THE MACKEY TOPOLOGY AND COMPLEMENTED SUBSPACES
OF LORENTZ SEQUENCE SPACES $d(w, p)$ FOR $0 < p < 1$

BY

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ABSTRACT. In this paper we continue the study of Lorentz sequence spaces $d(w, p)$, $0 < p < 1$, initiated by N. Popa [8]. First we show that the Mackey completion of $d(w, p)$ is equal to $d(v, 1)$ for some sequence $v$. Next, we prove that if $d(w, p) \not\subset l_1$, then it contains a complemented subspace isomorphic to $l_p$. Finally we show that if $\lim n^{-1}(\sum_{i=1}^{n} w_i)^{1/p} = \infty$, then every complemented subspace of $d(w, p)$ with symmetric bases is isomorphic to $d(w, p)$.

I. Introduction. A $p$-norm, $0 < p \leq 1$, on a vector space $X$ is a map $x \mapsto \|x\|$ such that:

1. $\|x\| > 0$ if $x \neq 0$.
2. $\|tx\| = |t| \|x\|$ for all $x \in X$ and all scalars $t$.
3. $\|x + y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$.

Let $B = \{x \in X : \|x\| \leq 1\}$; then the family \{rB\}_{r>0} is a base of neighbourhoods of zero for a Hausdorff locally bounded vector topology on $X$ (see [9]). If $X$ is complete, we say that $X$ is a $p$-Banach space.

The Mackey topology $\mu$ of a locally bounded space $X$ with separating dual is the strongest locally convex topology on $X$ which is weaker than the original one (see [10]). It is easy to see that this normable topology is generated by neighbourhoods \{rconv $B$\}_{r>0}. The Minkowski functional of the set $rconv B$ is called the Mackey norm on $X$. The completion of the space $(X, \mu)$ is called the Mackey completion of $X$ and denoted by $\hat{X}$. The extension of the Mackey norm to $\hat{X}$ is denoted by $\| \cdot \|$. For every subset $E$ of $\omega$ (= the space of all scalar sequences) we denote

$$E^+ = \{x = (x_i) \in E : x_i \geq 0 \text{ for } i = 1, 2, \ldots\}$$

and

$$E^{++} = \{x \in E^+ : x \text{ is nonincreasing}\}.$$

Let $0 < p < \infty$ and let $w = (w_i) \in l_1^\infty \setminus l_1$. For $x = (x_i) \in \omega$ we define

$$\|x\|_{w,p} = \sup_\pi \left( \sum_{i=1}^{\infty} |x_{\pi(i)}|^p w_i \right)^{1/p},$$

where $\pi$ ranges over all permutations of the positive integers. The space $d(w, p) = \{x \in \omega : \|x\|_{w,p} < \infty\}$ equipped with the locally bounded vector topology induced by $\| \cdot \|_{w,p}$ is called the Lorentz sequence space.

It is well known that $d(w, p)$ is a $p$-Banach space for $0 < p < 1$ and a Banach space for $p \geq 1$. Moreover, the sequence of unit vectors $(e_i)$ is a symmetric basis of...
d(w, p). From the assumption w ∈ l^+_∞ \setminus l_1 follows that d(w, p) ⊂ c_0. Therefore for every x = (x_i) ∈ d(w, p) there exists a nonincreasing rearrangement x^* = (x^*_i) of x (i.e. a nonincreasing sequence obtained from (|x_i|) by a suitable permutation of the integers) and ||x||_w,p = (∑_{i=1}^∞ x^*_i p w_i) \frac{1}{p}.

Observe that d(w, p) ≈ l_p if and only if w ∉ c_0 (cf. [6, p. 176]).

The first topic of the present paper is the Mackey topology of d(w, p), 0 < p < 1.

Using a representation of the dual of d(w, p), N. Popa [8] proved that the Mackey completion of d(w, p) (p = 1/k, k ∈ N, and w satisfies some additional conditions) is isomorphic to d(v, 1) for a suitable sequence v. In §3 we show that the above theorem holds for any Lorentz sequence space d(w, p), 0 < p < 1. Our result is obtained without determining any dual space.

The last part of our paper is devoted to the study of complemented subspaces of d(w, p), 0 < p < 1.

It is well known that every Lorentz sequence space d(w, p), p ≥ 1, has complemented subspace isomorphic to l_p (see [6, Proposition 4.e.3]). N. Popa [8] showed that unlike the case p ≥ 1 there are spaces d(w, p), 0 < p < 1, which contain no complemented subspaces isomorphic to l_p and conjectured that it is true for each d(w, p), 0 < p < 1. In §4 we prove that if inf_n n^{-1}(∑_{i=1}^n w_i)^{1/p} = 0 (i.e. d(w, p) ⊂ l_1, see Proposition 1), then d(w, p) has complemented subspace isomorphic to l_p. Moreover, if lim_{n→∞} n^{-1}(∑_{i=1}^n w_i)^{1/p} = ∞, then every complemented subspace of d(w, p) with symmetric basis is isomorphic to d(w, p).

