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RADIAL FUNCTIONS AND INVARIANT CONVOLUTION OPERATORS

BY

CHRISTOPHER MEANEY

ABSTRACT. For $1 < p < 2$ and $n > 1$, let $A_p(\mathbf{R}^n)$ denote the Figà-Talamanca-Herz algebra, consisting of functions of the form

$$(*) \qquad \sum_{k=0}^{\infty} f_k * g_k$$

with $\sum_k \|f_k\|_p \cdot \|g_k\|_{p'} < \infty$. We show that if $2n/(n+1) < p < 2$, then the subalgebra of radial functions in $A_p(\mathbf{R}^n)$ is strictly larger than the subspace of functions with expansions $(*)$ subject to the additional condition that f_k and g_k are radial for all k. This is a partial answer to a question of Eymard and is a consequence of results of Herz and Fefferman. We arrive at the statement above after examining a more abstract situation. Namely, we fix $G \in [FIA]_B^-$ and consider ${}^B A_p(G)$ the subalgebra of B-invariant elements of $A_p(G)$. In particular, we show that the dual of ${}^B A_p(G)$ is equal to the space of bounded, right-translation invariant operators on $L^p(G)$ which commute with the action of B.

Introduction. In his survey of the properties of the Figà-Talamanca-Herz algebras $A_p(G)$, Eymard asks the following question, [**Ey**, 9.3]. If $u \in A_p(\mathbf{R}^n)$ is *radial* does it have an expansion

$$u = \sum_{l=0}^{\infty} f_l * g_l$$

with not only the usual conditions $f_l \in L^p(\mathbf{R}^n)$, $g_l \in L^{p'}(\mathbf{R}^n)$, and

$$\sum_{l=0}^{\infty} \|f_l\|_p \|g_l\|_{p'} < \infty,$$

but also f_l and g_l *radial* for all l?

We use results of Herz and Fefferman to show that the answer is no when $n > 1$ and $2n/(n+1) < p < 2$.

A similar statement is possible for central functions in $A_p(G)$, where G is a compact, simply connected, simple Lie group.

It is possible to view the radial part of $A_p(\mathbf{R}^n)$ in a more general setting. Suppose that G is a locally compact group with a group of topological automorphisms B such that B contains all inner automorphisms of G and B is compact in $\mathrm{Aut}(G)$. We examine the subalgebra of B-invariant elements of $A_p(G)$, written ${}^B A_p(G)$, and

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show that its dual is the space of bounded right-translation invariant operators on $L^p(G)$ which commute with the action of B. Furthermore, let $\mathfrak{A}_p(G,B)$ be the image of $({}^BL^p(G))\hat{\otimes}({}^BL^{p'}(G))$ under the map $f \otimes g \mapsto g * f^\vee$. For $h \in L^1(G)$, denote by $\lambda(h)$ the operator $f \mapsto h*f$, acting on $L^p(G)$ and having norm $|||\lambda(h)|||_p$. If $h \in {}^BL^1(G)$ then $\lambda(h)\colon {}^BL^p(G) \to {}^BL^p(G)$ and we denote the norm of this operator by $N_p(f)$.

We arrive at our answer to Eymard's question via the following general principle. If there exists a sequence $\{h_n\}_n \in {}^BL^1(G)$ with $\{|||\lambda(h_n)|||_p\}_n$ unbounded and $\{N_p(h_n)\}_n$ bounded, then

$$ {}^BA_p(G) \neq \mathfrak{A}_p(G,B). $$

1. $[FIA]_B^-$ groups. If G is a locally compact group let $\mathrm{Aut}(G)$ be the group of topological automorphisms of G, equipped with the Birkhoff topology described in [**Br**] and [**PtSu**]. Throughout this paper we assume that there is a subgroup $B \subset \mathrm{Aut}(G)$ such that: (i) B contains all inner automorphisms of G; and (ii) B is compact in $\mathrm{Aut}(G)$. This is abbreviated by writing $G \in [FIA]_B^-$. For a list of the properties of the class $[FIA]_B^-$ see the survey article [**Pa**].

