A WEIGHTED INEQUALITY 
FOR THE MAXIMAL BOCHNER-RIESZ OPERATOR ON R^2 

BY 
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Abstract. For f ∈ ℳ(R^2), let (T*f)w(x) = (1 - |ξ|^2 R^2)w(ξ). It is a well-known theorem of Carleson and Sjölin that T*f defines a bounded operator on L^4 if a > 0. In this paper we obtain an explicit weighted inequality of the form
\[ \int \sup_{0<R<\infty} |T*f(x)|^2 w(x) dx \leq \int |f|^2 P_a w(x) dx, \]
with P_a bounded on L^2 if a > 0. This strengthens the above theorem of Carleson and Sjölin. The method gives information on the maximal operator associated to general suitably smooth radial Fourier multipliers of R^2.

1. Introduction. For f ∈ ℳ(R^2) and α > 0, let
\[ (T_R^\alpha f)^\wedge(ξ) = (1 - |ξ|^2 R^2)^\alpha \hat{f}(ξ), \]
where R ∈ R^+ and ^\wedge denotes the Fourier transform. Let
\[ T_R^\alpha f(x) = \sup_{0<R<\infty} |T_R^\alpha f(x)|. \]
In [1] it was shown that \|T_R^\alpha f\|_p \leq C_{p,\alpha} \|f\|_p for all α > 0 provided that 2 ≤ p ≤ 4 (a result which extended the theorem of Carleson and Sjölin [3]). In this note we establish a weighted inequality which provides a stronger form of the above result on the maximal Bochner-Riesz operator.

Theorem 1. Let 2 < q < \infty and α > 0. Then there exists an operator P = P_{q,α} which is bounded on L^q such that
\[ \int |T_R^\alpha f(x)|^2 w(x) dx \leq \int |f(x)|^2 P_w(x) dx \]
whenever w is a nonnegative function in L^q.

Several comments are in order.

1. It suffices to prove the theorem in the case q = 2. For if P_{2,α} fulfills the requirements of the theorem in the case q = 2, P_{q,α}, defined by P_{q,α}w = \|P_{2,α}(w^{q/2})\|_{q/2}^{2/q}, does so for L^q. That P_{q,α} is bounded on L^q is clear, and we may interpolate with change of measure between the estimates
\[ \int |T_R^\alpha f|^2 w_j(x) dx \leq \int |f|^2 (P_{2,α}w)^j, \quad j = 0, 1, \]
to obtain the inequality of Theorem 1. (This observation was pointed out to the author by Professor J. Garnett.)

2. The proof of Theorem 1 is constructive—that is, an explicit formula for $P_w$ in terms of $w$ is given. Previously, Rubio de Francia [7] had been able to show that for each nonnegative $w$ in $L^2$, there exists a nonnegative $w'$ in $L^2$ with $\|w'\|_2 \leq C \|w\|_2$ such that

$$\sup_{0 < R < \infty} \int |T_R^w f(x)|^2 w(x) \, dx \leq \int |f(x)|^2 w'(x) \, dx.$$

However, the proof gave no information on how to construct $w'$ from $w$. Since an earlier version of this paper was prepared, it has come to the author's notice that Córdoba [5] has constructed from each nonnegative $w$ in $L^2$ a nonnegative $w'$ in $L^2$, with $\|w'\|_2 \leq C \|w\|_2$, such that

$$\int |T_R^w f(x)|^2 w(x) \, dx \leq \int |f(x)|^2 w'(x) \, dx.$$

The definition of $w'$ given by Córdoba was not invariant under dilations, so the full strength of Rubio's inequality was not obtained.

3. It would be of some interest to establish a version of Theorem 1 in which the operator $P$ did not depend upon $q$. In particular, the conjecture (see [8, p. 7]) that $P$ may be realised as (essentially) $\sum_{k=0}^{\infty} 2^{-k} M_k^{\frac{1}{2}}$ remains unsolved. Here, as subsequently, $M_N$ denotes the maximal function corresponding to averages over rectangles in $N$ uniformly distributed directions in $\mathbb{R}^2$. The relevant fact about $M^N$ here is that $\|M^N f\|_2 \leq C (\log N)^\beta \|f\|_2$, where $C$ and $\beta$ are absolute constants. See Córdoba [4].

