NONVANISHING LOCAL COHOMOLOGY CLASSES

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ABSTRACT. We discuss the nonvanishing of a top-dimensional canonical cohomology class of the space \( BS_{\text{diff}} M \). We treat parallelizable and odd-dimensional stably parallelizable manifolds.

Conventions. The following hold unless stated otherwise:
1. \( M \) is a closed, oriented, \( n \)-dimensional smooth manifold with volume form \( \omega \).
2. All \( H^* \) and \( H_* \) are with \( \mathbb{R} \)-coefficients.
3. \( \deg f \) is the degree of \( f \).
4. \( [\eta] \) is the cohomology class of the closed form \( \eta \).

Introduction. Let \( \text{Diff}_\omega M \) denote the group of volume preserving (with respect to \( \omega \)) diffeomorphisms of \( M \) with the \( C^\infty \) topology. We will investigate certain local cohomology classes of \( \text{Diff}_\omega M \), originally defined by McDuff [M–1]. These classes are related to the Gelfand-Fuks cohomology of divergence free (with respect to \( \omega \)) vector fields on \( M \). These classes also, in a certain sense, measure how \( \text{Diff}_\omega M \) twists \( M \). In this paper we will show that the top-dimensional class in question is nonzero if \( M \) is parallelizable or if \( M \) is odd-dimensional and stably parallelizable.

Background. Let \( \text{Diff}^\delta M \) denote \( \text{Diff}_\omega M \) with the discrete topology. If \( \mathcal{G} \) is a topological group, then \( B\mathcal{G} \) denotes its classifying space. The inclusion \( i: \text{Diff}^\delta M \to \text{Diff}_\omega M \) passes to a continuous map

\[
i: B\text{Diff}^\delta M \to B\text{Diff}_\omega M.
\]

Let \( B\text{Diff}_\omega M \) be the homotopy theoretic fibre of \( (1) \). This space is well-defined up to homotopy type. By the local cohomology of \( \text{Diff}_\omega M \) we mean the (real) singular cohomology of \( B\text{Diff}_\omega M \).

The space \( B\text{Diff}_\omega M \) is in its own right a classifying space. It classifies globally trivialized \( M \)-bundles with a flat structure. Sitting over \( B\text{Diff}_\omega M \) we have the universal bundle \( B\text{Diff}_\omega M \times M \). Let us now choose a specific model for \( B\text{Diff}_\omega M \) [Ma].

Let \( \text{Sing}\text{Diff}_\omega M \) be the smooth singular complex of \( \text{Diff}_\omega M \). Note that \( \text{Diff}_\omega M \) acts freely on \( \text{Sing}\text{Diff}_\omega M \) by multiplication on the right, so it also acts freely on the...
geometric realization \([\text{Sing} \, \text{Diff}^c_* \, M]\). We can now form the quotient space \([\text{Sing} \, \text{Diff}^c_* \, M]/\text{Diff}^c_* \, M\). This will be our particular model for \(\text{B} \, \text{Diff}^c_* \, M\) \([\text{M}-1]\).

Thus \(\overline{\text{B} \, \text{Diff}^c_* \, M}\) has a PL-structure in which a \(k\)-simplex is a smooth map \(\Delta^k \to \text{Diff}^c_* \, M\), which is well defined up to composition on the right by an element of \(\text{Diff}^c_* \, M\). To get rid of this ambiguity we ask for the 0-vertex to go to the identity diffeomorphism;

\[(\Delta^k, 0) \to (\text{Diff}^c_* \, M, 1), \quad t \to h_t.\]

We are now in a position to define a foliation \(\mathcal{F}\) on \(\overline{\text{B} \, \text{Diff}^c_* \, M} \times M\).

To each \(k\)-simplex in \(\overline{\text{B} \, \text{Diff}^c_* \, M}\) we can associate a foliation on \(\Delta^k \times M\) with leaves

\[\mathcal{L}_m = \{(t, h_t(m)) : t \in \Delta^k\}.\]

This foliation is the pull-back of the point foliation on \(M\) by the map \(f : (t, m) \to h_t^{-1}(m)\). The foliation is volume preserving \([L]\) and has \(f^* \omega\) for its transverse volume form.

The foliations and transverse volume forms on each \((k\text{-simplex}) \times M\) fit together to give a codimension-\(n\) volume preserving (generalized) foliation \(\mathcal{F}\) on \(\overline{\text{B} \, \text{Diff}^c_* \, M} \times M\) with transverse volume form \(\Omega\).

