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PERIODIC SOLUTIONS OF HAMILTON'S EQUATIONS AND LOCAL MINIMA OF THE DUAL ACTION

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ABSTRACT. The dual action is a functional whose extremals lead to solutions of Hamilton's equations. Up to now, extremals of the dual action have been obtained either through its global minimization or through application of critical point theory. A new methodology is introduced in which local minima of the dual action are found to exist. Applications are then made to the existence of Hamiltonian trajectories having prescribed period.

1. Introduction. A classical and very interesting boundary-value problem is concerned with periodic solutions of Hamilton's equations:

$$-\dot{p}(t) = \nabla_x H(x(t), p(t)), \qquad \dot{x}(t) = \nabla_p H(x(t), p(t)), \tag{1.1}$$

where the function $H\colon R^n \times R^n \to R$ is called the *Hamiltonian.* Any such "Hamiltonian trajectory" (x, p) is such that $H(x(t), p(t))$ is constant; this constant is called the *energy* associated with (x, p). If $x(0) = x(T)$, $p(0) = p(T)$ for some $T > 0$, then (x, p) is said to have period T. One can ask for what energies and periods are their associated Hamiltonian trajectories. This and related issues have both a long history and an active present; we refer to Desolneux and Moulis [**16**], Mancini [**20**], and Rabinowitz [**23**] for surveys and a more extensive bibliography than would be appropriate here.

In this article, attention will be focused upon the case in which the Hamiltonian is a convex function, a situation which has drawn perhaps the most attention of late. We begin by reviewing briefly the basis for much of this recent work, a dual action principle introduced by Clarke in 1978 [**6**].

Let G denote the conjugate of H in the sense of convex analysis; i.e., the function

$$G(u, v) := \sup_{x,p}\{(u, v) \cdot (x, p) - H(x, p)\}, \tag{1.2}$$

where the supremum is taken over all (x, p) in $R^n \times R^n$. For a given $T > 0$, we consider a class of functions (x, p) mapping $[0, T]$ to $R^n \times R^n$, and we define an integral functional J_T on the class by the formula

$$J_T(x, p) := \int_0^T \{\langle \dot{p}(t), x(t)\rangle + G(-\dot{p}(t), \dot{x}(t))\}\, dt.$$

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It is J_T that we dub the *dual action*, and its utility is predicated on the fact that, by and large, when H is convex, and under homogeneous boundary conditions, extremals (i.e., critical points) of J_T correspond (modulo translation by a constant) to solutions (x, p) of Hamilton's equations (1.1) having period T. (A precise instance of this general principle appears in §3 as Proposition 3.1.) If then we are interested in solutions of Hamilton's equations, we may seek to produce and study extremals of J_T. So far this has been done in two ways: through global minimization of J_T and through the application of abstract critical point theory.

The first use of this approach occurred in the case in which H is the gauge function of some level set of the original Hamiltonian [**6, 8**]. (The intent was to give a simple proof of the theorem of Rabinowitz [**21**] and of Weinstein [**24**] of the existence of Hamiltonian trajectories of prescribed energy.) In this case the dual Hamiltonian G defined by (1.2) is a certain indicator function, a function having only the values 0 and $+\infty$, and the problem of minimizing J_T turns out to be an optimal control problem, one that has a solution. Since minimizers are extremals (here "extremal" is to be understood in the sense of [**10**, Definition 3.2.5]), the required Hamiltonian trajectory ensues.

In [**7**] Clarke gave an alternate derivation of the same result using only classical methods of the calculus of variations: the Legendre transform, the Tonelli existence theorem, the Hilbert multiplier rule for isoperimetric problems. In [**9**], again in an optimal control setting, and employing sensitivity analysis (see [**10**, Theorems 6.3.2 and 7.7.3]), the prescribed energy theorem was extended to nonperiodic boundary conditions.

Clarke and Ekeland [**11, 12**] employed the dual action in the case in which the issue is to find Hamiltonian trajectories of prescribed period. In this study, H was assumed to have subquadratic growth, in order that the function G (now finite) exhibit sufficient growth (coercivity) to allow, once again, global minimization of J_T. (A variant of this result, requiring growth limitation in one variable only, is given in [**10**, Theorem 8.8.1].)

The case in which H is superquadratic had been treated earlier by Rabinowitz [**22**] by entirely different means. It seemed at first to evade the purlieu of the dual action principle because of the fact that when H is superquadratic, the dual action J_T does not admit a minimum. As shown by Ekeland [**17**], however, the critical point theory of Ambrosetti and Rabinowitz could successfully be applied to J_T to get a relatively simple proof in the superquadratic case. (See Mancini [**20**] for further elaboration.)