On the other hand, it can be shown that the operator $P_{2,\alpha}$ of Theorem 1 satisfies $\|P_{2,\alpha} w\|_q \leq C_{q,\alpha} \|w\|_q$ when $2 \leq q \leq 4$. See the remark at the end of §2.

Theorem 1 follows from Theorem 2 below. For $\alpha > \frac{1}{2}$ let $\tilde{\phi}_\alpha(x) = (1 - |\xi|)^{-1} \phi_\alpha$. Let

$$\mathcal{G}_\alpha(f)^2(x) = \int_0^{\infty} |\phi_\alpha^* \ast f(x)|^2 \frac{dt}{t},$$

where if $\psi$ is a function defined on $\mathbb{R}^2$, $\phi_t(x) = \frac{1}{t^2} \psi(x/t)$.

**Theorem 2.** If $\alpha > \frac{1}{2}$ then there exists an operator $P$ bounded on $L^2$ such that

$$\int \mathcal{G}_\alpha(f)^2(x) w(x) \, dx \leq C_\alpha \int |f(x)|^2 Pw(x) \, dx$$

whenever $w$ is a nonnegative function in $L^2$.

In fact, Theorem 2 leads to a generalisation of Theorem 1. For $\alpha > \frac{1}{2}$ let $\mathcal{L}_\alpha^2$ denote the usual space of Bessel potentials on $\mathbb{R}$ (see, for example, [9, Chapter 5]) transformed under the change of variables $s \mapsto \exp s$. Thus, if $m$ is defined on $(0, \infty)$, $\|m\|_{\mathcal{L}_\alpha^2} = m(\|\log(\cdot)\|_{L^2})$. If we let

$$(S_R^m f)^\ast(x) = m(|x| R) \hat{f}(\xi) \text{ and } S_R^\ast f(x) = \sup_{0 < R < \infty} |S_R^m f(x)|,$$

we have the following inequality [2], valid when $\alpha > \frac{1}{2}$:

$$S_R^\ast f(x) \leq C_\alpha \|m\|_{\mathcal{L}_\alpha^2} \mathcal{G}_\alpha(f)(x).$$

Consequently, we obtain Theorem 3.
Theorem 3. Let \( \alpha > \frac{1}{2} \). Then there exists an operator \( P \) bounded on \( L^2 \) such that if \( m \) is a radial function on \( \mathbb{R}^2 \) whose restriction to \( (0, \infty) \) lies in \( \mathcal{L}_a^2 \), then
\[
\int |S_m^\alpha f(x)|^2 w(x) \, dx \leq C_a \|m\|_{\mathcal{L}_a^2} \int |f(x)|^2 Pw(x) \, dx
\]
whenever \( w \) is a nonnegative function in \( L^2 \).

Theorem 1 may be deduced from Theorem 3 as follows. Let
\[
(1 - |\xi|^2)^\lambda = \phi_0(|\xi|) + \left(1 - |\xi|^2\right)^\lambda - \phi_0(|\xi|),
\]
where \( \phi_0 \) is a \( C^\infty \) function of compact support in \( [-\frac{1}{4}, \frac{3}{4}] \), agreeing with \( (1 - r^2)^\lambda \) on \( [-\frac{1}{2}, \frac{1}{2}] \). Then the maximal operator corresponding to \( \phi_0 \) is dominated by the Hardy-Littlewood maximal function, while \( (1 - r^2)^\lambda - \phi_0(r) \) belongs to \( \mathcal{L}_a^2 \) if \( \alpha < \lambda + \frac{1}{2} \).

Theorem 2 follows from the following lemma, proved in §2 below.