For more background information see \([\text{Mo}]\).

**McDuff's classes.** The cohomology class \([\Omega]\) may be decomposed via the Künneth formula.

\[\[\Omega\] = \bigoplus_{i=0}^n [\Omega]_i, \quad \text{where} \quad [\Omega]_i \in H^{n-i}(\overline{\text{B} \, \text{Diff}^c_* \, M}) \otimes H^i(M).\]

The class \([\Omega]_i\) may be written as \(\sum_j \alpha_{n-i}^j \otimes \beta_i^j\), where \(j\) is indexed over the rank of \(H^i(M)\), \(\alpha_{n-i}^j \in H^{n-i}(\overline{\text{B} \, \text{Diff}^c_* \, M})\), and \(\beta_i^j \in H^i(M)\). Of course this representation is not unique.

We may now define the McDuff classes \(c_k(M) \in H^k(\overline{\text{B} \, \text{Diff}^c_* \, M}; H^{n-k}(M))\) by the formula

\[c_k(M) \kappa = \sum_j \langle \alpha_{n-i}^j, \kappa \rangle \cdot \beta_i^j,\]

where \(\kappa \in H_k(\overline{\text{B} \, \text{Diff}^c_* \, M})\). Note that the class \(c_k(M)\) is independent of the choices made. In fact \(c_k(M) \kappa\) is just, neglecting sign, the slant product of \([\Omega]_{n-k}\) with \(\kappa\). We will use absolute value signs to show that we are neglecting sign. Thus, we may also denote \(c_k(M) \kappa\) by

\[(4') \quad [\Omega]_i/|\kappa|].\]

In this paper we are concerned with the top class \(c_n(M)\). We may express \([\Omega]_0\) as

\[\alpha_n \otimes 1 \in H^n(\overline{\text{B} \, \text{Diff}^c_* \, M}) \otimes H^0(M) \equiv H^n(\overline{\text{B} \, \text{Diff}^c_* \, M}) \otimes \mathbb{R}.\]

So we may identify \(H^n(\overline{\text{B} \, \text{Diff}^c_* \, M})\) with \(H^n(\overline{\text{B} \, \text{Diff}^c_* \, M}) \otimes H^0(M)\) and thus identify \(c_n(M)\) with \([\Omega]_0\). The class \([\Omega]_0\) is the same as \([\Omega]\) restricted to \(\overline{\text{B} \, \text{Diff}^c_* \, M} \times \text{pt}\). Therefore,

\[c_n(M) \neq 0 \iff [\Omega]_0|_{\overline{\text{B} \, \text{Diff}^c_* \, M} \times \text{pt}} \neq 0.\]
Fundamental diagram. The diagram we use follows from [T]. Recall that $B\Gamma^n$ is the classifying space for codimension-$n$ volume preserving Haefliger structures. There is a special cohomology class $\bar{\mu} \in H^n(B\Gamma^n)$ called the universal transverse volume class. If $f : X \to B\Gamma^n$ classifies a codimension-$n$ volume preserving Haefliger structure $H$, then $f^*\bar{\mu} \in H^n(X)$ is called the transverse volume class of $H$. If $H$ is actually the Haefliger structure coming from a codimension-$n$ volume preserving foliation with transverse volume form $\lambda$, then $[\lambda] = f^*\bar{\mu}$. The class $\bar{\mu}$ can be constructed directly from the topological groupoid of germs $\Gamma^n$, or it can be defined by a functorial principle.

Associated to $B\Gamma^n$ we have the normal bundle of its universal Haefliger structure. This bundle can be classified by a map $d : B\Gamma^n \to B\text{SL}(n, \mathbb{R})$. The map $d$ has a homotopy theoretic fibre $B\Gamma^n$. We choose models and maps so that we get an actual Hurewicz fibration

$$B\Gamma^n \xrightarrow{i} B\Gamma^n \xrightarrow{d} B\text{SL}(n, \mathbb{R}).$$

The space $B\Gamma^n$ is the classifying space for codimension-$n$ volume preserving Haefliger structures whose normal bundle is framed. This space is $(n-1)$-connected with $\pi_{n-1}(B\Gamma^n) = \mathbb{R}$. The class $i^*\bar{\mu} \equiv \mu \in H^n(B\Gamma^n)$ corresponds to the identity homomorphism if we identify $H^n(B\Gamma^n)$ with $\text{Hom}_{\mathbb{Z}}(\mathbb{R}, \mathbb{R})$ in the canonical manner.

