

BILINEAR FORMS ON H^∞ AND BOUNDED BIANALYTIC FUNCTIONS

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ABSTRACT. Given an arbitrary Radon probability measure on the circle π , a generalization of the classical Cauchy transform is obtained. These projections are used to prove that each bounded linear operator from a reflexive subspace of L^1 or $L^1(\pi)/H^1$ into $H^\infty(D)$ admits a bounded extension. These facts lead to different variants of the cotype-2 inequality for $L^1(\pi)/H^1$. Applications are given to absolutely summing operators and the existence of certain bounded bianalytic functions. For instance, we derive the Hilbert space factorization of arbitrary bounded linear operators from $H^\infty(D)$ into its dual without an a priori approximation hypothesis, thus completing some of the work in [1]. Our methods give new information about the Fourier coefficients of $H^\infty(D \times D)$ -functions, thus improving a theorem in [6].

0. Introduction. The results of this paper are related to previous work in [1]. We are mainly concerned with lifting and extension properties, and their applications to absolutely summing operators on the disc algebra, and the existence of certain bianalytic functions.

One of the problems appearing in this framework is to find analogues of the classical Riesz-projection R satisfying L^p - L^p ($1 < p < \infty$) and L^1 - $L^{1,\infty}$ boundedness properties with respect to given measures. Since these measures are in general not weights, R itself cannot be used. In §1 it is shown that if Δ is a positive L^1 -function on the circle π , there exists an L^1 -function $\Delta_1 \geq \Delta$ and an "analytic" projection satisfying previously considered boundedness properties with respect to the measure $\Delta_1 \cdot dm$ ($m =$ Haar measure). This fact leads to a conceptual simplification of methods applied in [1] (see §2) to prove the cotype-2 and Grothendieck properties of L^1/H_0^1 . A way to formulate the cotype-2 property is that given any sequence (ϕ_j) of H^∞ -functions on the unit circle π satisfying the condition $\|\sum |\phi_j|^2\|_\infty \leq 1$, there exists a bounded function Φ on $\Omega \times \pi$ ($\Omega =$ Cantor group) which is H^∞ in the second variable and satisfies

$$\int_{\Omega} \Phi(\varepsilon, \theta) \varepsilon_j d\varepsilon = \phi_j(\theta) \quad \text{for each } j$$

(denoting the j th Rademacher function by ε_j). An explicit construction of the function Φ does not seem to be known. This fact means equivalently that if Y is the subspace of $L^1(\Omega)$ spanned by the Rademacher function and T is any bounded linear operator from Y into H^∞ , there exists a linear extension \hat{T} of T from $L^1(\Omega)$ into H^∞ , $\|\hat{T}\| \leq \text{const} \|T\|$. We generalize this principle to arbitrary reflexive

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subspaces Y of either L^1 (§2) or L^1/H_0^1 (§5). The L^1 -case provides for instance the inequality ($\|\cdot\|_*$ denotes the L^1/H_0^1 -norm)

$$\text{const} \int \left\| \sum \gamma_j(\omega)x_j \right\|_* d\omega \geq \sup \sum |\langle x_j, \phi_j \rangle|,$$

where (γ_j) are independent p -stable variables ($1 < p \leq 2$), (x_j) is a sequence in L^1/H^1 , and the supremum is taken over all H^∞ -sequences (ϕ_j) such that $\|\sum |\phi_j|^{p'}\| \leq 1$ ($p' = p/(p-1)$).

This fact leads to results concerning $(q, 1)$ -summing operators on the disc algebra which were not covered in [1]. They allow us to prove that bounded linear equators from $H^\infty(D)$ into any Banach space which has finite cotype is q -integral for some $q < \infty$. As a corollary the projective tensor algebra $H^\infty(D) \hat{\otimes} H^\infty(D)$ is closed in $L^\infty(\pi) \hat{\otimes} L^\infty(\pi)$. Notice that we obtain the H^∞ -version of U. Haagerup's theorem on C^* algebras (see [3]), although the approximation problem for H^∞ is still unsolved at the time this paper was written.

Since bounded operators from L^1/H_0^1 to H^∞ correspond to H^∞ -functions on the bidisc, the L^1/H_0^1 -extension result permits us to solve some interpolation problems for bianalytic functions (see §6). In this context these results seem the deepest obtained so far. The reader will find in §6 some related open questions and the failure of the so-called $(i_{p-\pi_p})$ -theorem for the bidisc algebra.

Most of this work was done while the author was visiting Mittag-Leffler Institute, which he thanks for their hospitality. The applications to summing operators arose partially from discussions with S. Kisliakov and G. Pisier at Paris shortly after. The reader is referred to [16 and 15] for the theory of absolutely summing and integral operators and to [2] for the basic 1-variable H^p -theory.

1. Existence of certain projections. If μ is a Radon probability measure on the unit circle π , let $H^p(\mu)$ denote the closure in $L^p(\mu)$ of the analytic polynomials ($0 < p < \infty$). Denote by m the Haar measure on π and let $d\mu = \Delta dm + d\mu_s$ be the Lebesgue decomposition of μ . As a consequence of peak-set theory ([1], e.g.), $H^p(\mu)$ decomposes as

$$H^p(\mu) = H^p(\Delta) \oplus L^p(\mu_s).$$

In this section we consider projections from $L^p(\mu)$ onto $H^p(\mu)$ for $1 < p < \infty$. The decomposition mentioned above reduces the problem to the case $d\mu = \Delta dm$, where Δ is in $L^1_+(\pi)$. In [15] the existence of a projection is shown for fixed $1 < p < \infty$ under the additional hypothesis that $\log \Delta$ is in $L^1(\pi)$. If Φ is an outer function with $|\Phi| = \Delta^{1/p}$ (on π), such a projection is given by the formula

$$P(\phi) = \Phi^{-1}R[\Phi\phi],$$

where R is the Riesz projection. Clearly this projection depends on p . In fact, for the questions in which we are interested, it suffices for given Δ in $L^1(\pi)$ to construct good projections with respect to some measure $\Delta_1 \cdot dm$, where $\Delta \leq \Delta_1$ and $\int \Delta_1 dm \leq C \int \Delta dm$. Now Δ_1 can be chosen such that $\Delta_1^{1/2}$ satisfies Muckenhoupt's A_1 -condition (see [14]) and, taking $|\Phi| = \Delta_1^{1/2}$, it can be shown that the projection defined above is p -bounded for all $1 < p \leq 2$ (see [7, §2]).

Several problems relative to the theory of absolutely summing operators on the disc algebra require in addition the $L^1(\Delta_1)$ - $L^{1,\infty}(\Delta_1)$ boundedness of the projection. The above operator P does not satisfy this property in general, and the

purpose of this section is to exhibit such projections. In [1] the cotype-2 and Grothendieck properties of L^1/H_0^1 were proved using operators (at least implicitly) satisfying a weak-type property but with respect to distinct measures. Using similar methods, the following fact will be shown.

THEOREM 1.1. *Given Δ in $L^1_+(\pi)$, $\int \Delta = 1$, there exist $\Delta_1 \geq \Delta$, $\int \Delta_1 \leq C$, and a projection P from $L^2(\Delta_1)$ onto $H^2(\Delta_1)$ which is $L^p(\Delta_1)$ - $L^p(\Delta_1)$ bounded for $1 < p < \infty$ and $L^1(\Delta_1)$ - $L^{1,\infty}(\Delta_1)$ bounded.*

In what follows, H^p -functions will always be considered as functions on π . C will be used to denote various constants. The proof of Theorem 1 depends on the following proposition.

PROPOSITION 1.2. *Assume f in $L^1(\pi)$, $f \geq 0$ and $\int f = 1$. Fix an integer $r \geq 1$. Then there exist positive scalars (c_i) and H^∞ -functions (θ_i) , (τ_i) and that*

1. $\|\theta_i\|_\infty \leq 1$ for each i ;
2. $\|\sum_i |\tau_i|\|_\infty \leq C$;
3. $\sum_i \theta_i \tau_i^r = 1$ a.e.

Defining $F = \sum c_i |\tau_i|$, we have

4. $f \leq F$,
5. $|\tau_i| F \leq C c_i$ for each i ,
6. $\|F\|_1 < C$,

where C is a constant depending on r .

The role of the power r will appear later. We assume $r > 2$ for convenience. Proposition 1 will be derived from

LEMMA 1.3. *If f is as in Proposition 1.2, there are H^∞ -functions $(\phi_i)_{i=1,2,\dots}$ and G in $L^1(\pi)$ satisfying*

1. $\|\sum |\phi_i|^{1/r}\|_\infty < C$,
2. $\sum \phi_i = 1$,
3. $f \leq G$,
4. $\sum M^i |\phi_i|^3 \leq G$,
5. $|\phi_i| G < C M^i$,
6. $\|G\|_1 < C$,

where $M > 2$ and C are constants.

DEDUCTION OF PROPOSITION 1.2 FROM LEMMA 1.3. We will replace the functions ϕ_i by functions $\hat{\phi}_i$ satisfying (1) and (2) of Lemma 1.3 and

- 4'. $\sum M^i |\hat{\phi}_i|^{1/r} \leq CG$,
- 5'. $|\hat{\phi}_i|^{1/r} G < C M^i$.

If one writes for each i ,

$$\hat{\phi}_i = \tau_i^r \theta_i \quad \text{with } |\theta_i| = 1 \text{ (on } \pi) \text{ and } \tau_i \text{ in } H^\infty,$$

then $\sum M^i |\tau_i| \leq CG$, and by (5'), (2),

$$C^{-1}G < \sum |\tau_i|^2 G < C \sum M^i |\tau_i|.$$

It follows by taking $c_i = C M^i$ (for some C) and $F = \sum c_i |\tau_i|$ that conditions (4)-(6) of the proposition hold.

