THE MACKEY TOPOLOGY AND COMPLEMENTED SUBSPACES OF LORENTZ SEQUENCE SPACES \( d(w, p) \) FOR \( 0 < p < 1 \)

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ABSTRACT. In this paper we continue the study of Lorentz sequence spaces \( d(w, p) \), \( 0 < p < 1 \), initiated by N. Popa [8]. First we show that the Mackey completion of \( d(w, p) \) is equal to \( d(v, 1) \) for some sequence \( v \). Next, we prove that if \( d(w, p) \not\subseteq l_1 \), then it contains a complemented subspace isomorphic to \( l_p \). Finally we show that if \( \lim n^{-1}(\sum_{i=1}^{n} w_i)^{1/p} = \infty \), then every complemented subspace of \( d(w, p) \) with symmetric bases is isomorphic to \( d(w, p) \).

I. Introduction. A \( p \)-norm, \( 0 < p \leq 1 \), on a vector space \( X \) is a map \( x \mapsto \|x\| \) such that:
1. \( \|x\| > 0 \) if \( x \neq 0 \).
2. \( \|tx\| = |t| \|x\| \) for all \( x \in X \) and all scalars \( t \).
3. \( \|x + y\|^p \leq \|x\|^p + \|y\|^p \) for all \( x, y \in X \).

Let \( B = \{x \in X : \|x\| \leq 1\} \); then the family \( \{rB\}_{r>0} \) is a base of neighbourhoods of zero for a Hausdorff locally bounded vector topology on \( X \) (see [9]). If \( X \) is complete, we say that \( X \) is a \( p \)-Banach space.

The Mackey topology \( \mu \) of a locally bounded space \( X \) with separating dual is the strongest locally convex topology on \( X \) which is weaker than the original one (see [10]). It is easy to see that this normable topology is generated by neighbourhoods \( \{r\text{conv} B\}_{r>0} \). The Minkowski functional of the set \( \text{conv} B \) is called the Mackey norm on \( X \). The completion of the space \((X, \mu)\) is called the Mackey completion of \( X \) and denoted by \( \hat{X} \). The extension of the Mackey norm to \( \hat{X} \) is denoted by \( \|\cdot\|\).

For every subset \( E \) of \( \omega \) (= the space of all scalar sequences) we denote
\[ E^+ = \{x = (x_i) \in E : x_i \geq 0 \text{ for } i = 1, 2, \ldots\} \]
and
\[ E^{++} = \{x \in E^+ : x \text{ is nonincreasing}\} \]

Let \( 0 < p < \infty \) and let \( w = (w_i) \in l_1^{\infty} \setminus l_1 \). For \( x = (x_i) \in \omega \) we define
\[ \|x\|_{w,p} = \sup_{\pi} \left( \sum_{i=1}^{\infty} |x_{\pi(i)}|pw_i\right)^{1/p} \]
where \( \pi \) ranges over all permutations of the positive integers. The space \( d(w, p) = \{x \in \omega : \|x\|_{w,p} < \infty\} \) equipped with the locally bounded vector topology induced by \( \|\cdot\|_{w,p} \) is called the Lorentz sequence space.

It is well known that \( d(w, p) \) is a \( p \)-Banach space for \( 0 < p < 1 \) and a Banach space for \( p \geq 1 \). Moreover, the sequence of unit vectors \( (e_i) \) is a symmetric basis of...
From the assumption $w \in l^{++}_{1} \setminus l_{1}$ follows that $d(w, p) \subset c_{0}$. Therefore for every $x = (x_{i}) \in d(w, p)$ there exists a nonincreasing rearrangement $x^{*} = (x^{*}_{i})$ of $x$ (i.e., a nonincreasing sequence obtained from $(|x_{i}|)$ by a suitable permutation of the integers) and $\|x\|_{w, p} = (\sum_{i=1}^{n} x_{i}^{p} w_{i})^{1/p}$.

Observe that $d(w, p) \approx l_{p}$ if and only if $w \notin c_{0}$ (cf. [6, p. 176]).

The first topic of the present paper is the Mackey topology of $d(w, p)$, $0 < p < 1$. Using a representation of the dual of $d(w, p)$, N. Popa [8] proved that the Mackey completion of $d(w, p)$ ($p = 1/k$, $k \in \mathbb{N}$, and $w$ satisfies some additional conditions) is isomorphic to $d(v, 1)$ for a suitable sequence $v$. In §3 we show that the above theorem holds for any Lorentz sequence space $d(w, p)$, $0 < p < 1$. Our result is obtained without determining any dual space.

The last part of our paper is devoted to the study of complemented subspaces of $d(w, p)$, $0 < p < 1$.

