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THE AMALGAMATION PROPERTY FOR VARIETIES OF LATTICES

BY

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ABSTRACT. There are precisely three varieties of lattices that satisfy the amalgamation property: trivial lattices, distributive lattices, and all lattices.

1. Introduction. In [**5**], Jónsson showed that the variety of all lattices satisfied the (strong) amalgamation property and in [**7**] Pierce proved the similar (weak) version for distributive lattices. Grätzer, Jónsson and Lakser supplied the first negative results in [**3**] by showing that the only varieties of modular lattices that satisfied the amalgamation property were varieties of distributive lattices (there are two such varieties, $\mathcal{T} = \mathcal{L}(x = y)$ and $\mathcal{D}$). This result is crucial in that it forces N_5, the pentagon, into any nondistributive $\mathcal{V}$ satisfying (AP). Using this fact and the description of primitive lattices from Ježek and Slavík [**4**], Slavík, [**8**], showed that such a nondistributive variety, $\mathcal{V}$, satisfying (AP) must contain all primitive lattices. In this paper we complete the process started by Slavík, though by slightly different methods, and show that $\mathcal{V} = \mathcal{L}$.[1]

Slavík's approach involved ingenious arguments using his notion of A-decomposability. This notion defines when a lattice, L, has to be the amalgamation of two proper sublattices, S_1 and S_2, thus providing an inductive procedure to force larger lattices into $\mathcal{V}$. Our first result is a complete characterization of this important idea. Slavík then used A-decomposability on the construction procedures for primitive lattices to produce his results. We apply it to $\mathcal{B}$, the class of so-called bounded lattices introduced by McKenzie in [**6**]. By Day [**1**], we have that $\mathbf{HSP}(\mathcal{B}) = \mathcal{L}$ and by Day [**2**], all members of $\mathcal{B}$ are generated by the interval construction from the lattice, **1**. Our second result implies that for $L \in \mathcal{B}$, if $L \in \mathcal{V}$ then $L[I] \in \mathcal{V}$, and this completes the proof that $\mathcal{V} = \mathcal{L}$.

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2. A-decomposibility. In this section we characterize Slavík's important notion of:

(2.1) DEFINITION [**8**]. Let L be a finite lattice and let S_1 and S_2 be *proper* sublattices of L. L is called *A-decomposable* by means of S_1 and S_2 [Notation: $L = A(S_1, S_2)$] if $L = S_1 \cup S_2$ and for any lattice, Z, and lattice monomorphisms $f_i : S_i \rightarrowtail Z$ with $f_1 \restriction S_1 \cap S_2 = f_2 \restriction S_1 \cap S_2$, $f = f_1 \cup f_2$ is a lattice monomorphism from L into Z.

Since the variety of all lattices satisfies (AP) by [**5**], the free amalgamation of $S_1 \cap S_2 \hookrightarrow S_i$, $i = 1, 2$, always exists. If $L = A(S_1, S_2)$ then L is this

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free amalgamation in $\mathcal{L}$. Moreover if $\mathcal{V}$ satisfies (AP) and $S_1, S_2 \in \mathcal{V}$, we obtain $L = A(S_1, S_2) \in \mathcal{V}$.

Given lattices S_1 and S_2 with $S = S_1 \cap S_2$, there are three relatively simple ways to amalgamate the diagram

$$\begin{array}{ccc} S & \hookrightarrow & S_1 \\ \downarrow & & \\ S_2 & & \end{array}$$

The most familiar method is, by [**5**], using the McNeil Completion of $(S_1 \cup S_2, \sqsubseteq)$ where $x \sqsubseteq y$ in $S_1 \cup S_2$ iff there exists a $z \in S$ with $x \le z$ in S_i and $z \le y$ in S_j for some $i, j \in \{0, 1\}$. Since $\mathrm{MC}(S_1 \cup S_2, \sqsubseteq)$ preserves existing joins and meets, there are lattice monomorphisms $f_i \colon S_i \rightarrowtail \mathrm{MC}(S_1 \cup S_2, \sqsubseteq)$ with $f_1 \restriction S = f_2 \restriction S$. This provides us with our first necessary condition for $L = A(S_1, S_2)$.