We now show how (4') and (5') are realized. First, expand

$$1 = \left(\sum \phi_i\right)^{3r} = \sum_i \Phi_i \phi_i,$$

where

$$\Phi_i = \sum_{i_1, \dots, i_{3r-1} \geq i} \alpha_{i_1, \dots, i_{3r-1}} \phi_{i_1} \cdots \phi_{i_{3r-1}}$$

and

$$1 \leq \alpha_{i_1, \dots, i_{3r-1}} \leq (3r)!$$

Defining $\phi'_i = \Phi_i \phi_i$, clearly $|\phi'_i| \leq C|\phi_i|$ and hence (1) and (5) still hold. Moreover, for each i ,

$$|\phi'_i|^{1/r} \leq C \left(\sum_{j \geq i} |\phi_j|\right)^3, \quad C = C(r),$$

and hence, using (4) of (1.3),

$$\begin{aligned} \sum M^i |\phi'_i|^{1/r} &\leq C \sum_i \left(\sum_{j \geq i} M^{-(j-i)/3} M^{j/3} |\phi_j|\right)^3 \\ &\leq C \sum_{i \leq j} \sum M^{(j-i)/3} M^j |\phi_j|^3 \leq CG; \end{aligned}$$

thus (4') holds with $\hat{\phi}_i$ replaced by ϕ'_i . (5') is obtained similarly, writing

$$1 = \left(\sum \phi'_i\right)^r = \sum \Phi'_i \phi'_i,$$

where

$$\Phi'_i = \sum_{i_1, \dots, i_{r-1} \leq i} \beta_{i_1, \dots, i_{r-1}} \phi'_{i_1} \cdots \phi'_{i_{r-1}}, \quad 1 \leq \beta_{i_1, \dots, i_{r-1}} \leq r^l,$$

and defining $\hat{\phi}_i = \Phi'_i \phi'_i$. Again $|\hat{\phi}_i| \leq C|\phi'_i|$ and (1), (2) and (4') are valid. Also

$$|\phi'_i|^{1/r} \leq C \sum_{j \leq i} |\phi'_j|, \quad |\hat{\phi}_i|^{1/r} G \leq C \sum_{j \leq i} |\phi'_j| G \leq CM^i.$$

PROOF OF LEMMA 1.3. We assume $\|f\|_\infty < \infty$. The general case then follows from a standard compactness argument left to the reader. Fix $0 < \varepsilon < \frac{1}{2}$, let $0 < \delta < \frac{1}{2}$ and $M > 2$ be constants to be made precise later, let j satisfy $f < M^j$, and let $A_i = [f \geq M^i]$ for $i = 0, 1, \dots, j$. Define inductively outer functions ψ_i with modulus given by

$$|\psi_j| = 1 - (1 - \varepsilon)\chi_{A_j}.$$

For $i = 0, \dots, j - 1$,

$$|\psi_i| = [1 - (1 - \varepsilon)\chi_{A_i}] \cdot [\sup\{1, \delta^{-1}|1 - \psi_{i+1}|, \dots, \delta^{-\{j-i\}}|1 - \psi_j|\}]^{-1}.$$

Since

$$\|1 - \alpha e^{iH(\log \alpha)}\|_2 \leq C(\|1 - \alpha\|_2 + \|\log \alpha\|_2), \quad H = \text{Hilbert transform},$$

it follows that

$$\|1 - \psi_i\|_2^2 \leq C(\log \varepsilon^{-1})^2 |A_i| + C \sum_{k=i+1}^j \delta^{-2(k-i)} \|1 - \psi_k\|_2^2.$$

Multiplication by M^i and summation yield

$$\sum_{i=0}^j M^i \|1 - \psi_i\|_2^2 \leq C(\log \varepsilon^{-1})^2 \sum M^i |A_i| + C \sum_{i < k} \sum \left(\frac{1}{\delta^2 M}\right)^{k-i} M^k \|1 - \psi_k\|_2^2.$$

Thus for $\delta^2 M$ large enough, we obtain the estimation

$$\sum M^i \|1 - \psi_i\|_2^2 < C(\log \varepsilon^{-1})^2 \sum M^i |A_i| < C(\log \varepsilon^{-1})^2.$$

By construction, $|\psi_i| < \varepsilon$ on A_i and

$$(*) \quad |\psi_i| |1 - \psi_k| < \delta^{k-i} \quad \text{for } k < i.$$

Define

$$G = M(1 - \varepsilon)^{-3} \sum M^i |1 - \psi_i|^3 + 1,$$

which clearly satisfies conditions (3) and (6) of the lemma. Take

$$\begin{aligned} \phi_{j+1} &= 1 - \psi_j, \\ \phi_j &= \psi_j(1 - \psi_{j-1}), \\ &\vdots \\ \phi_1 &= \psi_j \psi_{j-1} \cdots \psi_1(1 - \psi_0), \\ \phi_0 &= \psi_j \psi_{j-1} \cdots \psi_0. \end{aligned}$$

Thus (2) and (4) hold. It follows from (*) that for $i < k$,

$$\max(|\psi_i|, |1 - \psi_k|) < \delta^{(k-i)/2}.$$

From this it is easily seen that for any $\rho > 0$,

$$\sum_{i=0}^{j+1} |\phi_i|^\rho \leq 2 + \sum_{s=0}^j \delta^{\rho s/2} < C_\delta.$$

It remains to verify (5). Fix $i = 0, \dots, j - 2$. Then by (*)

$$\begin{aligned} |\phi_i|G &< CM^i + C \sum_{k=i+3}^j M^k |\psi_i| |\psi_{i+1}| |\psi_{i+2}| |1 - \psi_k|^3 \\ &< CM^i + C\delta^{-3} \sum_{k=i+3}^i M^k \delta^{3(k-i)} \\ &= CM^i \left(1 + \delta^{-3} \sum_{k>i} (M\delta^3)^{k-i} \right) < CM^i, \end{aligned}$$

if, moreover, δ, M are chosen s.t. $\delta^3 M < 1/2$. This condition is compatible with the previous requirement that $\delta^2 M$ be large.

PROOF OF THEOREM 1.1. We apply to $f = \Delta + 1$, taking $r = 5$. Define

$$\Delta_1 = F, \quad P(\phi) = \sum_i \theta_i \tau_i^4 R[\tau_i \phi].$$

Clearly $P_0(\phi) = \sum' \theta_i \tau_i^4 R[\tau_i \phi]$ makes sense (as an $L^{1,\infty}$ -function) for finite sums \sum' and in $L^1(\pi)$. For $1 < p < \infty$ the $L^p(\pi)$ -boundedness of R gives (using (2) and (5) of Proposition 1.2)

$$\begin{aligned} \int |P_0 \phi|^p \Delta_1 &\leq C \int \sum' |\tau_i| |R[\tau_i \phi]|^p \Delta_1 \\ &\leq C \sum' c_i \int |R[\tau_i \phi]|^p \leq c_p \int |\phi|^p \sum c_i |\tau_i|. \end{aligned}$$

Thus, for ϕ in $L^p(\Delta_1)$, the series defining P converges in $L^p(\Delta_1)$ and

$$\int |P\phi|^p \Delta_1 \leq c_p \int |\phi|^p \Delta_1.$$

It is easily seen that $P(\phi)$ is in $H^p(\Delta_1)$ by using the inclusion $H^\infty \subset H^p(\Delta_1)$ and approximation of R by $R * P_\rho$ ($0 < \rho < 1$), where P_ρ is the Poisson kernel. Also, if ϕ is in $H^p(\Delta_1)$, it follows from Proposition 1(3) that $P\phi = \phi$.

It follows from the weak-type property of R that if $\alpha \in L^1(\pi)$ and $\beta \in L^\infty(\pi)$, then

$$\int |R[\alpha]|^{1/2} |\beta| dm \leq C \|\alpha\|_1^{1/2} \|\beta\|_1^{1/2} \|\beta\|_\infty^{1/2}.$$

Hence, for any ω in $L^1_+(\pi)$, $\|\omega\|_\infty = 1$, again by the conditions listed in (1.2),

$$\begin{aligned} \int |P_0 \phi|^{1/2} \omega \Delta_1 &\leq \int \sum' |\tau_i|^2 |R[\tau_i \phi]|^{1/2} \omega \Delta_1 \\ &\leq C \sum' c_i \int |R[\tau_i \phi]|^{1/2} |\tau_i| \omega \leq C \sum' c_i \|\tau_i \phi\|_1^{1/2} \|\tau_i \omega\|_1^{1/2}, \end{aligned}$$

which by the Cauchy-Schwarz inequality is dominated by

$$c \left\{ \int |\phi| \left(\sum' c_i |\tau_i| \right) \right\}^{1/2} \left\{ \int \omega \Delta_1 \right\}^{1/2}.$$

Specifying $\omega = \chi_{\{|P_0 \phi| > \lambda\}}$, there follows now

$$\|P_0(\phi)\|_{L^{1,\infty}(\Delta_1)} \leq C \int |\phi| \left(\sum' c_i |\tau_i| \right).$$

If $\phi \in L^1(\Delta_1)$, then $P\phi = \lim P_0\phi$ in $L^{1,\infty}(\Delta_1)$ and $\|P\phi\|_{L^{1,\infty}(\Delta_1)} \leq C \|\phi\|_{L^1(\Delta_1)}$. This completes the proof.

REMARKS. 1. The results of this section admit generalizations in the frame of Dirichlet and log modular algebras. For instance, one may consider the $H^\phi(\Omega)$ -spaces defined using holomorphic Brownian martingales in [19]. On the other hand, generalized Cauchy projections for the bidisc-algebra $A(D^2)$ need not exist (see remarks at the end).

2. It may be an interesting question to determine for what functions Δ_1 the conclusion of Theorem 1 is valid, i.e. there exists a projection with the required properties.

3. Defining $H^p_0(\mu)$ as the closure in $L^p(\mu)$ of the analytic trigonometric polynomials of mean zero, Theorem 1.1 can be restated for $H^p_0(\Delta_1)$ (in fact, one need only replace R by $R - \int \cdot dm$).

2. Extension of operators defined on reflexive subspaces of L^1 . Recall that a normed space Y has type p ($1 \leq p \leq 2$) provided the inequality

$$\int \left\| \sum \varepsilon_i y_i \right\| d\varepsilon \leq C \left(\sum \|y_i\|^p \right)^{1/p}$$

holds for all finite sequences (y_i) in Y and some constant C (ε_i is the sequence of Rademacher functions). The smallest C for which the above inequality holds is denoted $T_p(Y)$. Recall also that a linear operator on a Banach space X is p absolutely summing ($0 < p < \infty$) provided its p -summing norm

$$\pi_p(u) = \sup \left\{ \left(\sum \|u(x_i)\|^p \right)^{1/p} \mid \left(\sum |\langle x_i, x^* \rangle|^p \right)^{1/p} \leq 1, \|x^*\|_{X^*} \leq 1 \right\}$$

is finite. The Pietsch factorization theorem (e.g. [16]) states that an operator $u: X \rightarrow Y$ is p -summing if and only if there is a factorization of the form $X \xrightarrow{i} C(K) \xrightarrow{\delta} L_p(K, \mu) \xrightarrow{\beta} L^\infty(H)$ for the operator $j \circ u$, where u is a probability measure, i is an embedding, δ is the canonical inclusion, and $L^\infty(H)$ is any space with Y embedded, $j: Y \rightarrow L^\infty(H)$.

In the range $1 < p < \infty$, we want to know later that a p -summing map is called p -integral if the above factorization holds with $L^\infty(H)$ replaced by Y^{**} . It is well known that a subspace Y of L^1 is reflexive iff it has type p for some $1 < p \leq 2$. Y can then be embedded in L^r for any $r < p$ [17, 11]. This embedding is obtained by a change of density: There is a nonnegative mean-1 function Δ such that

$$(*) \quad \left(\int \left(\frac{|y|}{\Delta} \right)^r \Delta \right)^{1/r} \leq C' \int |y| \quad \text{for all } y \in Y.$$

The constant C' depends on p, r and $T_p(Y)$. Denote by $i: Y \rightarrow L^1$ the embedding operator and $i^*: L^\infty \rightarrow Y^*$ its adjoint. The existence of a change of density follows from the fact that i^* is r' -summing and from the Pietsch theorem. We include for completeness the estimate for the r' -summing norm.

LEMMA 2.1. $\pi_{r'}(i^*) \leq C_{p,r} T_p(Y)$ ($r' = r/(r - 1)$).