It is well known that every Lorentz sequence space $d(w, p)$, $p \geq 1$, has complemented subspace isomorphic to $l_{p}$ (see [6, Proposition 4.e.3]). N. Popa [8] showed that unlike the case $p \geq 1$ there are spaces $d(w, p)$, $0 < p < 1$, which contain no complemented subspaces isomorphic to $l_{p}$ and conjectured that it is true for each $d(w, p)$, $0 < p < 1$. In §4 we prove that if $\inf_{n} n^{-1}(\sum_{i=1}^{n} w_{i})^{1/p} = 0$ (i.e., $d(w, p) \not\subset l_{1}$, see Proposition 1), then $d(w, p)$ has complemented subspace isomorphic to $l_{p}$. Moreover, if $\lim_{n \to \infty} n^{-1}(\sum_{i=1}^{n} w_{i})^{1/p} = \infty$, then every complemented subspace of $d(w, p)$ with symmetric basis is isomorphic to $d(w, p)$.

Throughout the paper we denote by $B_{w, p}$ the closed unit ball in $d(w, p)$, $\mathbb{R}^{n} = \text{span}\{e_{i}\}_{i=1}^{n}$, $B_{w, p}^{n} = B_{w, p} \cap \mathbb{R}^{n}$, $n = 1, 2, \ldots$. In addition we denote $S_{n}(x) = x_{1} + \cdots + x_{n}$, $n = 1, 2, \ldots$, $S_{0}(x) = 0$ for any sequence $x = (x_{i}) \in w$.

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II. Technical results. In this section we assume that $0 < p \leq 1$, $w = (w_{i}) \in l^{++}_{1} \setminus l_{1}$, $\sigma_{k} = S_{k}(w)^{1/p}$, $f_{k} = \sigma_{k}^{-1} \sum_{i=1}^{k} e_{i}$ for $k = 1, 2, \ldots$, and $f_{0} = 0$.

**Lemma 1.** Let $\| \cdot \|_{n}$ be the norm on $\mathbb{R}^{n}$ defined by
\[ \|x\|_{n} = \sum_{i=1}^{n} |x_{i}|(\sigma_{i} - \sigma_{i-1}) \quad \text{for} \quad x = (x_{i}) \in \mathbb{R}^{n}, \]
and let
\[ B^{n} = \{x = (x_{i}) \in \mathbb{R}^{n}: \|x\|_{n} \leq 1\}, \quad n \in \mathbb{N}. \]

Then:
\[(a) \quad (B^{n})^{++} = \text{conv}\{f_{k}: k = 0, 1, \ldots, n\}.\]
\[(b) \quad (B_{w, p}^{n})^{++} \subset (B^{n})^{++}.\]
\[(c) \quad \text{Let} 0 < p < 1 \text{ and } x = (x_{i}) \in (\mathbb{R}^{n})^{++}. \text{ Then} \|x\|_{n} = \|x\|_{w, p} = 1 \text{ if and only if} \]
\[x = f_{k} \text{ for some } k = 1, 2, \ldots, n.\]
\[(d) \quad \text{If } p = 1, \text{ then} \quad (B_{w, p}^{n})^{++} = (B^{n})^{++}.\]

**Proof.** (a) Every point $x \in \mathbb{R}^{n}$ may be written in the form
\[ x = \sum_{i=1}^{n-1} \sigma_{i}(x_{i} - x_{i+1})f_{i} + \sigma_{n}x_{n}f_{n}. \]
In addition, for each $x \in (\mathbb{R}^n)^+$,
\[
\|x\|_n = \sum_{i=1}^{n-1} \sigma_i(x_i - x_{i+1}) + \sigma_n x_n.
\]
Therefore every $x \in (\mathbb{R}^n)^+$ with $\|x\|_n = 1$ is a convex combination of the vector $f_1, \ldots, f_n$. It implies $(B^n)^+ \subset \text{conv}\{f_0, f_1, \ldots, f_n\}$.

We observe that $\|f_k\|_{w,p} = \|f_k\|_n = 1$ for $k = 1, 2, \ldots, n$. Both the sets $(\mathbb{R}^n)^+$ and $B^n$ are convex, so
\[
\text{conv}\{f_0, \ldots, f_k\} \subset B^n \cap (\mathbb{R}^n)^+ = (B^n)^+.
\]

(b) By the proof of (a) and by concavity of the function $x \mapsto \|x\|_{w,p}$ on $(\mathbb{R}^n)^+$,
\[
\|x\|_{w,p}^p \geq \sum_{i=1}^{n-1} \sigma_i(x_i - x_{i+1})\|f_i\|_{w,p}^p + \sigma_n x_n \|f_n\|_{w,p}^p = \|x\|_n
\]
for $x \in (\mathbb{R}^n)^+, \|x\|_n = 1$.

Thus (b) follows from homogeneity of the functionals $\|\cdot\|_n$ and $\|\cdot\|_{w,p}$.