Lemma. Let \( \Phi \) be a smooth real-valued bump function supported in \([-1,1] \). Let \( \phi(\xi) = \Phi((|\xi| - 1)/\delta) \) for small \( \delta > 0 \). Let \( \psi = \phi \). Then there exists an operator \( Q = Q_\delta \) such that
\[
(i) \quad \int_0^\infty \int |\psi \ast f(x)|^2 w(x) \, dx \, \frac{dt}{t} \leq \int |f(x)|^2 Q_\delta w(x) \, dx
\]
whenever \( w \) is a nonnegative test function, and
\[
(ii) \quad \|Q_\delta w\|_2 \leq C\delta (\log(3/\delta))^{\beta} \|w\|_2.
\]
(Here and subsequently, \( \beta \) will denote a positive absolute constant, and \( C \) will be a positive constant depending only possibly on \( \max_{0\leq j\leq 3} \|\Phi^{(j)}\|_\infty \); \( C \) and \( \beta \) may not be the same at each occurrence.)

To obtain Theorem 2 from this lemma, we merely write
\[
\hat{\Phi}^\alpha(\xi) = (1 - |\xi|^2)^{\alpha - 1} |\xi| = \sum_{k=0}^{\infty} 2^{-k(\alpha - 1)} \phi_k(|\xi|),
\]
with \( \phi_k \) smooth, supported in \([1 - 2^{-k}, 1 - 2^{-k - 2}] \) and satisfying \( \|\phi_k^{(j)}\|_\infty \leq C 2^{kj} \), and letting \( \hat{\psi}^k(\xi) = \phi_k(|\xi|) \), we observe that if \( \alpha > 1/2 \),
\[
\int \mathcal{G}^\alpha(f)^2 w = \int \int_0^\infty |\hat{\phi}^\alpha_t \ast f(x)|^2 w(x) \, dt \, dx
\]
\[
= \int \int_0^\infty \left( \sum_{k=0}^{\infty} 2^{-k(\alpha - 1)} \hat{\psi}_t^k \right) |f(x)|^2 w(x) \, dt \, dx
\]
\[
\leq \left\{ \sum_{k=0}^{\infty} 2^{-k(\alpha - 1)} \left( \int \int_0^\infty |\hat{\psi}_t^k \ast f(x)|^2 w(x) \, dt \, dx \right)^{1/2} \right\}^2
\]
\[
\leq \left\{ \sum_{k=0}^{\infty} 2^{-k(\alpha - 1)} \left( \int |f(x)|^2 Q_{\delta^{-1}} w(x) \, dx \right)^{1/2} \right\}^2
\]
\[
\leq \frac{C}{(\alpha - 1/2)} \int |f(x)|^2 \sum_{k=0}^{\infty} 2^{-k(\alpha - 3/2)} Q_{\delta^{-1}} w(x) \, dx.
\]
Thus $P$ is realised as $\sum_{k=0}^{\infty} 2^{-k(a-3/2)}Q_{2^{-k}}$, and
\[ \|Pw\|_2 \leq C \sum_{k=0}^{\infty} 2^{-k(a-3/2)}2^{-k\beta}\|w\|_2 \leq C\|w\|_2. \]

2. Proof of the lemma. From now on we consider $\Phi$ and $\delta$, hence $\phi$, to be fixed. We assume $\delta = 1/N^2$, where $N$ is a power of 2.