Throughout the paper we denote by B_{w,p} the closed unit ball in d(w, p), R^n = span{e_i}_{i=1}^n, B_{w,p}^n = B_{w,p} ∩ R^n, n = 1, 2, ..., In addition we denote S_n(x) = x_1 + \cdots + x_n, n = 1, 2, ..., S_0(x) = 0 for any sequence x = (x_i) ∈ w.

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II. Technical results. In this section we assume that 0 < p ≤ 1, w = (w_i) ∈ l^+_∞ \setminus l_1, σ_k = S_k(w)^{1/p}, f_k = σ_k^{-1} ∑_{i=1}^k e_i for k = 1, 2, ..., and f_0 = 0.

LEMMA 1. Let ||·||_n be the norm on R^n defined by

||x||_n = ∑_{i=1}^n |x_i| (σ_i - σ_{i-1}) for x = (x_i) ∈ R^n,

and let

B^n = \{x = (x_i) ∈ R^n: ||x||_n ≤ 1\}, n ∈ N.

Then:

(a) (B^n)^++ = conv{f_k: k = 0, 1, ..., n}.
(b) (B^n_{w,p})^++ ⊂ (B^n)^++.
(c) Let 0 < p < 1 and x = (x_i) ∈ (R^n)^++. Then ||x||_n = ||x||_{w,p} = 1 if and only if x = f_k for some k = 1, 2, ..., n.
(d) If p = 1, then (B^n_{w,p})^++ = (B^n)^++.

PROOF. (a) Every point x ∈ R^n may be written in the form

x = ∑_{i=1}^{n-1} σ_i (x_i - x_{i+1}) f_i + σ_n x_n f_n.
In addition, for each \( x \in (\mathbb{R}^n)^+ \),

\[
\|x\|_n = \sum_{i=1}^{n-1} \sigma_i(x_i - x_{i+1}) + \sigma_n x_n.
\]

Therefore every \( x \in (\mathbb{R}^n)^+ \) with \( \|x\|_n = 1 \) is a convex combination of the vector \( f_1, \ldots, f_n \). It implies \( (B^n)^+ \subset \text{conv}\{f_0, f_1, \ldots, f_n\} \).

We observe that \( \|f_k\|_{w,p} = \|f_k\|_n = 1 \) for \( k = 1, 2, \ldots, n \). Both the sets \( (\mathbb{R}^n)^+ \) and \( B^n \) are convex, so

\[
\text{conv}\{f_0, \ldots, f_k\} \subset B^n \cap (\mathbb{R}^n)^+ = (B^n)^+.
\]

(b) By the proof of (a) and by concavity of the function \( x \mapsto \|x\|_{w,p}^p \) on \( (\mathbb{R}^n)^+ \),

\[
\|x\|_{w,p}^p \geq \sum_{i=1}^{n-1} \sigma_i(x_i - x_{i+1}) \|f_i\|_{w,p}^p + \sigma_n x_n \|f_n\|_{w,p}^p = \|x\|_n
\]

for \( x \in (\mathbb{R}^n)^+ \), \( \|x\|_n = 1 \).

Thus (b) follows from homogeneity of the functionals \( \| \cdot \|_n \) and \( \| \cdot \|_{w,p} \).

(c) Since the function \( x \mapsto \|x\|_{w,p}^p \) is strictly concave on \( (\mathbb{R}^n)^+ \setminus \{0\} \), \( 0 < p < 1 \), the assertion (c) is clear.

(d) It is enough to observe that \( \|x\|_{w,1} = \|x\|_n \) for every \( x \in (\mathbb{R}^n)^+ \).

**COROLLARY 1.** If \( y = (y_i) \in (\mathbb{R}^n)^+ \) and \( S_k(y) \leq \sigma_k \) for \( k = 1, \ldots, n \), then

\[
\sum_{i=1}^{n} x_i y_i \leq \left( \sum_{i=1}^{n} x_i^p w_i \right)^{1/p} \quad \text{for every } x = (x_i) \in (\mathbb{R}^n)^+.
\]

**PROOF.** Corollary 1 follows immediately from Lemma 1(b).

**COROLLARY 2.** For every \( x = (x_i) \in (\mathbb{R}^n)^+ \),

\[
\left( \sum_{i=1}^{n-1} x_i^p w_i \right)^{1/p} + (\sigma_n - \sigma_{n-1}) x_n \leq \left( \sum_{i=1}^{n} x_i^p w_i \right)^{1/p}.
\]

**PROOF.** It suffices to apply Corollary 1 with \( \bar{w}_1 = S_{n-1}(w) \), \( \bar{w}_2 = w_n \), \( \bar{x}_1 = (\sum_{i=1}^{n-1} x_i^p w_i)^{1/p} \sigma_n \), \( \bar{x}_2 = x_n \), \( y_1 = \sigma_n - \sigma_{n-1} \), and \( y_2 = \sigma_n - \sigma_{n-1} \).