The foliation $\mathcal{F}$ on $B\text{Diff}_\omega M \times M$ can be classified by the map $\Phi$ into $B\Gamma^n$. Consider the following diagram:

$$B\text{Diff}_\omega M \times M \xrightarrow{\Phi} B\Gamma^n \xrightarrow{d} B\text{SL}(n, \mathbb{R}).$$

Here $\pi$ is projection, and $\tau$ classifies $TM$ (the choice of $\omega$ gives an $\text{SL}$-structure). The normal bundle $\nu(\mathcal{F})$ of $\mathcal{F}$ is just $B\text{Diff}_\omega M \times TM$ since $\mathcal{F}$ is transverse to the $M$-factors. We classify $\nu(\mathcal{F})$ by $d \circ \Phi$. Note that $\nu(\mathcal{F})$ is also classified by $\tau \circ \pi$. Everything is chosen so that (7) commutes and $d$ is a Hurewicz fibration.

We are interested in $L(M)$, the space of lifts of $\tau$. Let us define $\Pi : B\text{Diff}_\omega M \to L(M)$ by setting $\Pi(b) = \Phi(b, \cdot)$ for each $b \in B\text{Diff}_\omega M$. Since $M$ is compact and $n$-dimensional while the fibre of $d$ is $B\Gamma^n$, which is $(n-1)$-connected, the space $L(M)$ is not connected. Because $B\text{Diff}_\omega M$ is connected, its image under $\Pi$ is in one component, $L_0(M)$, of $L(M)$. McDuff [M–2], in the spirit of Thurston [T], has shown that $\Pi^*$ is a cohomology isomorphism. This enables us to view $c_n(M)$ as living in $H^n(L_0(M))$. Let us exploit this philosophy.

Define $\epsilon : L_0(M) \to B\Gamma^n$ as evaluation at the fixed point $m_0$ in $M$. Consider $\mathcal{F}\mid$, $\mathcal{F}$ restricted to $B\text{Diff}_\omega M \times m_0$. The bundle $\nu(\mathcal{F}\mid)$ is isomorphic to $B\text{Diff}_\omega M \times R^n$. This tells us that $\mathcal{F}\mid$ is a codimension-$n$ Haefliger structure with trivial normal bundle. Due to this $\Phi\mid$, the map classifying $\mathcal{F}\mid$, is homotopic to a map with image in $B\Gamma^n$. Without loss of generality we may assume $\Phi\mid$ actually maps into $B\Gamma^n$. The following diagram homotopy commutes.
Recall that $c_n(M)$ is $[\Omega]|_{\partial M \times M \times m_0}$. Therefore, $\Phi^*\mu$ is $c_n(M)$. Since $\Pi^*$ is an isomorphism,

(9) $c_n(M) \neq 0 \iff \epsilon^*\mu \neq 0$.

So the problem is now one of understanding the behavior of $\mathcal{L}_0(M) \rightarrow B\Gamma^n_s$. Inherent in $\mathcal{L}_0(M)$ is the twisting of $TM$. The simpler that $TM$ is, the easier it is to understand $\epsilon$. Such a case is provided when $M$ is stably parallelizable. By this we mean that $TM \oplus e^1 \cong e^{n+1}$. The most obvious example of such a manifold is $S^n$. In fact spheres "classify" stably parallelizable manifolds.

(10) **Proposition.** The manifold $M$ is stably parallelizable if and only if $TM \cong \gamma^*(TS^n)$ for some map $\gamma$ from $M \rightarrow S^n$.

Before proceeding further we need a technical lemma which follows easily from obstruction theory.

(11) **Lemma.** If $f_0$ and $f_1$ are two lifts in $\mathcal{L}(M)$, they are in the same component of $\mathcal{L}(M)$ if and only if $f_0^*\bar{\mu} = f_1^*\bar{\mu}$.

Since $\Pi(b)$ is $\Phi(b, \cdot)$ and $\Phi(b, \cdot)^*\bar{\mu}$ is $[\Omega]_{b \times M} = [\omega]$, we have

(12) $f \in \mathcal{L}_0(M) \iff f^*\bar{\mu} = [\omega]$.

(13) **Theorem.** If $M$ is an odd-dimensional stably parallelizable manifold, then $c_n(M) \neq 0$.

