ON HYPSERSINGULAR INTEGRALS
AND ANISOTROPIC BESSEL POTENTIAL SPACES

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ABSTRACT. In this paper we characterize anisotropic potential spaces in
terms of hypersingular integrals of mixed homogeneity with respect to a general
dilation matrix.

1. Introduction. The purpose of this paper is to give equivalent character-
izations of the anisotropic Bessel potential spaces $\mathcal{L}_p^\alpha$, $\alpha > 0$ and $1 < p < \infty$,
in terms of hypersingular integrals. Anisotropic Bessel potential spaces have been
earlier introduced by Lizorkin [4] and shown [5] to be equivalent with the $\mathcal{L}_p^\alpha$-
spaces considered here in the case of diagonal dilation matrices; for special in-
stances of these potential spaces see also Sadosky and Cotlar [8] and Torchinsky
[14]. The investigation of hypersingular integrals in the case of the standard Bessel
potentials (i.e., isotropic ones) has been carried out by Stein [10] and Wheeden
[15], in the case of anisotropic potential spaces with respect to a diagonal
dilation matrix by Lizorkin [5]. For related work see [9, 3, 7, 1]. Our methods
of proof essentially consist in using Fourier multiplier techniques. To fix ideas
let us give some notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with
elements $x, \xi, \ldots$ and scalar product $x \cdot \xi = x\xi = \sum_{j=1}^n x_j\xi_j$; $R_0^n = \mathbb{R}^n \setminus \{0\}$.
Let $P$ be a real $n \times n$ matrix whose eigenvalues $\lambda_j$ have positive real parts; set
$\alpha_m = \min_{j=1,...,n} \text{Re} \lambda_j$, $\alpha_M = \max_{j=1,...,n} \text{Re} \lambda_j$ and as a trace of $P$ set $\nu = \text{tr}(P)$.
As in Stein and Wainger [12] associate to $P$ the dilation matrix $A_t = tP$ and a
distance function $r$, defined by

$$r(x) = \frac{1}{t}, \quad BA_t x \cdot A_t x = 1, \quad x \neq 0,$$

where $B$ is real, positive definite, symmetric and defined by ($P'$ being the adjoint
of $P$)

$$B = \int_0^\infty e^{-tP}e^{-tP'} dt.$$

Analogously, the adjoint distance function $\rho$ is defined by

$$t\rho(x) = 1, \quad B^#A'_t x \cdot A'_t x = 1, \quad x \neq 0, \quad B^# = \int_0^\infty e^{-tP}e^{-tP'} dt.$$

Then it is shown in [12] that $r, \rho \in C(\mathbb{R}^n)$ are infinitely differentiable for $x \neq 0$.
Further, these distance functions satisfy

$$r(A_t x) = tr(x), \quad \rho(A'_t \xi) = t\rho(\xi)$$
and for any \( \varepsilon > 0 \) (see e.g. [12]) one has

\[
\max\{r(x), \rho(x)\} \leq C_\varepsilon \left\{ \begin{array}{ll}
|x|^{1/(\alpha_m - \varepsilon)}, & |x| \to \infty, \\
|x|^{1/(\alpha_M + \varepsilon)}, & |x| \to 0,
\end{array} \right.
\]

and

\[
\min\{r(x), \rho(x)\} \geq C_\varepsilon \left\{ \begin{array}{ll}
|x|^{1/(\alpha_M + \varepsilon)}, & |x| \to \infty, \\
|x|^{1/(\alpha_m - \varepsilon)}, & |x| \to 0.
\end{array} \right.
\]

(Here and in the following, \( C \) will denote constants, in general different from line to line, but always independent of \( f \) and \( x \).)

On \( S(\mathbb{R}^n) \), the space of rapidly decreasing \( C^\infty \)-functions, the Fourier transform \( \mathcal{F} \) is defined by

\[
\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int f(x)e^{-i\xi x} dx
\]

(where the integration is extended over all of \( \mathbb{R}^n \)); by \( \mathcal{F}^{-1} \) we denote its inverse, by \( [L^1(\mathbb{R}^n)]^\wedge \) the set of Fourier transforms of all \( L^1 \)-functions, and by \( M_p \) the set of bounded Fourier multipliers on \( L^p(\mathbb{R}^n) \). We cite from [2] the following useful

**LEMMA 1.** \( m \in BV_{N+1}, \ N = \lfloor n/2 + 1 \rfloor \), implies

\[
\|\mathcal{F}^{-1}[m(\rho(\cdot))]\|_1 \leq C\|m\|_{BV_{N+1}}.
\]

Here \( BV_{j+1} \) consists of all \( C[0, \infty) \)-functions vanishing at infinity which are sufficiently smooth and satisfy

\[
\|m\|_{BV_{j+1}} = \int_0^\infty t^j|dm^{(j)}(t)| < \infty.
\]