**PROPOSITION 1.** Let \( 0 < p \leq 1 \), \( w = (w_i) \) and \( v = (v_i) \) belong to \( l_\infty^{++} \setminus l_1^+ \). Then

\[
d(w, p) < d(v, 1) \quad \text{if and only if} \quad \inf_n \frac{S_n^{1/p}(w)}{S_n(v)} > 0.
\]

In particular

\[
d(w, p) < l_1 \quad \text{if and only if} \quad \inf_n n^{-1} S_n^{1/p}(w) > 0.
\]

**PROOF.** If \( d(w, p) < d(v, 1) \), then, by the closed graph theorem, the inclusion map is continuous. Moreover, \( \|f_n\|_{w,p} = 1 \) for \( n = 1, 2, \ldots \). Thus

\[
\sup_n \|f_n\|_{v,1} = \sup_n \frac{S_n(v)}{S_n^{1/p}(w)} < +\infty.
\]

If \( \inf_n S_n^{1/p}(w)/S_n(v) > 0 \), then, by Corollary 1, \( d(w, p) \subset d(v, 1) \).
LEMMA 2. Let \( \inf_n \sigma_n/n = 0 \). Then there exist an increasing sequence of integers \((n_k)\) and a sequence of positive numbers \( q = (q_n) \in c_0 \) such that:

(a) \( S_n(q) \leq \sigma_n \) for \( n = 1, 2, \ldots \)
(b) \( S_{n_k}(q) = \sigma_{n_k} \) for \( k = 1, 2, \ldots \)
(c) The sequence \( (S_n(q)/n) \) is nonincreasing.

PROOF. We define \((n_k)\) by induction taking \( n_1 = 1 \) and
\[
n_{k+1} = \inf \left\{ n > n_k : \frac{\sigma_n}{n} < \frac{\sigma_{n_k}}{n_k} \right\}, \quad k = 1, 2, \ldots
\]

Put \( Q_n = n_0 \sigma_{n_k}/n_k \) for \( n_k \leq n < n_{k+1}, \) \( k = 1, 2, \ldots \), \( q_n = Q_n - Q_{n-1} \) for \( n = 1, 2, \ldots \), and \( Q_0 = 0 \). The assertions (a), (b) and (c) follow immediately from the construction.

LEMMA 3. If \( q = (q_n) \in c_0^+ \) and \( (S_n(q)/n) \in \omega^{++} \), then \( S_n(q) \leq S_n(q^*) \leq 2S_n(q) \) for \( n = 1, 2, \ldots \).

PROOF. Evidently \( S_n(q) \leq S_n(q^*) \). We define
\[
A = \{ i \in \{1, \ldots, n\} : q^*_i = q_j \text{ for some } j > n \}.
\]

Since the sequence \( (S_n(q)/n) \) is nonincreasing, \( q_{n+1} \leq S_n(q)/n \) for \( n = 1, 2, \ldots \). Thus, if \( i \in A \) and \( q^*_i = q_j \) for \( j > n \), so \( q^*_i = q_j \leq S_{j-1}(q)/(j-1) \leq S_n(q)/n \). Therefore
\[
S_n(q^*) = \sum_{i \in A} q^*_i + \sum_{i \leq n} q_i \leq |A| \frac{S_n(q)}{n} + S_n(q) \leq 2S_n(q).
\]

LEMMA 4. Let \( \lim_{n \to \infty} \sigma_n/n = +\infty \) and let \( x_m = (x_{m_i}) \) be a normalized sequence in \( d(w,p) \). Then \( \lim_{m \to \infty} \|x_m\|_{c_0} = 0 \) implies \( \lim_{m \to \infty} \|x_m\|_{l_1} = 0 \).