Examples include locally compact abelian groups, with B trivial; central groups, with B equal to the group of inner automorphisms; and $G = \mathbf{R}^n$, $B = SO(n)$.

For $\alpha \in \mathrm{Aut}(G)$ and f a function on G we let ${}^\alpha f(x) = f(\alpha^{-1}(x))$, $x \in G$. The hypothesis $G \in [FIA]_B^-$ implies that G is unimodular and we fix a Haar measure m_G on G. In particular, m_G is B-invariant (see [**Br**, §IV.5]). The action of B extends to the Lebesgue spaces of G with respect to m_G. If f belongs to one of the spaces $L^p(G)$ $(1 \le p < \infty)$ or $C_0(G)$ then the map $\alpha \mapsto {}^\alpha f$ provides a strongly continuous representation of B by isometries [**Br**, p. 78]. If E is a Lebesgue space or a space of functions on G then we let

$$ {}^BE := \{f \in E\colon {}^\alpha f = f,\ \forall\, \alpha \in B\}. $$

Having equipped G with m_G we define convolution of functions on G as in [**HwRs**, §20]. Since m_G is B-invariant we see that

$$ {}^\alpha(\varphi * \psi) = ({}^\alpha\varphi) * ({}^\alpha\psi), \qquad \forall\, \alpha \in B, \tag{1.1} $$

whenever $\varphi * \psi$ makes sense. Note also that

$$ ({}^\alpha f)^\vee = {}^\alpha(f^\vee), \qquad \forall\, \alpha \in \mathrm{Aut}(G), $$

where $f^\vee(x) = f(x^{-1})$.

If $f \in C_0(G)$ or $L^p(G)$ $(1 \le p < \infty)$ we set

$$ Z_B f := \int_B ({}^\beta f)\, dm_B(\beta), \tag{1.2} $$

where the right-hand side is the Bochner integral with respect to the normalized Haar measure m_B of the compact group B. This is denoted $f^\#$ in [**Msk**]. The operator Z_B is obviously bounded and provides the projections $C_0(G) \to {}^BC_0(G)$ and $L^p(G) \to {}^BL^p(G)$. Since B contains all inner automorphisms of G, ${}^BL^1(G)$ is contained in the centre of $L^1(G)$.

The maximal ideal space of the commutative Banach algebra ${}^BL^1(G)$ is identified with $\mathfrak{X}_B$, the space of B-characters as defined in [**Msk**, §2]. These can be considered as the zonal spherical functions for the Gel'fand pair $(G \rtimes B,\ \{1\} \times B)$. Hence, $\mathfrak{X}_B$

can be equipped with a measure ν so that the Gel'fand transform $\mathcal{F}\colon {}^BL^1(G) \to C_0(\mathfrak{X}_B)$ extends to an isometric isomorphism $\mathcal{F}\colon {}^BL^2(G) \to L^2(\mathfrak{X}_B,\nu)$, see [**Go**]. The usual interpolation argument shows that if $1 < p < 2$ and $(1/p) + (1/p') = 1$ then $\mathcal{F}$ extends to be a bounded map

$$\mathcal{F}\colon {}^BL^p(G) \to L^{p'}(\mathfrak{X}_B,\nu).$$

For further details on analysis on $[FIA]_B^-$ groups see [**Ha, HHL, KS, LM, Mz, Msk, Pa, Pt, PtSu**].

2. Figà-Talamanca-Herz spaces. Fix $G \in [FIA]_B^-$ and $1 < p < \infty$. The action of G on $L^p(G)$ by right translation is denoted by

$$(\rho(x)f)(y) := f(yx), \qquad \forall\, x, y \in G,$$

and left translation is

$$(\lambda(x)f)(y) := f(x^{-1}y), \qquad \forall\, x, y \in G.$$

Furthermore, if $h \in L^1(G)$ and $f \in L^p(G)$ then we set

$$\lambda(h)f := h * f,$$

so that $\lambda(h)$ is a bounded linear operator on $L^p(G)$. The space of all bounded linear operators on $L^p(G)$ which commute with $\rho(G)$ is denoted $Cv_p(G)$ and is equipped with the operator norm $|||\cdot|||_p$. Clearly $\lambda\colon L^1(G) \to Cv_p(G)$ is a homomorphism of Banach algebras.