(2.2) LEMMA. *If $L = A(S_1, S_2)$, then S_1 and S_2 satisfy $O(S_1, S_2)$: $x \le y, x \in S_i$ and $y \in S_j$ imply there exists $z \in S$ with $x \le z$ and $z \le y$.*

The McNeil Completion is not the only way in which $(S_1 \cup S_2, \sqsubseteq)$ can be completed. Let $\mathrm{IC}(S_1, S_2)$ be the set of all (S_1, S_2)-ideals. That is: $I \in \mathrm{IC}(S_1, S_2)$ if (1) $x \in I$ and $y \sqsubseteq x$ imply $y \in I$ and (2) $x, y \in I \cap S_i$ imply $x \vee_i y \in I$. Clearly the intersection of (S_1, S_2)-ideals is again such and therefore $\mathrm{IC}(S_1, S_2)$ is indeed a lattice. It is easy to check that $f_i \colon S_i \to \mathrm{IC}(S_1, S_2)$ by $f_i(x) = {\downarrow}x = \{y \in S_1 \cup S_2 \colon y \sqsubseteq x\}$, $i = 1, 2$, are lattice monomorphisms with $f_1 \restriction S = f_2 \restriction S$. Dually we can define the (S_1, S_2)-filter completion, $\mathrm{FC}(S_1, S_2)$.

We need to describe certain joins and meets of $\mathrm{IC}(S_1, S_2)$ in the special case where $L = S_1 \cup S_2$ for sublattices S_1 and S_2 satisfying $O(S_1, S_2)$. Clearly ${\downarrow}x \cap {\downarrow}y = {\downarrow}x \wedge y$ for all $x, y \in L$. In order to calculate ${\downarrow}x \vee {\downarrow}y$ we need a technical lemma.

(2.3) LEMMA. *For $x \in S_1 \backslash S_2$ and $y \in S_2 \backslash S_1$, there exists $x' \in S_1$ and $y' \in S_2$ such that ${\downarrow}x \vee {\downarrow}y = {\downarrow}x' \vee {\downarrow}y'$ and there exists $z \in S$ with $z \le x' \wedge y'$.*

PROOF. Take $x \in S_1 \backslash S_2$ and $y \in S_2 \backslash S_1$. Since $x \wedge y \in L$ we have $x \wedge y \in S_1$ or $x \wedge y \in S_2$. Without loss of generality assume $x \wedge y \in S_1$. By $O(S_1, S_2)$, there exists $u \in S$ with $x \wedge y \le u \le y$ and by definition $x' = x \vee u \in {\downarrow}x \vee {\downarrow}y$. Now $x' \in S_1$ and by defining $y' = y$ we get ${\downarrow}x \vee {\downarrow}y = {\downarrow}x' \vee {\downarrow}y'$.

(2.4) DEFINITION. For $i \in \{1, 2\}$ and $S_i' = S_i \cup \{0, 1\}$ define $\alpha_i \colon L \to S_i'$ by $\alpha_i(x) = \bigvee\{u \in S_i' \colon u \le x\}$.

Note that for $x \in S_1 \backslash S_2$ and $y \in S_2 \backslash S_1$ we have:

(1) $x \wedge y \le \alpha_2(x) < x$ or $x \wedge y \le \alpha_1(y) < y$,

(2) $\alpha_2(x) < z \le x$ implies $z \in S_1 \backslash S_2$,

(3) $\alpha_2(x) \in \{0, 1\} \cup S$ (by 2.2).

(2.5) LEMMA. *For $x \in S_1 \backslash S_2$ and $y \in S_2 \backslash S_1$ define $x_0 = x$, $y_0 = y$ and $x_{n+1} = x_n \vee \alpha_1(y_n)$, $y_{n+1} = y_n \vee \alpha_2(x_n)$. Then ${\downarrow}x \vee {\downarrow}y = \bigcup\{{\downarrow}x_n \cup {\downarrow}y_n \colon n \ge 0\}$.*

PROOF. Easy induction gives $x_n \in S_1$ and $y_n \in S_2$ for each n as well as $x_n, y_n \in {\downarrow}x \vee {\downarrow}y$. However the union is clearly an (S_1, S_2)-ideal.