PROOF. Suppose (ϕ_j) is a sequence in L^∞ and (y_j) a sequence in Y such that

$$\left\| \sum |\phi_j|^{r'} \right\|_\infty \leq 1 \quad \text{and} \quad \sum \|y_j\|^r \leq 1.$$

If (γ_j) is a sequence of independent r -stable variables, we get

$$\begin{aligned} \sum |\langle \phi_j, y_j \rangle| &\leq \left\| \left(\sum |y_j|^r \right)^{1/r} \right\|_1 = \int \left\| \sum \gamma_j(\omega) y_j \right\|_1 d\omega \\ &\leq T_p(Y) \int \left(\sum \|y_j\|^p |\gamma_j(\omega)|^p \right)^{1/p} d\omega. \end{aligned}$$

If (δ_j) denotes a sequence of p -stable variables, the second factor in the right member is dominated by

$$\iint \left| \sum \|y_j\| \gamma_j(\omega) \delta_j(\omega') \right| d\omega d\omega' \leq \int \left(\sum \|y_j\|^r |\delta_j(\omega')|^r \right)^{1/r} d\omega' \leq C_{p,r}.$$

This proves the lemma.

Denote by $q: L^1(\pi) \rightarrow L^1/H_0^1$ the quotient map. The following lifting property holds.

PROPOSITION 2.2. Assume that Y is a reflexive subspace of a space $L^1(\Omega)$. say Y is of type $p > 1$. If (y_j) is a sequence in Y and (x_j) a sequence in L^1/H_0^1 , there exist $L^1(\pi)$ -functions (f_j) such that

$$q(f_j) = x_j \quad \text{for each } j,$$

and

$$\int \left\| \sum y_j(\omega) f_j \right\|_1 d\omega \leq C \int \left\| \sum y_j(\omega) x_j \right\| d\omega,$$

where C depends only on p and $T_p(Y)$.

PROOF. First, fixing some $1 < r < p$, we may assume the equivalence of the $L^1(\Omega)$ - and $L^r(\Omega)$ -norms on Y by changing the density. Further, since there will be uniform dependence of the constants, only finite sequences must be considered. Define

$$I = \inf \int \left\| \sum y_j(\omega) f_j \right\|_1 d\omega,$$

where the infimum is taken over all liftings.

Choose $(f_j) \subset L^1(\pi)$ realizing $2I$ and define Δ in $L_+^1(\pi)$ by

$$\Delta = I^{-1} \int \left\| \sum y_j(\omega) f_j \right\| d\omega.$$

Apply Theorem 1.1 to Δ , giving Δ_1 and the projection P onto $H_0^2(\Delta_1)$. Take an outer function Φ so that $|\Phi| = \Delta_1$ on π . Define

$$Q = \text{Id} - P \quad (Q \text{ annihilates } H_0^1),$$

$$f'_j = \Phi Q(\Phi^{-1} f_j) \quad \text{for each } j.$$

Since clearly $q(f'_j) = x_j = q(f_j)$, it follows that

$$I \leq \int \left\| \sum y_j(\omega) f'_j \right\|_1 d\omega.$$

Let $F_\omega = \sum y_j(\omega) f_j$ and G_ω an $L^1(\pi)$ -function satisfying $q(F_\omega) = q(G_\omega)$ for each ω . Take $0 < \alpha < 1$ and $0 < \theta < 1$ such that $1 = \theta\alpha + (1 - \theta)r$. By Hölder's inequality and the fact that $Q(\Phi^{-1} F_\omega) = Q(\Phi^{-1} G_\omega)$, we get

$$\begin{aligned} I &\leq \int \|Q(\Phi^{-1} G_\omega)\|_{L^1(\Delta_1)} d\omega \\ &\leq \left\{ \int \|Q(\Phi^{-1} G_\omega)\|_{L^\alpha(\Delta_1)} d\omega \right\}^{\theta\alpha} \left\{ \int \|Q(\Phi^{-1} G_\omega)\|_{L^r(\Delta_1)} d\omega \right\}^{1-\theta\alpha}. \end{aligned}$$

By the boundedness properties of P ,

$$\|Q(\Phi^{-1} G_\omega)\|_{L^\alpha(\Delta_1)} \leq C \|Q(\Phi^{-1} G_\omega)\|_{L^{1,\infty}(\Delta_1)} \leq C \|\Phi^{-1} G_\omega\|_{L^1(\Delta_1)} = C \|G_\omega\|_1$$

and

$$\|Q(\Phi^{-1} G_\omega)\|_{L^r(\Delta_1)} = \|Q(\Phi^{-1} F_\omega)\|_{L^r(\Delta_1)} \leq C_r \|\Phi^{-1} F_\omega\|_{L^r(\Delta_1)}.$$

Also, by hypothesis,

$$\int \|\Phi^{-1} F_\omega\|_{L^r(\Delta_1)} d\omega \leq \left\{ \iint |F_\omega|^r \Delta_1^{1-r} \right\}^{1/r} \leq CI \left\{ \int \Delta^r \Delta_1^{1-r} \right\}^{1/r}.$$

Collecting these inequalities, we obtain the estimate

$$I \leq C \left\{ \int \|G_\omega\|_1 d\omega \right\}^{\theta_\alpha} I^{1-\theta_\alpha}$$

and hence, by the definition of G_ω ,

$$I \leq C \int \left\| \sum y_j(\omega)x_j \right\| d\omega,$$

proving the proposition.

THEOREM 2.3. *Let Y be as in Proposition 2.2 and let $u: Y \rightarrow H^\infty$ be a bounded linear operator. Then u has an extension $\tilde{u}: L^1(\Omega) \rightarrow H^\infty$ with $\|\tilde{u}\| \leq C\|u\|$, where C depends on $p \geq 1$ and $T_p(Y)$.*

PROOF. Consider the subspace $Y \otimes L^1/H^1$ of $L^1(\Omega) \hat{\otimes} L^1/H_0^1$ on which we define the linear form $y \otimes x \rightarrow \langle u(y), x \rangle$. We claim that the linear form is bounded by $C\|u\|$. Let (y_j) (resp. (x_j)) be a finite sequence in Y (resp. L^1/H_0^1). Take liftings (f_j) in $L^1(\pi)$ satisfying Proposition 2.2; then

$$\begin{aligned} \int \left\| \sum y_j(\omega)x_j \right\| d\omega &\geq C^{-1} \int \left\| \sum y_j(\omega)f_j \right\|_1 d\omega \\ &\geq C^{-1} \|u\|^{-1} \int \left\| \sum f_j(\theta)u(y_j) \right\|_\infty d\theta \\ &\geq C^{-1} \|u\|^{-1} \sum \langle u(y_j), x_j \rangle, \end{aligned}$$

which proves the claim. Considering a Hahn-Banach extension, an element ξ in $L^\infty_{H^\infty}(\Omega)$ is obtained, where $\|\xi\| \leq C\|u\|$ and

$$\int \xi(\omega, \cdot)y(\omega) d\omega = u(y) \quad \text{for } y \text{ in } Y.$$

The extension \tilde{u} of u is then defined from ξ .

REMARKS. 1. For $1 < p \leq 2$, Theorem 2.3 implies in particular that any operator from a subspace Y of $L^p(\Omega)$ into H^∞ extends. This fact is equivalent to the $(i_p, i_{p'})$ -property of H^∞ (cf. [15, §9; and 9]). (A space X has the (i_p, π_p) -property if every p -summing map on X is p -integral.)

2. Theorem 2.3 does not hold for arbitrary subspaces Y of $L^1(\Omega)$. Take, for instance, $L^1(\Omega) = L^1(\pi)$ and $Y = H^1$. Define u by

$$u(y) = \sum_{j=1}^\infty j^{-1} \hat{y}(j) e^{ij\theta}.$$

Then u is bounded by Hardy's theorem, but does not admit an extension.

3. Applications. First we give the analogue of the cotype-2 inequality for L^1/H_0^1 , replacing the Rademacher functions by p -stable variables (in fact, we may consider any sequence in $L^1(\Omega)$ equivalent to the usual l^p -basis).

COROLLARY 3.1. *Denote by (γ_j) a sequence of independent p -stable variables on a probability space Ω ($1 < p \leq 2$). For (x_j) a sequence in L^1/H_0^1 ,*

$$c_p \sup \left| \sum \langle x_j, \phi_j \rangle \right| \leq \int \left\| \sum \gamma_j(\omega)x_j \right\| d\omega \leq \sup \left| \sum \langle x_j, \phi_k \rangle \right|,$$

where the supremum is taken over all sequences (ϕ_j) in H^∞ satisfying $\sum |\phi_j|^{p'} \leq 1$ and $p' = p/(p - 1)$.

PROOF. The left side follows from the lifting property stated in Proposition 2.2. For the right side, just observe that if $\xi \in L^\infty_{H^\infty}(\Omega)$ and $\phi_j = \int \xi(\omega, \cdot) \gamma_j(\omega) d\omega$, then

$$\left\| \left(\sum |\phi_j|^{p'} \right)^{1/p'} \right\|_\infty \leq \|\xi\|.$$

The next results are consequences of Theorem 2.3 and general operator ideal theory.

COROLLARY 3.2. 1. Assume that Y is a reflexive subspace of $L^1(\Omega)$ and $i: Y \rightarrow L^1(\Omega)$ is the embedding operator. If $u \in B(H^\infty, Y)$ and iu is nuclear, then u is nuclear. Moreover, $\nu_1(u) \leq C\nu_1(iu)$, where ν_1 denotes the nuclear norm and C depends on type and type-constant of Y .

2. If Y is a reflexive quotient of L^∞ , then $\Pi_1(Y, L^1/H_0^1) = N(Y, L^1/H_0^1)$. Related to (1), notice again the example of the identity operator $H^\infty \rightarrow H^1$ and the embedding $i: H^1 \rightarrow L^1(\pi)$. Then iu is integral, but u is not integral.

Corollary 3.1 permits us to extend certain properties of (p, q) -summing operators on $C(K)$ -spaces defined on H^∞ . Recall that a linear operator u between normed spaces E, F is (p, q) -summing provided that for some constant C and all finite sequences (x_j) in E ,

$$\left(\sum \|u(x_j)\|^p \right)^{1/p} \geq C \sup \left\{ \left(\sum |\langle x_i, x^* \rangle|^q \right)^{1/q}; x^* \text{ in } E^*, \|x^*\| \leq 1 \right\}$$

holds. The smallest constant satisfying the property is denoted $\pi_{p,q}(u)$.

In the sequel, u will be a linear operator from H^∞ into a Banach space Z . It will always be possible to assume that Z is finite dimensional and, by local reflexivity, that the range of adjoint operator u^* is contained in L^1/H_0^1 . The next fact was observed by G. Pisier (private communication).

COROLLARY 3.3. If $q > p \geq 2$ then $\pi_q(u) \leq C_{p,q} \pi_{p,2}(u)$.

PROOF. If (g_i) are 2-stable, we have

$$\int \left\| \sum g_i(\omega) u^*(z_i^*) \right\| d\omega \leq \pi_{p,2}(u) \left(\sum \|z_i^*\|^{p'} \right)^{1/p'}$$

for finite sequences (z_i^*) in Z^* .

Consider now sequences (ϕ_i) in H^∞ and (z_i^*) in Z^* such that $\sum |\phi_i|^q \leq 1$ and $\sum \|z_i^*\|^{q'} \leq 1$. If (γ_i) is a q' -stable sequence, Corollary 3.1 yields

$$\begin{aligned} \left| \sum \langle u(\phi_i), z_i^* \rangle \right| &\leq c_q \int \left\| \sum \gamma_i(\omega) u^*(z_i^*) \right\| d\omega \\ &\leq c'_q \pi_{p,2}(u) \int \left(\sum \|z_i^*\|^{p'} |\gamma_i(\omega)|^{p'} \right)^{1/p'} d\omega. \end{aligned}$$

The integral can be evaluated as in Lemma 2.1, using p' -stable variables. Hence the estimate $c_{p,q} \pi_{p,2}(u)$ follows.

