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PULL-BACKS OF C^*-ALGEBRAS AND CROSSED PRODUCTS BY CERTAIN DIAGONAL ACTIONS

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ABSTRACT. Let G be a locally compact group and $p\colon \Omega \to T$ a principal G-bundle. If A is a C^*-algebra with primitive ideal space T, the pull-back p^*A of A along p is the balanced tensor product $C_0(\Omega) \otimes_{C(T)} A$. If $\beta\colon G \to \operatorname{Aut} A$ consists of $C(T)$-module automorphisms, and $\gamma\colon G \to \operatorname{Aut} C_0(\Omega)$ is the natural action, then the automorphism group $\gamma \otimes \beta$ of $C_0(\Omega) \otimes A$ respects the balancing and induces the diagonal action $p^*\beta$ of G on p^*A. We discuss some examples of such actions and study the crossed product $p^*A \times_{p^*\beta} G$. We suggest a substitute D for the fixed-point algebra, prove $p^*A \times G$ is strongly Morita equivalent to D, and investigate the structure of D in various cases. In particular, we ask when D is strongly Morita equivalent to A—sometimes, but by no means always—and investigate the case where A has continuous trace.

Let B be a C^*-algebra and G a locally compact group acting on B as a strongly continuous automorphism group α. Our goal here is to study the crossed product C^*-algebra $B \times_\alpha G$ for two classes of diagonal actions for which the induced action of G on $\hat{B}$ is free. The first class includes actions of the form $\gamma \otimes \beta$ on $B = C_0(\Omega) \otimes A$, where $p\colon \Omega \to T$ is a principal G-bundle, γ is the dual action of G on $C_0(\Omega)$, and $\beta\colon G \to \operatorname{Aut} A$ is an action of G on another C^*-algebra A. We also consider diagonal actions on algebras which are the pull-backs of another algebra A along a principal bundle $p\colon \Omega \to T$: if A is a C^*-algebra with primitive ideal space T, then the pull-back p^*A is the balanced tensor product $C_0(\Omega) \otimes_{C_b(T)} A$. When $\beta\colon G \to \operatorname{Aut} A$ consists of $C(T)$-module automorphisms, the product action $\gamma \otimes \beta$ preserves the balancing, and the diagonal action $p^*\beta$ is, by definition, the induced action on p^*A.

In general, if $f\colon X \to Y$ and $q\colon \operatorname{Prim} A \to Y$ are continuous, then $C_b(Y)$ acts on $C_0(X)$ by composition with f, and on A by composition with q and the Dauns-Hofmann theorem. We can therefore define the pull-back f^*A of A along f as the C^*-algebraic tensor product $C_0(X) \otimes_{C_b(Y)} A$. The reason for the name is that when A is the algebra of sections of some C^*-bundle E over Y, there is a natural isomorphism of f^*A onto the algebra of sections of the pull-back f^*E. In §1 we discuss this and other basic properties of pull-backs and give some evidence to show they are likely to be of interest. In particular, we show that if G is abelian and $\alpha\colon G \to \operatorname{Aut} A$ is locally unitary in the sense of [**18**], then the crossed product

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$A\times_\alpha G$ is the pull-back of A along a canonical principal $\hat{G}$-bundle $p\colon (A\times_\alpha G)^\frown \to \hat{A}$. In fact, under this isomorphism the dual action of $\hat{G}$ is carried into the diagonal action $p^*\,\mathrm{id}$ as described above. This observation and the results we present here will be important in extending the results of [**18**] to nonabelian groups (see [**13**]).

Our main results concern diagonal actions on pull-backs along principal G-bundles or, more generally, along the orbit map $p\colon \Omega \to T$ for a free G-space Ω in which compact subsets of Ω are wandering [**10**, p. 88]. For such actions the orbit space $T = \Omega/G$ is locally compact and Hausdorff, and a theorem of Green [**10**, Theorem 14; or **22**, situation 2] asserts that in this case the crossed product $C_0(\Omega)\times G$ is strongly Morita equivalent to $C_0(T)$. We suppose Ω is such a G-space and A is a C^*-algebra for which there is a continuous map $q\colon \operatorname{Prim} A \to T$.

It is well known that the crossed product $B\times_\alpha G$ is closely related to the fixed-point algebra B^α (see, for example, [**3**]), and when G is compact, Green's theorem says that $C_0(\Omega)\times_\alpha G$ is strongly Morita equivalent to $C_0(\Omega)^\alpha = C_0(T)$. For noncompact G the fixed-point algebra is typically empty, but we can use the same idea provided we work in a large enough algebra. For the tensor product action $\alpha = \gamma\otimes\beta$, we use the algebra $GC(\Omega, A)$ of bounded continuous functions which "vanish at infinity on Ω/G" (see Definition 2.1); although it is rather messy to work with, the fixed point algebra $GC(\Omega,A)^\alpha$ proves quite tractable. In particular, its primitive ideal space turns out to be the orbit space for the natural action of G on $\Omega\times\operatorname{Prim} A$. For the diagonal action $p^*\beta$ on the pull-back p^*A, we use the quotient of $GC(\Omega,A)^\alpha$ by the ideal I corresponding to the diagonal in $(\Omega\times\operatorname{Prim} A)/G$—although it is possible to view this as a fixed-point algebra, we found it easier to work through $GC(\Omega,A)^\alpha$. Our main theorems (2.2 and 2.5) state that $C_0(\Omega,A)\times_{\gamma\otimes\beta} G$ and $p^*A\times_{p^*\beta} G$ are strongly Morita equivalent to $GC(\Omega,A)^\alpha$ and $GC(\Omega,A)^\alpha/I$ respectively. The imprimitivity bimodule X in Theorem 2.2 is similar to that constructed in Rieffel's proof of Green's theorem [**22**], and that of Theorem 2.5 is a quotient of X. These two results are proved in §2.

To complete our analysis of the crossed products we have to investigate the algebras $GC(\Omega,A)^\alpha$ and $GC(\Omega,A)^\alpha/I$, and we do this in §3. The latter has the same primitive ideal space as A, and in many cases it is isomorphic to A (for example, if β is trivial or if $\Omega = T\times G$ is the trivial bundle) or strongly Morita equivalent to A. In case $A = C_0(T)$ and $\beta = \mathrm{id}$, we recover Green's theorem, so this is not altogether unexpected. For nontrivial β and A, more interesting things can happen, and, in particular, $GC(\Omega,A)^\alpha/I$ need not be Morita equivalent to A (see Corollary 3.6). However, the algebras $GC(\Omega,A)^\alpha/I$ and A are closely related; we prove in Proposition 3.7 that their pull-backs along p are isomorphic. We conclude §3 with some comments on the case where A and, hence, $p^*A\times_{p^*\beta}G$ have continuous trace, and we discuss the relationship between their Dixmier-Douady invariants.