The Figà-Talamanca-Herz space $A_p(G)$ is the image of $L^p(G)\hat{\otimes}L^{p'}(G)$ under the map

$$P(f \otimes g) := g * f^\vee, \tag{2.1}$$

and the norm on $A_p(G)$ is the quotient norm on $(L^p(G)\hat{\otimes}L^{p'}(G))/\ker(P)$. Since G is amenable (see [**Pa**, diagram 1]) we can identify $A_p(G)^*$ with $Cv_p(G)$. For $T \in Cv_p(G)$ and $\varphi \in A_p(G)$, having series expansion $\varphi = P(\sum_{n=0}^\infty f_n \otimes g_n)$, the pairing is

$$\langle T, \varphi\rangle = \sum_{n=0}^{\infty} g_n * (Tf_n)^\vee(1). \tag{2.2}$$

In particular, if $h \in L^1(G)$ then

$$\langle \lambda(h), \varphi\rangle = \int_G \varphi h \, dm_G. \tag{2.3}$$

Herz has shown that $Cv_p(G)$ is the ultrastrong closure of $\lambda(C_c(G))$ [**Hz 3**, Theorem 5].

2.4 LEMMA. *If $T \in Cv_p(G)$ then there is a net $\{h_\gamma\}_\gamma \subset C_c(G)$ such that*

$$|||\lambda(h_\gamma)|||_p \le |||T|||_p, \qquad \forall\, \gamma,$$

and

$$\langle T, \varphi\rangle = \lim_\gamma \int_G \varphi h_\gamma \, dm_G, \qquad \forall\, \varphi \in A_p(G).$$

For details see [**Cw, Ey, FT, Hz 2, Hz 3, Rb**].

Herz also considered the map M, which takes functions on G to functions on $G \times G$ and is defined by

$$(Mh)(x,y) := h(xy^{-1}), \qquad \forall\, x,y \in G. \tag{2.5}$$

In particular, if $\varphi \in A_p(G)$ then $M\varphi$ is a multiplier of $L^p(G)\hat{\otimes}L^{p'}(G)$ and

$$P\left((M\varphi)\sum_{n=0}^{\infty} f_n \otimes g_n\right) = \varphi \cdot P\left(\sum_{n=0}^{\infty} f_n \otimes g_n\right).$$

This shows that $A_p(G)$ is a Banach algebra (see [**Ey**, Théorème 1]).

We next consider the action of B on $A_p(G)$. In fact, both B and $B \times B$ act on $L^p(G)\hat{\otimes}L^{p'}(G)$. For $f \in L^p(G)$, $g \in L^{p'}(G)$, and $\beta, \beta' \in B$ set

$$^{\beta}(f \otimes g) := (^{\beta}f) \otimes (^{\beta}g); \tag{2.7}$$

$$^{(\beta,\beta')}(f \otimes g) := (^{\beta}f) \otimes (^{\beta'}g). \tag{2.8}$$

Equation (2.7) (resp. (2.8)) defines a strongly continuous representation of B (resp. $B \times B$) on $L^p(G)\hat{\otimes}L^{p'}(G)$, acting as isometries. We need only remark that

$$\begin{aligned}\|f \otimes g - {}^{\beta}(f \otimes g)\| &= \|f \otimes g - (^{\beta}f) \otimes g + (^{\beta}f) \otimes g - (^{\beta}f) \otimes (^{\beta}g)\| \\ &\leq \|f - {}^{\beta}f\|_p \|g\|_{p'} + \|f\|_p \|g - {}^{\beta}g\|_{p'}.\end{aligned}$$

From equation (1.1) we see that if $h \in L^p(G)\hat{\otimes}L^{p'}(G)$ and if $\beta \in B$ then

$$P(^{\beta}h) = {}^{\beta}(Ph). \tag{2.9}$$

2.10 LEMMA. *If $f \in A_p(G)$ and $\beta \in B$ then $^{\beta}f \in A_p(G)$ and the map $\beta \mapsto {}^{\beta}f$ is a strongly continuous representation of B on $A_p(G)$, acting by isometries. Furthermore, $Z_B f \in A_p(G)$ and $\|Z_B f\|_{A_p(G)} \leq \|f\|_{A_p(G)}$.*

The case $p = 2$ was proved in [**PtSu**]. The following results were verified by Mosak [**Msk**, p. 284].