**Remark.** It is essential that $n$ be odd, for McDuff [M–1] has shown that $c_{2n}(S^{2n}) = 0$ and $c_{2n+1}(S^{2n+1}) \neq 0$.

**Proof.** Case 1. Suppose that $M$ is stably parallelizable but not parallelizable. Then by (10) $TM \cong \gamma^*(TS^n)$, and $\gamma$ has nonzero degree. Certainly we can give $M$ a volume form $\omega_M$ such that $[\omega_M] = \gamma^*[\omega_S]$, where $\omega_S$ is the standard volume form on $S^n$. Consider the following diagram.

\[
\begin{array}{ccc}
M & \rightarrow & S^n \\
\gamma \downarrow & & \downarrow \gamma \\
B\varGamma^n_s & \rightarrow & B\text{Sl}(n, R)
\end{array}
\]

The map $\gamma$ classifies $TS^n$ and therefore $\tau \circ \gamma$ classifies $TM$. Say $f \in \mathcal{L}_0(S^n)$; then $f^*\bar{\mu} = [\omega_S]$. It follows that $f \circ \gamma \in \mathcal{L}_0(M)$. Hence, we have a map $\eta: \mathcal{L}_0(S^n) \rightarrow \mathcal{L}_0(M)$ given by $\eta(f) = f \circ \gamma$. Consider the next diagram, where $\epsilon_m$ is evaluation at $m_0 \in M$ and $\epsilon_s$ is evaluation at $\gamma(m_0) \in S^n$.

\[
\begin{array}{ccc}
\mathcal{L}_0(S^n) & \eta & \mathcal{L}_0(M) \\
\downarrow \epsilon_s & & \downarrow \epsilon_m \\
\overline{B}\varGamma^n_s & \rightarrow & \overline{B}\Gamma^n_s
\end{array}
\]
Since $\varepsilon_m \circ \eta(f) = \varepsilon_m(f \circ \gamma) = (f \circ \gamma)(m_0) = \varepsilon_s \circ f$, the above diagram commutes.

Since $c_n(S^n) \neq 0$ for $n$ odd, (9) tells us that $\varepsilon^*_\mu \neq 0$; therefore $\varepsilon^*_\mu \neq 0$ and $c_n(M) \neq 0$.

Case 2. $M$ is parallelizable. Let $\tau$ now stand for a map classifying $TM$ into $BSO(n, R)$. Without loss of generality $\tau$ may be taken as a constant map. In this case (7) becomes

$$\begin{array}{ccc}
\overline{B} \text{Diff}_0 M \times M & \phi & \overline{B} \Gamma^n_{s1} \rightarrow \overline{B} \Gamma^n_{s1} \\
\downarrow \pi & & \downarrow d \\
M & \tau & \rightarrow BS O(n, R)
\end{array}$$

Since $M$ is parallelizable, $\mathcal{L}(M) = \text{Maps}(M, \overline{B} \Gamma^n_{s1})$ and we will designate the component corresponding to $\mathcal{L}_0(M)$ as $\text{Maps}_1(M, \overline{B} \Gamma^n_{s1})$. Therefore, $s \in \text{Maps}_1(M, \overline{B} \Gamma^n_{s1})$ if and only if $s^*\mu = [\omega]$. The evaluation map $\varepsilon: \mathcal{L}_0(M) \rightarrow \overline{B} \Gamma^n_{s1}$ becomes $\varepsilon: \text{Maps}_1(M, \overline{B} \Gamma^n_{s1}) \rightarrow \overline{B} \Gamma^n_{s1}$. We wish to show that $\varepsilon^*\mu \neq 0$. Choose $\omega$ so that $\langle [\omega], [M] \rangle = 1$.

Following McDuff’s proof [M-1] that $\pi_n(\overline{B} \Gamma^n_{s1}) \cong \mathbb{R}$, we choose a map $f: S^n \rightarrow \overline{B} \Gamma^n_{s1}$ such that $[f] = 1 \in \pi_n(\overline{B} \Gamma^n_{s1})$, i.e. $\langle f^*\mu, [S^n] \rangle = 1$. Let us give $S^n$ a volume form $\omega_S$ so that $f^*\mu = [\omega_S]$. We will say that $g: M \rightarrow S^n$ is of degree one if $g^*[\omega_S] = [\omega]$. Let $\text{Maps}_1(M, S^n)$ be all the maps of degree one. Now define a map $f: \text{Maps}_1(M, S^n) \rightarrow \text{Maps}_1(M, \overline{B} \Gamma^n_{s1})$ by setting $f(g) = f \circ g$. Let $\varepsilon$ and $\varepsilon'$ be evaluation at $m_0$. The following diagram commutes:

$$\begin{array}{ccc}
\text{Maps}_1(M, S^n) & \xrightarrow{f} & \text{Maps}_1(M, \overline{B} \Gamma^n_{s1}) \\
\downarrow \varepsilon' & & \downarrow \varepsilon \\
S^n & \xrightarrow{f} & \overline{B} \Gamma^n_{s1}
\end{array}$$