Following the above developments, a variety of extensions and applications to other problems were made in which the dual action was subject either to minimization or critical point analysis (see, for example, Ambrosetti and Mancini [**2**], Brézis, Coron and Nirenberg [**4**], Clarke and Ekeland [**13**], and Ekeland and Lasry [**18**]).

The purpose of this article is to introduce a new methodology by which *local* minima of the dual action J_T can be asserted to exist. Specifically, we will show that under certain hypotheses, for γ and T suitably related, the problem of minimizing $J_T(x, p)$ over all continuously differentiable functions (x, p) on $[0, T]$ satisfying

$$x(0) = x(T) = p(0) = p(T) = 0, \qquad |x(t)| < \gamma,$$

admits a (nontrivial) solution. (The case of a global minimum is then the special one in which $\gamma = +\infty$.)

We emphasize that our results apply in situations where J_T definitely fails to admit a global minimum. The approach serves to unify and extend the class of Hamiltonians for which periodic solutions can be characterized by a strong variational principle. A further feature of the method lies in the relative ease with which one can obtain the (often elusive) conclusion that the Hamiltonian trajectory it produces has *minimal* (or true) period T.

The *basic hypothesis* of this article, valid throughout, is that the Hamiltonian $H(x,p)$ is a continuously differentiable nonnegative convex function vanishing only at the origin, and satisfying the superlinearity condition

$$\lim_{|(x,p)|\to\infty} H(x,p)/|(x,p)| = \infty.$$

We now proceed to state the main results of the article concerning periodic trajectories. (The function H is said to be *radially increasing in* p if one has $\langle p, H_p(x,p)\rangle > 0$ for all (x,p) with $p \neq 0$.)

THEOREM 1.1. (i) *Suppose that* H *satisfies the condition*

$$(\mathrm{C}_0) \qquad |(x,p)| < \delta \quad \textit{implies } H(x,p) \geq a|x|^{1+r} + b|p|^{1+s},$$

where δ, a, b, r *and* s *are positive constants with* $rs < 1$. *Then for all* T *sufficiently small there is a solution of Hamilton's equations having period* T. *If* H *is radially increasing in* p, *there is a solution of minimal period* T.

(ii) *Suppose that* H *satisfies the condition*

$$(\mathrm{C}_\infty) \qquad |(x,p)| > \Delta \quad \textit{implies } H(x,p) \leq A|x|^{1+R} + B|p|^{1+S},$$

where Δ, A, B, R *and* S *are positive constants with* $RS < 1$. *Then for all* T *sufficiently large there is a solution of Hamilton's equations having minimal period* T.

(iii) *Suppose that* H *satisfies both* (C_0) *and* (C_∞). *Then for any* $T > 0$ *there is a solution of Hamilton's equations having minimal period* T.

The hypotheses above define a new class of Hamiltonians in the context of such a theorem. In (iii), for example, it is possible for H to exhibit at ∞ superquadratic growth in x (say) (i.e., $R > 1$) while H may behave subquadratically in x near 0 (i.e., $r < 1$). This is in contrast to those previous works in which H is either purely sub- or superquadratic. Note that the condition $RS < 1$ in (C_∞) requires that, on balance, H be more subquadratic in one variable than it is superquadratic in the other. The fully subquadratic case first treated by Clarke and Ekeland [**12**] is subsumed by part (iii) of the theorem by taking r, s, R and S all less than 1. (The stratification into three distinct statements is a new result.) The notable case that seems not to be amenable to our approach is the purely superquadratic one; see the references cited in the Introduction as well as the recent work of Girardi and Matzeu [**19**]. Finally we remark that in (i) and (ii) of the theorem, explicit estimates of how small or large T must be are immediate outgrowths of the proof (§4).

An important special case of the Hamiltonian is that in which $H(x,p) = |p|^2/2 + V(x)$, for then Hamilton's equations reduce to Newton's equation:

$$(\mathrm{N}) \qquad \ddot{x} = -\nabla V(x).$$

To assure that H satisfies the basic hypotheses, we suppose that V is C^1, convex, nonnegative, that $V(x) = 0$ iff $x = 0$, and that $\lim_{|x|\to\infty} V(x)/|x| = +\infty$.