(c) Since the function $x \mapsto \|x\|_{w,p}^p$ is strictly concave on $(\mathbb{R}^n)^+ \setminus \{0\}$, $0 < p < 1$, the assertion (c) is clear.

(d) It is enough to observe that $\|x\|_{w,1} = \|x\|_n$ for every $x \in (\mathbb{R}^n)^+$.

**Corollary 1.** If $y = (y_i) \in (\mathbb{R}^n)^+$ and $S_k(y) \leq \sigma_k$ for $k = 1, \ldots, n$, then
\[
\sum_{i=1}^n x_i y_i \leq \left( \sum_{i=1}^n x_i^p w_i \right)^{1/p}
\]
for every $x = (x_i) \in (\mathbb{R}^n)^+$.

**Proof.** Corollary 1 follows immediately from Lemma 1(b).

**Corollary 2.** For every $x = (x_i) \in (\mathbb{R}^n)^+$,
\[
\left( \sum_{i=1}^{n-1} x_i^p w_i \right)^{1/p} + (\sigma_n - \sigma_{n-1}) x_n \leq \left( \sum_{i=1}^n x_i^p w_i \right)^{1/p}.
\]

**Proof.** It suffices to apply Corollary 1 with $\tilde{w}_1 = S_{n-1}(w)$, $\tilde{w}_2 = w_n$, $\tilde{x}_1 = (\sum_{i=1}^{n-1} x_i^p w_i)^{1/p} \sigma_{n-1}$, $\tilde{x}_2 = x_n$, $y_1 = \sigma_{n-1}$, and $y_2 = \sigma_n - \sigma_{n-1}$.

**Proposition 1.** Let $0 < p \leq 1$, $w = (w_i)$ and $v = (v_i)$ belong to $l_1^+ \setminus l_1$. Then
\[
d(w,p) \subset d(v,1) \text{ if and only if } \inf_n \frac{S_{n^1/p}(w)}{S_n(v)} > 0.
\]

In particular
\[
d(w,p) \subset l_1 \text{ if and only if } \inf_n n^{-1} S_{n^1/p}(w) > 0.
\]

**Proof.** If $d(w,p) \subset d(v,1)$, then, by the closed graph theorem, the inclusion map is continuous. Moreover, $\|f_n\|_{w,p} = 1$ for $n = 1, 2, \ldots$. Thus
\[
\sup_n \|f_n\|_{v,1} = \sup_n \frac{S_n(v)}{S_{n^1/p}(w)} < +\infty.
\]
If $\inf_n S_{n^1/p}(w)/S_n(v) > 0$, then, by Corollary 1, $d(w,p) \subset d(v,1)$.
LEMMA 2. Let \( \inf_n \sigma_n / n = 0 \). Then there exist an increasing sequence of integers \( (n_k) \) and a sequence of positive numbers \( q = (q_n) \in c_0 \) such that:

(a) \( S_n(q) \leq \sigma_n \) for \( n = 1, 2, \ldots \).
(b) \( S_{n_k}(q) = \sigma_{n_k} \) for \( k = 1, 2, \ldots \).
(c) The sequence \( (S_n(q)/n) \) is nonincreasing.

PROOF. We define \( (n_k) \) by induction taking \( n_1 = 1 \) and

\[
n_{k+1} = \inf \left\{ n > n_k : \frac{\sigma_n}{n} < \frac{\sigma_{n_k}}{n_k} \right\}, \quad k = 1, 2, \ldots
\]

Put \( Q_n = \sigma_{n_k}/n_k \) for \( n_k \leq n < n_{k+1}, \) \( k = 1, 2, \ldots, q_n = Q_n - Q_{n-1} \) for \( n = 1, 2, \ldots, \) and \( Q_0 = 0 \). The assertions (a), (b) and (c) follow immediately from the construction.

LEMMA 3. If \( q = (q_n) \in c_0^+ \) and \( (S_n(q)/n) \in \omega^{++} \), then \( S_n(q) \leq S_n(q^*) \leq 2S_n(q) \) for \( n = 1, 2, \ldots \).

PROOF. Evidently \( S_n(q) \leq S_n(q^*) \). We define

\[
A = \{ i \in \{1, \ldots, n\} : q_i^* = q_j \text{ for some } j > n \}.
\]

Since the sequence \( (S_n(q)/n) \) is nonincreasing, \( q_{n+1} \leq S_n(q)/n \) for \( n = 1, 2, \ldots \). Thus, if \( i \in A \) and \( q_i^* = q_j \) for \( j > n \), so \( q_i^* = q_j \leq S_j(q)/(j - 1) \leq S_n(q)/n \). Therefore

\[
S_n(q^*) = \sum_{i \in A} q_i^* + \sum_{i < n, i \notin A} q_i \leq |A| \frac{S_n(q)}{n} + S_n(q) \leq 2S_n(q).
\]

LEMMA 4. Let \( \lim_{n \to \infty} \sigma_n / n = +\infty \) and let \( x_m = (x_{m_i}) \) be a normalized sequence in \( d(w, p) \). Then \( \lim_{m \to \infty} \|x_m\|_{c_0} = 0 \) implies \( \lim_{m \to \infty} \|x_m\|_{l_1} = 0 \).