In our last section we apply our results to extend a recent theorem of A. Wassermann [**24**, Theorem 5]. Let G be a separable locally compact group, Ω a principal G-bundle, and $\omega\colon G\times G\to \mathbf{T}$ a multiplier for G. For G a compact Lie group, Wassermann proves a version of our Theorem 2.2 [**24**, Corollary 1] and uses it to show that the twisted transformation group C^*-algebra $C^*(G,\Omega,\omega)$ has continuous trace and to compute its Dixmier-Douady class. We use Theorem 2.2 to prove a similar result for locally compact G.

Notation. If H is a Hilbert space, $K(H)$ denotes the algebra of compact operators on H, and $U(H)$ is the group of unitary operators on H. If A is a C^*-algebra, we write $M(A)$ for its multiplier algebra, $Z(A)$ for its centre, $U(A)$ for its unitary group, $\operatorname{Aut} A$ for the group of *-automorphisms and $\operatorname{Inn} A$ for the subgroup of inner automorphisms—that is, those of the form $\operatorname{Ad} u\colon a \to uau^*$ for some $u \in U(M(A))$. Automorphism groups $\alpha\colon G \to \operatorname{Aut} A$ will usually be strongly continuous, and we shall distinguish between an identity element 1_A and the identity automorphism id. If $\alpha\colon G \to \operatorname{Aut} A$ is an automorphism group, we denote the crossed product by $A \times_\alpha G$ and write $\pi \times U$ for the representation of $A \times_\alpha G$ corresponding to the covariant representation (π, U) of (A, G). The algebraic tensor product of two C^*-algebras A, B is denoted by $A \odot B$; in our context there will usually be a unique C^*-completion $A \otimes B$.

Given a locally compact (Hausdorff) space Ω, we denote by $C_b(\Omega), C_c(\Omega)$ and $C_0(\Omega)$ the algebras of continuous functions which, respectively, are bounded, are of compact support and vanish at infinity. If G is a topological group, then $\mathcal{G}$ will be the sheaf of germs of continuous G-valued functions. We write $H^i(\Omega, G)$ for Čech cohomology with coefficients in G, except in §4, where we write $\check{H}^i$ to distinguish it from the Moore cohomology groups which appear there. We variously describe the unit circle as S^1 (a topological space) or $\mathbf{T}$ (a compact group).

1. Pull-backs of C^*-algebras. Let A and B be C^*-algebras and suppose for convenience that at least one of A, B is nuclear, so that, among other things, there is a unique C^*-tensor product $A \otimes B$. Let Y be a locally compact Hausdorff space and suppose there are continuous maps $f\colon \operatorname{Prim} A \to Y$, $g\colon \operatorname{Prim} B \to Y$. Then by composing with f and g and invoking the Dauns-Hofmann theorem we can view A and B as modules over the algebra $C_b(Y)$ of continuous bounded functions. If Z is a subalgebra of $C_b(Y)$ the Z-balanced tensor product is by definition the quotient of $A \otimes B$ by the (two-sided closed) ideal I_Z generated by the set

$$\{a\phi \otimes b - a \otimes \phi b\colon a \in A,\ b \in B,\ \phi \in Z\}.$$

Since one of A, B is nuclear, the map $(J, K) \to J \otimes B + A \otimes K$ is a homeomorphism of $\operatorname{Prim} A \times \operatorname{Prim} B$ onto $\operatorname{Prim}(A \otimes B)$ [**2**, Theorem 3.3], and it follows easily that

$$\begin{aligned} I_{C_b(Y)} &= I_{C_0(Y)} \\ &= \bigcap \{J \otimes B + A \otimes K\colon J \in \operatorname{Prim} A,\ K \in \operatorname{Prim} B,\ f(J) = g(K)\}. \end{aligned}$$

Thus the tensor products $A \otimes_{C_b(Y)} B$ and $A \otimes_{C_0(Y)} B$ coincide, so we can write $A \otimes_{C(Y)} B$ without ambiguity, and we have the following lemma.

LEMMA 1.1. *Let A, B, Y, f, g be as above. Then* $\operatorname{Prim}(A \otimes_{C(Y)} B)$ *is homeomorphic to the space*

$$\Delta = \{(J, K) \in \operatorname{Prim} A \times \operatorname{Prim} B\colon f(J) = g(K)\}.$$

The spectrum of $A \otimes_{C(Y)} B$ has a similar description.

DEFINITION 1.2. Let X, Y be locally compact Hausdorff spaces, let A be a C^*-algebra, and suppose there are continuous maps $f\colon X \to Y$ and $g\colon \operatorname{Prim} A \to Y$. The *pull-back of A along f* is the balanced tensor product $C_0(X) \otimes_{C(Y)} A$, which we denote by f^*A.

Of course, as we have defined it, the pull-back f^*A depends just as much on the map g, but in the cases that interest us f will be the crucial ingredient—in fact, g will often be the identity. As a simple example to fix ideas, consider $A = C_0(Y, K(H))$. Then it follows from Lemma 1.1 that $f^*A \cong C_0(X, K(H))$.

Our notion of pull-back for a C^*-algebra is an extension of the usual notion of pull-back for bundles. Suppose that A is represented as the sections $\Gamma_0(E)$ of a C^*-bundle (E, p) over Y (cf. [**8**, §1 or **9**, §10]). The bundle pull-back f^*E is the bundle over X consisting of the pairs $(x, e) \in X \times E$ satisfying $f(x) = p(e)$, together with the obvious projection onto X. Note that the continuous sections of f^*E can be identified with the continuous functions $\phi\colon X \to E$ such that $p(\phi(x)) = f(x)$ for all $x \in X$. The following result justifies our terminology.

PROPOSITION 1.3. *Let E be a C^*-bundle over a locally compact space Y, and let $f\colon X \to Y$ be continuous. Then $f^*(\Gamma_0(E))$ is isomorphic to $\Gamma_0(f^*E)$.*

REMARK. There is a natural action of $C(Y)$ on $\Gamma_0(E)$, so the tensor product $C_0(X) \otimes_{C(Y)} \Gamma_0(E)$ makes sense. This also falls into the set-up above: every $\pi \in \Gamma_0(E)^\wedge$ factors through evaluation at some point $g(\ker \pi) \in Y$, the resulting map g is continuous, and the action of $C(Y)$ defined through g is the natural one.