Define the anisotropic Bessel potential kernel \( G_\alpha(x) = G_{\alpha, P}(x) \) by

\[
G_\alpha^\wedge(\xi) = (1 + \rho(\xi))^{-\alpha}.
\]

By Lemma 1 it is clear that \( G_\alpha \in L^1(\mathbb{R}^n) \) for \( \alpha > 0 \). The Bessel potential spaces \( \mathcal{L}_p^\alpha \) (with respect to \( P' \)) are now defined as

\[
\mathcal{L}_p^\alpha = \{ f \in L^p : f = G_\alpha \ast g, g \in L^p \}, \quad 1 \leq p \leq \infty, \ \alpha > 0,
\]

and normed by \( \|f\|_{p,\alpha} = \|g\|_p \). Clearly

\[
\|f\|_p \leq C\|f\|_{p,\alpha}, \quad 1 \leq p \leq \infty, \ \alpha > 0.
\]

By \( \chi \) denote a \( C^\infty(\mathbb{R}) \)-function which equals 0 if \( t \leq 1 \) and 1 if \( t \geq 2 \). The \( k \)th central difference operator \( \Delta_h^k f \) is defined by

\[
\Delta_h^k f(x) = f(x + h) - f(x - h), \quad \Delta_h^{k-1} f = \Delta_h(\Delta_h^{k-1} f).
\]

Our main results now read as follows.

**THEOREM 1.** Let \( 1 < p < \infty, \ \alpha > 0, \) and \( \kappa \) be an even integer greater than \( \alpha/\alpha_m \). The following norms are equivalent on \( \mathcal{L}_p^\alpha : \)

(i) \( \|f\|_{p,\alpha} \),

(ii) \( \|f\|_p + \sup_{\varepsilon > 0} \|D_\varepsilon^\alpha f\|_p, \quad D_\varepsilon^\alpha f = \int_{r(h) \geq \varepsilon} r(h)^{-\alpha - \nu} \Delta_h^\kappa f \, dh \),

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(iii) \[ \|f\|_p + \sup_{\varepsilon > 0} \left\| \int \chi \left( \frac{r(h)}{\varepsilon} \right) r(h)^{-\alpha - \nu} \Delta_h^k f(x) \, dh \right\|_p. \]

The equivalence of (i) and (ii) in the case of a diagonal dilation matrix \( A_t \) is already shown in Lizorkin [5], but the methods of proof there do not work in the general case. Apart from this, a careful reading of the proof presented here shows that the special distance function \( r \) may be replaced by a continuous, positive definite function \( d \) satisfying

\[ d(A_t \xi) = td(\xi), \quad d \in C^L(\mathbb{R}^n_0), \quad L > \max\{(N\alpha_M - \alpha)/\alpha_m; n\}. \]

Moreover, we may replace \( r(h)^{-\alpha - \nu} \) by \( \Omega(h)r(h)^{-\alpha - \nu} \), where \( \Omega \) is \( A_t \)-homogeneous of degree 0 and belongs to \( C^L(\mathbb{R}^n_0) \). Concerning the forward differences \( \Delta_h f(x) = f(x + h) - f(x) \) and \( \Delta_h^k f = \Delta_h(\Delta_h^{k-1} f) \), Lizorkin [5] showed that

\[ \Delta_h f = \frac{1}{2} \Delta_h^{2} f + \frac{1}{2} \Delta_h f. \]

Since \( r \) is an even function this implies, for \( \kappa > \alpha/\alpha_m \),

\[ \int_{r(h) \geq \varepsilon} r(h)^{-\nu - \alpha} \Delta_h^k f(x) \, dh = 2^{-\kappa - 1} \sum_{j=0}^{\kappa} \left( \begin{array}{c} \kappa \\ j \end{array} \right) (1 + (-1)^{\kappa + j}) \int_{r(h) \geq \varepsilon} r(h)^{-\nu - \alpha} \Delta_h^{2j} \Delta_h^{\kappa - j} f(x) \, dh. \]

Thus, our results also apply for the forward differences.

**COROLLARY 1.** If \( f \in L^p_0 \) then, under the hypotheses of Theorem 1,

(i) \[ \int_{r(h) \geq \varepsilon} r(h)^{-\alpha - \nu} \Delta_h^k f \, dh \]

converges in \( L^p \) for \( \varepsilon \to 0^+ \) and

(ii) \[ \int \chi \left( \frac{r(h)}{\varepsilon} \right) r(h)^{-\alpha - \nu} \Delta_h^k f(x) \, dh \]

converges for almost all \( x \in \mathbb{R}^n \) as \( \varepsilon \to 0^+ \).

Part (i) is proved in [5] for diagonal dilation matrices. For the proof of Theorem 1 we need the following technical lemmas.