For purposes of the following result, we set

$$v(\alpha) := \max\{V(x) \colon |x| \le \alpha\},$$
$$T_1 := \sup\{\gamma\alpha/[v(\gamma) + \alpha^2/2] \colon \gamma > 0,\ \alpha > 0\}.$$

THEOREM 1.2. *Suppose that for some positive ε and λ, V satisfies the condition*

$$(\mathrm{V}_0) \qquad V(x) \ge \varepsilon|x|^{2-\lambda} \quad \textit{for } x \textit{ near } 0.$$

Then, for every T in the interval $(0, T_1)$ there is a solution of Newton's equations (N) *of minimal period T.*

Newton's equation was studied by Clarke and Ekeland [**13**] under a global subquadricity assumption on V; see also Mancini [**20**].

The next section is devoted to proving the key result in our approach, the local existence theorem for P_T^γ, together with a new result on global existence. We make frequent use here of the methods of nonsmooth analysis [**10**], and of some recent techniques of Clarke and Vinter [**14, 15**]. §3 studies the issues of nontriviality and minimality of the period. The applications to Hamiltonian trajectories (and in particular the proofs of the two theorems stated above) appear in §4.

2. Existence of local and global minima. In this section we are dealing with a Hamiltonian H satisfying the basic hypotheses of §1. G continues to denote the conjugate of H (see (1.2)) and J_T (for $T > 0$) the dual action functional:

$$J_T(x, p) = \int_0^T \{\langle \dot{p}, x\rangle + G(-\dot{p}, \dot{x})\}\, dt. \tag{2.1}$$

The problem P_T^γ (for $0 < \gamma \le \infty$) is defined to be that of minimizing $J_T(x, p)$ over all continuously differentiable arcs $(x, p) \colon [0, T] \to R^n \times R^n$ which satisfy the boundary conditions

$$(x, p)(0) = (x, p)(T) = (0, 0) \tag{2.2}$$

as well as the constraint

$$|x(t)| < \gamma \quad \text{for all } t \text{ in } [0, T]. \tag{2.3}$$

Note that the inequality in (2.3) is strict, which tends to make more elusive the existence of a solution to P_T^γ. Our hypotheses do not imply that G is (quadratically) coercive, although it is a standard fact from convex analysis that G is finite, nonnegative and strictly convex.

We define a function

$$h \colon (0, \infty) \times (0, \infty) \to R$$

by

$$h(\gamma, \alpha) = \max\{H(x, p) \colon |x| \le \gamma,\ |p| \le \alpha\},$$

and we set

$$T_\gamma := \sup\{\gamma\alpha/h(\gamma, \alpha) \colon \alpha > 0\}.$$

2.1. *The main result*; γ *finite.*

THEOREM 2.1. *If* γ *is any number in* $(0, \infty)$, *and if* T *is any number in* $(0, T_\gamma)$, *then the problem* P_T^γ *admits a solution.*

PROOF. By assumption there exists $\alpha > 0$ such that $T < \gamma\alpha/h(\gamma, \alpha)$. We may choose $\tilde{\gamma} > \gamma$ such that

$$T < \gamma\alpha/\beta, \tag{2.4}$$

where $\beta := h(\tilde{\gamma}, \alpha)$. We pause to list some technical results.

LEMMA 1. $G(y, q) \geq \tilde{\gamma}|y| + \alpha|q| - \beta$ *for all* (y, q).

To see this, consider the convex function F defined by

$$F(x, p) = \begin{cases} H(x, p) & \text{if } |x| \leq \tilde{\gamma},\ |p| \leq \alpha, \\ +\infty & \text{otherwise.} \end{cases}$$

We have $F \geq H$, whence $F^* \leq H^*$ $(= G)$ by conjugacy. We observe

$$\begin{aligned} F^*(y, q) &:= \sup\{(y, q) \cdot (x, p) - H(x, p) \colon |x| \leq \tilde{\gamma},\ |p| \leq \alpha\} \\ &\geq \sup\{(y, q) \cdot (x, p) - \beta \colon |x| \leq \tilde{\gamma},\ |p| \leq \alpha\} \\ &= \tilde{\gamma}|y| + \alpha|q| - \beta, \end{aligned}$$

which leads to the estimate in the lemma. Let the quantity $[\alpha^{-1} + (\tilde{\gamma} - \gamma)^{-1}]\beta$ be signified by Δ.

LEMMA 2. *There exists a number* r_0 *such that one has* $|\nabla H(z)| > \Delta$ *whenever* $|z| > r_0$.

If this were not the case, there would exist a sequence z_i with $|z_i| \to \infty$ such that $|\nabla H(z_i)|$ is bounded (by M, say). The convexity of H gives

$$H(0) - H(z_i) \geq \langle \nabla H(z_i), -z_i \rangle,$$

which in turn implies

$$H(z_i)/|z_i| \leq M.$$

This contradicts the basic growth hypothesis on H and proves the lemma.