PROOF. We define a map Φ from $C_0(X) \odot \Gamma_0(E)$ to $\Gamma_0(f^*E)$ by

$$\Phi(\phi \otimes a)(x) = \phi(x)a(f(x)).$$

Because Φ is the tensor product of the obvious embeddings of $C_0(X)$ and $\Gamma_0(E)$ into $M(\Gamma_0(f^*E))$, it extends to a homomorphism on the (unique) C^*-tensor product. The kernel of Φ contains $I_{C(Y)}$, and hence Φ defines a homomorphism of $f^*\Gamma_0(E)$ into $\Gamma_0(f^*E)$. The range of Φ is a $C_b(X)$-submodule of $\Gamma_0(f^*E)$, and $\{\Phi(\phi \otimes a)(x)\}$ equals $(f^*E)_x = E_{f(x)}$ for each x, so the proof of [**8**, Proposition 1.7] shows that the range of Φ is dense. Of course, Φ is a homomorphism between C^*-algebras, so it must actually be surjective.

According to Lemma 1.1, every irreducible representation of $f^*\Gamma_0(E)$ is equivalent to one of the form $\varepsilon_x \otimes \pi$, where ε_x denotes evaluation at $x \in X$, and $\pi \in \Gamma_0(E)^\wedge$ satisfies $g(\ker \pi) = f(x)$. By definition of g (see the remark above) there exists $\rho \in (E_{f(x)})^\wedge$ with $\pi(a) = \rho(a(f(x)))$, and then we have

$$(\varepsilon_x \otimes \pi)(\phi \otimes a) = \rho(\Phi(\phi \otimes a)(x)).$$

In other words, every irreducible representation of $f^*\Gamma_0(E)$ factors through Φ, which is therefore an isomorphism on $f^*\Gamma_0(E)$. □

Our next two propositions describe two ways in which pull-backs of C^*-algebras arise. The first concerns continuous trace C^*-algebras. Here it is possible to decide, at least up to stable isomorphism, whether a given algebra B with spectrum X is a pull-back along a given map $f\colon X \to Y$ by inspecting the Dixmier-Douady invariant of B. Our second proposition concerns the crossed product of a C^*-algebra A with spectrum Y by a locally unitary abelian automorphism group $\alpha\colon G \to \operatorname{Aut} A$ [**18**]. In this case, $A \times_\alpha G$ is the pull-back of A along the fibres of a certain canonically defined principal $\hat{G}$-bundle over Y.

PROPOSITION 1.4. *Let X, Y be locally compact spaces and $f\colon X \to Y$ a continous map.*

(1) *If A is a continuous trace C^*-algebra with spectrum Y, then f^*A is a continuous trace C^*-algebra with spectrum X, and the Dixmier-Douady invariant $\delta(f^*A)$ is the image of $\delta(A)$ under the induced homomorphism $f^*\colon H^3(Y,\mathbf{Z}) \to H^3(X,\mathbf{Z})$.*

(2) *If B is a separable continuous trace C^*-algebra with spectrum X, then B is stably isomorphic to f^*A for some continuous trace C^*-algebra A with spectrum Y if and only if $\delta(B) \in f^*(H^3(Y,\mathbf{Z}))$.*

(3) *If B is a separable and stable continuous trace C^*-algebra with spectrum X, then B is isomorphic to f^*A for some continuous trace C^*-algebra A with spectrum Y if and only if $\delta(B) \in f^*(H^3(Y,\mathbf{Z}))$. In this case, $B \cong f^*A$ whenever A is a stable continuous trace C^*-algebra with $f^*(\delta(A)) = \delta(B)$.*

PROOF. Suppose that A is a continuous trace C^*-algebra with spectrum Y, and notice that the map $x \to (x, f(x))$ is a homeomorphism of X onto $\Delta = (f^*A)\hat{}$. By applying the techniques of [**17**, Lemma 2.6] to the single algebra A, we can choose an open cover $\{N_i\}$ of Y and elements p_i, a_{ij} of A such that $p_i(y)$ is a rank one projection if $y \in N_i$, and

$$a_{ij}^*(y)a_{ij}(y) = p_i(y), \qquad a_{ij}(y)a_{ij}^*(y) = p_j(y)$$

if $y \in N_{ij} = N_i \cap N_j$. The functions $u_{ijk}\colon N_{ijk} \to S^1$ defined by

$$(a_{ik}^* a_{jk} a_{ij})(y) = u_{ijk}(y)p_i(y)$$

form a 2-cocycle with values in the sheaf $\mathcal{S}$ of continuous S^1-valued functions on Y. Let $\gamma(A)$ denote the corresponding cohomology class in $H^2(Y,\mathcal{S})$; then the Dixmier-Douady class $\delta(A)$ is the image of $\gamma(A)$ in $H^3(Y,\mathbf{Z}) \cong H^2(Y,\mathcal{S})$ (see [**17**, §2] for this description of $\delta(A)$). For each i choose an open cover $\{W_q^i\}$ of $f^{-1}(N_i)$ by relatively compact sets and functions $f_q^i \in C_c(X)$ such that f_q^i is identically one on W_q^i. We define $p_{i,q}$ and $a_{iq,jr}$ in $C_0(X) \odot A$ by

$$p_{i,q} = f_q^i \otimes p_i \quad \text{and} \quad a_{iq,jr} = f_q^i \otimes a_{ij}.$$

Then for $y \in N_{ijk}$ and $x \in W_q^i$ we have

$$a_{iq,ks}(x,y)^* a_{jr,ks}(x,y) a_{iq,jr}(x,y) = u_{ijk}(y)p_{i,q}(x,y).$$

In particular, this holds for (x,y) of the form $(x, f(x))$. Thus, if we replace the p's and a's by their images in f^*A, we see that $\delta(f^*A)$ is represented by the 2-cocycle $\{W_q^i, u_{ijk}\circ f\}$. But the cover $\{W_q^i\}$ is a refinement of $f^{-1}(N_i)$, so this cocycle defines the same class as $\{f^{-1}(N_i), u_{ijk}\circ f\}$, which, by definition, is $f^*(\delta(A))$. This proves (1) as well as the "only if" parts of (2) and (3), since δ is a stable isomorphism invariant (by, for example, [**6**, Théorème 1 and **16**, Lemma 1.11]). The other implication in (2) follows from the fact that δ classifies separable continuous trace C^*-algebras up to stable isomorphism (see, for example, [**17**, Corollary 1.5]). The rest of (3) will follow once we prove that if A is stable, then so is f^*A.