**LEMMA 2.** If one defines the function \( J(\xi) \) by

\[ J(\xi) = \int \chi(r(h)) r(h)^{-\alpha - \nu} e^{i\xi h} \, dh, \]

then \( J \in C^\infty(\mathbb{R}^n_0) \) and \( J \) is rapidly decreasing at infinity. In particular, \( \chi(\rho(\xi)) \times \rho(\xi)^{-\alpha} J(\xi) \) is an \( S \)-function.

There is an analogous partial result in [5] stating that \( \int r(h)^{-\nu - \alpha} \sin^k \xi h \, dh \) is infinitely differentiable except at the origin. But the method of proof given in §3 differs from that in [5].
LEMMA 3. Denote by \( D_{\#}^\gamma f = \mathcal{F}^{-1}[(1 + |\xi|^2)^{\gamma/2}f] \) the Bessel derivative of \( f \in S' \) of order \( \gamma > 0 \). Let \( 1 < q \leq 2 \) and \( m_j(\xi) = m(A'_2, \xi) \); assume \( D_{\#}^\gamma m_j \in L^q_\text{loc}(\mathbb{R}^n) \) for \( \gamma q > n \) with
\[
BV_{q, \gamma}[m] := \sum_{j \in Z} \|D_{\#}^\gamma(m_j \phi)\|_q < \infty,
\]
where \( \phi \in C^\infty(\mathbb{R}^n) \) is a bump function with support contained in \( \{\xi : \frac{1}{2} \leq \rho(\xi) \leq 2\} \). Then \( m \in [L^1(\mathbb{R}^n)]' \) and, if \( q < p \leq \infty \),
\[
\sup_{t>0} \left\| \sup_{t>0} |(\mathcal{F}^{-1}m)_t * f| \right\|_p \leq CBV_{q, \gamma}[m]\|f\|_p, \quad f \in L^p,
\]
where we use the notation \( g_t(x) = t^\nu g(A_t x) \).

We recall that for integer \( \beta \), \( 1 < p < \infty \), one can identify the classical Bessel potential space with the corresponding Sobolev space (cf. \([11, \text{p. 135}]\)); hence
\[
\sum_{j \in Z} \sum_{|\sigma| \leq \beta} \left( \int_{1/2 \leq \rho(\xi) \leq 2} |D^\sigma m_j(\xi)|^q d\xi \right)^{1/q} < \infty
\]
enures \( BV_{q, \gamma}[m] < \infty \) for \( \beta \geq \gamma \).

Lemma 3 is a variant of the well-known Bernstein theorem and in some sense an extension insofar as it states that the maximal function generated via convolution with \( (\mathcal{F}^{-1}m)_t \) is a bounded operator on \( L^p, 1 < q < p \leq \infty \). The same method of proof given in \( \S 3 \) also yields the following variant.

LEMMA 3'. Let \( m \in L^q_\text{loc}(\mathbb{R}^n), 1 < q \leq 2, A_t = \text{diag}(t^{\alpha_1}, \ldots, t^{\alpha_n}) \) and
\[
\sum_{j \in Z} \left\| \mathcal{F}^{-1} \left[ \prod_{k=1}^n (1 + \xi_k^2)^{\gamma/2} \right] * (m_j \phi) \right\|_q = c_{m, \gamma} < \infty, \quad \gamma q > 1.
\]
Then
\[
\sup_{t>0} \left| (\mathcal{F}^{-1}m)_t * f(x) \right| \leq CC_{m, \gamma}(M(|f|^q)(x))^{1/q},
\]
where \( Mf \) is the classical Hardy-Littlewood maximal function with respect to rectangles (cf. \([13, \text{p. 53}]\)) having sides parallel to the axes.

Another variant of Lemma 3' with a particularly simple proof will follow in (2.14).

2. Proof of Theorem 1. (i) and (ii) are equivalent norms. Suppose \( f \in L^q_\alpha \cap S \); then
\[
(D_{\#}^\gamma f)(\xi) = (I^{-\alpha}f)(\xi)K(\xi)A'_e(\xi),
\]
where
\[
(I^{-\alpha}f)(\xi) = \rho(\xi)^\alpha f(\xi), \quad K(\xi) = \rho(\xi)^{-\alpha} \int_{\rho(h) \geq 1} k(\xi, h) dh, \quad k(\xi, h) = (e^{i\theta \xi} - e^{-i\theta \xi})^{\kappa}(h)^{-\alpha - \nu}.
\]
Now decompose $K = K_1 + K_2 + K_3$, where

$$K_1(\xi) = \chi(\rho(\xi))\rho(\xi)^{-\alpha} \int_{r(h) \geq 1} k(\xi, h) dh,$$

$$K_2(\xi) = -(1 - \chi(\rho(\xi)))\rho(\xi)^{-\alpha} \int_{r(h) \leq 1} k(\xi, h) dh,$$