2.11 LEMMA. (i) *If $f \in L^1(G)$ and $g \in {}^{B}L^1(G)$ then $Z_B(f * g) = (Z_B f) * g$ and $Z_B(g * f) = g * (Z_B f)$.*

(ii) *If $f, h \in L^1(G)$ then $Z_B(f * h) = Z_B(h * f)$.*

(iii) *If $1 < p < \infty$, $f \in L^p(G)$ and $g \in L^{p'}(G)$ then $Z_B(f * g) = Z_B(g * f)$.*

2.12 COROLLARY. *If $1 < p < \infty$ then $^{B}A_p(G) = {}^{B}A_{p'}(G)$ with equality of norms.*

Note that $^{B}A_p(G)$ is a closed subalgebra of $A_p(G)$. If $h \in {}^{B}A_p(G)$ and $\beta \in B$ then $(Mh)(\beta(x), \beta(y)) = h(\beta(xy^{-1})) = Mh(x,y)$, so that Mh is a multiplier of the invariant subspace

$$^{B}(L^p(G) \otimes L^{p'}(G)).$$

The action of $B \times B$ does not fit in with P, for if $f \in L^p(G)$, $g \in L^{p'}(G)$, and $\beta, \beta' \in B$ then

$$P(^{\beta}f \otimes {}^{\beta'}g) = (^{\beta'}g) * (^{\beta}f)^{\vee} = {}^{\beta}P(f \otimes {}^{\beta^{-1}\cdot\beta'}g).$$

In fact, $F \in L^p(G)\hat{\otimes}L^{p'}(G)$ is B-invariant if and only if

$$F = \int_B ({}^\beta F)\, dm_B(\beta), \qquad \text{(Bochner integral)}$$

while it is $B \times B$-invariant if and only if it belongs to $({}^B L^p(G))\hat{\otimes}({}^B L^{p'}(G))$. Eymard's question [**Ey**, 9.3] asks if

$$ {}^B A_p(G) = P(({}^B L^p(G))\hat{\otimes}({}^B L^{p'}(G)))? \tag{2.13}$$

Peters [**Pt**] has shown that the answer is yes when $p = 2$.

Let us use the abbreviation

$$\mathfrak{A}_p(G, B) := P(({}^B L^p(G))\hat{\otimes}({}^B L^{p'}(G))). \tag{2.14}$$

This is the analogue of a Figà-Talamanca-Herz space for the hypergroup of B orbits in G (see [**Ha, HHL**]).

2.15 REMARK. We cannot use the technique of [**Hz 2**] to verify whether $\mathfrak{A}_p(G, B)$ is an algebra. For if $h \in \mathfrak{A}_p(G, B) \subseteq {}^B A_p(G)$ then Mh is a multiplier of ${}^B(L^p(G)\hat{\otimes}L^{p'}(G))$ but not necessarily of $({}^B L^p(G))\hat{\otimes}({}^B L^{p'}(G))$, since it need not be $B \times B$-invariant. The best we can say is that the function

$$(x, y) \mapsto \int_B h(x \cdot \beta(y^{-1}))\, dm_B(\beta)$$

is a multiplier of $({}^B L^p(G))\hat{\otimes}({}^B L^{p'}(G))$.