We therefore fix a stable continuous trace C^*-algebra with spectrum Y and choose an isomorphism Φ from $A\otimes K(H)$ to A such that the induced map on spectra is the identity (see [**18**, Lemma 4.3]). Then Φ is a $C_0(Y)$-module isomorphism, and $\mathrm{id}\otimes\Phi$ carries the ideal $I_{C(Y)}$ into the corresponding ideal in $C_0(X)\otimes(A\otimes K(H))$.

Of course, this ideal is just $I_{C(Y)} \otimes K(H)$. Thus

$$\begin{aligned} f^*A = C_0(X) \otimes_{C(Y)} A &\cong C_0(X) \otimes_{C(Y)} (A \otimes K(H)) \\ &= [C_0(X) \otimes A \otimes K(H)]/I_{C(Y)} \otimes K(H) \\ &\cong [C_0(X) \otimes A/I_{C(Y)}] \otimes K(H) = f^*A \otimes K(H), \end{aligned}$$

and f^*A is stable. The result follows. $\square$

This proposition can give useful information in specific situations. For example, let T be a compact space, $X = \mathbf{R} \times T$, $Y = S^1 \times T$ and $f\colon X \to Y$ be the identity on T and the quotient map on $\mathbf{R}$. Then $f^*\colon H^3(Y) \to H^3(X)$ is surjective, so every stable continuous trace C^*-algebra with spectrum X has the form f^*A; this representation is not necessarily unique since f^* can have nontrivial kernel (take $T = S^2$, say). On the other hand, if $f\colon S^3 \to S^2$ is the Hopf fibration, then $H^3(S^2) = 0$ and no continuous trace C^*-algebra B with spectrum S^3 and $\delta(B) \neq 0$ is a pull-back.

Now let A be a type I C^*-algebra, let G be a locally compact abelian group and suppose that $\alpha\colon G \to \operatorname{Aut} A$ is locally unitary in the sense of [**18**]. If $\pi \in (A \times_\alpha G)\widehat{\ }$ and $\overline{\pi}$ is its extension to $M(A \times_\alpha G)$, then the restriction of $\overline{\pi}$ to the subalgebra A of $M(A \times_\alpha G)$ is still irreducible: this defines a continuous map p, called the restriction map, of $(A \times_\alpha G)\widehat{\ }$ onto $\hat{A}$ [**18**, Proposition 2.1]. Further, Theorem 2.2 of [**18**] shows that p and the dual action of $\hat{G}$ make $(A \times_\alpha G)\widehat{\ }$ into a locally trivial principal $\hat{G}$-bundle over $\hat{A}$.

PROPOSITION 1.5. *Let A be a C^*-algebra with paracompact spectrum Y, let $\alpha\colon G \to \operatorname{Aut} A$ be a locally unitary automorphism group, and let $p\colon (A \times_\alpha G)\widehat{\ } \to Y$ be the restriction map. Then $A \times_\alpha G$ is isomorphic to the pull-back p^*A. Further, there is such an isomorphism which carries the dual action of $\hat{G}$ on $A \times_\alpha G$ into the action*

$$\gamma \otimes_{C(Y)} \mathrm{id}\colon \hat{G} \to \operatorname{Aut}[C_0((A \times_\alpha G)\widehat{\ }) \otimes_{C(Y)} A] = \operatorname{Aut} p^*A,$$

where γ is the action of $\hat{G}$ on $C_0((A \times_\alpha G)\widehat{\ })$ induced by the dual action.

PROOF. Let $m\colon C_b((A \times_\alpha G)\widehat{\ }) \to M(A \times_\alpha G)$ denote the embedding guaranteed by the Dauns-Hofmann theorem, and let $R_A\colon A \to M(A \times_\alpha G)$ be the embedding of A as multiples of the Dirac δ-function. As the ranges of m and R_A commute they define a homomorphism

$$\Phi\colon C_0((A \times_\alpha G)\widehat{\ }) \otimes A \to M(A \times_\alpha G).$$

It is straightforward to check that the ideal $I_{C(Y)}$ is contained in the kernel of Φ, so we have a homomorphism

$$\Psi\colon p^*A = C_0((A \times_\alpha G)\widehat{\ }) \otimes_{C(Y)} A \to M(A \times_\alpha G).$$

We shall prove that Ψ is an isomorphism of p^*A onto $A \times_\alpha G$, but first we must prove that the range of Ψ is contained in $A \times_\alpha G$.

We therefore need to show $\Phi(\phi \otimes a) \in A \times_\alpha G$ for any $a \in A$, $\phi \in C_c((A \times_\alpha G)\widehat{\ })$. Using a partition of unity, we may write $\phi = \sum \phi_i$, where the ϕ_i have small support. Since it will suffice to prove that $\Phi(\phi_i \otimes a) \in A \times_\alpha G$, we may assume that $\operatorname{supp} \phi \subset$

$p^{-1}(N)$, where N is open in Y and α is implemented over N by $u\colon G \to M(A)$. By Theorem 2.2 of [**18**], we then have

$$p^{-1}(N) = \{\pi \times \gamma\overline{\pi}(u)\colon \pi \in N,\ \gamma \in \hat{G}\}.$$

Let $n\colon C_0(\hat{A}) \to M(A)$ be the usual embedding. If $\lambda \in C_c(G)$ and $f \in C_0(N)$, then for each $\pi \in N$, $s \to \overline{\pi}(\lambda(s)n(f))$ is a continuous, compactly supported function with values in $\overline{\pi}(ZM(A)) = \mathbf{C}1$. In particular, we may define $\phi(\lambda, f) \in C_0(p^{-1}(N))$ by

$$\phi(\lambda, f)(\pi \times \gamma\overline{\pi}(u))1 = \int_G \overline{\pi}(\lambda(s)n(f))\gamma(s)\,ds.$$

We also remark that by [**18**, Theorem 2.2] the map $h_u\colon (\pi, \gamma) \to \pi \times \gamma\overline{\pi}(u)$ is a homeomorphism of $N \times \hat{G}$ onto $p^{-1}(N)$, so that $h_u^*\colon \phi \to \phi \circ h_u$ is an isomorphism of $C_0(p^{-1}(N))$ onto $C_0(N \times \hat{G})$. A straightforward calculation shows that

$$\phi(\lambda, f) \circ h_u = \hat{\lambda} \otimes f,$$

where $\hat{\lambda}$ is the usual Fourier transform of λ. Since the image of $C_c(G)$ is dense in $C_0(\hat{G})$ and h_u^* is an isomorphism, we conclude that the $\phi(\lambda, f)$ span a dense subspace of $C_0(p^{-1}(N))$. Thus it will be enough for us to show that $\Phi(\phi(\lambda, f) \otimes a)$ belongs to $A \times_\alpha G$ for every $\lambda \in C_c(G)$, $f \in C_0(N)$ and $a \in A$.