PROOF. We can assume that \( x_m = x^*_m \) for every \( m \in \mathbb{N} \). Fix \( \varepsilon > 0 \). There is \( n_0 \in \mathbb{N} \) such that \( 2n/\varepsilon \leq \sigma_n \) for every \( n \geq n_0 \). Let
\[
y_i = \begin{cases} 0 & \text{if } i < n_0, \\ 2/\varepsilon & \text{if } i \geq n_0. \end{cases}
\]

Then \( S_k(y) \leq \sigma_k \) for every \( k \in \mathbb{N} \). From Corollary 1 follows
\[
\sum_{i=1}^n x_{m_i} y_i \leq \left( \sum_{i=1}^n x_{m_i}^{p_i} w_i \right)^{1/p} \leq \|x_m\|_{w,p} = 1, \quad n, m = 1, 2, \ldots.
\]

Thus
\[
\frac{2}{\varepsilon} \sum_{i=n_0}^{\infty} x_{m_i} \leq 1 \quad \text{for } m = 1, 2, \ldots.
\]

Finally
\[
\sum_{i=n_0}^{\infty} x_{m_i} \leq \frac{\varepsilon}{2} \quad \text{for } m = 1, 2, \ldots.
\]
LEMMA 5. Let $0 < p < 1$ and $x = (x_i) \in d(w,p)^{++}$. If $\|x\|_{w,p} = \|x\|_{w,p} = 1$, then $x = f_k$ for some $k = 1, 2, \ldots$.

**Proof.** Let $x^{(n)} = \sum_{i=1}^{n} x_i e_i$ and let $\|\cdot\|_n$ be as in Lemma 1. Every point $f_k$ is of the form $f_k = (\alpha, \alpha, \ldots, \alpha, 0, \ldots)$ for some $\alpha > 0$. Suppose that $x \neq f_k$ for $k = 1, 2, \ldots$. Then there is $l \in \mathbb{N}$ such that $x_{l-1} > x_l > 0$. Therefore by Lemma 1(b) $\|x^{(l)}\|_l \leq \|x^{(l)}\|_{w,p}$ and by Lemma 1(c) we see that the equality cannot hold. Thus for some $\varepsilon > 0$ we have

$$\|x^{(l)}\|_l \leq \|x^{(l)}\|_{w,p} - \varepsilon.$$  

From this, using Corollary 2, we get by induction

$$\|x^{(n)}\|_{w,p} \leq \|x^{(n)}\|_n \leq \|x^{(n)}\|_{w,p} - \varepsilon \quad \text{for } n \geq 1.$$  

Thus $\|x\|_{w,p} \leq \|x\|_{w,p} - \varepsilon$.

### III. The Mackey topology of $d(w,p)$, $0 < p < 1$.

**Theorem 1.** Let $0 < p < 1$ and $w = (w_i) \in l^+_\infty \setminus l_1$. Then there exists a sequence $v = (v_i) \in l^+_\infty \setminus l_1$ such that $d(w, p) \subset d(v, 1)$ and the Mackey topology of $d(w, p)$ is induced from $d(v, 1)$.

**Proof.** If $\inf_n n^{-1} S_n^{1/p}(w) > 0$, then by Proposition 1 $d(w, p) \subset l_1 = d(v, 1)$ for $v = (1, 1, \ldots)$. By [8, Proposition 3.4], the Mackey topology of $d(w, p)$ is induced from $l_1$.

Let $\inf_n n^{-1} S_n^{1/p}(w) = 0$. We choose sequences $(n_k) \subset \mathbb{N}$ and $(q_n)$ according to Lemma 2. Put $v_n = q_n^*, n = 1, 2, \ldots$.

We will show that

$$B_{v,1}^n \subset \text{conv } B_{w,p}^n \subset 2B_{v,1}^n \quad \text{for every } n \in \mathbb{N}.$$  

Indeed, by Lemma 3, $S_k(v) = S_k(q^*) \leq 2S_k(q) \leq 2S_k^{1/p}(w), \text{ for } k = 1, 2, \ldots$. Thus, using Corollary 1 with $y_k = \frac{1}{k} v_k$, we obtain $S_k^{1/p}(w) = 2B_{v,1}^{n+1}$. Hence the right inclusion follows from the convexity of $B_{v,1}$.

It is obvious that if $(B_{v,1}^{n+1})^{++} \subset \text{conv } B_{w,p}^n$, then the left inclusion holds. Since $(B_{v,1}^{n+1})^{++} = \text{conv } \{g_j: j = 0, 1, \ldots, n\}$, where $g_j = S^{-1}_j(v) \sum_{i=1}^{j} e_i, g_0 = 0$ (see Lemma 1(a) and (b)), it suffices to prove that $g_j \in \text{conv } B_{w,p}^n$ for $j = 1, \ldots, n$.