LEMMA 3. *For each number* $r > 0$ *there exists a strictly convex function* $G_r \colon R^n \times R^n \to R$ *with the following properties*:

(a) $G_r(y, q) \geq \tilde{\gamma}|y| + \alpha|q| - \beta$ *for all* (y, q) (*cf. Lemma* 1).

(b) $G_r(y, q) = G(y, q)$ *if* $|(y, q)| \leq r$.

(c) *For some number* $\rho(r)$ *and for all* (y, q) *satisfying* $|(y, q)| > \rho(r)$, *one has* $G_r(y, q) = -\beta + T[|(y, q)|^2 - r^2]$.

For a proof of this needing only slight modification, the reader is referred to Clarke and Vinter [**15**, Lemma 5.1] (the functions L, L_r of that reference correspond to the functions $G + \beta$, $G_r + \beta$ of this article). Note the implication that G is strictly convex, a property always possessed by the conjugate of a differentiable convex function.

We now turn to the proof of the theorem, in which the first step consists of considering the following variational problem: to minimize (for given $r > 0$) the functional

$$J_T^r(x,p) := \int_0^T \{\langle \dot{p}, x\rangle + G_r(-\dot{p}, \dot{x})\}\, dt$$

over all (not necessarily smooth) absolutely continuous (x,p) satisfying the boundary conditions (2.2) and the constraint

$$|x(t)| \leq \gamma. \tag{2.5}$$

It is not difficult to prove directly through examination of a minimizing sequence that a solution (x_r, p_r) of this problem exists. The condition (c) plays the central role in the argument, which is given in detail in Clarke and Ekeland [**12**, Lemma 2] (the constraint (2.5) was not present in that reference, but necessitates no changes whatever).

Note that the zero arc is feasible for the above problem, whence $J_T^r(x_r,p_r) \leq J_T^r(0,0) = 0$. Together with (a) of Lemma 3 this yields

$$\alpha \int_0^T |\dot{x}_r|\, dt + (\tilde{\gamma} - \gamma) \int_0^T |\dot{p}_r|\, dt \leq \beta T,$$

which gives

$$\int_0^T |(\dot{x}_r, \dot{p}_r)|\, dt \leq T\Delta, \tag{2.6}$$

as well as

$$\int_0^T |\dot{x}_r|\, dt \leq \beta T/\alpha,$$

which in turn implies $|x_r(t)| \leq \beta T \alpha < \gamma$ (the latter by (2.4)). It follows that (x_r, p_r) provides a (strong) local minimum for the functional J_T^r (subject to the boundary conditions). This allows us to apply Clarke and Vinter [**14**, Proposition 3.2 and Theorem 2.1(ii)] to deduce that (x_r, p_r) is continuously differentiable. It follows now that (x_r, p_r) satisfies the Euler-Lagrange inclusion (Clarke [**5**]), which here reduces to:
there exists an arc (q, w) such that

$$\dot{q} = \dot{p}_r, \qquad \dot{w} = 0, \qquad (x_r - w, q) \in \partial G_r(-\dot{p}_r, \dot{x}_r) \quad \text{a.e.} \tag{2.7}$$

(in invoking [**5**] we have used the fact that G_r, as a finite convex function, is locally Lipschitz [**10**, Proposition 2.2.6]).

Let us set

$$h_0 := \max\{H(z) \colon |z| \leq r_0\},$$
$$\sigma := \max\{|\nabla H(z)| \colon z \text{ such that } H(z) \leq h_0\},$$

where r_0 was introduced in Lemma 2.

LEMMA 4. *If $r > \max[\sigma, \Delta]$, then (x_r, p_r) satisfies*

$$|(\dot{x}_r, \dot{p}_r)| \leq \sigma \quad \textit{for all } t \textit{ in } [0,T].$$

To see this, observe first that the continuous function $(\dot{x}_r, \dot{p}_r)$ has norm bounded above by Δ at least some of the time, in view of (2.6). Let $[a,b]$ be any subinterval of $[0,T]$ in which $|(\dot{x}_r, \dot{p}_r)|$ is strictly less than r, and in which $|(\dot{x}_r, \dot{p}_r)|$ is bounded above by Δ at least at one point. We shall prove that $|(\dot{x}_r, \dot{p}_r)|$ is actually bounded above by σ on $[a,b]$, which will prove the lemma. On $[a,b]$, in view of condition (b) of Lemma 3, one has