Let $z \in C_c(G, A)$, $\pi \in N$ and $\gamma \in \hat{G}$. Then we compute

$$\begin{aligned}
(\pi \times \gamma\overline{\pi}(u))(m(\phi(\lambda, f))R_A(a)z) &= \int_G \int_G \pi(\lambda(s)n(f)R_A(a)z(r)u_r)\gamma(sr)\,dr\,ds \\
&= \int_G \int_G \pi(\lambda(s)n(f)R_A(a)z(s^{-1}r)u_{s^{-1}r})\gamma(r)\,dr\,ds, \\
&= \int_G \pi\left[\int_G \lambda(s)n(f)au_s^{-1}\alpha_s(z(s^{-1}r))\,ds\right]\gamma(r)\overline{\pi}(u_r)\,dr \\
&\qquad (\text{since } z(s^{-1}r)u_s^{-1} = u_s^{-1}\alpha_s(z(s^{-1}r))), \\
&= (\pi \times \gamma\overline{\pi}(u))(y * z),
\end{aligned}$$

where $y(s) = \lambda(s)n(f)au_s^{-1}$ is in $C_c(G, A)$. Now every irreducible representation of $A \times_\alpha G$ has the form $\pi \times U$ for some $\pi \in \hat{A}$, and if $\pi \notin N$, then both $m(\phi(\lambda, f))R_A(a)z$ and $y * z$ are in the kernel of $\pi \times U$. Thus we can deduce that

$$\Phi(\phi(\lambda, f) \otimes a) = m(\phi(\lambda, f))R_A(a) = y,$$

and we have shown that Φ and, hence, Ψ map into $A \times_\alpha G$.

If $\pi \times U \in (A \times_\alpha G)^\wedge$, then

$$\pi \times U(\Phi(\phi \otimes a)) = [\pi \times U(m(\phi))\pi \times U(R_A(a))] = \phi(\pi \times U)\pi(a);$$

in other words, $(\pi \times U) \circ \phi$ is the representation $(\pi \times U, \pi)$ in $(A \times_\alpha G)^\wedge \times \hat{A}$. By Lemma 1.1 this implies that all irreducible representations of p^*A factor through Φ, and hence that Ψ is an isomorphism. Also, this shows that the range of Ψ is a rich subalgebra of $A \times_\alpha G$, and since $A \times_\alpha G$ is liminal (being the section algebra of a locally trivial bundle over a Hausdorff space) the range must be all of $A \times_\alpha G$. Thus Ψ is an isomorphism of p^*A onto $A \times_\alpha G$.

It only remains for us to check that Ψ carries $\gamma \otimes_{C(Y)} \mathrm{id}$ into the dual action. However, for $z \in C_c(G, A)$ and $\chi \in \hat{G}$ we have

$$(\pi \times U)(\hat{\alpha}_\chi(z)) = (\pi \times \overline{\chi}U)(z),$$

and this formula works equally well on the multiplier algebra, so

$$\begin{aligned}(\pi \times U)(\Phi(\gamma_\chi\phi \otimes a)) &= (\gamma_\chi\phi)(\pi \times U)[\pi \times U(R_A(a))] = \phi(\pi \times \overline{\chi}U)\pi(a) \\ &= (\pi \times \overline{\chi}U)(\Phi(\phi \otimes a)) = (\pi \times U)(\hat{\alpha}_\chi(\Phi(\phi \otimes a))).\end{aligned}$$

This completes the proof of Proposition 1.5. $\square$

REMARK. We observe that the results of [**18**, §4] follow from Proposition 1.5 and the general result on pull-backs of continuous trace C^*-algebras in Proposition 1.4.

2. Crossed products by diagonal actions. Let G be a locally compact group acting freely on a locally compact (Hausdorff) space Ω in such a way that compact subsets of Ω are wandering [**10**, p. 80], and let $\gamma\colon G \to \operatorname{Aut} C_0(\Omega)$ be the corresponding action. Then $T = \Omega/G$ is also a locally compact Hausdorff space [**10**, first part of Theorem 14]; let $p\colon \Omega \to T$ be the orbit map. To begin with we shall be interested in automorphism groups of the form $\alpha = \gamma \otimes \beta\colon G \to \operatorname{Aut} C_0(\Omega) \otimes A$, where β is a strongly continuous automorphism group of a C^*-algebra A. If we view $C_0(\Omega) \otimes A$ as $C_0(\Omega, A)$, then α is defined by

$$\alpha_t(f)(x) = \beta_t(f(t^{-1}x)) \qquad (f \in C_0(\Omega, A),\ x \in \Omega,\ t \in G).$$

As we pointed out in the introduction, there will often be no fixed points for this action, but we can rectify this by working in a slightly larger algebra.

DEFINITION 2.1. Let $GC(\Omega, A)$ be the C^*-algebra of bounded continuous functions $\phi\colon \Omega \to A$ with the property that

$$p(\{x \in \Omega\colon \|\phi(x)\| \geq \varepsilon\})$$

is relatively compact for every $\varepsilon > 0$.

Each automorphism α_t extends to $GC(\Omega, A)$, so we can talk about the fixed point algebra $GC(\Omega, A)^\alpha$. (Note that this action of G will not always be strongly continuous, however.) When G is compact, $GC(\Omega, A) = C_0(\Omega, A)$ and this is the fixed-point algebra for α in the usual sense. Our interest in $GC(\Omega, A)^\alpha$ stems from our first main result.

THEOREM 2.2. *Let G, Ω, A, β and α be as above. Then the crossed product $C_0(\Omega, A) \times_\alpha G$ is strongly Morita equivalent to $GC(\Omega, A)^\alpha$.*

As is more or less standard by now, we shall in fact construct an imprimitivity bimodule between the dense subalgebra $E = C_c(G \times \Omega, A)$ of $C_0(\Omega, A) \times_\alpha G$ and $B = GC(\Omega, A)^\alpha$. The imprimitivity bimodule will be $X = C_c(\Omega, A)$, with the module actions given by

$$\begin{aligned} z.\phi(x) &= \int_G z(s,x)\beta_s(\phi(s^{-1}x))\Delta(s)^{1/2}\,ds, \\ \phi.b(x) &= \phi(x)b(x) \end{aligned} \tag{2.1}$$

for $z \in E$, $b \in B$ and $\phi \in X$—note that by the wandering hypothesis the integrand has compact support. In order to define the B- and E-valued inner products we need a simple lemma.