Fix $j \in \{1, \ldots, n\}$. We find $n_k$ such that $n_k \leq j < n_{k+1}$. Let $C$ be the family of all subsets of cardinality $n_k$ in the set $\{1, \ldots, j\}$. We define

$$x_C = S_{n_k}^{-1/p}(w) \sum_{i \in C} e_i \quad \text{for some } C \in C.$$
We have \( \|x_C\|_{w,p} = 1 \) and

\[
\frac{1}{|C|} \sum_{C \in C} x_C = \left( \frac{j}{n_k} \right)^{-1} S_{n_k}^{-1}(w) \sum_{C \in C} S_{n_k}^{-1/p}(w) \sum_{i=1}^{j} e_i
\]

\[
= \left( \frac{j}{n_k} \right)^{-1} \left( \frac{j - 1}{n_k - 1} \right) S_{n_k}^{-1/p}(w) \sum_{i=1}^{j} e_i
\]

\[
= \frac{n_k}{j} S_{n_k}^{-1/p}(w) \sum_{i=1}^{j} e_i = S_j^{-1}(q) \sum_{i=1}^{j} e_i
\]

\[
= \frac{S_j(q^*)}{S_j(q)} g_j.
\]

Thus \( (S_j(q^*)/S_j(q))g_j \in \text{conv}B^{n}_{w,p} \). Since \( S_j(q) \leq S_j(q^*) \) and the set \( \text{conv}B^{n}_{w,p} \) is balanced, \( g_j \in \text{conv}B^{n}_{w,p} \). Therefore the assertion (*) holds. Thus the Mackey topology of \( d(w,p) \) and the \( d(v,1) \)-topology coincide on the subspace of all finitely supported sequences. Since this subspace is dense in \( d(w,p) \), these two topologies coincide on \( d(w,p) \).

If \( \inf_n n^{-1} S_n^{1/p}(w) = 0 \), then \( v \in C_0 \) by Lemma 2.

As a simple application of Theorem 1 we obtain the representation of the dual \( d(w,p)^\prime \) of \( d(w,p) \), \( 0 < p < 1 \).

**Corollary 3.** Let \( 0 < p < 1, w = (w_i) \in l_+^+ \backslash l_1 \). Then

(a) \( d(w,p)^\prime = l_\infty \) if \( \inf_n \frac{S_n^{1/p} w}{n} > 0 \);

(b) \( d(w,p)^\prime = \{ y \in C_0 : \sup_n \frac{S_n(y^*)}{S_n^{1/p}(w)} < +\infty \} =: E(w,p) \) if \( \inf_n \frac{S_n^{1/p}(w)}{n} = 0 \).

**Proof.** If \( \inf_n S_n^{1/p}(w)/n > 0 \), then by Theorem 1 \( d(\bar{w},p) = l_1 \), so \( d(w,p)^\prime = l_\infty \). Let \( \inf_n S_n^{1/p}(w)/n = 0 \). Then by Theorem 1 there exists \( v = (v_i) \in C_0^+ \backslash l_1 \) such that \( d(\bar{w},p) = d(v,1) \). Therefore by Proposition 1 \( \sup_n S_n(v)/S_n^{1/p}(w) < +\infty \). By [4, Theorem 11], \( d(v,1) = \{ y \in C_0 : \sup_n S_n(y^*)/S_n(v) < +\infty \} \). Hence \( d(w,p)^\prime = d(v,1)^\prime \subset E(w,p) \).

The inclusion \( E(w,p) \subset d(w,p)^\prime \) follows directly from Corollary 1.

**Remark 1.** Theorem 1 and Corollary 3 are respectively extensions of Theorem 6.3 and Proposition 6.1 in [8].

**IV. Complemented subspaces of** \( d(w,p) \), \( 0 < p < 1 \).

**Theorem 2.** Let \( 0 < p < 1 \) and let \( w = (w_i) \in C_0^+ \backslash l_1 \). If \( \inf_n S_n^{1/p}(w)/n = 0 \), then there is a positive continuous projection from \( d(w,p) \) onto a sublattice order isomorphic to \( l_p \).

**Proof.** First we construct by induction an increasing sequence of integers \( \{n_k\}_{k=0}^\infty \) and a sequence \( q = (q_i) \in \omega^+ \) such that the following conditions are
satisfied for all $k \geq 0$:

1. \[ \left( \sum_{i=n_k+1}^{j} w_i \right)^{1/p} \leq \sum_{i=n_k+1}^{j} q_i \quad \text{for } n_k < j \leq n_{k+1}; \]

2. \[ k \leq \left( \sum_{i=n_k+1}^{n_{k+1}} w_i \right)^{1/p} = \sum_{i=n_k+1}^{n_{k+1}} q_i; \]