3. Invariant convolution operators. Maintain the notation and hypotheses of §2. The compact group B acts on $Cv_p(G)$ via conjugation. If $T \in Cv_p(G)$ and $\beta \in B$ let ${}^\beta T$ be the bounded linear transformation on $L^p(G)$ defined by

$$({}^\beta T)f := {}^{\beta^{-1}}(T({}^\beta f)), \qquad \forall f \in L^p(G).$$

An elementary calculation confirms that ${}^\beta T \in Cv_p(G)$ and clearly

$$|||{}^\beta T|||_p = |||T|||_p.$$

3.1 DEFINITION. *We set* ${}^B Cv_p(G) = \{T \in Cv_p(G) \colon {}^\beta T = T, \forall \beta \in B\}$. From (2.2) and (1.1) it follows that if $\varphi \in A_p(G)$ can be written as $P(\sum_{n=0}^\infty f_n \otimes g_n)$ and if $T \in Cv_p(G)$ then

$$\begin{aligned} \langle {}^\beta T, \varphi\rangle &= \sum_{n=0}^\infty g_n * ({}^{\beta^{-1}}(T({}^\beta f_n)))^\vee(1) \\ &= \sum_{n=0}^\infty ({}^\beta g_n) * ((T({}^\beta f_n)))^\vee(1) \\ &= \langle T, {}^\beta\varphi\rangle, \qquad \forall \beta \in B. \end{aligned} \tag{3.2}$$

We wish to show that ${}^B Cv_p(G)$ is the closure of $\lambda({}^B C_c(G))$.

3.3 LEMMA. *If* $\varphi \in C_c(G)$ *then* $|||\lambda(Z_B\varphi)|||_p \le |||\lambda(\varphi)|||_p$.

PROOF. We know that

$$|||\lambda(Z_B\varphi)|||_p = \sup \left| \int_G f(Z_B\varphi)\, dm_G \right|,$$

where the supremum is taken over $\{f \in A_p(G) \colon \|f\|_{A_p} \leq 1\}$. However,

$$\left|\int_G f(Z_B\varphi)\,dm_G\right| = \left|\int_B\int_G ({}^\beta f)\varphi\,dm_G\,dm_B\right|, \tag{3.4}$$

since m_G is B-invariant, and the right-hand side is less than or equal to

$$\int_B |||\lambda(\varphi)|||_p \|{}^\beta f\|_{A_p}\,dm_B.$$

Now apply Lemma 2.10. Q.E.D.

Fix $T \in {}^BCv_p(G)$ and let $\{h_\gamma\}_\gamma \subset C_c(G)$ be a net as described in Lemma 2.4. We have just seen that

$$|||\lambda(Z_Bh_\gamma)|||_p \leq |||T|||_p, \qquad \forall\gamma. \tag{3.5}$$

Furthermore, for every $\varphi \in A_p(G)$,

$$\begin{aligned}\langle T,\varphi\rangle = \int_B \langle {}^\beta T,\varphi\rangle\,dm_B(\beta) &\overset{(3.2)}{=} \langle T, Z_B\varphi\rangle = \lim_\gamma (Z_B\varphi) * h_\gamma^\vee(1)\\ &\overset{(3.4)}{=} \lim_\gamma \varphi * (Z_Bh_\gamma^\vee)(1) = \lim_\gamma\langle\lambda(Z_Bh_\gamma),\varphi\rangle.\end{aligned} \tag{3.6}$$

3.7 LEMMA. *If $T \in {}^BCv_p(G)$ then there exists a net $\{h_\gamma\} \subset {}^BC_c(G)$ such that $|||\lambda(h_\gamma)|||_p \leq |||T|||_p$, $\forall\gamma$, and*

$$\langle T,\varphi\rangle = \lim_\gamma \int_G \varphi h_\gamma\,dm_G, \qquad \forall\varphi \in A_p(G).$$

This shows that ${}^BCv_p(G)$ is the image $Z_B^*Cv_p(G)$ where Z_B^* is the adjoint of $Z_B \colon A_p(G) \to A_p(G)$. From [**Mz**, p. 67], it follows that ${}^BCv_p(G) \cong {}^BA_p(G)^*$.

3.8 PROPOSITION. *The dual of the Banach space ${}^BA_p(G)$ is equal to ${}^BCv_p(G)$, with the pairing as in* (2.2).

3.9 COROLLARY. *For $1 < p < \infty$ and $G \in [FIA]_B^-$ we have ${}^BCv_p(G) = {}^BCv_{p'}(G)$ and ${}^BCv_p(G) \subset {}^BCv_2(G)$.*

PROOF. The first part follows from Corollary 2.12 and the second form from the Riesz-Thorin convexity theorem. Q.E.D.