LEMMA 2.3. *Let $\psi \in C_c(\Omega, A)$. Then the function $\phi\colon \Omega \to A$ defined by*

$$\phi(x) = \int_G \beta_s(\psi(s^{-1}x))\, ds$$

is continuous.

PROOF. Let $K = \operatorname{supp}\psi$, $x \in \Omega$ and $\varepsilon > 0$. If N is a compact neighbourhood of x in Ω, then

$$L = \{s \in G\colon \psi(s^{-1}y) \neq 0 \text{ for some } y \in N\}$$

is a subset of

$$\{s \in G\colon s(N \cup K) \cap (N \cup K) \neq \varnothing\},$$

and hence is relatively compact by the wandering hypothesis. A standard compactness argument gives a neighbourhood W of x such that $W \subset N$ and

$$\|\psi(t^{-1}y) - \psi(t^{-1}x)\| < \varepsilon/\mu(L) \quad \text{for } t \in L,\ y \in W.$$

Since $\psi(t^{-1}y) = 0$ if $t \notin L$, it follows that ϕ is continuous at x. □

We can now define the B-valued inner product on X by

$$\langle \phi, \psi\rangle_B(x) = \int_G \beta_s(\phi(s^{-1}x)^*\psi(s^{-1}x))\, ds, \tag{2.2}$$

for the lemma shows that $\langle \phi, \psi\rangle_B$ is a continuous function on Ω, and it follows from the left invariance of Haar measure that $\langle \phi, \psi\rangle_B$ is invariant under α. In particular, $\|\langle \phi, \psi\rangle_B(x)\|$ is constant on G-orbits and vanishes on orbits outside $p(\operatorname{supp}\phi^*\psi)$, so $\langle \phi, \psi\rangle_B$ does belong to $GC(\Omega, A)$. An argument similar to the first part of the above proof shows that if $\psi \in C_c(\Omega, A)$ then the function $(s, x) \to \psi(s^{-1}x)$ has compact support modulo G, so we may define the E-valued inner product by

$$\langle \phi, \psi\rangle_E(s, x) = \Delta(s)^{-1/2}\phi(x)\beta_s(\psi(s^{-1}x)^*). \tag{2.3}$$

We have to show that with the module actions (2.1) and the inner products (2.2), (2.3), X is an E-B imprimitivity bimodule.

The purely algebraic axioms can be easily verified, and we have only to establish (a) the positivity of the inner product, (b) the density of the spans of the ranges of the inner products, and (c) the continuity of the module actions. We verify (a) and (b) by constructing approximate identities of a special kind: our construction is based on that of [**11**, Lemma 2 and **22**, pp. 306–308].

LEMMA 2.4. (1) *There is a net $\{f_k\}$ in E which is an approximate identity both for $E \subset C_0(\Omega, A) \times_\alpha G$ and for the E-module X in the B-norm and which consists of finite sums of elements of the form $\langle \phi, \phi\rangle_E$.*

(2) *There is a net $\{g_l\}$ in B with the corresponding properties.*

PROOF. (1) Let N be a neighbourhood of e in G, let D be a compact subset of Ω, and let $\varepsilon > 0$. The construction of [**22**, pp. 307–308] (with the subgroup $H = \{e\}$) shows we can choose positive functions $g_i \in C_c(\Omega)$ such that if

$$\Phi_{(N,D,\varepsilon)}(s, x) = \sum_i \Delta(x)^{1/2} g_i(x)\overline{g_i(s^{-1}x)},$$

then

$$\Phi_{(N,D,\varepsilon)}(s, x) \begin{cases} \geq 0 & \text{for } s \in N, \\ = 0 & \text{for } s \notin N, \end{cases} \tag{1}$$

$$\left| \int_G \Delta(s)^{1/2} \Phi_{(N,D,\varepsilon)}(s,x)\, ds - 1 \right| < \varepsilon \quad \text{for all } x \in D. \tag{2}$$

We then choose an approximate identity $\{a_\gamma \colon \gamma \in \Gamma\}$ for A such that $0 \le a_\gamma \le 1$ for all γ and set

$$\Psi_{(N,D,\varepsilon,\gamma)}(s,x) = \Phi_{(N,D,\varepsilon)}(s,x) a_\gamma \beta_s(a_\gamma).$$

We aim to prove that the $\Psi_{(N,D,\varepsilon,\gamma)}$, indexed over reverse inclusion on N, inclusion on D, decreasing ε and Γ, form approximate identities for $E = C_c(G, C_c(\Omega, A))$ and the E-module X with respect to the inductive limit topologies. It then follows from routine arguments that they form approximate identities with respect to the necessary norms. Further, we observe that if we set $z_i(x) = g_i(x) a_\gamma$, then

$$\Psi_{(N,D,\varepsilon,\gamma)} = \sum_i \langle z_i, z_i \rangle_E,$$

so this approximate identity has the required form.

Let $f \in E$ be fixed. We have to show that given $\varepsilon > 0$ we can find $(N_0, D_0, \delta_0, \gamma_0)$ such that

$$\|\Psi_{(N,D,\delta,\gamma)} f - f\|_\infty < \varepsilon \quad \text{whenever } N \subset N_0,\ D \supset D_0,\ \delta < \delta_0,\ \gamma \ge \gamma_0.$$

If $a \in A$ then a routine calculation shows that for each $\delta > 0$ there are $\gamma_1 \in \Gamma$ and a compact neighbourhood N_1 of e in G such that

$$\|a_\gamma \beta_t(a_\gamma) a - a\| < \delta \quad \text{whenever } \gamma \ge \gamma_1 \text{ and } t \in N_1.$$

The range of the function $f \colon G \times \Omega \to A$ is compact, so a standard compactness argument implies that there are a neighbourhood N_2 of e and $\gamma_0 \in \Gamma$ such that

$$\|a_\gamma \beta_t(a_\gamma) f(s,x) - f(s,x)\| < \frac{\varepsilon}{4(1+\varepsilon)} \quad \text{for } t \in N_2,\ \gamma \ge \gamma_0,\ (s,x) \in G \times \Omega.$$