If a is covered by b in L we call a (resp. b) a lower (resp. upper) neighbour of b (resp. a). We let $\mathrm{LN}(a)$ (resp. $\mathrm{UN}(a)$) be the set of all lower (resp. upper) neighbours of a.

(2.6) THEOREM. *Let L be a finite lattice with $L = S_1 \cup S_2$ for proper sublattices S_1 and S_2 and define $S = S_1 \cap S_2$. L is A-decomposable by means of S_1 and S_2 if and only if S_1 and S_2 also satisfy:*

$O(S_1, S_2)$: *$x \in S_i$, $y \in S_j$ and $x \le y$ imply the existence of $z \in S$ with $x \le z$ and $z \le y$.*

$\mathrm{LN}(S_1, S_2)$: *For all $x \in S$, $\mathrm{LN}(x) \subseteq S_1$ or $\mathrm{LN}(x) \subseteq S_2$.*

$\mathrm{UN}(S_1, S_2)$: *For all $x \in S$, $\mathrm{UN}(x) \subseteq S_1$ or $\mathrm{UN}(x) \subseteq S_2$.*

PROOF. Assume firstly that $L = A(S_1, S_2)$. By Lemma (2.3), $O(S_1, S_2)$ holds and by definition the map $x \mapsto {\downarrow}x$ of L into $\mathrm{IC}(S_1, S_2)$ must be a lattice monomorphism. If there existed an element $u \in S$ with $x \in \mathrm{LN}(u) \backslash S_2$ and $y \in \mathrm{LN}(u) \backslash S_1$, ${\downarrow}x \cup {\downarrow}y$ would be an (S_1, S_2)-ideal (observe that $S_2 \cap ({\downarrow}x \cup {\downarrow}y) \subseteq {\downarrow}y$) and therefore ${\downarrow}x \vee {\downarrow}y \neq {\downarrow}(x \vee y)$. Therefore $\mathrm{LN}(S_1, S_2)$ and dually $\mathrm{UN}(S_1, S_2)$ hold.

Conversely assume the three conditions hold and take lattice monomorphisms $f_i \colon S_i \rightarrowtail Z$ with $f_1 \restriction S = f_2 \restriction S$. We must show $g = f_1 \cup f_2$ is a monomorphism.

Claim 1. $x < y$ implies $g(x) < g(y)$.

Assume $x \in S_1 \backslash S_2$ and $y \in S_2 \backslash S_1$. By $O(S_1, S_2)$ there is a $z \in S$ with $x \le z \le y$. Moreover one of these inequalities must be strict. Therefore $g(x) \le g(z) \le g(y)$ with one of these inequalities strict, hence $g(x) < g(y)$.

Claim 2. $g(x \vee y) = g(x) \vee g(y)$.

We need only consider the case where $x \in S_1 \backslash S_2$ and $y \in S_2 \backslash S_1$. By easy induction we obtain $g(x) \vee g(y) = g(x_n) \vee g(y_n)$ for all n.

Now if $x_n, y_n < x \vee y$ for all n, there exists k with $x_{k+1} = x_k \in S_1 \backslash S_2$ and $y_{k+1} = y_k \in S_2 \backslash S_1$. Therefore $\alpha_1(y_k) \le x_k$ and $\alpha_2(x_k) \le y_k$. By 2.4(1) we get $x_k \wedge y_k \in \{\alpha_1(y_k), \alpha_2(x_k)\}$. Hence $x_k \wedge y_k \in S$ by 2.4(3). Now 2.4(2) supplies the contradiction to $\mathrm{UN}(S_1, S_2)$. Therefore for some n, $x_n = x \vee y \ge y_n$ and $g(x) \vee g(y) = g(x \vee y)$.

Claim 3. $g(x \wedge y) = g(x) \wedge g(y)$.

By duality.

Therefore $g \colon L \to Z$ is indeed a lattice monomorphism and $L = A(S_1, S_2)$.

The above characterization makes it trivial to obtain certain properties of A-decomposable lattices.