3. the sequence \[ \left( \sum_{i=n_k+1}^{j} \frac{q_i}{j-n_k} \right)_{j=n_k+1}^{n_{k+1}} \] is nonincreasing;

4. \[ \left( \sum_{i=n_k+1}^{n_{k+1}-n_k} w_i \right)^{1/p} \leq 2 \left( \sum_{i=n_k+1}^{n_{k+1}} w_i \right)^{1/p}. \]

We start with $n_0 = 0$, $q_0 = 0$. Suppose that $n_k$ has been already defined for some $k \geq 0$. Since $w \not\in l_1$, there is $r \in \mathbb{N}$, $r \geq n_k$ such that for every $n > r$

\[ \left( \sum_{i=n_k+1}^{n} w_i \right)^{1/p} \leq 2 \left( \sum_{i=n_k+1}^{n} w_i \right)^{1/p}. \]

Applying Lemma 2 to the sequence $(w_i)_{i=n_k+1}^{\infty}$ we can find $n_{k+1} > r$ and $(q_i)_{i=n_k+1}^{n_{k+1}}$ such that (1), (2) and (3) hold. As $n_{k+1} > r$, the same is true of (4).

Let

\[ f_k = \left( \sum_{i=n_k+1}^{n_{k+1}} w_i \right)^{-1/p} \sum_{i=n_k+1}^{n_{k+1}} e_i, \quad k = 0, 1, 2, \ldots. \]

It follows from (4) that $\|f_k\|_{w,p} \leq 2$.

Now we define the projection $P: d(w,p) \to \text{span}\{f_k\}_{k=0}^{\infty}$ by

\[ P(x) = \sum_{k=0}^{\infty} \left( \sum_{i=n_k+1}^{n_{k+1}} x_i q_i \right) f_k, \quad \text{where } x = (x_i) \in d(w,p). \]

Let $x = (x_i) \in d(w,p)$ and let $(\hat{x}_i)_{i=n_k+1}^{n_{k+1}}$ and $(\hat{q}_i)_{i=n_k+1}^{n_{k+1}}$, $k = 0, 1, \ldots$, be respectively nonincreasing rearrangements of the sequences $(|x_i|)_{i=n_k+1}^{n_{k+1}}$ and $(q_i)_{i=n_k+1}^{n_{k+1}}$. Using (3) and Lemma 3 we have

\[ \frac{1}{l} \sum_{i=n_k+1}^{l} \hat{q}_i \leq 2 \sum_{i=n_k+1}^{1} q_i, \quad l = n_k + 1, \ldots, n_{k+1}. \]

Thus by (1) and Corollary 1 we get

\[ \|Px\|_{w,p}^p \leq 2^p \sum_{k=0}^{\infty} \left( \sum_{i=n_k+1}^{n_{k+1}} x_i q_i \right)^p \leq 2^p \sum_{k=0}^{\infty} \left( \sum_{i=n_k+1}^{n_{k+1}} \hat{x}_i \hat{q}_i \right)^p \]

\[ \leq 2^{p+1} \sum_{k=0}^{\infty} \left( \sum_{i=n_k+1}^{n_{k+1}} \hat{x}_i^p w_i \right) \leq 2^{p+1} \sum_{i=1}^{\infty} x_i^p w_i = 2^{p+1} \|x\|_{w,p}^p. \]
Thus $P$ is continuous. By (2) and [8, Lemma 3.1] there is a strictly increasing sequence $(j_k)$ such that $(f_{j_k})$ is equivalent to the canonical basis of $l_p$. Therefore the desired result follows from unconditionality of the basic sequence $(f_k)$.

**Remark 2.** Theorem 2 solves Problems 3 and 3a in [8].

**Corollary 4.** If $\inf_n n^{-1} S_n^{1/p}(w) = 0$, then $d(w, p) \oplus l_p$ is isomorphic to $d(w, p)$, $0 < p < 1$.

**Proof.** By Theorem 2, $d(w, p) = X \oplus l_p$ for some $F$-space $X$. Therefore

$$d(w, p) = X \oplus l_p = X \oplus l_p \oplus l_p = d(w, p) \oplus l_p.$$  

**Corollary 5.** Let $0 < p < 1$, and $\inf_n n^{-1} S_n^{1/p}(w) = 0$. Then $d(w, p)$ has uncountably many mutually nonequivalent unconditional bases.

**Proof.** It is enough to know that $d(w, p)$ has at least two mutually nonequivalent bases (cf. [6, p. 118]). Thus our result follows from Corollary 4.

In the proof of the next theorem we use the same ideas as in [7, Theorem 2.3].