This is different from the case of all of $Cv_p(G)$, for there are examples [**Hz 4, Hz 5, Lh, Ob**] of values p and groups G with $Cv_p(G) \neq Cv_{p'}(G)$. Corollary 3.9 is very well known for various special cases, such as locally compact abelian groups and compact groups, with B the group of inner automorphisms.

3.10 REMARKS. Recall the notation of §1. Hartmann [**Ha**] has shown that

$${}^BA_2(G) = \mathfrak{A}_2(G,B) \cong L^1(\mathfrak{X}_B,\nu),$$

where the isomorphism is the "inverse Fourier transform" $\mathcal{F}^{-1}$. Hence, ${}^BCv_2(G) \cong L^\infty(\mathfrak{X}_B,\nu)$, so that elements of ${}^BCv_2(G)$ can be viewed as multipliers. That is, if $T \in {}^BCv_2(G)$ then there is $\mathcal{F}T \in L^\infty(\mathfrak{X}_B,\nu)$ such that

$$\langle T,\varphi\rangle = \int_{\mathfrak{X}_B} (\mathcal{F}T)(\mathcal{F}\varphi)\,d\nu,$$

for all $\varphi \in {}^BA_2(G) \cap C_c(G)$.

Conversely, $m \in L^\infty(\mathfrak{X}_B, \nu)$ is equal to $\mathcal{F}T$ for some $T \in {}^B Cv_p(G)$ if and only if

$$\left| \int_{\mathfrak{X}_B} m \cdot \mathcal{F}\varphi \cdot d\nu \right| \leq \text{const} \cdot \|\varphi\|_{A_p(G)} \tag{3.11}$$

for all $\varphi \in {}^B A_p(G) \cap C_c(G)$. We could also use ${}^B A_2(G) \cap C_c(G)$, equipped with $\|\cdot\|_{A_p(G)}$, as a test space in (3.11). See [**Cw**].

This line of reasoning suggests a means of sometimes distinguishing $A_p(G)$ and $\mathfrak{A}_p(G, B)$.

Observe that ${}^B L^1(G)$ acts on ${}^B L^p(G)$ via convolution. Let us denote by $N_p(f)$ the norm of $\lambda(f)\colon {}^B L^p(G) \to {}^B L^p(G)$, where $f \in {}^B L^1(G)$. Clearly, from (2.14) we know that

$$N_p(f) = \sup\left\{ \left| \int_G f\varphi\, dm_G \right| : \varphi \in \mathfrak{A}_p(G, B), \|\varphi\|_{\mathfrak{A}_p} \leq 1 \right\}.$$

If one could show that $N_p(f) \neq |||\lambda(f)|||_p$, for some $f \in {}^B L^1(G)$, then it would follow that $\mathfrak{A}_p(G, B) \neq {}^B A_p(G)$, since

$$|||\lambda(f)|||_p = \sup\left\{ \left| \int_G f\varphi\, dm_G \right| : \varphi \in {}^B A_p(G),\ \|\varphi\|_{A_p} \leq 1 \right\}.$$

We shall demonstrate this for special cases in the next sections.

4. Radial multipliers. In this section we let $G = \mathbf{R}^n$, for fixed $n > 1$, and $B = SO(n)$, so that ${}^B L^p(G)$ is the subspace of radial elements of $L^p(\mathbf{R}^n)$. We use $\hat{f}$ to denote the usual Fourier transform of an integrable function f on $\mathbf{R}^n$. The Schwartz space is denoted by $\mathcal{S}(\mathbf{R}^n)$ and $\mathcal{D}(\mathbf{R}^n)$ is the space of C^∞-functions with compact support.