$$K_3(\xi) = (1 - \chi(\rho(\xi)))\rho(\xi)^{-\alpha} \int k(\xi, h) dh.$$ 

If we can show that

$$K_1, K_2 \in L^1(\mathbb{R}^n), \quad K_3 \in M_p, \quad 1 < p < \infty,$$

then it follows from (2.1) that for all $f \in S$ we have

$$\|D^\alpha f\|_p \leq C\|I^{-\alpha} f\|_p \leq C\|\mathcal{F}^{-1}[(1 + \rho(\xi))^\alpha f]\|_p = C\|f\|_{p, \alpha}$$

since $1 - t^\alpha(1 + t)^{-\alpha}$ satisfies the hypotheses of Lemma 1, and therefore, $\mathcal{F}^{-1}[\rho(\xi)^\alpha(1 + \rho(\xi))^{-\alpha}]$ is a bounded measure. Now $S$ is dense in $L^p_\alpha$ so that (2.4) holds for all $f \in L^p_\alpha$ uniformly in $\varepsilon > 0$. This in combination with (1.5) proves that (ii) is a weaker norm than (i) provided we can establish (2.3). Concerning $K_1$ first observe that $K_1$ is continuous with $K_1(0) = 0$. Next, since $\chi(\rho(\xi))\rho(\xi)^{-\alpha} \in [L^1(\mathbb{R}^n)]$ by Lemma 1, and

$$\left\| \int_{r(h) \geq 1} r(h)^{-\alpha - \nu} \Delta^\nu f dh \right\|_1 \leq C\|f\|_1$$

for all $f \in L^1$, one has obviously $K_1 \in L^1(\mathbb{R}^n)$. Concerning $K_2$ note that for a sufficiently high difference order $\kappa$ one has $D^\alpha K_2 \in L^2(\mathbb{R}^n)$ for all $|\sigma| \leq N = [n/2] + 1$ so that $K_2 \in L^1(\mathbb{R}^n)$ by the Carlson-Beurling inequality. If one is interested in small difference orders $\kappa$, $\kappa > \alpha/\alpha$, one can e.g. use the first part of Lemma 3. Concerning $K_3(\xi) = (1 - \chi(\rho(\xi)))m(\xi)$, observe that

$$m(\xi) = \int (e^{ih\xi'} - e^{-ih\xi'})^\kappa r(h)^{-\alpha - \nu} dh, \quad \xi' = A_{1/\rho(\xi)}\xi,$$

shows that $m(\xi) \neq 0$ for all $\xi \neq 0$ ($\kappa$ is an even integer) and $m$ is $A_1$-homogeneous of degree 0. Hence, by Proposition 4 in [6], we obtain $m \in M_p$, $1 < p < \infty$, once we can prove $D^\alpha m \in C(\mathbb{R}^n_\alpha)$ for all $|\sigma| \leq N$. Since $\rho(\xi)^{-\alpha} \in C^\infty(\mathbb{R}^n_\alpha)$, we have to consider

$$\rho(\xi)^\alpha m(\xi) = \int (1 - \chi(\rho(\xi)))k(\xi, h) dh$$

$$+ \sum_{l=0}^k \binom{k}{l} \int \chi(\rho(\xi))r(h)^{-\alpha - \nu} e^{i(2l-k)\xi h} dh.$$ 

The first term as well as the contribution $l = k/2$ of the second one on the right side are clearly $C^\infty$-functions of $\xi$. The remaining ones are of type of Lemma 2. Thus $m \in M_p$, $1 < p < \infty$, and since $1 - \chi(\rho(\xi)) \in S$ we have $K_3 \in M_p$ and (2.3) is proved.
Conversely, let the expression in (ii) be finite. First we observe that \( 1 - t^\alpha(1 + x(t)t^\alpha)^{-1} \in BV_{N+1} \) and hence, by Lemma 1 and the convolution theorem,

\[
\|f\|_{p,\alpha} \leq C\{\|f\|_p + \|\mathcal{F}^{-1}[\chi(\rho(\xi))\rho(\xi)^\alpha f]\|_p \}
\]

(2.6)

\[
\leq C\left\{\|f\|_p + \sup_{\varepsilon > 0} \|\mathcal{F}^{-1}[\chi(\rho(\xi))\rho(\xi)^\alpha \phi^\varepsilon(\lambda_x\xi)f]\|_p \right\}
\]

for an arbitrary \( S \)-function \( \phi \) with \( \phi^\varepsilon(0) = 1 \); here the latter inequality holds since \( \phi^\varepsilon = e^{-\nu} \phi(A_1/e) \) is an approximate identity for \( \varepsilon \to 0^+ \) and thus, in particular,

\[
\|g\|_p = \lim_{\varepsilon \to 0^+} \|\phi\varepsilon * g\|_p \leq \sup_{\varepsilon > 0} \|\phi\varepsilon * g\|_p.
\]

Now

\[
\|f\|_p = \sup_{\varepsilon > 0} \|\phi\varepsilon * f\|_p.
\]