**Theorem 3.** Let $0 < p < 1$, $w = (w_i) \in l_{++1}^1$. If $\lim_{n \to \infty} S_n^{1/p}(w)/n = \infty$, then each infinite-dimensional complemented subspace of $d(w, p)$ contains a subspace $Y$ which is isomorphic to $d(w, p)$ and complemented in $d(w, p)$.

**Proof.** Let $P$ be a continuous projection from $d(w, p)$ onto an infinite-dimensional subspace $X$ of $d(w, p)$. Since $\lim_{n \to \infty} S_n^{1/p}(w)/n = \infty$, by Theorem 1 $d(w, p) = l_1$. Because $X$ is complemented in $d(w, p)$, so is its Mackey topology is also induced from $l_1$. Since the $l_1$-closure of $\text{conv}\{P(e_i) : i \in \mathbb{N}\}$ is a neighbourhood of zero in $X$, the set $\{P(e_i) : i \in \mathbb{N}\}$ is not precompact in $l_1$. Therefore, using the standard gliding hump method, we can construct a strictly increasing sequence of the integers $(n_k)$ and sequences of vectors $(y_k)$ and $(z_k)$ such that:

1. $y_k = P(e_{n_{2k+1}} - e_{n_{2k}})$;
2. $z_k = \sum_{i \in \mathbb{N}, k} t_i e_i$ is a block basic sequence;
3. $\sum_{k=1}^{\infty} \|y_k - z_k\|_{l_p} < 1$;
4. $0 < C_1 \leq \|z_k\|_{l_i} \leq \|z_k\|_{w, p} \leq C_2$ for $k \in \mathbb{N}$, where $C_1, C_2$ are some constants.

By Lemma 4 we have $\inf_k \max_{i \in \mathbb{N}} |t_i| > 0$. Since $(e_k)$ is symmetric and $P$ is continuous, the sequence $(z_k)$ is equivalent to $(e_k)$. Thus, as in [3], we may define a continuous projection $Q$ by

$$Q(x) = \sum_{n=1}^{\infty} \frac{x_{i_n}}{t_{i_n}} z_n \quad \text{if} \quad x = (x_i) \in d(w, p),$$  

where $i_n \in A_n$ and $|t_{i_n}| = \max\{|t_i| : i \in A_n\}$, $n = 1, 2, \ldots$. Using a stability theorem (cf. [6, Proposition 1.4] and [7, Proposition 1.2]) we conclude that $\text{span}\{P(e_{n_{2k+1}}) - P(e_{n_{2k}})\}_{k \geq k_0}$ is isomorphic to $d(w, p)$ and complemented in $d(w, p)$.

Our next result is an easy consequence of Theorem 3 and Pelczyński's decomposition method.

**Corollary 6.** Let $0 < p < 1$ and $w = (w_i) \in l_{++1}^1$. If $\lim_{n \to \infty} S_n^{1/p}(w)/n = \infty$, then every infinite-dimensional complemented subspace of $d(w, p)$ with symmetric basis is isomorphic to $d(w, p)$.  

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COROLLARY 7. Let $0 < p < 1$, $w = (w_i) \in c_0^{++} \setminus l_1$ and $\lim_{n \to \infty} S_{n}^{1/p}(w)/n = \infty$. Then $d(w, p)$ contains a closed subspace $X$ nonisomorphic to $l_p$ and $d(w, p)$ such that $X \approx l_1$.

PROOF. It follows from Corollary 6 that $d(w, p) \hat{\oplus} l_p \not\approx d(w, p)$. Moreover $d(w, p) \hat{\oplus} d(w, p)$ is isomorphic to some subspace $Z$ of $d(w, p) \hat{\oplus} d(w, p) \approx d(w, p)$. Since $l_p \oplus d(w, p) = l_1 \oplus l_1 \approx l_1$ we get $\hat{Z} \approx l_1$.

REMARK 3. Corollary 7 solves partially Problem 2 in [8].

PROPOSITION 2. Let $0 < p < 1$, $w = (w_i) \in l_\infty^{++} \setminus l_1$ and $w_1 < S_{n}^{1/p}(w)/n$ for $n > 1$. If $P : d(w, p) \to d(w, p)$ is a constructive projection, then $Y = \text{span}\{e_i : i \in A\}$ for some set $A \subset N$.

PROOF. We can assume that $w_1 = 1$. Since $1 < n^{-1}S_{n}^{1/p}(w)$, by Theorem 1 and Corollary 1, we have $d(\overline{w}, p) = l_1$ and $(B_{w,p}^n)^{++} \subset B_{l_1}$, $n = 1, 2, \ldots$. Thus $B_{w,p} \subset B_{l_1}$ and

$$\hat{B} = \text{conv}^{l_1}B_{w,p} \subset B_{l_1} = \text{conv}^{l_1}\{e_i : i = 1, 2, \ldots\} \subset \text{conv}^{l_1}B_{w,p} = \hat{B},$$

where $\hat{B} = \{x \in l_1 : \|x\|_{w,p} \leq 1\}$.