In fact, this result has the following improvement (cf. [12]).

THEOREM 3.4. *If $q > p \geq 2$ then $\pi_q(u) \leq c_{p,q}\pi_{p,1}(u)$.*

From this fact the results on H^∞ -bilinear forms mentioned in the introduction will be derived. Of course one can assume $p > 2$. By Corollary 3.3 it will suffice to majorize $\pi_{p,2}(u)$. We proceed by the extrapolation method. To avoid problems of approximation by finite rank operators, however, we proceed on a bounded number of vectors. For fixed $n = 1, 2, \dots$, let $\pi_{p,q}^{(n)}(u)$ denote the (p, q) -summing norm of the operators $u: E \rightarrow F$ with respect to n vectors, i.e. the smallest constant $c > 0$ satisfying

$$\left(\sum_{i=1}^n \|u(x_i)\|^p \right)^{1/p} \leq c \sup_{\|x^*\| \leq 1} \left(\sum_{i=1}^n |\langle x_i, x^* \rangle|^q \right)^{1/q}$$

whenever $(x_i)_{i=1}^n$ in E . With this notation we have the following analogue to Corollary 3.3 (we continue to use the $'$ -notation for the conjugate index).

LEMMA 3.5. *If $p > r \geq 2$ then $\pi_{p,r}^n(u) \leq c_{p,r}\pi_{p,2}^{(n)}(u)$.*

PROOF. Again, by Corollary 3.1, $K_{p,s}(u) \leq \pi_{p,s}^{(n)}(u) \leq CK_{p,s}(u)$, where $K_{p,s}(u)$ is the smallest constant such that

$$(†) \quad \int \left\| \sum_{i=1}^n \gamma_i(\omega) u^*(z_i^*) \right\| d\omega \leq K_{p,s}(u) \left(\sum_{i=1}^n \|z_i^*\|^{p'} \right)^{1/p'}$$

for all finite sequences $(z_i^*)_{i=1}^n$ in Z^* , and (γ_i) is an s' -stable sequence. To estimate $K_{p,r}(u)$, consider r' -stable variables (γ_i) and introduce a Rademacher-average in the left member of (†) to obtain the majorization; since $p' < r'$,

$$K_{p,2}(u) \int \left(\sum \|z_i^*\|^{p'} |\gamma_i(\omega)|^{p'} \right)^{1/p'} d\omega \leq c_{p,r} K_{p,2}(u) \left(\sum \|z_i^*\|^{p'} \right)^{1/p'}$$

proving the lemma.

To interpolate, the following fact will be used, whose proof we momentarily postpone.

LEMMA 3.6. *Fix $r > 2$. For $\delta > 0$ there is a constant C_δ such that if (ϕ_i) is a sequence of H^∞ -functions satisfying $\sum |\phi_i|^2 \leq 1$, there is a decomposition of each ϕ_i as a sum of H^∞ -functions $\phi_i = \phi'_i + \phi''_i$, where $\sum |\phi''_i|^r \leq \delta^r$ and $\sum |\phi'_i| \leq C_\delta$.*

PROOF OF THEOREM 3.4. Fix $2 < r < p$. To estimate $\pi_{p,2}^{(n)}(u)$ consider a finite sequence $(\phi_i)_{i=1}^n$ of H^∞ -functions satisfying the hypothesis of Lemma 3.6. Choose $\delta > 0$ and consider a decomposition as in 3.6. Then

$$\begin{aligned} \left(\sum \|u(\phi_i)\|^p \right)^{1/p} &\leq \left(\sum \|u(\phi'_i)\|^p \right)^{1/p} + \left(\sum \|u(\phi''_i)\|^p \right)^{1/p} \\ &\leq C_\delta \pi_{p,1}^{(n)}(u) + \delta \pi_{p,r}^{(n)}(u) \end{aligned}$$

and, hence,

$$\pi_{p,2}^{(n)}(u) \leq C_\delta \pi_{p,1}^{(n)}(u) + \delta \pi_{p,r}^{(n)}(u).$$

Thus, by 3.5, it follows that $\pi_{p,2}^{(n)}(u) \leq C\pi_{p,1}^{(n)}(u)$ for $\delta > 0$ small enough. Since n is arbitrary, this means $\pi_{p,2}(u) \leq C\pi_{p,1}(u)$.

Lemma 3.6 is a consequence of the following (χ stands for indicator function).

LEMMA 3.7. Assume $B < \infty$ and (A_i) are measurable subsets of π satisfying $\|\chi_{A_i}\|_\infty \leq B$. For $\delta > 0$ there exist H^∞ -functions τ_i satisfying:

1. $|1 - \tau_i| < \delta$ on A_i for each i .
2. $\|\tau_i\|_\infty < C$ for each i .
3. $\|\sum |\tau_i|\|_\infty < C(B, \delta)$.

DEDUCTION OF 3.6 FROM 3.7. Defining $A_i = [|\phi_i| > \delta_i]$, it follows from the hypothesis of 3.6 that $\|\sum \chi_{A_i}\|_\infty < \delta_1^{-2}$. Apply 3.7 with δ replaced by δ_1 . For each i define $\phi'_i = \phi_i \tau_i$ and $\phi''_i = \phi_i(1 - \tau_i)$. Clearly, by (1) and (2) of 3.7, $\|\phi''_i\|_\infty < C\delta_1$ while, by (3) of 3.7, $\|\sum |\phi'_i|\|_\infty < C(\delta_1)$.

Consequently, $\sum |\phi''_i|^r < C\delta_1^{r-2}$ and it remains to choose δ_1 small enough.

The proof of 3.7 is essentially contained in [1] (see §4). It is based on the following result concerning interpolating sequences in the disc (see [1, Corollary 3.7] and [2] for basic theory).

LEMMA 3.8. Given $\eta > 0$ there is a constant $M(\eta)$ such that if E_1, E_2, \dots, E_N is a partition of an η -separated sequence in the disc, there are H^∞ -functions ψ_1, \dots, ψ_N satisfying:

1. $\psi_i(z) = 1$ for each z in E_i and $i = 1, \dots, N$.
2. $\|\psi_i\|_\infty \leq C$ for each $i = 1, \dots, N$.
3. $\|\sum |\psi_i|\|_\infty \leq M(\eta)$.

The interest of the lemma is that the constant C appearing in (2) does not depend on η . The next reasoning is well known and will therefore be presented without all of the details (cf. [1, Lemma 4.3]).

PROOF OF 3.7. We proceed by a separation argument in $(\sum_{i=1}^N L^\infty(A_i))_\infty$ (alternatively, a compactness argument could be used). Thus it must be shown that if for each $i = 1, \dots, N$ we take α_i in $L^1(\pi)$ supported on A_i and $\sum \|\alpha_i\|_1 \leq 1$, then there exist H^∞ -functions ψ_i ($1 \leq i \leq N$) verifying (2) and (3) of 3.7 such that

$$(*) \quad \sum_{i=1}^N \left| \int \alpha_i - \int \alpha_i \psi_i \right| < \delta.$$

Fix $\delta_1 > 0$. Standard approximation arguments (discretization of the Poisson integral) permit us to write

$$\left\| \alpha_i - \sum_{z \in E_i} a_z P_z \right\|_1 < \delta_1 \|\alpha_i\|_1 \quad (i = 1, \dots, N),$$

where E_1, \dots, E_N is a partition of an η -separated sequence E in the disc and $\eta = \eta(\delta_1, B)$. Consider the functions ψ_i given by 3.8. Then for $i = 1, \dots, N$,

$$\begin{aligned} \left| \int \alpha_i - \sum_{z \in E_i} a_z \right| &< \delta_1 \|\alpha_i\|_1, \\ \left| \int \alpha_i \psi_i - \sum_{z \in E_i} a_z \right| &< \delta_1 C \|\alpha_i\|_1 \quad ((2), (3) \text{ of } 3.8), \end{aligned}$$

and hence (*) holds for δ_1 small enough.

REMARK The estimate obtained for $C(B, \delta)$ in 3.7 is of the form $BC^{1/\delta}$ ($C =$ constant). This leads to an exponential dependence of $C_\delta = C(\delta)$ in Lemma 3.6. One can, however, fix some δ_0 and consider functions τ_i^0 given by 3.7. Fix some integer v and define $\tau_i = 1 - (1 - \tau_i^0)^v$. Then $|1 - \tau_i| < \delta_0^v$ on A_i , $\|\tau_i\|_\infty < C^v$ and $\|\sum |\tau_i|\|_\infty < C(\delta_0)C^v B$. For appropriate choices of δ_0 and v , one can thus ensure the conditions:

- 1'. $|1 - \tau_i| < \delta$ on A_i ;
 - 2'. $\|\tau_i\|_\infty < \delta^{-\kappa}$;
 - 3'. $\|\sum |\tau_i|\|_\infty < C\delta^{-\kappa} B$,
- where $\kappa > 0$ is any arbitrary fixed number.

For $1 \leq p \leq \infty$ define now for $F = (f_1, \dots, f_N)$,

$$\|F\|_p = \inf_{h_i \in H_0^1} \int \left(\sum |f_i + h_i|^p \right)^{1/p}.$$

If $1 \leq p < q < \infty$ and $1 - \theta = p/q$, one has, for $0 < \phi < \theta$, the interpolation inequality

$$\|F\|_q \leq C_\phi \|F\|_p^{1-\phi} \|F\|_\infty^\phi.$$

It is clear from the definition of the $\pi_{p,1}$ -norms that if $u: X \rightarrow Y$ is a bounded linear operator and Y has a finite cotype $p \geq 2$ constant, say $C_p(Y)$, then $\pi_{p,1}(u) \leq C_p(Y)\|u\|$. Combining this remark, Theorem 3.4 and the $(i_p-\pi_p)$ equivalence ($1 < p < \infty$) for operators defined on H^∞ (see [5]), the next corollary follows.

COROLLARY 3.9. *If Y is an arbitrary Banach space of cotype $p \geq 2$, then $B(H^\infty, Y) = I_q(H^\infty, Y)$ whenever $q > p$. This means that any bounded linear operator u from $H^\infty(D)$ to Y factors through some identity $L^\infty(\mu) \rightarrow L^q(\mu)$ for some Borel probability measure μ on the spectrum of H^∞ . In view of [10], one can restate the result by saying that any u in $B(H^\infty, Y)$ has a bounded linear extension \tilde{u} in $B(L^\infty(\pi), Y)$, considering $H^\infty(D)$ as subspace of $L^\infty(\pi)$.*

Since L^1/H_0^1 has the cotype-2 property and H^∞ identifies with its dual space, the local reflexivity principle implies that $(H^\infty)^*$ also has cotype-2. Hence, since any operator from an L^∞ space into a space with cotype-2 is 2-summing.

COROLLARY 3.10. *$B(H^\infty, (H^\infty)^*) = \pi_2(H^\infty, (H^\infty)^*)$ and, in particular, those operators factor through Hilbert space. Equivalently, bounded bilinear forms on $H^\infty(D)$ are the restrictions of bounded bilinear forms on $L^\infty(\pi)$.*

Let us give some formulations of the latter fact in terms of projective tensor algebras. If A, B are commutative Banach algebras, the Banach space projective tensor product $A \hat{\otimes} B$ again has a natural structure of a commutative Banach algebra and is called the projective tensor algebra of A and B . More details on this can be found in [8] and in the introduction of [10].