$$(x_r - w, q) \in \partial G(-\dot{p}_r, \dot{x}_r) \quad \text{a.e.},$$

which implies by conjugacy

$$(-\dot{p}_r, \dot{x}_r) = \nabla H(x_r - w, q). \tag{2.8}$$

Since $(x_r - w, q)$ and (x_r, p_r) have the same derivative, the Hamiltonian equation (2.8) implies

$$H(x_r - w, q) = \text{constant} =: c \quad \text{on } [a,b]. \tag{2.9}$$

Together (2.8) and (2.9) imply $c \leq h_0$, in view of Lemma 2, and because $|(-\dot{p}_r, \dot{x}_r)|$ is bounded above by Δ at least once in $[a,b]$. From (2,8) again, together now with the definition of σ, we deduce

$$|(\dot{x}_r, \dot{p}_r)| \leq \sigma \quad \text{on } [a,b],$$

which completes the proof of the lemma.

Let us now set $(\hat{x}, \hat{p}) := (x_s, p_s)$, where $s := \max[\sigma, \Delta] + 1$. We claim that $(\hat{x}, \hat{p})$ solves the problem P_T^γ (it is clear that $(\hat{x}, \hat{p})$ is feasible for the problem). Let (x,p) be any other arc feasible for P_T^γ, and choose any $r > s$ such that $|(\dot{x}, \dot{p})|$ is bounded above by r on $[0,T]$. Then, keeping Lemma 4 in mind, we calculate

$$\begin{aligned} J_T(\hat{x}, \hat{p}) &= J_T^s(\hat{x}, \hat{p}) \quad (\text{since } G \text{ and } G_s \text{ agree along } (\hat{x}, \hat{p})) \\ &\leq J_T^s(x_r, p_r) \quad (\text{since } (\hat{x}, \hat{p}) = (x_s, p_s)) \\ &= J_T^r(x_r, p_r) \quad (\text{since } G_r, G_s, G \text{ all agree along } (x_r, p_r)) \\ &\leq J_T^r(x, p) \quad (\text{by the optimality of } (x_r, p_r)) \\ &= J_T(x, p) \qquad (\text{since } G_r,\ G \text{ agree along } (x, p)). \end{aligned}$$

This completes the proof of the theorem. □

2.2. *A criterion for the existence of a global minimum.* We now give conditions under which P_T^γ admits a solution when $\gamma = +\infty$. (It is easy to produce examples in which this fails to be the case under merely the hypotheses of Theorem 2.1.)

PROPOSITION 2.1. *Suppose that for all x and p sufficiently large one has*

$$H(x,p) \leq A|x|^{1+R} + B|p|^{1+S},$$

where A, B, R and S are positive constants with $RS < 1$. Then for every $T > 0$ the problem P_T^∞ admits a solution.

PROOF. Let $\theta := 1 + R$, $\psi := 1 + S$. The hypothesis implies the existence of a constant m such that

$$H(x,p) \leq A|x|^\theta + B|p|^\psi + m \quad \text{for all } (x,p). \tag{2.10}$$

This is equivalent to

$$G(u,v) \geq \tilde{A}|u|^{\theta^*} + \tilde{B}|v|^{\psi^*} - m \quad \text{for all } (u,v), \tag{2.11}$$

where θ^*, for example, is the exponent conjugate to θ: $1/\theta + 1/\theta^* = 1$. The hypothesis $RS < 1$ implies the existence of a number σ satisfying $\theta^* > \sigma > \psi$; it follows that we have $\sigma^* < \psi^*$.

Now let (x, p) be any absolutely continuous function satisfying the boundary conditions (2.2). We calculate

$$\int_0^T \langle \dot{p}, x \rangle \, dt \le \int_0^T \{c_1 |\dot{p}|^\sigma + c_2 |x|^{\sigma^*}\} \, dt$$

(for certain constants c_1, c_2, by Young's inequality)

$$\le c_1 \int_0^T |\dot{p}|^\sigma \, dt + c_3 \left\{ \int_0^T |\dot{x}| \, dt \right\}^{\sigma^*}$$

(using $x(t) = \int_0^t \dot{x}(s)\, ds$)

$$\le c_4 \left\{ \int_0^T |\dot{p}|^{\theta^*} \right\}^{\sigma/\theta^*} + c_5 \left\{ \int_0^T |\dot{x}|^{\psi^*} \right\}^{\sigma^*/\psi^*}.$$