(2.7)

where

\[
K^\varepsilon(\xi) = \rho(\xi)^{-\alpha} \int_{r(h) \geq 1} k(\xi, h) \, dh = m(\xi) - \mu(\xi)
\]

with \( m(\xi) \) defined as in (2.5) and

\[
\mu(\xi) = \rho(\xi)^{-\alpha} \int_{r(h) \leq 1} k(\xi, h) \, dh = \int r(h) \leq \rho(\xi) k(\xi', h) \, dh, \quad \xi' = A_1^{1/\rho(\xi)} \xi.
\]

We now choose \( \phi \in S \) in such a way that on the compact support of \( \phi^\varepsilon \) the function \( K^\varepsilon \) does not vanish. Note that \( m(\xi) \neq 0 \) for all \( \xi \in \mathbb{R}_0^n \), \( m \in C(\mathbb{R}_0^n) \), and is \( A_1 \)-homogeneous of degree zero so that

\[
\inf_{\xi \in \mathbb{R}_0^n} |m(\xi)| = \inf\{|m(\xi')| : \rho(\xi') = 1\} = \delta > 0.
\]

Since \( k(\xi', h) \) is locally integrable with respect to \( h \) it is clear that \( \lim_{\varepsilon \to 0^+} \mu(\xi) = 0 \); further, the first representation of \( \mu \) shows \( \mu \in C(\mathbb{R}_0^n) \). Hence choose \( \phi \in S \) so that \( |\mu(\xi)| \leq \delta/2 \) for all \( \xi \in \text{supp} \phi^\varepsilon \). Then, with \( \psi(\xi) = 1 \) on \( \text{supp} \phi^\varepsilon \), \( \psi \in C(\mathbb{R}_0^n) \) having appropriate compact support,

\[
\frac{\phi^\varepsilon(\xi)}{K^\varepsilon(\xi)} = \frac{1}{m(\xi)} \frac{\phi^\varepsilon(\xi)}{1 - \psi(\xi)\mu(\xi)/m(\xi)} = \frac{M(\xi)}{m(\xi)}.
\]

By Lemma 3 there holds \( \psi(\xi)\mu(\xi)/m(\xi) \in [L^1(\mathbb{R}^n)]^\varepsilon \) so that by Wiener's theorem \( M \in [L^1(\mathbb{R}^n)]^\varepsilon \) and since, by [6], also \( 1/m \in M_p \), \( 1 < p < \infty \), we conclude from (2.7) that

\[
\|\mathcal{F}^{-1}[\chi(\rho(\xi))\rho(\xi)^\alpha \phi^\varepsilon f]\|_p \leq C\|\mathcal{F}^{-1}[\chi(\rho(\cdot)) * D^\varepsilon f]\|_p
\]

\[
\leq C \sup_{\varepsilon > 0} \|D^\varepsilon f\|_p,
\]

i.e., in combination with (2.6) the assertion for all \( f \in S \) and hence for all \( f \in L^p \) with finite norm (ii) in Theorem 1.

(i) and (iii) are equivalent norms. First suppose \( f \in L^p\). For \( \varepsilon > 0 \) define on \( S \)

\[
E^\varepsilon f = \int \chi\left(\frac{r(h)}{\varepsilon}\right)r(h)^{-\alpha - \nu}\Delta^\varepsilon f \, dh.
\]

(2.9)
Analogously to (2.1) we have
\[(E_{\varepsilon}^{\alpha} f)^{\sim}(\xi) = (I^{-\alpha} f)^{\sim}(\xi)K^{\sim}(A_{\varepsilon}^{-1} \xi),\]
where this time
\[K^{\sim}(\xi) = \rho(\xi)^{-\alpha} \int \chi(r(h))k(\xi, h) \, dh = \sum_{j=1}^{3} K_{j}^{\sim}(\xi),\]
\[K_{1}^{\sim}(\xi) = \chi(\rho(\xi))\rho(\xi)^{-\alpha} \int \chi(r(h))k(\xi, h) \, dh,\]
\[K_{2}^{\sim}(\xi) = -(1 - \chi(\rho(\xi)))\rho(\xi)^{-\alpha} \int (1 - \chi(r(h)))k(\xi, h) \, dh,\]
\[K_{3}^{\sim}(\xi) = (1 - \chi(\rho(\xi)))\rho(\xi)^{-\alpha} \int k(\xi, h) \, dh.\]

By the same methods as above one can show \(K_{1}, K_{2} \in L^{1}(\mathbb{R}^{n})\) and \(K_{3}(\xi) = \phi^{\sim}(\xi)m(\xi)\) with \(\phi \in S\), \(\phi^{\sim}(0) = 1\) and \(m \in C^{\infty}(\mathbb{R}^{n})\) being \(A_{-}^{-}\)-homogeneous of degree 0. Also, defining \(T_{m}\) via \((T_{m}\psi)^{\sim} = m(\xi)\psi^{\sim}\), \(\psi \in S\), we have
\[(2.10) \quad E_{\varepsilon}^{\alpha} f = K_{1,\varepsilon} * I^{-\alpha} f + K_{2,\varepsilon} * I^{-\alpha} f + \phi_{\varepsilon} * T_{m} I^{-\alpha} f.\]