COROLLARY 3.11. *If i denotes the natural imbedding of $H^\infty(D)$ into $L^\infty(\pi)$, then $i \hat{\otimes} i$ is an isomorphic embedding of the projective tensor algebra $H^\infty(D) \hat{\otimes} H^\infty(D)$ into $L^\infty(\pi) \hat{\otimes} L^\infty(\pi)$. Thus $H^\infty(D) \hat{\otimes} H^\infty(D)$ is a closed subalgebra of $L^\infty(\pi) \hat{\otimes} L^\infty(\pi)$.*

Notice that Corollary 3.11 is an *isomorphic*, and not an *isometric*, result, as pointed out in [4]. Since the identity map from $L^\infty(\pi) \hat{\otimes} L^\infty(\pi)$ to the injective tensor product $L^\infty(\pi) \hat{\otimes} L^\infty(\pi)$ is one-to-one, (3.11) implies

COROLLARY 3.12. *The natural map associating to an element of $H^\infty(D) \hat{\otimes} H^\infty(D)$ a bounded bianalytic function on the bidisc, i.e. an element of $H^\infty(D \times D)$, is one-to-one.*

Notice that this result would be formal if we knew that $H^\infty(D)$ has the bounded approximation property.

COROLLARY 3.13. *If the two-variable function induced by an element ζ of $L^\infty(\pi) \hat{\otimes} L^\infty(\pi)$ belongs to $H^\infty(D \times D)$, then ζ is an element of $H^\infty(D) \hat{\otimes} H^\infty(D)$. More correctly, if $(f_\alpha), (g_\alpha)$ are sequences in $L^\infty(\pi)$, $\sum \|f_\alpha\|_\infty \cdot \|g_\alpha\|_\infty < \infty$, and $\sum f_\alpha(\theta)g_\alpha(\psi)$ is the boundary value of a bounded bianalytic function on $D \times D$, then there is a representation*

$$\sum f_\alpha(\theta)g_\alpha(\psi) = \sum F_\alpha(e^{i\theta})G_\alpha(e^{i\psi}) \quad \text{for } \theta, \psi \in \pi,$$

where $F_\alpha, G_\alpha \in H^\infty(D)$ and $\sum \|F_\alpha\| \|G_\alpha\| < \infty$.

PROOF. By (3.11), if ζ is not in $H^\infty \hat{\otimes} H^\infty$, there is a bilinear form β on L^∞ which annihilates the cross product $H^\infty \otimes H^\infty$ and $\langle \zeta, \beta \rangle \neq 0$. By Grothendieck's theorem, β extends to a bounded bilinear form on $L^2(\mu)$ for some probability measure μ on the spectrum of $L^\infty(\pi)$. Using a suitable finite rank orthogonal projection on $L^2(\mu)$, β can be replaced by an element β_1 of the cross-product $(L^\infty)^* \otimes (L^\infty)^*$ annihilating $H^\infty \otimes H^\infty$ such that $\langle \zeta, \beta_1 \rangle \neq 0$ still holds. From the reflexivity principle, β_1 may be chosen of the form $\beta_1 = \sum'_\tau u_\tau \otimes v_\tau$, with u_τ, v_τ in $L^1(\pi)$ and either $u_\tau \perp H^\infty$ or $v_\tau \perp H^\infty$ for each τ . But, clearly, this contradicts the assumption that ζ induces an element of $H^\infty(D \times D)$.

4. Interpolation inequalities involving Riesz projections. Let R_- denote the Riesz-projection on the (strictly) negative integers. The method of proving the next inequality is related to [6, §4]. However, the exponent θ will be important for our purposes.

PROPOSITION 4.1. *If $1 < p < q$ and $1/p = \theta + (1 - \theta)/q$, then*

$$\|R_- \phi\|_p \leq C \|\phi\|_{1,\infty}^\theta \|R_-[\phi]\|_q^{1-\theta}$$

holds.

PROOF. Fix $\lambda > 0$. We truncate ϕ by multiplication with the outer function τ with boundary value

$$\tau = \alpha e^{iH|\log \alpha|} \quad \text{where } \alpha = (1 \vee \lambda^{-1}|\phi|)^{-1}.$$

Thus $\phi = \tau\phi + (1 - \tau)\phi$ and clearly

$$\begin{aligned} R_-[\phi] &= R_-[\tau\phi] + R_-[(1 - \tau)R_-[\phi]], \\ \|R_-[\phi]\|_p &\leq C_p \|\tau\phi\|_p + C_p \|(1 - \tau)R_-[\phi]\|_p, \\ \|\tau\phi\|_p^p &\leq \int_{\{|\phi| \leq \lambda\}} |\phi|^p + \lambda^p m\{|\phi| > \lambda\}; \end{aligned}$$

by Hölder's inequality

$$\|(1 - \tau)R_-[\phi]\|_p^p \leq \left\{ \int |1 - \tau|^{pq/(q-p)} \right\}^{1-p/q} \left\{ \int |R_-[\phi]|^q \right\}^{p/q}.$$

Also

$$\begin{aligned} |1 - \tau|^{pq/(q-p)} &\leq C|1 - \tau|^p \leq Cm[|\phi| > \lambda] + C_p \int_{\{|\phi| > \lambda\}} \log^p \frac{|\phi|}{\lambda} \\ &\leq C'_p \lambda^{-1} \|\phi\|_{1,\infty}. \end{aligned}$$

Collecting inequalities yields

$$\|R_-[\phi]\|_p^p \leq C_p \{ \lambda^{p-1} \|\phi\|_{1,\infty} + (\lambda^{-1} \|\phi\|_{1,\infty})^{1-p/q} \|R_-[\phi]\|_p^p \}.$$

Taking

$$\lambda = \|\phi\|_{1,\infty}^{-q'/q} \|R_-[\phi]\|_q^{q'}$$

gives the required result.

The proof of 4.1 also gives the following.

COROLLARY 4.2. *Let Ω be a probability space and ϕ be defined on $\pi \times \Omega$ equipped with the product measure. Denote by $R_-^{(1)}$ the R_- -projection with respect to the first variable. Then*

$$\|R_-^{(1)}[\phi]\|_p \leq C \|\phi\|_{1,\infty}^\theta \|R_-^{(1)}[\phi]\|_q^{1-\theta}.$$

Taking $\Omega = \pi$, m we have, from the weak-type property of the Riesz projection

COROLLARY 4.3. *Define $\Lambda = \{(m, n) \text{ in } \mathbf{Z} \times \mathbf{Z}; m \leq 0 \text{ and } n < 0\}$.*

1. *If K is the orthogonal projection $\hat{\Lambda}$, that is, $K = R_- \otimes R_-$, then for functions ϕ on $\pi \times \pi$,*

$$\|K[\phi]\|_p \leq C \|\phi\|_1^\theta \|K[\phi]\|_q^{1-\theta}.$$

2. *Assume $\text{Spec } \phi \subset \Lambda$. Then*

$$\|\phi\|_p \leq C \left(\inf_f \|\phi + f\|_1 \right)^\theta \|\phi\|_q^{1-\theta},$$

where the infimum is taken over functions f with $\text{Spec } f \subset (\mathbf{Z} \times \mathbf{Z}) \setminus \Lambda$. ($\text{Spec } \phi$ denotes the spectrum of ϕ .)

Corollary 4.3 can be generalized using the projections considered in §1.

PROPOSITION 4.4. *For $i = 1, 2$ consider Δ_i in $L^1_+(\pi)$, $\int \Delta_i = 1$. Let $\bar{\Delta}_i \geq 1 + \Delta_i$ be the L^1 -majorant and P_i the projection on $H^2_0(\Delta_i)$ obtained in Theorem 1.1. Define $Q_i = \text{Id} - P_i$ ($i = 1, 2$). Equip $\pi \times \pi$ with the measure $\bar{\Delta}_1(x) dx \bar{\Delta}_2(y) dy$ and let $Q = Q_1 \otimes Q_2$. Then*

$$\|Q[\phi]\|_p \leq C \left(\inf_f \|\phi + f\|_1 \right)^\theta \|\phi\|_q^{1-\theta},$$

where again the infimum is taken over functions f with $\text{Spec } f \subset (\mathbf{Z} \times \mathbf{Z}) \setminus \Lambda$.

PROOF. Taking $\Omega = \pi$, $\bar{\Delta}_2(y) dy$ in 4.2 and using the weak-type property of Q_2 ,

$$(\dagger) \quad \|R_-^{(1)}[Q_2\phi]\|_{L^p(\bar{\Delta}_2(y) dx dy)} \leq C \|\phi\|_{L^1(\bar{\Delta}_2(y) dx dy)}^\theta \|R_-^{(1)}[Q_2\phi]\|_{L^q(\bar{\Delta}_2(y) dx dy)}^{1-\theta}.$$

By the proof of Theorem 1.1, Q_1 is explicitly given by

$$Q_1[\alpha] = \sum \theta_i \tau_i^4 R_-[\tau_i \alpha].$$

Hence, for ϕ a function on $\pi \times \pi$,

$$Q[\phi] = \sum \theta_i(x) \tau_i(x)^4 R_-^{(1)}[\tau_i(x) Q_2[\phi]],$$

$$|Q[\phi]|^p \leq C \sum |\tau_i(x)| |(R_-^{(1)} \times Q_2)[\tau_i(x)\phi]|^p.$$

Continuing with the notations used in the proof of Theorem 1.1, it follows from (†) that

$$\iint |Q[\phi]|^p \bar{\Delta}_1(x) \bar{\Delta}_2(y) \leq C \sum c_i \iint |(R_-^{(1)} \otimes Q_2)[\tau_i(x)\phi]|^p \bar{\Delta}_2(y)$$

$$\leq C \sum c_i \|\tau_i(x)\phi\|_{L^1(\bar{\Delta}_2(y) dx dy)}^{p\theta} \|(R_-^{(1)} \otimes Q_2)[\tau_i(x)\phi]\|_{L^q(\dots)}^{p(1-\theta)}.$$

Notice that $\text{Spec } f \subset (\mathbf{Z} \times \mathbf{Z}) \setminus \Lambda$ implies

$$Q[f] = 0 = (R_-^{(1)} \otimes Q_2)[f].$$

By this observation and the $L^1(\bar{\Delta}_2)$ -boundedness of Q_2 ,

$$\|Q[\phi]\|_p^p \leq C \sum c_i \|\tau_i(x)(\phi + f)\|_{L^1(\bar{\Delta}_2(y) dx dy)}^{p\theta} \|\tau_i(x)\phi\|_{L^q(\dots)}^{p(1-\theta)}.$$

Since $p\theta + p(1 - \theta)/q = 1$, Hölder's inequality and the fact that $\bar{\Delta}_1 = \sum c_i |\tau_i|$ give

$$\|Q[\phi]\|_p^p \leq C \|\phi + f\|_1^{p\theta} \|\phi\|_q^{p(1-\theta)},$$

thus proving the proposition.

We end this section with some inequalities involving only L^p ($p \leq 1$) and $L^{1,\infty}$. They will not be applied later on.