We now observe that the action of G on $C_0(G \times \Omega, A)$ defined by

$$t.f(s,x) = \beta_t(f(t^{-1}s, t^{-1}x))$$

is strongly continuous, so we can choose N_0 such that $N_0 \subseteq N_2$ and

$$\|\Delta(t)^{-1/2} \beta_t(f(t^{-1}s, t^{-1}x)) - f(s,x)\| < \frac{\varepsilon}{4(1+\varepsilon)}$$

for all $\gamma \ge \gamma_0$, $t \in N_0$ and $(s,x) \in G \times \Omega$. Finally, we let D_1 be a compact subset of Ω containing $\{x \in \Omega \colon (t,x) \in \operatorname{supp} f \text{ for some } t\}$ and take $D_0 = N_0 D_1$. Then for $(N, D, \delta, \gamma) \ge (N_0, D_0, \varepsilon/2, \gamma_0)$ we have

$$\begin{aligned}
&\|\Psi_{(N,D,\delta,\gamma)} f(s,x) - f(s,x)\| \\
&\quad = \left\| \int_G \Phi_{(N,D,\delta)}(t,x) a_\gamma \beta_t(a_\gamma) \beta_t(f(t^{-1}s, t^{-1}x))\, dt - f(s,x) \right\| \\
&\quad \le \int_G \Delta(t)^{1/2} \Phi(t,x) \|\Delta(t)^{-1/2} a_\gamma \beta_t(a_\gamma) \beta_t(f(t^{-1}s, t^{-1}x)) - f(s,x)\|\, dt + \varepsilon/2.
\end{aligned}$$

If $x \notin D_0$ the integrand vanishes; if $x \in D_0$ we have

$$\|\Psi f(s,x) - f(s,x)\| \le \frac{\varepsilon}{2(1+\varepsilon)}(1+\delta) + \frac{\varepsilon}{2} < \varepsilon.$$

A similar, but slightly easier, argument shows that if $z \in X$ then

$$\sup_{x\in\Omega} \|\Psi_{(N,D,\varepsilon,\gamma)} z(x) - z(x)\|_A \to 0 \quad \text{as } (N, D, \varepsilon, \gamma) \to .$$

This gives (1).

(2) We first note that Lemma 2.3 (with $\beta = \mathrm{id}$) shows that for any $\lambda \in C_c(\Omega)$ the formula

$$\tilde{\lambda}(G.x) = \int_G \lambda(s^{-1}x)\, ds$$

defines a function in $C_c(\Omega/G)$. In fact every function in $C_c(\Omega/G)$ has this form: for if we are given $f \in C_c(\Omega/G)$ with support K, we can choose $g \in C_c(\Omega)$ such that $p(\{x\colon g(x) \neq 0\})$ contains K, and then take $\lambda(x) = g(x)f(G.x)/\tilde{g}(G.x)$. Thus for each compact subset D of Ω we can choose $\lambda_D \in C_c(\Omega)$ such that $0 \le \tilde{\lambda}_D \le 1$ and $\tilde{\lambda} \equiv 1$ on $p(D)$. We then define

$$\theta_{(D,\gamma)} = \langle \lambda_D^{1/2} a_\lambda^{1/2}, \lambda_D^{1/2} a_\lambda^{1/2} \rangle_B,$$

where $\{a_\gamma\}$ is an approximate identity for A with $a_\gamma \ge 0$ and $\|a_\gamma\| \le 1$. Arguments like those we used to establish (1) show that if $\phi \in B$ and $z \in X$, then

$$\|(\theta_{(D,\gamma)}\phi)(x) - \phi(x)\|_A \to 0 \quad \text{and} \quad \|(\theta_{(D,\gamma)}z)(x) - z(x)\|_A \to 0$$

uniformly in $x \in \Omega$ as (D, γ) run through, respectively, increasing compact subsets of Ω and Γ. It follows that $\{\theta_{(D,\gamma)}\}$ is an approximate identity as claimed. □

We can use Lemma 2.4 to prove properties (a) and (b) exactly as Rieffel does in [**22**, p. 308], and it remains to check the continuity condition (c). If B does not have an identity, we can extend the action of B on X to an action of B^+, and then it is straightforward to check that

$$\langle \phi, \psi b\rangle_E = \langle \phi b^*, \psi\rangle_E \quad \text{for } \phi, \psi \in X,\ b \in B^+.$$

Since $\|b\|^2 - b^*b \ge 0$ in B^+, it follows from the positivity of the inner product that

$$\langle \phi(\|b\|^2 - b^*b), \phi\rangle_E \ge 0,$$

or, in other words, that

$$\langle \phi b, \phi b\rangle_E \le \|b\|^2 \langle \phi, \phi\rangle_E.$$

It remains to prove the analogous result for the B-valued inner product. For any state p of B, we consider the pre-inner product on X defined by

$$(\phi, \psi)_p = p(\langle \phi, \psi\rangle_B^*),$$

and denote the (Hausdorff) completion of X in this inner product by V_p. For $s \in G$, we define $V(s) \in U(V_p)$ by

$$V(s)\phi(x) = \Delta(s)^{1/2}\beta_s(\phi(s^{-1}x)).$$

If M is the representation of $C_0(\Omega, A)$ on V_p by pointwise multiplication, then (V, M) is a covariant representation of $(G, C_0(\Omega, A))$, and a simple calculation shows that the integrated form of this representation satisfies

$$(M \times V)(z)\phi = z.\phi \quad \text{for } z \in E,\ \phi \in X.$$

In particular, we have

$$p(\langle z\phi, z\phi\rangle_B) \le \|z\|^2 p(\langle \phi, \phi\rangle_B)$$

for each state p of B. Therefore

$$\langle z\phi, z\phi\rangle_B \leq \|z\|^2\langle \phi, \phi\rangle_B$$

in B. This completes the proof of Theorem 2.2. $\square$

We now consider diagonal actions on pull-backs of a C^*-algebra A. Suppose $p\colon \Omega \to T$ is the orbit map for a G-space as before, let γ be the corresponding action of G on $C_0(\Omega)$, and let $q\colon \operatorname{Prim} A \to T$ be continuous. This implies as usual that $C_b(T)$ acts on A: we suppose that $\beta\colon G \to \operatorname{Aut} A$ consists of automorphisms which preserve this action. It is then easy to see that the ideal $I_{C(T)}$ is invariant under $\gamma\otimes\beta$. The diagonal action $p^*\beta$ is the automorphism group induced by $\gamma\otimes\beta$ on the quotient $p^*A = C_0(\Omega)\otimes A/I_{C(T)}$. We have already met some interesting examples: Proposition 1.5 shows that the dual actions of locally unitary abelian automorphism groups have this form. Here are a few more.