It is well known that $T \in {}^{SO(n)} Cv_p(\mathbf{R}^n)$ corresponds to an element $\mathcal{F}T \in L^\infty([0, \infty))$ such that

$$(Tf)(x) = \int_{\mathbf{R}^n} \hat{f}(\xi) \mathcal{F}T(|\xi|) e^{ix\cdot\xi}\, d\xi,$$

for all $f \in \mathcal{S}(\mathbf{R}^n)$. Conversely, we have seen in §3 that $m \in L^\infty([0, \infty))$ is of the form $m = \mathcal{F}T$, for some $T \in {}^{SO(n)} Cv_p(\mathbf{R}^n)$, provided

$$\left| \int_{\mathbf{R}^n} \hat{f}(\xi) m(|\xi|)\, d\xi \right| \leq \text{const}.\|f\|_{A_p(\mathbf{R}^n)} \tag{4.1}$$

for all $f \in {}^{SO(n)} \mathcal{D}(\mathbf{R}^n)$.

For each $r > 0$ let T_r° be the operator defined by

$$(T_r^\circ f)(x) = \int_{|\xi| \leq r} \hat{f}(\xi) e^{i\xi\cdot x}\, d\xi,$$

for all $f \in \mathcal{S}(\mathbf{R}^n)$. We recall the following results of Herz [**Hz 1**] and Fefferman [**Ff**].

4.2 LEMMA. (a) *For $n > 0$, $r > 0$, and $2n/(n+1) < p \leq 2$ the operator T_r° is bounded on ${}^{SO(n)} L^p(\mathbf{R}^n)$ and the norm is independent of r.*

(b) *For $n > 1$, $r > 0$, and $p \neq 2$, $T_r^\circ \notin Cv_p(\mathbf{R}^n)$.*

The following lemma was shown to me by Michael Cowling.

4.3 LEMMA. *Let $\psi \in C^\infty([0,\infty))$ be compactly supported and have $\psi' \le 0$. Furthermore let T_ψ be the operator defined by*

$$(T_\psi f)^\wedge(\xi) = \psi(|\xi|)\hat{f}(\xi), \qquad \forall\, \xi \in \mathbf{R}^n,\ f \in \mathcal{S}(\mathbf{R}^n).$$

Then for each $2n/(n+1) < p < 2$ there is a constant $c_p > 0$ such that

$$\|T_\psi f\|_p \le c_p \|f\|_p \psi(0), \qquad \textit{for all } f \in {}^{SO(n)}\mathcal{S}(\mathbf{R}^n).$$

PROOF. For $f, g \in {}^{SO(n)}\mathcal{S}(\mathbf{R}^n)$ we see that

$$(T_\psi f) * g(0) = \int_{\mathbf{R}^n} \hat{f}(\xi)\hat{g}(\xi)\psi(|\xi|)\, d\xi.$$

Integrating by parts we see that this is equal to

$$-\int_0^\infty \psi'(r) \int_{|\xi|\le r} \hat{f}(\xi)\hat{g}(\xi)\, d\xi\, dr = -\int_0^\infty \psi'(r)((T_r^\circ f) * g)(0)\, dr.$$

Now apply the preceding lemma. Q.E.D.

Note that we could also have used [**GT**, p. 238] and [**Ig**].

We can now give a partial answer to [**Ey**, 9.3].

4.4 THEOREM. *For $n > 1$ and $2n/(n+1) < p < 2$,*

$${}^{SO(n)}A_p(\mathbf{R}^n) \ne P({}^{SO(n)}L^p(\mathbf{R}^n) \hat{\otimes} {}^{SO(n)}L^{p'}(\mathbf{R}^n)).$$

PROOF. Suppose ${}^{SO(n)}A_p(\mathbf{R}^n) = \mathfrak{A}_p(\mathbf{R}^n, SO(n))$. The open mapping theorem implies equivalence of norms

$$\|f\|_{A_p} \le \|f\|_{\mathfrak{A}_p} \le \kappa \|f\|_{A_p}.$$

Fix a smooth, compactly supported function ψ on $[0,\infty]$ such that:

(i) $\psi(t) = 1$ if $t \le 1$;
(ii) $0 \le \psi(t) < 1$ if $t > 1$; and
(iii) $\psi'(t) \le 0$, $\forall\, t \ge 0$.