PROPOSITION 4.5. *For $0 < p < 1$ we have*

$$\|R_-[\phi]\|_p \leq C_p \|\phi\|_{1,\infty}^{1/2} \|R_-[\phi]\|_1^{1/2}.$$

PROOF. Let τ be an H^∞ -function to be defined later. Clearly

$$R_-[\phi] = R_-[\phi\tau] + (1 - \bar{\tau})R_-[\phi] + R_-[(1 - \tau)\bar{\tau}R_-[\phi]] - R_-[(1 - \bar{\tau})\tau R_-[\phi]].$$

From Kolmogorov's inequality it follows that

$$(*) \quad \|R_-[\phi]\|_p \leq C_p \|\phi\tau\|_1 + C_p \|(1 - \tau)\bar{\tau}R_-[\phi]\|_1 + \|(1 - \tau)R_-[\phi]\|_p.$$

Fix a function $0 < \omega < 1$ and define inductively

$$\omega_0 = \omega, \quad \omega_{k+1} = |H[\omega_k]| \quad (H = \text{Hilbert transform}).$$

If we let $\omega_\infty = \sum_{k=0}^\infty 2^{-k} \omega_k$ then, by construction,

$$|H[\omega_\infty]| \leq 2\omega_\infty.$$

Moreover,

$$\|\omega_\infty\|_2 \leq 2\|\omega\|_2.$$

Choose $\omega \geq \sigma = \max(0, \log(|\phi|/\lambda))$ and let τ be the outer function with boundary value

$$\tau = e^{-\omega_\infty - iH[\omega_\infty]}.$$

Hence $|\tau| = e^{-\omega_\infty} \leq e^{-\sigma}$ and $|1 - \tau| \leq 3\omega_\infty$ on π . For this choice, (*) becomes

$$\begin{aligned} \|R_-[\phi]\|_p &\leq C_p \left\{ \int_{\|\phi\| \leq \lambda} |\phi| + \lambda m[\|\phi\| > \lambda] \right\} \\ &\quad + \left(\int |1 - \tau|^{p/1-p} \right)^{(1-p)/p} \|R_-[\phi]\|_1 + C_p \|\omega_\infty e^{-\omega_\infty} R_-[\phi]\|_1. \end{aligned}$$

Choosing $0 < \delta < 1$, the first term can be estimated by $\delta\lambda + (\log 1/\delta)\|\phi\|_{1,\infty}$. Fix a positive integer J . ω will be one of the functions $2^0\sigma, 2^1\sigma, \dots, 2^J\sigma$. Since for any positive number γ ,

$$\sum_{j=0}^J 2^j \gamma e^{-2^j \gamma} < 5,$$

an averaging argument shows that, for some choice of ω ,

$$\|\omega_\infty e^{-\omega_\infty} R_-[\phi]\|_1 < CJ^{-1} \|R_-[\phi]\|_1.$$

Without restriction, we may assume $p \geq 2(1 - p)$. Thus, by the definition of ω, σ ,

$$\begin{aligned} \int |1 - \tau|^{p/1-p} &\leq C \int |1 - \tau|^2 \leq C \int \omega_\infty^2 \leq C \int \omega^2 \\ &\leq C4^J \int \sigma^2 \leq C4^J \frac{\|\phi\|_{1,\infty}}{\lambda}. \end{aligned}$$

Collecting the previous inequalities, one gets for any $0 < \delta < 1$,

$$C_p^{-1} \|R_-[\phi]\|_p \leq \delta\lambda + \left(\log \frac{1}{\delta} \right) \|\phi\|_{1,\infty} + \left(J^{-1} + 2^J \left(\frac{\|\phi\|_{1,\infty}}{\lambda} \right)^{(1-p)/p} \right) \|R_-[\phi]\|_1.$$

Take $\lambda = K^J \|\phi\|_{1,\infty}$ and $\delta = K^{-J}$, where K is large enough. Thus

$$C_p^{-1} \|R_-[\phi]\|_p \leq J\|\phi\|_{1,\infty} + J^{-1} \|R_-[\phi]\|_1$$

and 4.5 follows for an appropriate choice of J .

Again, 4.5 applies to the double-Riesz transform $K = R_- \otimes R_-$ and yields

COROLLARY 4.6. For $0 < p < 1$, $\|K[\phi]\|_p \leq C_p \|\phi\|_1^{1/2} \|K[\phi]\|_1^{1/2}$ holds.

(4.6) has the following formal consequence.

COROLLARY 4.7. For functions Φ on $\pi \times \pi$,

$$\|K[\phi]\|_{1,\infty} \leq C \|\phi\|_1^{1/2} \|K[\phi]\|_1^{1/2}.$$

We rely on the following probabilistic lemma.

LEMMA 4.8. Assume that X is a random variable on a probability space Ω such that $\|X\|_{1,\infty} < \infty$. Then there exists an average Y of independent copies of the symmetrization X_s of X such that, for $0 < p < 1$,

$$C_p^{-1} \|X\|_{1,\infty} \leq \|Y\|_p \leq C_p \|X\|_{1,\infty}.$$

The proof is standard. Fix $\lambda < 0$ and use the fact that $\|Y\|_p$ dominates the L^p -norm of the square function of independent copies of $\lambda\chi_{\{|X_s| < \lambda\}}$. The number of copies is of order $\|X_s\|_{1,\infty}^{-1}$. We leave the details to the reader.

To deduce (4.7), let $\phi = \phi(x, y)$ be a polynomial on $\pi \times \pi$ and consider (almost) independent copies of ϕ of the form $\phi(x, y) = \phi(n_j x, n_j y)$, where (n_j) is a suitable sequence of positive integers. Observe that $K[\phi](n_j x, n_j y) = K[\phi_j](x, y)$. It remains to combine the left inequality in (4.8) with (4.6).

REMARK. The exponent $1/2$ in (4.6) and (4.7) is sharp. To see this, let P_r, Q_r ($0 < r < 1$) be the Poisson kernel and conjugate Poisson kernel, respectively. Take $\phi = P_r \otimes P_r$.

Thus

$$4K[\phi] = (\alpha - Q_r \otimes Q_r) + i(P_r \otimes Q_r + Q_r \otimes P_r).$$

A simple computation shows that ($P = m \times m$)

$$P[(|Q_r| \otimes |Q_r|) > (1 - r)^{-1}] \sim (1 - r) \log(1 - r)^{-1}$$

and, therefore,

$$\|K[\phi]\|_{1, \infty} \gtrsim \log(1 - r)^{-1} \quad \text{while} \quad \|K[\phi]\|_1 \sim (\log(1 - r)^{-1})^2.$$

5. Extension of operators defined on reflexive subspaces of L^1/H_0^1 . In this section we use the results of §4 to prove the following theorem.

THEOREM 5.1. *Assume Y is a reflexive subspace of L^1/H_0^1 of type $p > 1$ with constant $T_p(Y)$. Then, given a bounded linear operator u from Y into H^∞ , there is an extension \bar{u} of u to L^1/H_0^1 such that $\|\bar{u}\| \leq C\|u\|$, where the constant C only depends on p and $T_p(Y)$.*

Clearly (5.1) includes (2.3). Since the bounded operators from L^1/H_0^1 into H^∞ also corresponds to bounded analytic functions on the bidisc ($A(D \times D) = A(D) \hat{\otimes} A(D)$), Theorem 5.1 has following reformulation:

COROLLARY 5.2. *Take Y as in (5.1) and let $u \in B(Y, H^\infty)$. Then there exists Φ in $H^\infty(D \times D)$ satisfying*

$$\|\Phi\| \leq C\|u\|, \quad u(y)(\psi) = \int \Phi(\theta, \psi)y(\theta) d\theta \quad (y \text{ in } Y).$$

Applications of (5.2) will be given in the next section. Let us first recall the following lifting principle for reflexive subspaces of L^1/H_0^1 (see [1, §2] for the proof).

PROPOSITION 5.3. *Assume Y as in (5.1) and let $q: L^1 \rightarrow L^1/H_0^1$ be the quotient map. Then there exists a subspace \bar{Y} of $L^1(7)$ such that $q\bar{Y}$ is an isomorphism between \bar{Y} and Y , and, more precisely,*

$$\|\bar{y}\|_1 \leq C\|q(\bar{y})\| \quad \text{for } \bar{y} \text{ in } \bar{Y},$$

where $C = C(p, T_p(Y))$.

Let \bar{y} in \bar{Y} be the element corresponding to y in Y (thus $q(\bar{y}) = y$). Again take $\Lambda = \{(m, n) \text{ in } \mathbf{Z} \times \mathbf{Z}; m \leq 0, n \leq 0\}$. For ϕ in $L^1(\pi \times \pi)$, define

$$\begin{aligned} \|\phi\|_* &= \inf_{\text{Sp}(f) \cap \Lambda = \emptyset} \|\phi + f\|_1 \\ &= \sup \left\{ \iint \phi \xi; \xi \text{ in } H^\infty(D \times D) \text{ and } \|\xi\|_\infty \leq 1 \right\}. \end{aligned}$$

Proceeding as in §2, Theorem 5.1 will follow from

PROPOSITION 5.4. *Given (y_j) in Y and (x_j) in L^1/H_0^1 , there are functions (f_j) in $L^1(\pi)$ such that $q(f_j) = x_j$ for each j and*

$$(†) \quad \iint \left| \sum \bar{y}_j(\psi) f_j(\theta) \right| d\theta d\psi \leq C \left\| \sum x_j \otimes y_j \right\|_*.$$

PROOF. Choose $1 < q < r < p$, $q^{-1} = \theta + (1 - \theta)r^{-1}$. Take Δ_2 in $L_+^1(\pi)$, $\int \Delta_2 = 1$ such that

$$\int \left(\frac{\bar{y}}{\Delta_2} \right)^r \Delta_2 \leq C \|\bar{y}\|_1 \quad \text{for } \bar{y} \text{ in } \bar{Y}.$$

Let I be the infimum of the left member in (†) and $(f_j) \subset L^1(\pi)$ so that

$$\iint \left| \sum \bar{y}_j(\psi) f_j(\theta) \right| \leq 2I.$$

Define

$$\Delta_1(\theta) = I^{-1} \int \left| \sum \bar{y}_j(\psi) f_j(\theta) \right| d\psi.$$

Aply Proposition 4.4 to the pair Δ_1, Δ_2 . For $i = 1, 2$, let Φ_i in $H^1(D)$ be an outer function with $|\Phi_i|$ of boundary value $\bar{\Delta}_i$. For each j define

$$f'_j = \Phi_1 Q_1 [\Phi_1^{-1} f_j].$$

Since clearly $q(f_j) = q(f'_j)$,

$$I \leq \iint \left| \sum \bar{y}_j(\psi) f'_j(\theta) \right| d\theta d\psi.$$

Analogously, if we let $y'_j = \Phi_2 Q_2 [\Phi_2^{-1} \bar{y}_j]$, then $q(\bar{y}_j) = q(y'_j)$ and it follows by construction (fixing first θ) that

$$I \leq C \iint \left\| \sum y'_j(\psi) f'_j(\theta) \right\| d\theta d\psi.$$

Thus

$$I \leq C \iint |Q[\Phi_1^{-1}(\theta)\Phi_2^{-1}(\psi)F]| \bar{\Delta}_1(\theta)\bar{\Delta}_2(\psi) d\theta d\psi,$$

where $F(\theta, \psi) = \sum \bar{y}_j(\psi) f_j(\theta)$ and $Q = Q_1 \otimes Q_2$.