Since $f^*\mu = [\omega_S]$, if we can show that $\varepsilon'^*[\omega_S] \neq 0$, we will have shown that $\varepsilon^*\mu \neq 0$.

$\text{Maps}_1(S^n, S^n)$ is the space of maps from $S^n$ to $S^n$ such that $f^*[\omega_S] = [\omega_S]$ (as before). Let $\xi$ be a fixed element of $\text{Maps}_1(M, S^n)$ such that $\xi(m_0) = s$, the south pole of $S^n$. Define $\xi: \text{Maps}_1(S^n, S^n) \rightarrow \text{Maps}_1(M, S^n)$ by $\xi(h) = h \circ \xi$. The following diagram, with $\varepsilon''$ being evaluation at $s$, commutes.

$$\begin{array}{ccc}
\text{Maps}_1(S^n, S^n) & \xrightarrow{\xi} & \text{Maps}_1(M, S^n) \\
\downarrow \varepsilon'' & & \downarrow \varepsilon' \\
S^n & \xrightarrow{\text{id}} & S^n
\end{array}$$

If we can show that $\varepsilon''*[\omega_S] \neq 0$, then we will have shown that $\varepsilon^*[\omega_S] \neq 0$ and we will be done. The left-hand side of (17) is from the fibration

$$\Omega^n S^1_n \rightarrow \text{Maps}_1(S^n, S^n) \rightarrow S^n,$$

where $\Omega^n S^1_n$ is the obvious component. Using the spectral sequence of (18) and the fact that $n$ is odd, we have $\varepsilon''*[\omega_S] \neq 0$. Q.E.D.
We will now prove

(19) **Theorem.** If $M$ is parallelizable then $c_n(M) \neq 0$.

**Remark.** Here we have removed the condition of $n$ being odd but yet we still use $S^n$ in our argument.

**Proof.** Since $M$ is parallelizable it may be immersed in $\mathbb{R}^{n+1}$ [H]. We will consider $M$ as being immersed in $\mathbb{R}^{n+1}$ with a fixed immersion. We will also freely identify $M$ with its image in $\mathbb{R}^{n+1}$. This causes no trouble, as will become apparent. Our goal is to define a map $\Psi: M \to \text{Maps}_n(M, S^n)$ such that if $\epsilon'$ is evaluation at $m_0$ then $\epsilon' \circ \Psi: M \to S^n$ has nonzero degree. If this is true then $\epsilon'[\omega_S]$ is nonzero, where $\omega_S$ is the volume form on $S^n$. Let $\tilde{\Psi}$ be the composite $\epsilon' \circ \Psi$.

\[
\begin{array}{ccc}
M & \xrightarrow{\Psi} & \text{Maps}_n(M, S^n) \\
\downarrow \epsilon' & & \\
S^n \end{array}
\]

Choose positive $\epsilon$ less than the injectivity radius of $M$ and small enough so that every (open) ball $B(p, \epsilon)$ with center $p \in M$ and radius $\epsilon$ is embedded by the immersion. Consider $\text{Exp}^{-1}: B(p, \epsilon) \to TM_p$. In fact, the image is in $B(O_p, \epsilon) \subset TM_p$. Let $\gamma$ be the Gauss map from $M$ to $S^n$. By parallel translation in $\mathbb{R}^{n+1}$ we get a congruence from $TM_p \to TS^n_{\gamma(p)}$. Remembering that $\pi$ is the injectivity radius of the sphere we see that $\text{Exp}$ maps $B(\bar{O}_{\gamma(p)}, \pi) \subset TS^n_{\gamma(p)}$ diffeomorphically onto $S^n - A(\gamma(p))$, where $A$ is the antipodal map on $S^n$. Let $\xi: TS^n \to TS^n$ by $\xi(\bar{v}) = \pi/\epsilon \cdot \bar{v}$. We are now in a position to define $\Psi: M \to \text{Maps}_n(M, S^n)$.