(Holder's inequality used twice). When combined with (2.11), this estimate leads to

$$J_T(x,p) \ge \tilde{A} \|\dot{p}\|_{\theta^*}^{\theta^*} - c_4 \|\dot{p}\|_{\theta^*}^{\sigma} + \tilde{B} \|\dot{x}\|_{\psi^*}^{\psi^*} - c_5 \|\dot{x}\|_{\psi^*}^{\sigma^*} - m,$$

where $\|f\|_s$ denotes the norm of an element f of $L^s([0,T], R^n)$. Since $\sigma < \theta^*$ and $\sigma^* < \psi^*$, it follows from standard arguments involving minimizing sequences and weak compactness (see, for example, [**12**, Lemma 2]) that J_T admits a global minimum over all (x,p) as above. Now [**14**, Proposition 3.2 and Theorem 2.1(ii)] implies that the minimizing (x,p) is in fact C^1, and so constitutes a solution of P_T^∞. □

3. Properties of solutions to P_T^γ.

3.1. *The dual variational principle.* As always, we are dealing with a Hamiltonian satisfying the basic hypotheses of §1. The essence of our approach is the following

PROPOSITION 3.1. *If (x,p) solves the problem P_T^γ for some T in $(0,\infty)$ and some γ in $(0,\infty]$, then there exist constants c and k in R^n such that on $[0,T]$ the arcs $y := x + c$, $q := p + k$ satisfy Hamilton's equations*

$$(-\dot{q}, \dot{y}) = \nabla H(y, q).$$

PROOF. The Euler-Lagrange "equation" for this situation involves generalized gradients, since G is not differentiable. As shown in [**5**], the necessary condition is that an arc (u, v) exists such that

$$(\dot{u}, \dot{v}, u, v) \in \partial L(x, p, \dot{x}, \dot{p}) \quad \text{a.e.},$$

where L is the integrand in J_T. This reduces to

$$\dot{u} = \dot{p}, \qquad \dot{v} = 0, \qquad (x - v, u) \in \partial G(-\dot{p}, \dot{x}) \quad \text{a.e.}$$

It follows that for constants c and k, one has $x - v = x + c$, $u = p + k$. It is a standard result of convex analysis that z belongs to $\partial G(\varsigma)$ iff $\varsigma = \nabla H(z)$; the proposition follows immediately. □

COROLLARY. *Suppose that the problem* P_T^γ *admits a solution, and that* $\inf P_T^\gamma < 0$. *Then there exists a nontrivial solution of Hamilton's equations having period* T.

Of course the arc (y, q) of the proposition (extended to $-\infty < t < \infty$) is the required solution of period T of Hamilton's equations. Clearly it is nonconstant (i.e., nontrivial) iff the solution (x, p) to P_T^γ is not identically zero (certainly the case if $\inf P_T^\gamma < 0$), and clearly T is the *minimal* period of (y, q) iff it is the minimal period of (x, p). We proceed to study these two issues in turn.

3.2. *Nontriviality of the solution.* Our eventual goal is to obtain a *nontrivial* periodic solution of Hamilton's equations from the solution z $(= (x, p))$ to the minimization problem P_T^γ; this will necessitate that z not be identically zero, a fact certainly implied by the condition $\inf P_T^\gamma < 0$. A simple way to ensure this is available in terms of the following quantity:

$$t_0 := \pi \limsup_{|z| \to 0} |z|^2 / H(z). \tag{3.1}$$

PROPOSITION 3.2. *If* $T > t_0$, *then* $\inf P_T^\gamma < 0$ *for any* $\gamma > 0$.

The proof of this fact proceeds by simply exhibiting a smooth feasible arc (x, p) of arbitrarily small supremum norm for which $J_T(x, p)$ is negative; details are given in [**12**, Lemma 3]. □

PROPOSITION 3.3. *Suppose that, for all* (x, p) *sufficiently small, one has*

$$H(x, p) \geq a|x|^{1+r} + b|p|^{1+s},$$

where a, b, r *and* s *are positive constants with* $rs < 1$. *Then for all positive* γ *and* T *one has* $\inf P_T^\gamma < 0$.

PROOF. Set $\theta := 1 + r$, $\psi := 1 + s$. Then (cf. Proposition 2.1) one has

$$(\theta^* - 1)(\psi^* - 1) > 1,$$

whence we may choose $\lambda > 0$ such that

$$\theta^* - 1 > \lambda > (\psi^* - 1)^{-1}.$$

An elementary argument (given in [**12**, Lemma 1]) shows that the condition on H implies that for all (u, v) sufficiently near $(0, 0)$ one has

$$G(u, v) \leq \tilde{a}|u|^{\theta^*} + \tilde{b}|v|^{\psi^*}$$

for certain constants $\tilde{a}, \tilde{b}$.