(2.7) COROLLARY. *If $L = A(S_1, S_2)$ and $S_i \le T_i < L$, $i = 1, 2$, then $L = A(T_1, T_2)$.*

(2.8) COROLLARY. *If there exists $0 < a \le b < 1$ in L with $L = {\uparrow}a \cup {\downarrow}b$, then $L = A({\uparrow}a, {\downarrow}b)$.*

3. $\mathcal{V} = \mathcal{L}$. Let $\mathcal{V}$ be a nondistributive variety of lattices satisfying the amalgamation property. By [**3**] we have $N_5 \in \mathcal{V}$. We wish to show $\mathcal{V} = \mathcal{L}$.

(3.1) LEMMA. *Let L be a finite lattice with $I = [u, v] \le L$. If there exists $\theta \in \mathrm{Con}(L)$ with $I = [I]\theta$, then $L[I]$ is a sublattice of a product of L and $L/\theta[I/\theta]$.*

PROOF Let $\psi = \mathrm{Ker}\, f$ for the canonical $f \colon L[I] \twoheadrightarrow L$ and define $\overline{\theta} \in \mathrm{Con}(L[I])$ by $x\overline{\theta}y$ iff $x, y \in L \backslash I$ and $x\theta y$ or $x, y \in I \times \{i\}$ and $f(x)\theta f(y)$ for $i \in \mathbf{2}$. Easy calculations show that $\overline{\theta} \in \mathrm{Con}(L[I])$, $L[I]/\overline{\theta} \le (L/\theta)[u/\theta, v/\theta]$ and $\psi \wedge \overline{\theta} = \Delta_{L[I]}$.

(3.2) COROLLARY. *If L is semidistributive, $0 \prec u \leq v \prec 1$, and $I = [u,v]$, then $L[I]$ is a sublattice of a product of L and N_5.*

PROOF. Let $\kappa(u) = \bigvee\{x \in L : x \wedge u = 0\}$ and $\lambda(v) = \bigwedge\{y \in L : y \vee v = 1\}$. Then we have a homomorphism $f: L \to \mathbf{2}^2$ with congruence classes $[u,v]$, $[\lambda(v), \kappa(u)]$, $[0, v \wedge \kappa(u)]$ and $[u \vee \lambda(v), 1]$.

We would have liked a direct proof that $L \in \mathcal{B} \cap \mathcal{V}$ implies $L[I] \in \mathcal{B} \cap \mathcal{V}$ but this seems impossible. The following variation, however, does do the job.

(3.3) LEMMA. *For $L \in \mathcal{B} \cap \mathcal{V}$ and $I = [u,v] \leq L$, $i \in \mathbf{2}$, then $(L \times \mathbf{2})[(u,i),(v,i)] \in \mathcal{V}$ (and $\mathcal{B}$).*

PROOF. By induction on $|L|$. Assume $i = 1$. If $v = 1$, then $(L \times \mathbf{2})[(u,1),(v,1)] \leq L \times \mathbf{3} \in \mathcal{V}$. If $v < 1$, there is a co-atom m, $v \leq m \prec 1$, and for $p = \lambda(m)$, $L = \downarrow m \cup \uparrow p$. Therefore $L \times \mathbf{2}$ can be pictured as in Figure (i). Since $J = [(0,1),(m,1)]$ is a congruence class of the homomorphism $f: L \times \mathbf{2} \to \mathbf{2}^2$, we can double J to produce a lattice that is a subdirect product of $L \times \mathbf{2}$ and N_5, hence a lattice in $\mathcal{V}$. The congruence classes modulo the homomorphism $g: (L \times \mathbf{2})[J] \twoheadrightarrow N_5$ produce the diagram in Figure (ii). Again since B_0 is a congruence class of this lattice, we can double this interval to produce a lattice $M \in \mathcal{V}$ as in Figure (iii). Now let J be the interval I considered as lying in the congruence class labelled B in Figure (iii), and consider the lattice $M[J] = A \cup B_0 \cup B_1 \cup C \cup D \cup B[J]$. By defining $S_1 = A \cup B_0 \cup B_1 \cup C \cup D$ and $S_2 = B_0 \cup B_1 \cup B[J]$, we obtain $M[J] = A(S_1, S_2)$. Since S_1 is the lattice of Figure (ii), $S_1 \in \mathcal{V}$. Since $S_2 = A(B_0 \cup B[J], B_1 \cup B[J])$, $S_2 \in \mathcal{V}$ if and only if these two lattices belong to $\mathcal{V}$. But $B = [0,m]$ with $|B| < |L|$, and these two lattices are $B \times \mathbf{2}$ with the interval $I \times \{i\}$ split upstairs and downstairs respectively. By induction then $S_2 \in \mathcal{V}$ and hence $M[J] \in \mathcal{V}$. Since $(L \times \mathbf{2})[(u,1),(v,1)] \cong A \cup B[J] \cup C \cup D \leq M[J]$, this lattice is in $\mathcal{V}$.