Application of (4.4) gives (with respect to the measure $(\Delta_1 \times \Delta_2) d\theta d\psi$ on π^2)

$$I \leq C \|\Phi_1^{-1}(\theta)\Phi_2^{-1}(\psi)F + f\|_1^\theta \|\Phi_1^{-1}(\theta)\Phi_2^{-1}(\psi)F\|_r^{1-\theta}$$

whenever $\text{Spec } f \subset (\mathbf{Z} \times \mathbf{Z}) \setminus \Lambda$.

Clearly f can be replaced by $\Phi_1^{-1}(\theta)\Phi_2^{-1}(\psi)f$ in the first factor. A suitable choice of f proves that

$$I \leq C \|F\|_*^\theta \left\{ \iint |F(\theta, \psi)|^r \bar{\Delta}_1(\theta)^{1-r} \bar{\Delta}_2(\psi)^{1-r} \right\}^{(1-\theta)/r}.$$

Exploiting (†), the second factor is dominated by

$$CI^{1-\theta} \left\{ \int \Delta_1(\theta)^r \bar{\Delta}_1(\theta)^{1-r} \right\}^{(1-\theta)/r}.$$

Therefore, $I \leq C \|F\|_*$, completing the proof.

6. Consequences. Recall that a subset Λ of $\mathbf{Z} \times \mathbf{Z}$ is a Λ_2 -set provided the L^1 - and L^2 -norms are equivalent on polynomials with spectrum contained in Λ .

The proof of the next fact is immediate from Corollary 5.2.

COROLLARY 6.1. *Assume that Λ is a Λ_2 -subset of the positive integers and that $(\phi_k)_{k \in \Lambda}$ is a sequence of H^∞ -functions which is weakly 2-summable, i.e.*

$$\sup_{z \in D} \sum_{k \in \Lambda} |\phi_k(z)|^2 < \infty.$$

Then there exists Φ in $H^\infty(D \times D)$ satisfying, for each k in Λ ,

$$\hat{\Phi}_\theta(k) = \int \Phi(\theta, \psi) e^{ik\theta} d\theta = \phi_k(\psi).$$

COROLLARY 6.2. *If $(f_j)_{j=0,1,\dots}$ is a sequence of L^1 -functions on π , then the operator from $H^\infty(D \times D)$ into l^1 mapping Φ into the sequence $|\langle \hat{\Phi}_\theta(j), f_j \rangle|_{j=0,1,2,\dots}$ is bounded iff*

$$(*) \quad \inf_{h_j \in H_0^1} \int \left(\sum |f_j + h_j|^2 \right)^{1/2} < \infty.$$

PROOF. The ‘‘if’’ part is trivial. Denote the Cantor group $\{1, -1\}^N$ by Ω and let Y be the Hilbertian subspace of $L^1_{(L^1/H_0^1)}(\Omega)$ generated by the functions $\varepsilon_j \otimes e^{-ij\theta}$ ($j = 0, 1, 2, \dots$). Since L^1/H_0^1 and its direct sum in the l^1 -sense, $(L^1/H_0^1) \oplus_1 (L^1/H_0^1) \oplus_1 \dots$, are isomorphic as Banach spaces, Corollary 5.2 applies as well to $L^1_{(L^1/H_0^1)}(\Omega)$, which has the same local structure as the L^1 -direct sum. Thus for any weakly 2-summable sequence (ϕ_j) in H^∞ , there is a function Φ in $L^\infty(\Omega \times D \times D)$ satisfying

$$\Phi(\varepsilon, \cdot) \text{ is } H^\infty(D \times D) \text{ for each } \varepsilon \text{ in } \Omega,$$

and

$$\int_{\Omega \times D} \Phi(\varepsilon, \theta, \psi) \varepsilon_j^{-ij\theta} d\theta d\varepsilon = \phi_j(\psi), \quad \text{for each } j.$$

Hence, for fixed ε ,

$$\sum_{j=0}^\infty \left| \iint \Phi(\varepsilon, \theta, \psi) e^{ij\theta} f_j(\psi) d\theta d\psi \right| \leq C \|\Phi\|_\infty.$$

Inserting ε_j in the j th term and integrating in ε ,

$$\sum_{j=0}^\infty |\langle f_j, \phi_j \rangle| \leq C \|\phi\|_\infty.$$

Taking the supremum of the left-member over suitable $(\phi_j) \subset H^\infty$ gives (*).

Next, we have, as a consequence of (6.2), S. Kisliakov’s characterization of the l^1 -multipliers on the bidisc-algebra $A(D \times D)$ (see [6]).

COROLLARY 6.3. *The norm of the l^1 -multiplier on $A(D \times D)$ gives by the matrix $(a_{mn})_{0 \leq m, n < \infty}$ is equivalent to $(\sum_{m, n} |a_{mn}|^2)^{1/2}$.*

PROOF. Fix a sequence $\varepsilon_n = \pm 1$ and let

$$f_m(\psi) = \sum_{n=0}^\infty \varepsilon_n a_{mn} e^{-in\psi} \quad \text{for each } m.$$

Then the operator-norm of $\Phi \rightarrow ((\hat{\Phi}_\theta(m), f_m))$ ranging in l^1 is bounded by the multiplier norm. By (6.2)

$$\sum_{m=0}^{\infty} \left\| \sum_{n=0}^{\infty} \varepsilon_n a_{mn} e^{-in\psi} \right\|^2 \leq C \quad (\| \cdot \| = L^1/H_0^1\text{-norm}),$$

and it remains to integrate with respect to ε , after using the weak-type minorization for the negative Riesz-transform of the L^1/H_0^1 -norm.

REMARKS. 1. (6.1)–(6.3) rely only on the weaker version of (5.1) for translation-invariant Y . In this setting one could in fact even impose on the bianalytic functions Φ the additional condition that for fixed ψ , $\Phi(\theta, \psi)$ has a uniformly convergent Fourier series in θ .

2. There are many problems related to the results we discussed before.

It is not known to what extent the Banach space properties of $A(D)$ generalize to $A(D \times D)$. One may ask the questions:

- PROBLEM 1. (a) Does $A(D \times D)^*$ have cotype-2?
- (b) Does $A(D \times D)^*$ have finite cotype-9? Thus

$$\int \left\| \sum \varepsilon_j f_j \right\|_* d\varepsilon \geq C \left(\sum \|f_j\|_*^q \right)^{1/q}$$

for all finite sequences (f_j) , where

$$\|f\|_* = \sup_{\substack{\phi \in A(D^2) \\ \|\phi\|_\infty \leq 1}} |\langle \phi, f \rangle| \quad \text{for } f \text{ on } \pi \times \pi.$$

(5.2) answers (a) affirmatively if one restricts to $A(D \times D)^*$ -elements of the form $\sum (x_\alpha \otimes y_\alpha)$, where $x_\alpha \in A(D)^*$ and y_α is taken in a fixed reflexive subspace of $A(D)^*$.

The constant appearing in the inequality depends (a priori) on the type and the type-constant of this reflexive space.

Despite the fact that $R_- \otimes R_-$ has no regularity with respect to the $L^1(\pi \times \pi)$ -norm, certain results for the bidisc-algebra could be obtained from the interpolation inequalities given in §4. Inequalities of this nature do not seem to be known for the 3-fold projection $R_- \otimes R_- \otimes R_-$. The problem of whether or not the analogue to Kisliakov's theorem on l^1 -multipliers holds for $A(D^3)$ is open. Thus

PROBLEM 2. (a) Define $K = R_- \otimes R_- \otimes R_-$. Is there an analogue of (4.3), (4.6) and (4.7)? Does there exist $0 < p < q < \infty$ and $\theta > 0$ such that $\|K[f]\|_p \leq C \|f\|_1^\theta \|K[f]\|_q^{1-\theta}$?

(b) Let $(a_{k,1,m})_{k,1,m \geq 0}$ be a positive matrix such that $\sum a_{k,1,m} |\hat{\phi}(k, 1, m)| \leq C \|\phi\|_\infty$ holds for ϕ in $A(D^3)$. Is it true that $\sum a_{k,1,m}^2 < \infty$?

Obviously, any affirmative version of (a) implies (b).

3. If X is a Banach space and $1 < p < \infty$, one defines $k_p(X) = \sup(i_p(u)/\pi_p(u))$, i.e. the ratio of the p -integral to the p -summing norm, where the supremum is taken over all linear operators u from X into an arbitrary Banach space. If $X = A(D)$ or $X = H^\infty(D)$, it is known that $k_p(X) \leq Cp^2/(p-1)$. The reader will find an exposition and applications of these results in [16]. In fact, this estimate on $k_p(X)$ for $X = A(D)$ is also a corollary to the projection theorem presented in §1. Our purpose here is to prove that if $X = A(D)^2$, then $k_p(X) = \infty$ whenever

$p \neq 2$. Of course, this implies the nonexistence of generalized Riesz operators for the polydisc-algebras in more than one variable. In particular, Theorem 1.1 will not generalize to algebras of more than one variable. The technique described below permits us to disprove the i_p - π_p property ($p \neq 2$) for the ball-algebras $A(B_n)$, $m > 1$. Details for the ball-algebra will appear elsewhere. Our argument shows the existence of an L^1 -function Δ on π^2 such that the natural map $A(D^2) \rightarrow H^p(\Delta \cdot d\sigma)$ ($\sigma = \text{Haar measure on } \pi^2$) is not p -integral whenever $p \neq 2$.

The first observation is

LEMMA 6.4. *Assume X is a linear subspace of a $C(K)$ -space ($K = \text{compact}$) and $k_p(X) < \infty$, where p is some value in $[1, \infty[$. Let $\varepsilon > 0$ and $(\phi_j)_{j=1}^n$ a sequence in the unit ball of X such that the sets $[|\phi_j| > \varepsilon] = \{t \in K \mid |\phi_j(t)| > \varepsilon\}$ are mutually disjoint. Then there exists a decomposition $\phi_j = \phi'_j + \phi''_j$, ϕ'_j and ϕ''_j in X , satisfying*

- (i) $\|\phi'_j\|_\infty \leq 3k_p(X)n^{\max(2,p)^{-1}}\varepsilon$,
- (ii) $\|\sum |\phi''_j|\|_\infty \leq 3k_p(X)n^{\max(2,p')^{-1}}$.

PROOF. Denote by A and B the respective numbers in (i) and (ii). If (ϕ_j) does not decompose, then either

$$(1) \quad \|\max |\phi_j - \psi_j|\|_\infty > A \quad \text{or} \quad \left\| \sum |\psi_j| \right\|_\infty > B$$

whenever $\Psi = (\psi_1, \dots, \psi_n)$ is an n -sequence in X . A separation argument shows the existence of a probability measure μ on k satisfying

$$(2) \quad \|\max |\phi_j - \psi_j|\|_{L^1(\mu)} > A/2 \quad \text{and} \quad \left\| \sum |\psi_j| \right\|_{L^1(\mu)} \geq B/2.$$

The details are standard and left to the reader. Consider the case $p \geq 2$ (the case $p < 2$ is similar). By hypothesis there is thus a probability measure ν on K and an operator T , $\|T\| \leq k_p(X)$, factoring

$$\begin{array}{ccc} X & \rightarrow & X^p(\mu) \\ \downarrow & & \uparrow T \\ C(K) & \rightarrow & L^p(\nu) \end{array}$$

where $X^p(\mu)$ is the closure of X in $L^p(\mu)$ and the maps other than T are identity maps. This is, of course, the Pietsch theorem from above. Clearly (2) remains valid for systems $\Psi = (\psi_1, \dots, \psi_n)$ where $\psi_j \in X^p(\mu)$. Define $\psi_j = T(\alpha_j)$, where $\alpha_j = \phi_j \chi_{[|\phi_j| > \varepsilon]}$ ($1 \leq j \leq n$). Then

$$\begin{aligned} \|\max |\phi_j - \psi_j|\|_{L^1(\mu)} &\leq \left[\sum \|T(\psi_j - \alpha_j)\|_{L^p(\mu)}^p \right]^{1/p} \\ &\leq \|T\| \left\| \left(\sum |\phi_j - \alpha_j|^p \right)^{1/p} \right\|_{L^p(\nu)} \end{aligned}$$

and hence is dominated by $k_p(X)n^{1/p}\varepsilon$.