Now fix T and γ, and set

$$x(t) := \tau^2 \sin(2\pi t/T), \qquad p(t) := -q\tau \cos(2\pi t/T) + q\tau,$$

where q and τ are positive constants chosen small enough so that (x, p) is feasible for P_T^γ (i.e., $\tau^2 < \gamma$) and so that the bound on G holds along $(-\dot{p}(t), \dot{x}(t))$. We calculate

$$\begin{aligned} J_T(x, p) &\leq -\pi q \tau^{1+\lambda} + \int_0^T \{\tilde{a}(2\pi q \tau/T)^{\theta^*} + \tilde{b}(2\pi\tau^2/T)^{\psi^*}\}\, dt \\ &= \tau^{1+\lambda}\{-\pi q + c_1 q^{\theta^*} \tau^{\theta^* - 1 - \lambda} + c_2 \tau^{\lambda(\psi^* - 1) - 1}\}, \end{aligned}$$

for certain constants c_1 and c_2. Since $\theta^* - 1 - \lambda$ and $\lambda(\psi^* - 1) - 1$ are positive and θ^* exceeds 1, the dominant term in the braces, for q and τ small, is the first. Thus $J_T(x, p)$ is negative for q and τ sufficiently small, whence $\inf P_T^\gamma < 0$. □

PROPOSITION 3.4. *For all T sufficiently large, one has* $\inf P_T^\infty < 0$.

PROOF. The basic hypotheses imply the existence of positive constants m and k such that

$$H(z) \geq m|z| - k \quad \text{for all } z \text{ in } R^{2n}.$$

By conjugacy this is equivalent to

$$G(\varsigma) \leq k \quad \text{for } |\varsigma| \leq m. \tag{3.2}$$

Let us set

$$x(t) := Tm\cos(2\pi t/T)/(2\pi) - Tm, \qquad p(t) := Tm\,\sin(2\pi t/T)/(2\pi).$$

Routine calculation together with (3.2) gives

$$J_T(x,p) \leq -T^2m^2/(4\pi) + kT.$$

The right side is negative for T sufficiently large. □

3.3. *Minimality of the period.* When we use a solution z of P_T^γ to obtain a periodic Hamiltonian trajectory, the question of whether T is the *minimal* (or true) period amounts to asking whether for some integer $k > 1$, the arc z is T/k periodic on $[0,T]$ (i.e., whether $z(0) = z(iT/k), \dot{z}(0) = \dot{z}(iT/k)$ for $i = 1, 2, \ldots, k$). When this is not the case, we shall say that z has minimal period T.

We now define a new condition on H that turns out to bear upon the question of minimality of the period. We say that H is *radially increasing in p* if

$$\langle p, H_p(x,p)\rangle > 0 \quad \text{for all } x \text{ and for all } p \neq 0. \tag{3.3}$$

A simple situation implying (3.3) is the following

PROPOSITION 3.5. *Suppose that for each x, the function $p \to H(x,p)$ attains a unique minimum at* 0. *Then H is radially increasing in p.*

PROOF. For any $p \neq 0$, we have

$$0 > H(x,0) - H(x,p) \geq -\langle p, H_p(x,p)\rangle. \quad \square$$

We remark that an important case in which the Hamiltonian satisfies (3.3) is when $H(x,p) = V(x) + p^2/2$, for which Hamilton's equations reduce to Newton's equation $\ddot{x} = -\nabla V(x)$.

PROPOSITION 3.6. *Suppose that* $\inf P_T^\gamma < 0$ *and that H is radially increasing in p. Then any solution to P_T^γ has minimal period T.*

PROOF. We require the following fact.

LEMMA 5. *Let vectors u, v in R^n $(v \neq 0)$ and a scalar s in $(0,1)$ be given. Then one has $G(u, sv) < G(u,v)$.*

To see this, observe

$$\begin{aligned} G(u,sv) &= \sup_{x,p}\{(u,sv)\cdot(x,p) - H(x,p)\} \\ &= \sup_{x,q}\{(u,v)(x,q) - H(x,q/s)\} \\ &\leq \sup_{x,q}\{(u,v)\cdot(x,q) - H(x,q) + (1-s^{-1})q\cdot H_p(x,q)\} \\ &\qquad\qquad\qquad \text{(by the subgradient inequality)} \\ &\leq \sup_{x,q}\{(u,v)\cdot(x,q) - H(x,q)\} \quad \text{(by (3.3))} \\ &= G(u,v). \end{aligned}$$

Equality will hold only if in the above the maximizing (z,q) and (x,p) are such that $q = p = 0$; i.e., only if $(u,sv) = \nabla H(x,0)$. This is equivalent to $(x,0) \in \partial G(u,sv)$, which implies $0 \in \partial_v G(u,sv)$, which in turn implies that $G(u,\cdot)$ attains a global minimum at sv. Since G is strictly convex and $v \neq sv$, the lemma follows.