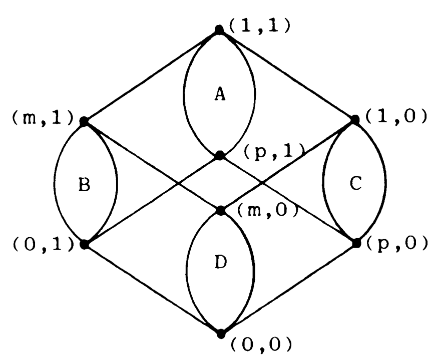

FIGURE (i)

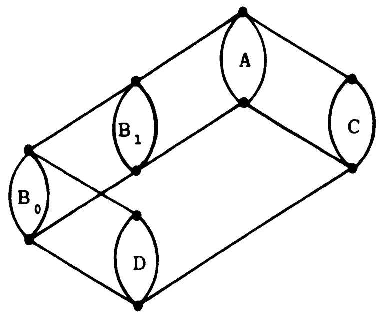

FIGURE (ii)

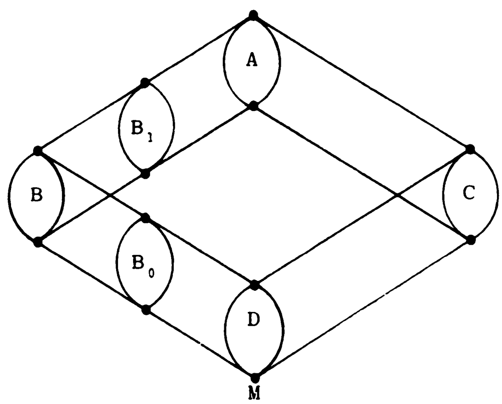

FIGURE (iii)

The proof for $i = 0$ follows by symmetry.

(3.4) THEOREM. *The only varieties of lattices that satisfy the amalgamation property are $\mathcal{T}$, $\mathcal{D}$, and $\mathcal{L}$.*

PROOF. If $\mathcal{V}$ is a nondistributive variety satisfying satisfying (AP), then by [**3**], $N_5 \in \mathcal{V}$. Lemma (3.3) implies that for every $L \in \mathcal{B}$, if $L \in \mathcal{V}$ then $L[I] \in \mathcal{V}$ since $L[I] \leq (L \times \mathbf{2})[I \times \{i\}]$. By [**2**], $\mathcal{B} \subseteq \mathcal{V}$ and by [**1**], $\mathcal{L} = \mathbf{HSP}(\mathcal{B}) \subseteq \mathcal{V}$.

Since our proof requires only that $N_5 \in \mathcal{V}$ it would be of interest to have an elementary proof (as opposed to [**3**]) that if $\mathcal{V}$ satisfies (AP) and $M_3 \in \mathcal{V}$ then $N_5 \in \mathcal{V}$. Such a proof is not known to the authors.

(3.5) COROLLARY. *$\mathcal{L}$ is the only variety of lattices satisfying the strong amalgamation property.*

PROOF. $\mathcal{D}$ does not satisfy (SAP). Whether or not $\mathcal{T}$ has the strong amalgamation property depends directly on whether or not the empty lattice, ϕ, is allowed. If $\phi \in \mathcal{T}$, then

$$\begin{tikzcd} \phi \arrow[r, hook] \arrow[d, hook] & \{x\} \\ \{y\} & \end{tikzcd}$$

has no strong amalgamation in $\mathcal{T}$.

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