We need to recall that in [1] we constructed a covering of $\mathbb{R}^2 - \{0\}$ by rectangles $\{S_k\}$ such that $\sum_{k \in \mathbb{Z}} |S_k| \leq 25$, together with a partition of unity $\{\beta_k\}$ subordinate to that covering. If the distance between the centre of $S_k$ and the origin is $d$, then $S_k$ has sidelengths comparable to $\delta d$ and $\delta^{1/2}d$, and is oriented so that the direction of the longer side is approximately perpendicular to the line joining the origin to the centre of $S_k$. Thus $S_k$ lies in an annulus $A_k$ centred at the origin of width approximately $\delta 2^k$ and subtends an angle of approximately $\delta^{1/2}$ at the origin. In each large annulus $\{2^l \leq |\xi| \leq 2^{l+1}\}$, $l \in \mathbb{Z}$, there are $1/\delta$ smaller annuli $A_k$. Let $\gamma_k$ be a smooth multiplier supported in $2S_k$ (the rectangle with the same centre and orientation as $S_k$ but twice the sidelengths) and satisfying the same estimates as $\beta_k$, and let $(B_k f)(\xi) = \gamma_k(\xi)\hat{f}(\xi)$. Let $\hat{\rho}(\xi_1, \xi_2)$ be a smooth bump function of $\xi_1$, supported in $[1/4, 4] \cup [-4, -1/4]$ and identically one on $[1,2] \cup [-2, -1]$, and let $g(f)(x) = (\sum_{k \in \mathbb{Z}} |p_{2k} \ast f(x)|^2)^{1/2}$. The proof of Proposition 4 of [1] shows us that
\[ (1) \quad \int \sum_{j, k} |B_{kj}f|^2 w \leq \frac{C}{(s-1)^\beta} \int g(f)^2 (M_1^2 M_N w^s)^{1/s} \]
whenever $s > 1$. (In that proof $M_N$ appeared raised to the third power, but since we are using smooth cutoff functions $\gamma_k$, we can dispose of two of these powers.) The operator $g$, being a vector-valued singular integral, satisfies the inequality
\[ (2) \quad \int g(f)^2 u \leq \frac{C}{(s-1)^\beta} \int |f|^2 (Mu^s)^{1/s} \]
for all $s > 1$, since $(Mu^s)^{1/s}$, $s > 1$, is an $A_1$ weight with constant not exceeding $C/(s-1)^\beta$. (Here $M$ is the Hardy-Littlewood maximal function which is dominated by the strong maximal function $M_1$.) Combining (1) and (2) yields
\[ (3) \quad \int \sum_{j, k} |B_{kj}f|^2 w \leq \frac{C}{(s-1)^\beta} \int |f|^2 (M_1^3 M_N w^s)^{1/s} \]
for $s > 1$.

Construction of $Q$. For $f$ and $w$ in the Schwartz class, say, and $w$ nonnegative, we see that
\[ \int_0^\infty \int |\psi_t \ast f(x)|^2 w(x) \, dx \, \frac{dt}{t} \]
\[ = \int_0^\infty \int \psi_t \ast f(x) \psi_t \ast \overline{f(x)} w(x) \, dx \, \frac{dt}{t} \]
\[ = \int_0^\infty \int \sum_{j, j', k, k'} T_{kj}f(x) \overline{T_{kj'}f(x)} w(x) \, dx \, \frac{dt}{t} \]
\[ = \int_0^\infty \int \sum_{j, j', k, k'} T_{kj}f(x) \overline{T_{kj'}f(x)} R_{kk'}w(x) \, dx \, \frac{dt}{t} , \]
where \((T_{k,t}^\delta)^*(\xi) = \phi(t\xi)\beta(t)\tilde{f}(\xi)\), and \((R_{k,k}^{j,j'}w)^*(\xi) = \hat{w}(\xi)\) if \(\xi \in S_k - S_k'\). (We have left a certain latitude in the definition of \(R_{k,k}^{j,j'}\) which we exploit later.) Observe now that, for a given \(t\), there are at most three consecutive values of \(k\) for which \(T_{k,t}\) is not the zero operator for all \(j\). Applying Parseval’s relation once more, and the Cauchy-Schwarz inequality in \(j, j', k, k'\), we have

\[
\int_0^\infty \int_0^\infty \int f(x) \phi(t^*f(x))^2 w(x) dx \frac{dt}{t} = \int_0^\infty \sum_{k, k'} \sum_{|j-j'| \geq 2} T_{k,t}^\delta T_{k',t}^\delta f R_{k,k}^{j,j'} w \frac{dt}{t} + \int_0^\infty \sum_{k, k'} \sum_{|j-j'| \leq 1} T_{k,t}^\delta T_{k',t}^\delta f w dx \frac{dt}{t}
\]

\[
\leq 3 \int_0^\infty \int \left( \sum_{j, j', k, k'} |T_{k,t}^\delta T_{k',t}^\delta f|^2 \right)^{1/2} \left( \sup_{|k-k'| \leq 2} \sum_{|j-j'| \geq 2} |R_{k,k}^{j,j'}|^2 \right)^{1/2} dx \frac{dt}{t} + 9 \int_0^\infty \sum_{j, k} |T_{k,t}^\delta f|^2 w dx \frac{dt}{t}
\]

\[
= 3 \int_0^\infty \int \sum_{j, k} |T_{k,t}^\delta f(x)|^2 A w(x) dx \frac{dt}{t},
\]

where

\[
Aw(x) = \left( \sup_{|k-k'| \leq 2} \sum_{|j-j'| \geq 2} |R_{k,k}^{j,j'}|^2 \right)^{1/2} + 3w(x).
\]