Since \(I^{-\alpha} : L^{p}_{\alpha} \to L^{p}\) is continuous, the representation (2.10) holds for all \(f \in L^{p}_{\alpha}\) almost everywhere. In particular,
\[(2.11) \quad \sup_{\varepsilon > 0} |E_{\varepsilon}^{\alpha} f(x)| \leq \sup_{\varepsilon > 0} |K_{1,\varepsilon} * I^{-\alpha} f(x)| + \sup_{\varepsilon > 0} |K_{2,\varepsilon} * I^{-\alpha} f(x)| + \sup_{\varepsilon > 0} |\phi_{\varepsilon} * T_{m} I^{-\alpha} f(x)|.\]

Clearly, there exists a nonnegative decreasing majorant \(L(\xi) = C(1 + |\xi|)^{-n-1}\) of \(\phi\); then, by [6] the last term is majorized by the maximal function
\[(2.12) \quad M_{\varepsilon}(x) = \sup_{t > 0} (L_{t} * |g|)(x), \quad g = T_{m} I^{-\alpha} f,\]
and
\[(2.13) \quad ||\sup_{\varepsilon > 0} |\phi_{\varepsilon} * T_{m} I^{-\alpha} f(x)||_{p} \leq C ||T_{m} I^{-\alpha} f||_{p} \leq C ||f||_{p,\alpha}.\]

Concerning \(K_{1}\), we note that
\[K_{1}^{\sim}(\xi) = \chi(\rho(\xi))\rho(\xi)^{-\alpha} \left( \frac{\kappa}{\kappa / 2} \right) (-1)^{\kappa / 2} \int \chi(r(h))r(h)^{-\alpha - \nu} \, dh\]
\[+ \chi(\rho(\xi))\rho(\xi)^{-\alpha} \sum_{l \neq \kappa / 2} \left( \frac{\kappa}{l} \right) (-1)^{\kappa - l} \int \chi(r(h))r(h)^{-\alpha - \nu} e^{i(2l - \kappa) \xi h} \, dh.\]

By Lemma 2, the second term on the right side is an \(S\)-function so that an estimate analogous to (2.13) holds. Concerning the first term, consider first for \(\lambda, \Lambda \in \mathbb{N}, \lambda = n + 2, \Lambda > (n + 2)\alpha M - \nu,\)
\[R_{\lambda,\Lambda,t}(x) = \mathcal{F}^{-1}[(1 - \rho(\xi)^{A}/t^{\lambda})^{\frac{1}{\Lambda}}](x);\]
obviously, since $R_{\lambda,A} \in C^{n+1}(\mathbb{R}^n)$, there holds $R_{\lambda,A}(x) \leq C(1 + |x|)^{-n-1}$, i.e., $R_{\lambda,A}$ has a radial integrable majorant. Further,

\begin{equation}
(2.14) \quad |\mathcal{F}^{-1}[m(\rho(\xi)^A)] * f(x)| = C \left| \int_0^\infty R_{\lambda,A,s} * f(x)s^\lambda m^{(\lambda+1)}(s) \, ds \right| \leq C \|m\|_{BV_{\lambda+1}} Mf(x)
\end{equation}

(see (2.12)) by [6]. But certainly $\chi(t^{1/A}t^{-\alpha/A}) \in BV_{\lambda+1}$ and hence

$$\sup_{\varepsilon > 0} |K_{1,\varepsilon} * I^{-\alpha}f(x)| \leq CM(I^{-\alpha}f)(x).$$

Finally $K_2$ satisfies the hypotheses of Lemma 3 which can be seen as follows. $K_2$ has compact support so that we have only to show $\sum_{i=-\infty}^L B_i < \infty$ for some finite $L$, where

$$B_i = \sum_{|\varepsilon| \leq n} \left( \int_{1/2 \leq \rho(\xi) \leq 2} |D^\sigma K_2(A_2',\xi)|^q d\xi \right)^{1/q}.$$

But this is obvious since for all $\xi$, $1/2 \leq \rho(\xi) \leq 2$, there holds

$$|D^\sigma K_2(A_2',\xi)| \leq C \min \{2^{l[\chi(\alpha_m-\varepsilon)-\alpha]_2(\alpha_m-\varepsilon)}, \quad l \in \mathbb{N}, \sigma \in \mathbb{N}^0.$$

Summarizing we obtain the desired result by taking $L^p$-norms in (2.11) and using the second inequality in (2.4). Conversely, let (iii) be finite. Then

$$\|f\|_{p,\alpha} \leq C \left\{ \|f\|_p + \sup_{\varepsilon > 0} \|E_{\varepsilon}^\alpha f\|_p \right\} \leq C \left\{ \|f\|_p + \| \sup_{\varepsilon > 0} |E_{\varepsilon}^\alpha f| \|_p \right\},$$

where the first inequality is analogous to the proof of (ii)→(i) and the second one is obvious. Thus all is proved.