For each $k \ge 1$ let Ψ_k be the element of ${}^{SO(n)}\mathcal{S}(\mathbf{R})$ such that

$$\hat{\Psi}_k(\xi) = (\psi(|\xi|))^k, \qquad \forall\, \xi \in \mathbf{R}^n.$$

For an arbitrary pair $f, g \in \mathcal{D}(\mathbf{R}^n)$ our hypothesis implies that there exists sequences $\{F_l\}_{l\ge0}$ and $\{G_l\}_{l\ge0}$ contained in ${}^{SO(n)}\mathcal{D}(\mathbf{R}^n)$ and satisfying

$$Z_{SO(n)}(f * g) = \sum_{l=0}^\infty F_l * G_l$$

and $\sum_{l=0}^\infty \|F_l\|_p \|G_l\|_{p'} \le 2\kappa \|f\|_p \|g\|_{p'}$. This involves Lemma 2.10 and the density of ${}^{SO(n)}\mathcal{D}(\mathbf{R}^n)$ in ${}^{SO(n)}L^p(\mathbf{R}^n)$. We now examine

$$\begin{aligned}|\langle \lambda(\Psi_k), f * g\rangle| &= |\langle \lambda(\Psi_k), Z_{SO(n)}(f * g)\rangle| \\ &= \left|\sum_{l=0}^\infty \Psi_k * F_l * G_l(0)\right| \le \sum_{l=0}^\infty c_p \|F_l\|_p \|G_l\|_{p'}\end{aligned}$$

on account of Lemma 4.3.

However, this shows that for all $k \ge 1$,

$$\left|\int_{\mathbf{R}^n} \hat{f}(\xi)\hat{g}(f)(\psi(|\xi|))^k\, d\xi\right| \le 2\kappa c_p \|f\|_p \|g\|_{p'}. \tag{4.5}$$

The left-hand side converges to $|\langle T_1^\circ, f * g\rangle|$ as $k \to \infty$ and so (4.5) contradicts Fefferman's solution of the multiplier problem for the ball, [**Ff**]. Q.E.D.

5. Central multipliers. Let G be a d-dimensional, compact, simply connected, simple Lie group of rank r, with a fixed maximal torus T. In this case $G \in [FIA]_B^-$, with B the group of inner automorphisms of G, and Z_B is the operation of centralization,

$$Z_B f(x) = \int_G f(yxy^{-1})\, dm_G(y).$$

Hence, ${}^B A_p(G)$ is the subalgebra of central functions in $A_p(G)$ and ${}^B L^p(G)$ is the subspace of central elements of $L^p(G)$.

We use some results of Stanton and Tomas, [**SnTo**], to show that ${}^B A_p(G) \neq \mathfrak{A}_p(G, B)$ for certain values of p. Fix a Weyl group-invariant polyhedron R in the Lie algebra of T and let $\{D_n \colon n \geq 1\}$ be the Dirichlet kernels for summation of Fourier series on G, as described in [**SnTo**, p. 478]. There is a constant $p(R)$, satisfying

$$2d/(d+r) \leq p(R) \leq (2d - 2r + 2)/(d - r + 2) < 2,$$

such that for all $p(R) < p \leq 2$ and $n \geq 1$

$$\|D_n * f\|_p \leq \text{const.}_p \|f\|_p, \qquad \forall f \in {}^B L^p(G).$$

However, if $p \neq 2$ then

$$\sup_{n \geq 1} |||\lambda(D_n)|||_p = \infty.$$

5.1 THEOREM. *For G, B, R and $p(R)$ as above and $p(R) < p < 2$, we have*

$${}^B A_p(G) \neq P({}^B L^p(G) \hat{\otimes} {}^B L^{p'}(G)).$$

This is an immediate consequence of §3 and [**SnTo**, Theorems D and E].

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DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF ADELAIDE, G.P.O. BOX 498, ADELAIDE, SOUTH AUSTRALIA 5001, AUSTRALIA

Current address: Department of Mathematics, University of Texas, Austin, Texas 78712