Now the \(dt/t\) measure of \(\{t \in \mathbb{R} : T_{k,t}^\delta \neq 0\}\) is dominated by \(\delta\) for each fixed \((k, j)\), and integrating by parts as in [1] shows that \(|T_{k,t}^\delta f(x)| \leq CL_k \phi|f(x)|\), with \(C\) independent of \(t\) and depending only on \(\max_{0 \leq k \leq 3} \|\Phi^{(j)}\|_{\infty}\). Here, \(\|L_k\| \leq 1\) and \(L_k \phi|f(x)| \leq CM_N f(x)\). Let \(\gamma_k^\delta\) be a smooth bump function supported in \(2S_k\) and identically one on \(S_k\), so that if \((B_k f)^*(\xi) = \gamma_k^\delta(\xi)\tilde{f}(\xi)\) then \(T_{k,t}^\delta f(x) = T_{k,t}^\delta B_k f(x)\). Thus,

\[
\int_0^\infty \int_0^\infty |\psi_t \star f(x)|^2 w(x) dx \frac{dt}{t} \leq C \delta \int \sum_{j, k} \left( L_k \phi |B_k f|^2 (x) \right) A w(x) dx
\]

\[
\leq C \delta \int \sum_{j, k} \left( L_k \phi |B_k f|^2 (x) \right) A w(x) dx
\]

\[
= C \delta \int \sum_{j, k} |B_k f|^2 (x) L_k \phi A w(x) dx
\]

\[
\leq C \delta \int \sum_{j, k} |B_k f|^2 M_N A w
\]

\[
\leq \frac{C \delta}{(s - 1)^{\beta}} \int |f|^2 \left( M_1^3 M_N M_N A w \right)^s \}
\]

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for each \( s > 1 \), by (3). We now choose \( s = 1 + 1 / \log(3/\delta) \), so that
\[
\int_0^\infty \left| \int \psi_t * f(x) \right|^2 w(x) \frac{dt}{t} dx \\
\leq \int |f|^2 \left[ C\delta \left( \log \frac{3}{\delta} \right)^\beta \left\{ M_N^2 M_N (M_N A w)^{1 + (\log(3/\delta))^{-1}} \right\}^{1/(1 + (\log(3/\delta))^{-1})} \right]
\]
This completes our construction of \( Q \) satisfying condition (i) of the lemma.

**Boundedness of \( Q \).** A very simple interpolation argument shows that since \( M_N \) satisfies
\[
\| M_N f \|_2 \leq C(\log 3N)^\beta \| f \|_2 \quad \text{and} \quad \| M_N f \|_3/2 \leq CN \| f \|_3/2,
\]
then \( M_N \) is bounded on \( L^p \) with \( p = 2/(1 + (\log(3/\delta))^{-1}) \) with norm still no larger than \( O((\log 3N)^\beta) \). To complete the proof of the lemma, then, it remains to show that \( \| Aw \|_2 \leq C(\log(3/\delta))^{\beta} \| w \|_2 \), to which task we now turn ourselves; without loss of generality it suffices to obtain a similar estimate for
\[
A'w(x) = \left( \sup_{k} \sum_{|j-j'| \geq 2} \left| R_{jk}' w(x) \right|^2 \right)^{1/2},
\]
where, for notational simplicity, we have written \( R_{jk}' \) as \( R_{jk} \).