On the other hand, by the square function property

$$\begin{aligned} \left\| \sum |\psi_j| \right\|_{L^1(\mu)} &\leq n^{1/2} \left\| \left(\sum |T(\alpha_j)|^2 \right)^{1/2} \right\|_{L^p(\mu)} \\ &\leq n^{1/2} \|T\| \left\| \left(\sum |\alpha_j|^2 \right)^{1/2} \right\|_{L^p(\nu)} \leq n^{1/2} k_p(X) \end{aligned}$$

since the α_j 's are disjointly supported. A contradiction of (2) results. This proves the lemma.

The key observation is the following property.

LEMMA 6.5. *There exists for each $\varepsilon > 0$ and $n = 1, 2, \dots$ a system $(\phi_j)_{j=1}^n$ in the unit ball of $A(D^2)$ such that $\{x \in \pi^2 \mid |\phi_j(x)| > \varepsilon\}$ are disjoint subsets of π^2 , while $\|\sum |\psi_j|\|_\infty > cn\varepsilon$ whenever $(\psi_j)_{j=1}^n$ in $A(D^2)$ is an approximation $\|\phi_j - \psi_j\| < c$ ($1 \leq j \leq n$) (c a positive numerical constant).*

Momentarily postponing the proof, we show how the lemmas give the required conclusion, i.e. the (i_p, π_p) failure of $A(D^2)$ for any $p \neq 2$. Fix $p \neq 2$ and choose

$$\varepsilon = \frac{1}{4}ck_p(X)^{-1}n^{-\max(2,p)^{-1}},$$

where c is the constant appearing in (6.5). Apply (6.4) to the sequence $(\phi_j)_{j=1}^n$ given by (6.5). Hence, a decomposition $\phi_j = \phi'_j + \phi''_j$ in $A(D^2)$ is obtained, where ϕ'_j, ϕ''_j satisfy (i), (ii) of (6.1). Hence $\|\phi'_j\|_\infty < c$, implying $(\psi_j = \phi'_j)$

$$\begin{aligned} 3k_p(X)n^{\max(2,p')^{-1}} &\geq c^2n\frac{1}{4}k_p(X)^{-1}n^{-\max(2,p)^{-1}}, \\ k_p(X)^2 &\geq (c^2/12)n^{1/2-\max(p,p')^{-1}}. \end{aligned}$$

By letting n tend to infinity, the finiteness of $k_p(X)$ is contradicted.

The proof of Lemma 6.5 uses the spaces of homogeneous polynomials $\mathcal{P}_N = [z^j w^{N-j} \mid j = 0, 1, \dots, N]$ on D^2 . The ϕ_j of (6.2) will be homogeneous polynomials of degree N_j . We claim that it suffices to consider only systems $(\psi_j)_{j=1}^n$, where $\psi_j \in \mathcal{P}_{N_j}$ for each $j = 1, \dots, n$. Indeed, the conditions $\|\phi_j - \psi_j\|_\infty < c$ ($1 \leq j \leq n$) and $\|\sum_{j=1}^n |\psi_j|\|_\infty \leq c$ in ε satisfied by some system (ψ_j) in $A(D^2)$ are preserved if ψ_j is replaced by $\hat{\psi}_j \in \mathcal{P}_{N_j}$, defined by

$$\hat{\psi}_j(z, w) = \frac{1}{2\pi} \int_0^{2\pi} \psi_j(e^{i\theta}z, e^{i\theta}w)e^{-iN_j\theta} d\theta.$$

This reduces (6.5) to the following problem.

LEMMA 6.6. *There is a sequence $(p_j)_{j=1}^n$ of analytic trigonometric polynomials of degree N_j such that the sets $\{\theta \in \pi \mid |p_j(\theta)| > \varepsilon\}$ are mutually disjoint and $\|\sum_{j=1}^n |q_j|\|_\infty > c$ in ε whenever $(g_j)_{j=1}^n$ is a sequence of analytic polynomials satisfying $\|p_j - q_j\|_\infty < c$ and $\text{degree}(q_j) \leq \text{degree}(p_j)$ ($1 \leq j \leq n$).*

The reduction is carried out by restricting, for example, ϕ_j to $\pi \times \pi$ and writing

$$\phi_j(z, w) = w^{N_j} \sum_{k=0}^{N_j} c_k \left(\frac{z}{w}\right)^k \equiv w^{N_j} p_j\left(\frac{z}{w}\right).$$

We outline the proof, which is rather elementary. Let us agree to call an analytic trigonometric polynomial on π a polynomial and use the notation $d(\)$ for the degree. We use

LEMMA 6.7. *There exists a polynomial p , $\|p\|_\infty = 1$ and $\hat{p}(0) = 0$, and an interval I in π such that $\{|p| < \varepsilon\} \subset I$, and for each $\theta \in [\sigma, 2\pi]$ there is a measure $\mu \in M(\pi \setminus I)$ satisfying $\|\mu\| \leq 6\varepsilon^{-1}$ and $\hat{\mu}(j) = e^{ij\theta}$ for $j = 0, 1, \dots, d(p) + 1$.*

PROOF. The existence of the measure will be ensured by the property $\|q\|_{L^\infty(\pi \setminus I)} \geq (\varepsilon/6)\|q\|_\infty$ if $d(q) \leq d(p) + 1$. For each integer N define

$$\alpha(N, \varepsilon) = \sup\{\alpha > 0 \mid \|p\|_{L^\infty(\pi \in [-\alpha, \alpha])} > \varepsilon\|p\|_\infty \text{ if } d(p) \leq N, \hat{p}(0) = 0\}.$$

Of course $\alpha(N, \varepsilon) \rightarrow 0$ for $N \rightarrow \infty$ and N can be chosen with $\alpha(N, \varepsilon) = o(\varepsilon)$. p is a minimizing polynomial of degree N , and I some interval of length 2α . The required property is easily seen to hold.

PROOF OF 6.6. Let p and I be as in (6.7). Consider a (nonanalytic) polynomial τ satisfying $\|\tau\|_\infty = 1$, $|1 - \tau| < \gamma$ on $\pi \setminus I$ and $|\tau| < \gamma$ if $|p| \geq 2\varepsilon$, $\gamma = \gamma(n)$ being some small number.

Then define

$$\begin{aligned} p_1(\theta) &= p(\theta), \\ p_2(\theta) &= \tau(\theta)p(k_1\theta), \\ p_3(\theta) &= \tau(\theta)\tau(k_1\theta)p(k_2\theta), \\ &\vdots \\ p_n(\theta) &= \tau(\theta) \cdots \tau(k_{n-2}\theta)p(k_{n-1}\theta), \end{aligned}$$

where (k_t) is an increasing sequence constructed inductively. In particular, it can be chosen such that the p_t are analytic, and $d(p_t) \leq k_{t-1}(d(p) + 1/2)$. Clearly $\|p_t\|_\infty \leq 1$ and $\{|p_t| \geq 2\varepsilon\} \subset \{|\tau(k_{t-1}\theta)| < \gamma\}$, making the sets $\{|p_t| > 2\varepsilon\}$ disjoint. Fix a sequence of polynomials $(q_t)_{t=1}^n$, $d(q_t) \leq d(p_t)$ and $\|p_t - q_t\|_\infty < 1/10$. We exhibit a point where $\sum |q_t|$ is large, using an argument of successive small perturbations. Assume that $\theta_t \in [0, 2\pi[$ was defined and the conditions

$$(1) \quad |q_s(\theta_t)| > \varepsilon/10 \quad (s = 1, \dots, t), \quad |1 - \tau(k_{s-1}\theta_t)| < \gamma$$

satisfied. Since $p_{t+1}(\theta) = \prod_{s < t} \tau(k_s\theta)p(k_t\theta)$ and k_t is large, we may essentially assume (1) valid for a point θ_t that satisfies, moreover, $p(k_t\theta_t) \approx 1$. Thus $|p_{t+1}(\theta_t)| \approx 1$, hence $|q_{t+1}(\theta_t)| > 4/5$. Consider (see (6.4)) a measure μ on π , $\|\mu\| \leq 6\varepsilon^{-1}$, μ supported by $\{|1 - \tau| < \gamma\}$ and $\hat{\mu}(j) = e^{ij k_t \theta_t}$ for $j = 0, 1, \dots, d(p) + 1$. Define μ_1 by $\hat{\mu}_1(k_t j) = \hat{\mu}(j)$, $\hat{\mu}_1(k) = 0$ otherwise. Let $\hat{\nu} = \hat{\mu}_1 * \hat{F}$, where

$$F(\theta) = \sum_{|j| \leq k_t} \frac{k_t - |j|}{k_t} e^{ij(\theta_t + \theta)}$$

is the θ_t -translate of the k_t -Féjer kernel. One then obtains $\|\nu\| = \|\mu_1\| = \|\mu\| \leq 6\varepsilon^{-1}$ and $\text{supp } \nu = \text{supp } \mu_1 \{|1 - \tau(k_t\theta)| < \gamma\}$,

$$\hat{\nu}(j) = e^{ij\theta_t} \quad \text{for } j = 0, 1, \dots, (d(p) + 1)k_t.$$

Let $v(\theta) = 2^{-R}(1 + e^{i(\theta - \theta_t)})^R$, where $R < \frac{1}{2}k_t$. Since

$$d(q_{t+1}v) \leq k_t(d(p) + \frac{1}{2}) + \frac{1}{2}k_t = k_t(d(p) + 1),$$

we have

$$\|vq_{t+1}\|_{L^\infty(\{|1 - \tau(k_t\theta)| < \gamma\})} \geq \|vq_{t+1}\|_{L^\infty(\text{supp } \nu)} \geq \frac{\varepsilon}{6} \left| \int vq_{t+1} d\nu \right| = \frac{\varepsilon}{6} |q_{t+1}(\theta_t)|,$$

implying the existence of a point θ_{t+1} so that

$$(2) \quad |v(\theta_{t+1})q_{t+1}(\theta_{t+1})| > 2\varepsilon/15, \quad |1 - \tau(k_t\theta_{t+1})| < \gamma.$$

Since $|1 + e^{i(\theta_{t+1} - \theta_t)}| > 2(\varepsilon/10)^{1/R}$, R large with respect to k_1, \dots, k_{t-1} , θ_{t+1} is sufficiently close to θ_t to preserve (1) and, in addition, (2) yields the properties for $s = t + 1$.

The point θ_n obtained at the end verifies

$$\sum_{t=1}^n |q_t(\theta_n)| \geq \frac{n}{10} \varepsilon.$$

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