3. Proof of Corollary 1. (i) By (2.1) and (2.2) we have

$$D_\varepsilon^\alpha f = \sum_{i=1}^3 \mathcal{F}^{-1}[K_3(A_i',\xi)] * I^{-\alpha}f,$$

where $I^{-\alpha}f \in L^p$ since $f \in L^p$ (see (2.4)). By the properties of $K_1$ and $K_2$ discussed in §2 it is clear that

$$\lim_{\varepsilon \to 0+} \|K_{1,\varepsilon} * I^{-\alpha}f\|_p = 0, \quad i = 1, 2$$

(cf. [13, p. 11]). We recall $m(A_i',\xi) = m(\xi)$ and, therefore,

$$K_3(A_i',\xi) = (1 - \chi(\rho(A_i',\xi)))m(\xi).$$

Since $\mathcal{F}^{-1}(1 - \chi(\rho(A_i',\xi)))$ is an approximate identity for $\varepsilon \to 0+$ it is clear that

$$\mathcal{F}^{-1}[K_3(A_i',\xi)] * I^{-\alpha}f \to T_m I^{-\alpha}f$$

in $L^p$ for $\varepsilon \to 0+$, i.e., the assertion.

(ii) This is an immediate consequence of Theorem 1 in combination with Theorem 3.12, Chapter II in [13], since by (2.10) we have for $f \in S$ with $0 \notin \text{supp} \hat{f}$ that

$$\lim_{\varepsilon \to 0+} K_{1,\varepsilon} * I^{-\alpha}f(x) = 0, \quad i = 1, 2,$$
and

\[ \lim_{\varepsilon \to 0} \mathcal{F}^{-1}(1 - \chi(\rho(A_{\varepsilon} \xi))) * T_m I^{-\alpha} f(x) = T_m I^{-\alpha} f(x) \quad \text{a.e.} \]

**Proof of Lemma 2.** By definition it is clear that \( J \) is continuous on \( \mathbb{R}^n \) and vanishes at infinity by the Riemann-Lebesgue lemma. Let \( \xi \neq 0 \), in particular \( \xi_j \neq 0 \); denote by \( \xi^{(j)} = (\xi_1, \ldots, \xi_j - 1, \xi_{j+1}, \ldots, \xi_n) \in \mathbb{R}^{n-1} \); let \( \sigma \in \mathbb{N}_0^p \) be arbitrary, \( |\sigma| = l \), choose \( L > (l \alpha_m - \alpha)/\alpha_m \), and integrate partially \( L \) times with respect to \( h_j \):

\[
J(\xi) = \int_{\mathbb{R}^{n-1}} e^{i\xi^{(j)} h^{(j)}} \int_{-\infty}^{\infty} e^{i\xi_j h_j \chi(r(h)) r(h)^{-\alpha - \nu}} dh_j dh^{(j)}
\]

\[= \frac{1}{(i\xi_j)^L} \int_{\mathbb{R}^n} e^{i\xi h} \left( \frac{\partial}{\partial h_j} \right)^L \{\chi(r(h)) r(h)^{-\alpha - \nu}\} dh
\]

\[= \frac{1}{(i\xi_j)^L} \sum_{l=0}^{L-1} \left( \begin{array}{c} L \\ l \end{array} \right) \int_{\mathbb{R}^n} e^{i\xi h} \left( \frac{\partial}{\partial h_j} \right)^{l} r(h)^{-\alpha - \nu} \left( \frac{\partial}{\partial h_j} \right)^{L-l} \chi(r(h)) dh
\]

\[+ \frac{1}{(i\xi_j)^L} \int_{\mathbb{R}^n} e^{i\xi h} \chi(r(h)) \left( \frac{\partial}{\partial h_j} \right)^L r(h)^{-\alpha - \nu} dh
\]

\[= J_1(\xi) + J_2(\xi).
\]

In the case \( l < L \) the function \((\partial/\partial h_j)^{L-l} \chi(r(h))\) has compact support away from the origin; hence \( J_1 \in C^\infty (\mathbb{R}_0^p) \). Since

\[ \left( \frac{\partial}{\partial h_j} \right)^L r(h)^{-\alpha - \nu} \leq C e \tau(h)^{-\alpha - \nu - L(\alpha_m - \varepsilon)}, \quad |h| \to \infty,
\]

and since for \( \sigma' \in \mathbb{N}_0^p \), \( |\sigma'| \leq l \), there holds \( |h^{\sigma'}| \leq |h| \leq C e \tau(h)^{l(\alpha_m + \varepsilon)} \) for \( |h| \to \infty \), we have