Decompose $M$ as $B(p, \epsilon) \cup \{M - B(p, \epsilon)\}$:

\[
\begin{array}{ccc}
B(p, \epsilon) & \xrightarrow{\text{Exp}^{-1}} & \bar{B}(\bar{O}_p, \epsilon) \\
& \xrightarrow{|=\text{trans}|} & \bar{B}(\bar{O}_{\gamma(p)}, \epsilon) \\
& \downarrow \xi & \\
& B(\bar{O}_{\gamma(p)}, \pi) & \\
& \downarrow \text{Exp} & \\
S^n - A(\gamma(p)) & & \\
\end{array}
\]

If $m \in B(p, \epsilon)$ define $\Psi(p)m$ to be the image of $m$ under the above composition. At this stage we want the orientation on $B(p, \epsilon)$, induced from the orientation of $M$, to go to the usual orientation on $S^n - A(\gamma(p))$. If it does not we give $M$ a different orientation and volume form. If $m \notin B(p, \epsilon)$ set $\Psi(p)m = A(\gamma(p))$.

As we vary $p$ we get a continuous map $\Psi: M \to \text{Maps}_n(M, S^n)$. Each $\Psi(p)$ is of degree one, for $\Psi(p)$ is just a standard collapsing map of degree one.

(22) **Lemma.** $\deg \tilde{\Psi} = (-1)^{n+1} \deg \gamma + (-1)^n$, where $\deg$ stands for degree of the map.

**Proof of Lemma.** We just sketch the proof since the techniques are standard. If necessary, the first thing we do is adjust $\epsilon$ in our definition of $\Psi$ to make sure that in a neighborhood $D$ containing $B(m_0, \epsilon)$ we can slightly deform $M$ so that $D$ is flat.
Thus, the Gauss map $\gamma$ is constant on $D$, hence $\gamma(D) = \gamma(m_0)$. On $M - D$, $\overline{\gamma}$ sends $x$ to $\overline{\Psi}(x)m_0 = A(\gamma(x))$. If $x \in D$ then we must be careful. If $x \in D - B(M_0, \epsilon)$, then $\overline{\Psi}(x) = \Psi(x)m_0 = A(\gamma(x)) = A(\gamma(m_0))$ since $D$ is flat. The map $\overline{\gamma}$ restricted to $B(m_0, \epsilon)$ has degree $(-1)^n$. $\overline{\gamma}$ is not smooth on $\partial B(m_0, \epsilon)$. Therefore, we replace it by a map $\overline{\gamma}'$ in the same homotopy class and very close to $\overline{\gamma}$ and calculate the Brouwer degree. The $(-1)^{n+1}\deg \gamma$ part comes from $A \circ \gamma$, while $(-1)^n$ is due to the fact that $\overline{\gamma}$, near $m_0$, is a local diffeomorphism. By a judicious choice of a regular value it is easy to show that $\overline{\gamma}'$, and hence $\overline{\gamma}$, has the proper degree. The proof of the lemma is complete.

Remember that we are trying to show that if $M$ is parallelizable then $c_n(M) \neq 0$. By (16) if we can show that $\epsilon^*[\omega_j] \neq 0$ we will be done. By the previous lemma $\overline{\gamma}$ has degree $(-1)^n + (-1)^{n+1}\deg \gamma$. If we can show that this is nonzero, then $\epsilon^*[\omega_j] \neq 0$. This is our plan. We will vary our immersion of $M$ so that we get $\deg \gamma$ to our liking. To accomplish this we appeal to some results of Hopf [Ho, Mi].

(23) Theorem (Hopf). Let $L: M^n \to \mathbb{R}^{n+1}$ be an immersion with corresponding Gauss map $\gamma(L): M^n \to S^n$.

(a) If $n$ is even, $\deg \gamma(L) = \frac{1}{2}\chi(M)$, where $\chi(M)$ is the Euler characteristic of $M$.

(b) If $n$ is odd and $\deg \gamma(L) = k$, then given any $m$ one can find an immersion $j = j(m)$ such that $\deg \gamma(j) = k + 2m$.

Thus $\overline{\gamma}$ can be taken to have nonzero degree. Q.E.D.

Remark. It is worth pointing out that all 3-manifolds and products of spheres with one factor being odd are parallelizable and therefore have $c_n(M) \neq 0$.

References


