lambda^i \bmod p\}$ and then $P_{1*} = \prod_{\lambda_r \in \Lambda_0}(x - \lambda_r) P_{2*} = \prod_{\lambda_r \in \Lambda_1}(x - \lambda_r)$ satisfy the conditions of Theorem A. If f: $\hat{X} \to X$ realizes $\bigcup_r \ker P_{1*}^r(\pi_*(T) \otimes Z_p)$ by 4.2.1, one can compute $H^*(\hat{X}, F_p)$ as follows: Ωf: $\Omega\hat{X} \to \Omega X$ realizes $\bigcup_r \ker P_{1*}^r(\pi_*(T) \otimes Z_p)$ and as $H^*(\Omega X, F_p)$ is an exterior algebra on odd-dimensional generators, σ^*: $QH^*(X, F_p) \to QH^{*-1}(\Omega X, F_p)$ is an isomorphism and $H^*(\Omega\hat{X}, F_p)$, by 4.2.1, is an exterior algebra on $\bigcup_r \ker P_{1*}^r QH^*(\Omega T, F_p)$, $H^*(\hat{X}, F_p)$ is a polynomial algebra on $\ker P_{1*}QH^*(T, F_p) = \bigoplus QH^t(X, F_p)$, the sum taken over those t for which $\lambda_t \in \Lambda_0$. The natural examples are $X = BG$, G a compact simple Lie group $T = \psi_\lambda$: $BG \to BG$ Adams-Sullivan maps.

If $X = \mathrm{BSU}(n)$, $T = \psi_\lambda$ (λ representing an element of order q) $q|p-1$, in $F_p - \{0\}$, then $\hat{X}$ is the Quillen-Stasheff realization of $F_p[x_{2q}, x_{4q}, \ldots, x_{2mq}]$, $m = [n/q]$ (see [**Quillen**], [**Stasheff**]).

Similarly, one has

$$\psi_2: BE_8 \to BE_8, \quad p = 5, \quad \hat{X} \to BE_8,$$

realizing $\bigcup_r \ker(\pi_*(T) \otimes Z_p - 1)^r$ satisfies $H^*(X; F_5) = F_5[x_{16}, x_{24}, x_{40}, x_{48}]$ (not on the [**Clark and Ewing**] list).

Friedlander has constructed a map $\tilde{\psi}$: $(BF_4)_{1/2} \to (BF_4)_{1/2}$ for which Wilkerson computed $H^4(\tilde{\psi}, F_3)$ to be -1. If $X = BF_4$, $T = \tilde{\psi}$ one can see that $\hat{X}$ realizing $\bigcup_r \ker(\pi_*(T) \otimes Z_3 - 1)^r$ satisfies $H^*(\hat{X}, F_3) = F_3[x_{12}, x_{16}]$. $F_4 \approx \Omega\hat{X} \times X(3)$ corresponds to a well-known factorization (due to Harper, Cooke, Ewing and others). Note that $QH^{12}(BF_4, F_3) \to QH^{12}(\hat{X}, F_3)$ is zero; thus $H^*(BF_4, F_3) \to H^*(\hat{X}, F_3)$ is not surjective.

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