For \( |j - j'| \geq 2 \), let
\[
r_{k}(j, j') = \sup \{|\xi - \xi'|: \xi, \xi' \in S_{k} - S_{k}', \text{ with } \xi \text{ and } \xi' \text{ lying on the same line segment passing through the origin}\},
\]
and let
\[
t_{k}(j, j') = \sup \{|\xi - \xi'|: \xi, \xi' \in S_{k} - S_{k}', \text{ with } \xi \text{ and } \xi' \text{ on the same line segment which is tangent to a circle centred at the origin}\}.
\]
(These are the ‘radial’ and ‘tangential’ lengths of \( S_{k} - S_{k}' \).) Let the eccentricity \( e_k(j, j') \) of \( S_{k} - S_{k}' \) be \( t_k(j, j')/r_k(j, j') \). For each \( k \in \mathbb{Z} \) and each \( m \in \mathbb{Z} \) let \( \mathcal{B}^m = \{(j, j') | 2^m < e_k(j, j') \leq 2^{m+1}\} \). Notice that \( \mathcal{B}^m \) is independent of \( k \). Observe that the eccentricities range between \( 1/N \) and \( N \); thus \( \mathcal{B}^m \) is nonempty only when \( |m| \leq \log N \). Now
\[
\int \left( \sup_{(j, j')} |R_{jk}' w|^2 \right) dx = \int \sup_k \left( \sum_m \sum_{(j, j') \in \mathcal{B}^m} |R_{jk}' w|^2 \right) \frac{dx}{|m| < \log N} \\
\leq C(\log N)^\beta \max_{|m| < \log N} \int \sup_k \left( \sum_{(j, j') \in \mathcal{B}^m} |R_{jk}' w|^2 \right) dx.
\]
Thus it is sufficient to obtain a bound for
\[
\int \sup_k \left( \sum_{(j, j') \in \mathcal{B}^m} |R_{jk}' w|^2 \right) dx
\]
which is independent of \( m \).
For each $m$, then, let $\mathcal{F}^m = \{A_\lambda | \lambda \in \Lambda_m\}$ be the collection of 'rectangles' formed by the concentric circles \(||\xi|| = 2^{p/2mN}\}_{p \in \mathbb{Z}}\) and the uniformly distributed rays \(\{\arg \xi = 2\pi p/N\}, p = 0, 1, \ldots, N - 1\). Let \((A_\lambda g)\hat{(}\xi) = X_{A_\lambda}(\xi)\hat{g}(\xi)\).

We now make use of our freedom in defining $R_k^{j,j'}w$. If \((j, j') \in \mathcal{B}^m\), let \((R_k^{j,j'}w)\hat{(}\xi) = \sum X_{A_\lambda}(\xi)\hat{w}(\xi)\), the sum being taken over all $\lambda \in \Lambda_m$ such that \(A_\lambda \cap (S_k^j - S_k^{j'}) \neq \emptyset\). By the construction of \(\{A_\lambda\}\), there will be at most four nonzero terms in this sum, and C. Fefferman's observation \([6]\) that for a given $k$, the sets \(\{S_k^j - S_k^{j'}\}_{j, j'}\) are “almost disjoint” in the sense that $\sum_{j, j'}X_{S_k^j - S_k^{j'}} \leq C$ yields

$$\sum_{(j, j') \in \mathcal{B}^m} |R_k^{j,j'}w|^2 \leq C \sum_{\lambda \in \Lambda_m} |A_\lambda w|^2$$

for all $k$. If we now take the supremum over $k \in \mathbb{Z}$, we obtain the desired result since

$$\left\| \left( \sum_{\lambda \in \Lambda_m} |A_\lambda w|^2 \right)^{1/2} \right\|_2 = \|w\|_2,$$

$\sum X_{A_\lambda}$ being identically equal to one almost everywhere on $\mathbb{R}^2$.

**Remark.** The above computations concerning the boundedness of $Q$ show that

$$\|Qw\|_q \leq C \delta (\log (3/\delta))^{1/2} \|w\|_q$$

will hold provided that $q \geq 2$ and

$$\left\| \left( \sum_{\lambda \in \Lambda_m} |A_\lambda w|^2 \right)^{1/2} \right\|_q \leq C (\log N)^{1/2} \|w\|_q$$

with $A_\lambda$ as above and $|m| \leq \log N$. In the case $q = 2$ this is trivial; however, it remains true when $2 < q < 4$. The endpoint case $m = \log N$ is in \([1]\) (in fact inequality (3) above is the required estimate) and the case $m = -\log N$ is implicit in the article of Córdoba in the proceedings of the conference in harmonic analysis in honour of Antoni Zygmund held at Chicago (W. Beckner et al., eds., Wadsworth, 1982). The other cases are handled similarly; the middle case $m = 0$ is substantially simpler.

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