\[
D^\sigma J_2(\xi) = \sum_{\sigma = \sigma' + \sigma''} D^\sigma'' \left( \frac{1}{i\xi_j} \right)^L D^\sigma' \int_{\mathbb{R}^n} e^{i\xi h} \chi(r(h)) \left( \frac{\partial}{\partial h_j} \right)^L r(h)^{-\alpha - \nu} dh
\]

\[\leq C \sum_{\sigma = \sigma' + \sigma''} \left| D^\sigma'' \left( \frac{1}{i\xi_j} \right)^L \int_{\mathbb{R}^n} |h^{\sigma'}| \chi(r(h)) \left( \frac{\partial}{\partial h_j} \right)^L r(h)^{-\alpha - \nu} \right| dh
\]

\[\leq C \sum_{\sigma = \sigma' + \sigma''} \left| D^\sigma'' \left( \frac{1}{i\xi_j} \right)^L \int_{r(h) \geq 1} r(h)^{-\alpha - \nu - L\alpha_m + l\alpha_n} (L+1) \varepsilon dh
\]

which for small \( \varepsilon > 0 \) converges because \( l\alpha_m - \alpha - L\alpha_m < 0 \) by our choice of \( L \). It is clear that by integrating partially \( kL \) times we may produce an arbitrary decrease in \( \xi_j \) at infinity. Since \( \xi_j \) and the order of differentiation, namely \( l \), was arbitrary, there holds

\[
\sum_{j=1}^n \xi_j^l D^\sigma J(\xi) = \mathcal{O}(1), \quad |\xi| \to \infty,
\]

i.e. \( J \in C^\infty (\mathbb{R}_0^p) \) and \( J \) is rapidly decreasing. Since \( \chi(\rho(\xi))\rho(\xi)^{-\alpha} \in C^\infty (\mathbb{R}^n) \) with support away from the origin is only slowly increasing, the last assertion of Lemma 2 is obvious.
Proof of Lemma 3. Madych proved in [6, Theorem 5] that $BV_{2,\beta}[m] < \infty$ for $\beta > n/2$ is sufficient for $m$ to belong to $[L^1(\mathbb{R}^n)]^\sim$. If now $\gamma > n/q$ then we can find $\beta > n/2$ such that $\gamma - n/q \geq \beta - n/2$. Hence, by [11, pp. 119 and 133], $BV_{2,\beta}[m] \leq CBV_{q,\gamma}[m]$ and thus $m \in [L^1(\mathbb{R}^n)]^\sim$. Let $\varphi \in C^\infty(\mathbb{R}^1)$ with $\text{supp} \, \varphi \subset [\frac{1}{2}, 2]$ and $\sum_{j \in \mathbb{Z}} \varphi(2^{-j}t) = 1$ for $t > 0$; set $\varphi(2^{-j} \rho(\xi)) = \phi_j(\xi), \phi_0 = \phi$. Since $m \in [L^1(\mathbb{R}^n)]^\sim$ we have, for all $f \in L^p$,

$$\tag{3.1} |(\mathcal{F}^{-1}m)_t \ast f(x)| \leq \sum_{j \in \mathbb{Z}} |(\mathcal{F}^{-1}m\phi_j)_t \ast f(x)| = \sum_{j \in \mathbb{Z}} I_j(x).$$

Now observe that $\mathcal{F}^{-1}m\phi_j(x) = 2^{j\nu} \mathcal{F}^{-1}m\phi_j(A_2, x)$ and apply Hölder’s inequality to obtain

$$\tag{3.2} I_j(x) \leq \left\{ \left(2^j \nu \right)^q \int \left(1 + |A_2t(x - y)|^2 \right)^{\gamma/2} \mathcal{F}^{-1}m\phi_j(A_2t(x - y)) \right\}^{q/q'} dy \cdot \left\{ \left(2^j \nu \right)^q \int \left(1 + |A_2t(x - y)|^2 \right)^{-\gamma/2} |f(y)|^q dy \right\}^{1/q}
= \left\| \left(1 + |\cdot|^2 \right)^{\gamma/2} \mathcal{F}^{-1}m\phi_j \right\|_{q'} \left\{ K_{2t} \ast |f|^q(x) \right\}^{1/q},$$

where $K(x) = (1 + |x|^2)^{-\gamma/2}$. It is easy to verify that $K$ and $A_t$ satisfy the assumptions of Theorem 3 in [6]; thus, the maximal operator

$$M(g) = \sup_{t > 0} |K_t \ast g(x)|$$

is bounded from $L^p$ into itself if $1 < p \leq \infty$. Using this and the Hausdorff-Young inequality we conclude from (3.1) and (3.2), if $q < p \leq \infty$, that

$$\left\| \sup_{t > 0} |(\mathcal{F}^{-1}m)_t \ast f| \right\|_p \leq \sum_{j \in \mathbb{Z}} \left\| D^\gamma_m \phi_j \right\|_{q'} \left\| M(|f|^q) \right\|_{p/q} \leq CBV_{q,\gamma}[m] \left\| f \right\|_p.$$

References

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