PROOF. We can assume that \( x_m = x_m^* \) for every \( m \in \mathbb{N} \). Fix \( \varepsilon > 0 \). There is \( n_0 \in \mathbb{N} \) such that \( 2n/\varepsilon \leq \sigma_n \) for every \( n \geq n_0 \). Let

\[
y_i = \begin{cases} 0 & \text{if } i < n_0, \\ 2/\varepsilon & \text{if } i \geq n_0. \end{cases}
\]

Then \( S_k(y) \leq \sigma_k \) for every \( k \in \mathbb{N} \). From Corollary 1 follows

\[
\sum_{i=1}^n x_{m_i}y_i \leq \left( \sum_{i=1}^n x_{m_i}^p w_i \right)^{1/p} \leq \|x_m\|_{w, p} = 1, \quad n, m = 1, 2, \ldots.
\]

Thus

\[
\frac{2}{\varepsilon} \sum_{i=n_0}^\infty x_{m_i} \leq 1 \quad \text{for } m = 1, 2, \ldots.
\]

Finally

\[
\sum_{i=n_0}^\infty x_{m_i} \leq \frac{\varepsilon}{2} \quad \text{for } m = 1, 2, \ldots.
\]
LEMMA 5. Let $0 < p < 1$ and $x = (x_i) \in d(w, p)^{++}$. If $\|x\|_{w, p} = \|x\|_{\wedge, p} = 1$, then $x = f_k$ for some $k = 1, 2, \ldots$.

PROOF. Let $x^{(n)} = \sum_{i=1}^{n} x_i e_i$ and let $\|\cdot\|_n$ be as in Lemma 1. Every point $f_k$ is of the form $f_k = (\alpha, \alpha, \ldots, \alpha, 0, \ldots)$ for some $\alpha > 0$. Suppose that $x \neq f_k$ for $k = 1, 2, \ldots$. Then there is $i \in \mathbb{N}$ such that $x_{i-1} > x_i > 0$. Therefore by Lemma 1(b) $\|x^{(l)}\|_l \leq \|x^{(l)}\|_{w, p}$ and by Lemma 1(c) we see that the equality cannot hold. Thus for some $\varepsilon > 0$ we have

$$\|x^{(l)}\|_l \leq \|x^{(l)}\|_{w, p} - \varepsilon.$$ From this, using Corollary 2, we get by induction

$$\|x^{(n)}\|_{\wedge, p} \leq \|x^{(n)}\|_n \leq \|x^{(n)}\|_{w, p} - \varepsilon \quad \text{for } n \geq 1.$$ Thus $\|x\|_{\wedge, p} \leq \|x\|_{w, p} - \varepsilon$.

III. The Mackey topology of $d(w, p)$, $0 < p < 1$.

THEOREM 1. Let $0 < p < 1$ and $w = (w_i) \in l_1^{\frac{1}{p}} \setminus l_1$. Then there exists a sequence $v = (v_i) \in l_1^{\frac{1}{p}} \setminus l_1$ such that $d(w, p) \subset d(v, 1)$ and the Mackey topology of $d(w, p)$ is induced from $d(v, 1)$.

The sequence $v \in c_0$ if and only if $\inf_n n^{-1} S^{1/p}_n(w) = 0$.

PROOF. If $\inf_n n^{-1} S^{1/p}_n(w) > 0$, then by Proposition 1 $d(w, p) \subset l_1 = d(v, 1)$ for $v = (1, 1, \ldots)$. By [8, Proposition 3.4], the Mackey topology of $d(w, p)$ is induced from $l_1$.

Let $\inf_n n^{-1} S^{1/p}_n(w) = 0$. We choose sequences $(n_k) \subset \mathbb{N}$ and $(q_n)$ according to Lemma 2. Put $v_n = q_n$, $n = 1, 2, \ldots$.

We will show that

$$B_{v, 1}^n \subset \text{conv } B_{w, p}^n \subset 2B_{v, 1}^n \quad \text{for every } n \in \mathbb{N}.$$ Indeed, by Lemma 3, $S_k(v) = S_k(q^*) \leq 2S_k(q) \leq 2S^{1/p}_k(w)$, for $k = 1, 2, \ldots$. Thus, using Corollary 1 with $y_k = \frac{1}{n} v_k$, we obtain $(B_{w, p}^n)^{++} \subset 2(B_{v, 1}^n)^{++}$. Hence the right inclusion follows from the convexity of $B_{v, 1}$.