Therefore $\| \|_{w,p} = \| \cdot \|_{l_1}$.

Hence a continuous extension $\hat{P}$ of $P$ is a contractive projection in $l_1 = d(\overline{w}, p)$. By [5, Chapter 6, §17, Theorem 3] (see also [6, Theorem 2.a.4]),

$$\hat{P}(x) = \sum_{j=1}^{m} h_j(x)u_j,$$

where $\{u_j\}_{j=1}^{m}$ are vectors of norm 1 in $l_1$ ($m = \dim Y$ is either an integer or $\infty)$, $u_j = \sum_{i \in A_j} t_je_i$, with $A_j \cap A_k = \emptyset$ for $j \neq k$ and $\{h_j\}_{j=1}^{m} \subset l_1$ satisfy $\|h_j\|_{\infty} = h_j(u_j) = 1$, $j = 1, 2, \ldots$.

Since for every $x \in d(w, p)$ and $j = 1, 2, \ldots$,

$$\|x\|_{w,p} \geq \|Px\|_{w,p} = \|\hat{P}x\|_{w,p} \geq \|h_j(x)u_j\|_{w,p},$$

so $u_j \in d(w, p)$ and $Q_j(x) := h_j(x)u_j$ is a contractive projection from $d(w, p)$ onto a one-dimensional subspace $\text{span}\{u_j\}$.

Therefore $\|u_j\|_{w,p} = \|u_j\|_{w,p} = 1$. By Lemma 5, $u_j^* = f_k$ for some $k = 1, 2, \ldots$.

Since $1 < S_{n}^{1/p}(w)/n$ for $n > 1$, $\|f_k\|_{w,p} = \|f_k\|_{w,p}$ if $k > 1$. Thus $u_j^* = e_1$, $j = 1, 2, \ldots$.

COROLLARY 8. Let $0 < p < 1$, $w = (w_i) \in c_0^{++} \setminus l_1$ and $w_1 < S_{n}^{1/p}(w)/n$ for $n > 1$. Then $l_p$ is not isomorphic to the range of a contractive projection in $d(w, p)$.

REMARK 4. Corollary 8 is an extension of Theorem 5.5 in [8].

V. OPEN PROBLEMS AND REMARKS. If $\lim_{n \to \infty} S_{n}^{1/p}(w)/n = 0$, then by Theorem 2 there exists a continuous projection $P$ from $d(w, p)$ onto a subspace isomorphic to $l_p$. Moreover, if $\lim_{n \to \infty} S_{n}^{1/p}(w)/n = \infty$, then by Theorem 3 no subspace isomorphic to $l_p$ is complemented in $d(w, p)$.

PROBLEM 1. Let $0 < p < 1$ and $0 < \lim_{n \to \infty} S_{n}^{1/p}(w)/n < \infty$. Is there a continuous projection from $d(w, p)$ onto a subspace isomorphic to $l_p$?
Problem 2. Let $0 < p < 1$ and $\lim_{n \to \infty} S_n^{1/p}(w)/n = 0$. Is there a contractive projection from $d(w, p)$ onto a subspace isomorphic to $l_p$?

The next result is an extension of Theorem 3.8 in [8].

Proposition 3. Each symmetric basis $(y_k)$ of $d(w, p)$ ($0 < p < 1$) is equivalent to the canonical basis $(e_k)$ of $d(w, p)$.

Proof. Using the standard gliding hump method we can find a strictly increasing sequence of natural numbers $(n_k)$ such that the sequence $x_k = y_{n_{2k}} - y_{n_{2k+1}}$ is equivalent to a block basic sequence $z_k = \sum_{i \in A_k} b_i e_i$. Since $x_k$ is symmetric and equivalent to $(y_k)$, by [8, Lemma 3.1] $\inf_k \max_{i \in A_k} |b_i| > 0$. Hence $(y_k)$ dominates $(e_k)$. If we interchange the roles of $(e_k)$ and $(y_k)$ we deduce the equivalence of these bases.

If $\lim_{n \to \infty} S_n^{1/p}(w)/n = 0$, then $d(w, p)$ has uncountably many mutually non-equivalent unconditional bases. However, the above proposition and Corollary 6 suggest the following

Problem 3. Let $0 < p < 1$ and $\lim_{n \to \infty} S_n^{1/p}(w)/n = \infty$. Are every two unconditional bases in $d(w, p)$ equivalent?

References


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