To prove the proposition, let us suppose that (x,p) is a solution to P_T^γ, and that (x,p) is T/k periodic for an integer $k > 1$. Let us define a (smooth) arc (y,q) on $[0,T]$ as follows:

$$y(t) := x(t/k), \qquad q(t) := kp(t/k).$$

Observe that (y,q) is feasible for P_T^γ. Note also that x is not identically zero (for otherwise we would have $\inf P_T^\gamma \geq 0$, in view of the fact that G is nonnegative). We calculate

$$\begin{aligned} J_T(y,q) &= \int_0^T \{\dot{p}(t/k)\cdot x(t/k) + G(-\dot{p}(t/k), \dot{x}(t/k)/k)\}\,dt \\ &< \int_0^T \{\dot{p}(t/k)\cdot x(t/k) + G(-\dot{p}(t/k), \dot{x}(t/k))\}\,dt \quad \text{(by the preceding lemma)} \\ &= k\int_0^{T/k} \{\dot{p}(s)\cdot x(s) + G(-\dot{p}(s), \dot{x}(s))\}\,ds \quad \text{(by a change of variables)} \\ &= J_T(x,p) \end{aligned}$$

(since (x,p) is T/k periodic). This contradicts the fact that (x,p) solves P_T^γ, and proves the proposition. □

In the case of a global minimum ($\gamma = +\infty$), it follows much as above, but with no additional hypotheses on H, that any nontrivial solution to P_T^∞ has minimal period T (see [**12**, Lemma 5]):

PROPOSITION 3.7. *If* $\inf P_T^\infty < 0$, *then any solution to* P_T^∞ *has minimal period* T.

We now apply the results of this section concerning solutions to P_T^γ to the study of periodic solutions of the Hamiltonian equations.

4. Hamiltonian trajectories of prescribed period. A simple example of how to combine the results of previous sections to derive the existence of periodic Hamiltonian trajectories is provided by the following. (The hypotheses are the basic ones of the introduction; the number t_0 is defined by (3.1).)

THEOREM 4.1. *If T satisfies*

$$t_0 < T < \sup_{\gamma,\alpha>0} \{\gamma\alpha/h(\gamma,\alpha)\},$$

then there exists a solution of Hamilton's equations of period T. If in addition H is radially increasing in p, then there is a solution having minimal period T.

PROOF. There is a number γ such that

$$T < \sup_{\alpha>0} \gamma\alpha/h(\gamma,\alpha) \qquad (= T_\gamma),$$

so that P_T^γ admits a solution by Theorem 2.1. By Proposition 3.2 one has $\inf P_T^\gamma < 0$, so the first assertion of the theorem follows from the Corollary to Proposition 3.1. The second assertion is a consequence of Proposition 3.6. □

It is natural to inquire about supplementary hypotheses on H that would guarantee certain things about the set of values T obtainable as periods, for example that it be nonempty, that it contain all small (or large) T, etc. A variety of such results can be obtained through considering P_T^γ. In Theorem 1.1 (stated in §1), a number of these are grouped.

PROOF OF THEOREM 1.1. To prove (i), pick any $\gamma > 0$ and then any $T > 0$ for which $T < T_\gamma$. Then $\inf P_T^\gamma < 0$ (Proposition 3.3) and a solution to P_T^γ exists (Theorem 2.1). The assertion then follows from the Corollary to Proposition 3.1; the extra statement regarding minimality is a consequence of Proposition 3.6.

To prove (ii), merely apply to the problem P_T^∞ the results of Proposition 3.1 (existence), Proposition 3.4 (nontriviality for large T) and Proposition 3.7 (minimality of the period).

Assertion (iii) is proven the same way as (ii), except that Proposition 3.3 is invoked for nontriviality instead of Proposition 3.4. This removes the need to consider only large T, whence the global conclusion. □

PROOF OF THEOREM 1.2. For T in the given range, there exists $\gamma > 0$ such that

$$T < T_\gamma := \sup_{\alpha>0} \gamma\alpha/[v(\gamma) + \alpha^2/2].$$

Then P_T^γ admits a solution by Theorem 2.1. The Hamiltonian H in this case satisfies (C_0) with $a = \varepsilon$, $r = 1 - \lambda$, $b = 1/2$ and $s = 1$, whence $\inf P_T^\gamma < 0$ by Proposition 3.3. The result follows by the Corollary to Proposition 3.1 together with Proposition 3.6, since H is radially increasing in p. □

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