It is obvious that if $(B_{v, 1}^n)^{++} \subset \text{conv } B_{w, p}^n$, then the left inclusion holds. Since $(B_{v, 1}^n)^{++} = \text{conv } \{g_j : j = 0, 1, \ldots, n\}$, where $g_j = S_j^{-1}(v) \sum_{i=1}^{j} e_i$, $g_0 = 0$ (see Lemma 1(a) and (b)), it suffices to prove that $g_j \in \text{conv } B_{w, p}^n$ for $j = 1, \ldots, n$.

Fix $j \in \{1, \ldots, n\}$. We find $n_k$ such that $n_k \leq j < n_{k+1}$. Let $\mathcal{C}$ be the family of all subsets of cardinality $n_k$ in the set $\{1, \ldots, j\}$. We define

$$x_C = S_{n_k}^{-1/p}(w) \sum_{i \in C} e_i \quad \text{for some } C \in \mathcal{C}.$$
We have \( \|x_C\|_{w.p} = 1 \) and
\[
\frac{1}{|C|} \sum_{C \in C} x_C = \left( \begin{array}{c} j \\ n_k \end{array} \right)^{-1} \sum_{C \in C} \sum_{i \in C} e_i \\
= \left( \begin{array}{c} j \\ n_k \end{array} \right)^{-1} \left( \begin{array}{c} j-1 \\ n_k-1 \end{array} \right) \sum_{i=1}^{j} e_i \\
= \frac{n_k}{j} \sum_{i=1}^{j} e_i = S_j^{-1}(q) \sum_{i=1}^{j} e_i \\
= \frac{S_j(q^*)}{S_j(q)} g_j.
\]
Thus \( (S_j(q^*)/S_j(q))g_j \in \text{conv} \mathcal{B}_E \). Since \( S_j(q) \leq S_j(q^*) \) and the set \( \text{conv} \mathcal{B}_E \) is balanced, \( g_j \in \text{conv} \mathcal{B}_E \). Therefore the assertion (*) holds. Thus the Mackey topology of \( d(w,p) \) and the \( d(v,1) \)-topology coincide on the subspace of all finitely supported sequences. Since this subspace is dense in \( d(w,p) \), these two topologies coincide on \( d(w,p) \).

If \( \inf_n n^{-1} S_n^{1/p}(w)/n = 0 \), then \( v \in C_0 \) by Lemma 2.

As a simple application of Theorem 1 we obtain the representation of the dual \( d(w,p)' \) of \( d(w,p), 0 < p < 1 \).

**Corollary 3.** Let \( 0 < p < 1, w = (w_i) \in l_1^{++} \setminus l_1 \). Then

(a) \( d(w,p)' = l_\infty \) if \( \inf_n \frac{S_n^{1/p} w}{n} > 0 \);

(b) \( d(w,p)' = \left\{ y \in C_0 : \sup_n \frac{S_n(y^*)}{S_n^{1/p}(w)} < +\infty \right\} =: E(w,p) \) if \( \inf_n \frac{S_n^{1/p}(w)}{n} = 0 \).

**Proof.** If \( \inf_n S_n^{1/p}(w)/n > 0 \), then by Theorem 1 \( d(w,p) = l_1 \), so \( d(w,p)' = l_\infty \). Let \( \inf_n S_n^{1/p}(w)/n = 0 \). Then by Theorem 1 there exists \( v = (v_i) \in C_0^{++} \setminus l_1 \) such that \( d(w,p) = d(v,1) \). Therefore by Proposition 1 \( \sup_n S_n(v)/S_n^{1/p}(w) < +\infty \). By [4, Theorem 11], \( d(v,1) = \{ y \in C_0 : \sup_n S_n(y^*)/S_n(v) < +\infty \} \). Hence \( d(w,p)' = d(v,1)' \subset E(w,p) \).

The inclusion \( E(w,p) \subset d(w,p)' \) follows directly from Corollary 1.

**Remark 1.** Theorem 1 and Corollary 3 are respectively extensions of Theorem 6.3 and Proposition 6.1 in [8].

**IV. Complemented subspaces of** \( d(w,p), 0 < p < 1 \).

**Theorem 2.** Let \( 0 < p < 1 \) and let \( w = (w_i) \in C_0^{++} \setminus l_1 \). If \( \inf_n S_n^{1/p}(w)/n = 0 \), then there is a positive continuous projection from \( d(w,p) \) onto a sublattice order isomorphic to \( l_p \).

**Proof.** First we construct by induction an increasing sequence of integers \( \{n_k\}_{k=0}^\infty \) and a sequence \( q = (q_i) \in \omega^+ \) such that the following conditions are...
satisfied for all $k \geq 0$:

1. \[
\left( \sum_{i=n_k+1}^{j} w_i \right)^{1/p} \geq \sum_{i=n_k+1}^{j} q_i \quad \text{for } n_k < j \leq n_{k+1};
\]

2. \[
k \leq \left( \sum_{i=n_k+1}^{n_{k+1}} w_i \right)^{1/p} = \sum_{i=n_k+1}^{n_{k+1}} q_i;
\]

3. The sequence \[
\left( \sum_{i=n_k+1}^{j} \frac{q_i}{j-n_k} \right)_{j=n_k+1}^{n_{k+1}}
\]
is nonincreasing;

4. \[
\left( \sum_{i=1}^{n_{k+1}-n_k} w_i \right)^{1/p} \leq 2 \left( \sum_{i=n_k+1}^{n_{k+1}} w_i \right)^{1/p}.
\]

We start with $n_0 = 0$, $q_0 = 0$. Suppose that $n_k$ has been already defined for some $k \geq 0$. Since $w \notin l_1$, there is $r \in \mathbb{N}$, $r \geq n_k$ such that for every $n > r$

\[
\left( \sum_{i=n_k+1}^{n} w_i \right)^{1/p} \leq 2 \left( \sum_{i=n_k+1}^{n} w_i \right)^{1/p}.
\]

Applying Lemma 2 to the sequence $(w_i)_{i=n_k+1}^{\infty}$ we can find $n_{k+1} > r$ and $(q_i)_{i=n_k+1}^{n_{k+1}}$ such that (1), (2) and (3) hold. As $n_{k+1} > r$, the same is true of (4).

Let \[
f_k = \left( \sum_{i=n_k+1}^{n_{k+1}} w_i \right)^{-1/p} \sum_{i=n_k+1}^{n_{k+1}} e_i, \quad k = 0, 1, 2, \ldots
\]

It follows from (4) that $\|f_k\|_{w,p} \leq 2$.

Now we define the projection $P: d(w,p) \to \text{span} \{ f_k \}_{k=0}^{\infty}$ by

\[
P(x) = \sum_{k=0}^{\infty} \left( \sum_{i=n_k+1}^{n_{k+1}} x_i q_i \right) f_k, \quad \text{where } x = (x_i) \in d(w,p).
\]

Let $x = (x_i) \in d(w,p)$ and let $(\hat{x}_i)_{i=n_k+1}^{n_{k+1}}$ and $(\hat{q}_i)_{i=n_k+1}^{n_{k+1}}$, $k = 0, 1, \ldots$, be respectively nonincreasing rearrangements of the sequences $(|x_i|)_{i=n_k+1}^{n_{k+1}}$ and $(q_i)_{i=n_k+1}^{n_{k+1}}$.

Using (3) and Lemma 3 we have

\[
\sum_{i=n_k+1}^{1} \hat{q}_i \leq 2 \sum_{i=n_k+1}^{1} q_i, \quad l = n_k + 1, \ldots, n_{k+1}.
\]

Thus by (1) and Corollary 1 we get

\[
\|Px\|_{w,p}^p \leq 2^p \sum_{k=0}^{\infty} \left( \sum_{i=n_k+1}^{n_{k+1}} x_i q_i \right)^p \leq 2^p \sum_{k=0}^{\infty} \left( \sum_{i=n_k+1}^{n_{k+1}} \hat{x}_i \hat{q}_i \right)^p
\]

\[
\leq 2^{p+1} \sum_{k=0}^{\infty} \left( \sum_{i=n_k+1}^{n_{k+1}} \hat{x}_i^p w_i \right)^p \leq 2^{p+1} \sum_{i=1}^{\infty} \hat{x}_i^p w_i = 2^{p+1} \|x\|_{w,p}^p.
\]
Thus $P$ is continuous. By (2) and [8, Lemma 3.1] there is a strictly increasing sequence $(j_k)$ such that $(f_{j_k})$ is equivalent to the canonical basis of $l_p$. Therefore the desired result follows from unconditionality of the basic sequence $(f_k)$.

**Remark 2.** Theorem 2 solves Problems 3 and 3a in [8].

**Corollary 4.** If $\inf_n n^{-1} S_n^{1/p}(w) = 0$, then $d(w,p) \oplus l_p$ is isomorphic to $d(w,p)$, $0 < p < 1$.

**Proof.** By Theorem 2, $d(w,p) = X \oplus l_p$ for some $F$-space $X$. Therefore

$$d(w,p) = X \oplus l_p = X \oplus l_p \oplus l_p = d(w,p) \oplus l_p.$$ 

**Corollary 5.** Let $0 < p < 1$, and $\inf_n n^{-1} S_n^{1/p}(w) = 0$. Then $d(w,p)$ has uncountably many mutually nonequivalent unconditional bases.

**Proof.** It is enough to know that $d(w,p)$ has at least two mutually nonequivalent bases (cf. [6, p. 118]). Thus our result follows from Corollary 4.

In the proof of the next theorem we use the same ideas as in [7, Theorem 2.3].

**Theorem 3.** Let $0 < p < 1$, $w = (w_i) \in l^{++}_{1,1}$. If $\lim_{n \to \infty} S_n^{1/p}(w)/n = \infty$, then each infinite-dimensional complemented subspace of $d(w,p)$ contains a subspace $Y$ which is isomorphic to $d(w,p)$ and complemented in $d(w,p)$.

**Proof.** Let $P$ be a continuous projection from $d(w,p)$ onto an infinite-dimensional subspace $X$ of $d(w,p)$. Since $\lim_{n \to \infty} S_n^{1/p}(w)/n = \infty$, by Theorem 1 $d(w,p) = l_1$. Because $X$ is complemented in $d(w,p)$, so its Mackey topology is also induced from $l_1$. Since the $l_1$-closure of $\text{conv}\{P(e_i) : i \in \mathbb{N}\}$ is a neighborhood of zero in $X$, the set $\{P(e_i) : i \in \mathbb{N}\}$ is not precompact in $l_1$. Therefore, using the standard gliding hump method, we can construct a strictly increasing sequence of the integers $(n_k)$ and sequences of vectors $(y_k)$ and $(z_k)$ such that:

1. $y_k = P(e_{n_{2k+1}} - e_{n_{2k}})$;
2. $z_k = \sum_{i \in A_k} t_i e_i$ is a block basic sequence;
3. $\sum_{k=1}^\infty \|y_k - z_k\|_{l_1} < 1$;
4. $0 < C_1 \leq \|z_k\|_{l_1} \leq \|z_k\|_{w_1,p} \leq C_2$ for $k \in \mathbb{N}$, where $C_1, C_2$ are some constants.

By Lemma 4 we have $\inf_k \max_{i \in A_k} |t_i| > 0$. Since $(e_k)$ is symmetric and $P$ is continuous, the sequence $(z_k)$ is equivalent to $(e_k)$. Thus, as in [3], we may define a continuous projection $Q$ by

$$Q(x) = \sum_{n=1}^\infty \frac{x_i}{t_i} z_n \text{ if } x = (x_i) \in d(w,p),$$

where $i_n \in A_n$ and $|t_i| = \max\{|t_i| : i \in A_n\}$, $n = 1, 2, \ldots$. Using a stability theorem (cf. [6, Proposition 1.a.9] and [7, Proposition 1.2]) we conclude that $\text{span}\{P(e_{n_{2k+1}}) - P(e_{n_{2k}})\}_{k \geq k_0}$ is isomorphic to $d(w,p)$ and complemented in $d(w,p)$.

Our next result is an easy consequence of Theorem 3 and Pelczýński's decomposition method.

**Corollary 6.** Let $0 < p < 1$ and $w = (w_i) \in l^{++}_{1,1}$. If $\lim_{n \to \infty} S_n^{1/p}(w)/n = \infty$, then every infinite-dimensional complemented subspace of $d(w,p)$ with symmetric basis is isomorphic to $d(w,p)$.
COROLLARY 7. Let $0 < p < 1$, $w = (w_i) \in c_0^{++} \setminus l_1$ and $\lim_{n \to \infty} S_n^{1/p}(w)/n = \infty$. Then $d(w, p)$ contains a closed subspace $X$ nonisomorphic to $l_p$ and $d(w, p)$ such that $X \approx l_1$.

PROOF. It follows from Corollary 6 that $d(w, p) \oplus l_p \not\approx d(w, p)$. Moreover $d(w, p) \oplus l_p$ is isomorphic to some subspace $Z$ of $d(w, p) \oplus d(w, p) \approx d(w, p)$. Since $l_p \oplus d(w, p) = l_1 \oplus l_1 \approx l_1$ we get $Z \approx l_1$.

REMARK 3. Corollary 7 solves partially Problem 2 in [8].

PROPOSITION 2. Let $0 < p < 1$, $w = (w_i) \in l_1^{++} \setminus l_1$ and $w_1 < S_n^{1/p}(w)/n$ for $n > 1$. If $P: d(w, p) \to Y \subset d(w, p)$ is a constructive projection, then $Y = \text{span}\{e_i : i \in A\}$ for some set $A \subset N$.

PROOF. We can assume that $w_1 = 1$. Since $1 < n^{-1}S_n^{1/p}(w)$, by Theorem 1 and Corollary 1, we have $d(\overline{w}, p) = l_1$ and $(B_{w,p}^n)^{++} \subset B_{l_1}$, $n = 1, 2, \ldots$. Thus $B_{w,p} \subset B_{l_1}$ and

$$B = \text{conv}^l_{l_1}B_{w,p} \subset B_{l_1} = \text{conv}^l_{l_1}\{e_i : i = 1, 2, \ldots\} \subset \text{conv}^l_{l_1}B_{w,p} = \hat{B},$$

where $\hat{B} = \{x \in l_1 : \|x\|_{w,p} \leq 1\}$.

Therefore $\|\cdot\|_{w,p} = \|\cdot\|_{l_1}$.

Hence a continuous extension $\hat{P}$ of $P$ is a contractive projection in $l_1 = d(\overline{w}, p)$. By [5, Chapter 6, §17, Theorem 3] (see also [6, Theorem 2.4.3]),

$$\hat{P}(x) = \sum_{j=1}^{m} h_j(x)u_j,$$

where $\{u_j\}_{j=1}^{m}$ are vectors of norm 1 in $l_1$ ($m = \dim Y$ is either an integer or $\infty$), $u_j = \sum_{i \in A_j} t_i e_i$, with $A_j \cap A_k = \emptyset$ for $j \neq k$ and $\{h_j\}_{j=1}^{m} \subset l_1$ satisfy $\|h_j\|_{l_1} = h_j(u_j) = 1$, $j = 1, 2, \ldots$.

Since for every $x \in d(w, p)$ and $j = 1, 2, \ldots$,

$$\|x\|_{w,p} \geq \|Px\|_{w,p} = \|\hat{P}x\|_{w,p} \geq \|h_j(x)u_j\|_{w,p},$$

so $u_j \in d(w, p)$ and $Q_j(x) := h_j(x)u_j$ is a contractive projection from $d(w, p)$ onto a one-dimensional subspace $\text{span}\{u_j\}$.

Therefore $\|u_j\|_{w,p} = \|u_j\|_{\hat{w},p} = 1$. By Lemma 5, $u_j^* = f_k$ for some $k = 1, 2, \ldots$.

Since $1 < S_n^{1/p}(w)/n$ for $n > 1$, $\|f_k\|_{\hat{w},p} < \|f_k\|_{w,p}$ if $k > 1$. Thus $u_j^* = e_1$, $j = 1, 2, \ldots$.

COROLLARY 8. Let $0 < p < 1$, $w = (w_i) \in c_0^{++} \setminus l_1$ and $w_1 < S_n^{1/p}(w)/n$ for $n > 1$. Then $l_p$ is not isomorphic to the range of a contractive projection in $d(w, p)$.

REMARK 4. Corollary 8 is an extension of Theorem 5.5 in [8].

V. Open problems and remarks. If $\lim_{n \to \infty} S_n^{1/p}(w)/n = 0$, then by Theorem 2 there exists a continuous projection $P$ from $d(w, p)$ onto a subspace isomorphic to $l_p$. Moreover, if $\lim_{n \to \infty} S_n^{1/p}(w)/n = \infty$, then by Theorem 3 no subspace isomorphic to $l_p$ is complemented in $d(w, p)$.

PROBLEM 1. Let $0 < p < 1$ and $0 < \lim_{n \to \infty} S_n^{1/p}(w)/n < \infty$. Is there a continuous projection from $d(w, p)$ onto a subspace isomorphic to $l_p$?
PROBLEM 2. Let $0 < p < 1$ and $\lim_{n \to \infty} S_{n}^{1/p}(w)/n = 0$. Is there a contractive projection from $d(w,p)$ onto a subspace isomorphic to $l_p$?

The next result is an extension of Theorem 3.8 in [8].

PROPOSITION 3. Each symmetric basis $(y_k)$ of $d(w,p)$ ($0 < p < 1$) is equivalent to the canonical basis $(e_k)$ of $d(w,p)$.

PROOF. Using the standard gliding hump method we can find a strictly increasing sequence of natural numbers $(n_k)$ such that the sequence $x_k = y_{n2k} - y_{n2k+1}$ is equivalent to a block basic sequence $z_k = \sum_{i \in A_k} b_ik$. Since $x_k$ is symmetric and equivalent to $(y_k)$, by [8, Lemma 3.1] $\inf_{k} \max_{i \in A_k} |b_i| > 0$. Hence $(y_k)$ dominates $(e_k)$. If we interchange the roles of $(e_k)$ and $(y_k)$ we deduce the equivalence of these bases.

If $\lim_{n \to \infty} S_{n}^{1/p}(w)/n = 0$, then $d(w,p)$ has uncountably many mutually non-equivalent unconditional bases. However the above proposition and Corollary 6 suggest the following

PROBLEM 3. Let $0 < p < 1$ and $\lim_{n \to \infty} S_{n}^{1/p}(w)/n = \infty$. Are every two unconditional bases in $d(w,p)$ equivalent?

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