EXAMPLES. (1) Let T be a compact space with $H^1(T) = 0$, $H^2(T) \neq 0$, $H^3(T) \neq 0$, let Ω be a nontrivial principal $\mathbf{T}$-bundle over T, and let A be a continuous trace C^*-algebra with $\delta(A) \neq 0$. The Gysin sequence for the bundle $p\colon \Omega \to T$ shows that p^* is injective on $H^3(T)$, so $\delta(p^*A) \neq 0$ by Proposition 1.4. Then $p^*\operatorname{id}$ is an action of $\mathbf{T}$ on the nontrivial continuous trace algebra p^*A which induces the original $\mathbf{T}$-action on $(p^*A)\widehat{\ } = \Omega$. (Note that, in general, actions on the spectrum need not lift to actions of the algebra: this will be discussed in [**19**].)

(2) Let B a nontrivial n-homogeneous C^*-algebra with spectrum $\mathbf{T}^2$. Then B can be realized as a pull-back along a variety of finite coverings $p\colon \mathbf{T}^2 \to \mathbf{T}^2$ (see [**4**]), and for each such realisation B carries an action of the corresponding group of deck transformations.

We shall relate the crossed product $p^*A \times_{p^*\beta} G$ to the quotient of $GC(\Omega, A)^\alpha$ by the ideal

$$I = \{b \in GC(\Omega, A)^\alpha\colon \pi(b(x)) = 0 \text{ whenever } q(\ker\pi) = p(x)\};$$

in other words, $GC(\Omega, A)^\alpha/I$ is our analogue of the fixed point algebra in this case. At least when G is compact it is easy to see that $GC(\Omega, A)^\alpha/I$ is isomorphic to the fixed point algebra for $p^*\beta$: for then $GC(\Omega, A) = C_0(\Omega, A)$, the ideal $I_{C(T)}$ corresponds to the closed subset $\{(x, J)\colon p(x) = q(J)\}$ of $\Omega\times\operatorname{Prim} A$, and the quotient map gives an isomorphism of $C_0(\Omega, A)^\alpha/I$ onto $(p^*A)^{p^*\beta}$. In general it is possible to realise $GC(\Omega, A)^\alpha/I$ as the fixed point algebra for an extension of $p^*\beta$ to a larger algebra. However, we have found it easier to work with $GC(\Omega, A)^\alpha$ and its quotient, so we omit the details.

THEOREM 2.5. *Let A be a C^*-algebra, G a locally compact group, and suppose that G acts freely on a locally compact space Ω in such a way that compact subsets of Ω are wandering. Suppose also that there is a continuous map $q\colon \operatorname{Prim} A \to T = \Omega/G$, so that $C_b(T)$ acts on A and $\beta\colon G \to \operatorname{Aut} A$ consists of $C_b(T)$-module automorphisms. Let $p\colon \Omega \to T$ be the orbit map, and let $GC(\Omega, A)^\alpha$, I be as above. Then the crossed product $p^*A\times_{p^*\beta} G$ is strongly Morita equivalent to $GC(\Omega, A)^\alpha/I$.*

In the course of proving this theorem we need to know what the spectrum of $GC(\Omega, A)^\alpha$ looks like.

LEMMA 2.6. *For $x \in \Omega$, $\pi \in \hat{A}$ let $M(x, \pi)$ be the representation of $GC(\Omega, A)^\alpha$ defined by $M(x, \pi)(b) = \pi(b(x))$. Each $M(x, \pi)$ is irreducible, and $M(x, \pi)$ is equivalent to $M(y, \rho)$ if and only if $y = s.x$ and ρ is equivalent to $\pi\circ\beta_s^{-1} = s.\pi$ for some*

$s \in G$. *Further, every irreducible representation of* $GC(\Omega, A)^\alpha$ *is equivalent to some* $M(x, \pi)$.

PROOF. All except for the last sentence follows easily from the fact that $A = \{b(x)\colon b \in GC(\Omega, A)^\alpha\}$ for each $x \in \Omega$. To see this, note that if $\phi \in C_c(\Omega)$ has support in a small neighbourhood of x and satisfies $\int \phi(s^{-1}x)\,ds = 1$ then

$$\tilde{\phi}(y) = \int_G \phi(s^{-1}y)\beta_s(a)\,ds$$

defines an element of $GC(\Omega, A)^\alpha$ with $\tilde{\phi}(x)$ close to a.

On the other hand, suppose ρ is an irreducible representation of $B = GC(\Omega, A)^\alpha$. There is a natural homomorphism of $C_0(T)$ into the centre of $M(B)$, and this action of $C_0(T)$ on B is nondegenerate, so the canonical extension of ρ to $M(B)$ defines a nonzero complex homomorphism $\overline{\rho}$ on $C_0(T)$. Therefore $\overline{\rho}(f) = f(G.x)$ for every $f \in C_0(T)$ and some $x \in \Omega$. We claim that if $b \in B$ satisfies $b(x) = 0$, then $\rho(b) = 0$. To see this let $\varepsilon > 0$. Since $x \to \|b(x)\|$ defines an element of $C_0(\Omega/G)$, we can find $f \in C_0(\Omega/G)$ with $\|fb - b\| < \varepsilon$ and $f(G.x) = 0$. But $\rho(fb) = \overline{\rho}(f)\rho(b) = 0$, so this implies $\|\rho(b)\| < \varepsilon$ and justifies our claim. We can now define a representation π of A by $\pi(b(x)) = \rho(b)$ (recall that $A = \{b(x)\}$). Then π is irreducible, and $\rho = M(x, \pi)$ by definition. □

PROOF OF THEOREM 2.5. We first note that under the natural isomorphism of $C_0(\Omega) \otimes A$ onto $C_0(\Omega, A)$, the ideal $I_{C(T)}$ is carried onto

$$K = \{\phi \in C_0(\Omega, A)\colon \pi(\phi(x)) = 0 \text{ whenever } q(\ker\pi) = p(x)\},$$

and the group $\gamma \otimes \beta$ is carried into α. By [**11**, Proposition 12(ii)] the algebra $(C_0(\Omega, A)/K) \times_\alpha G$ is isomorphic to the quotient of $C_0(\Omega, A) \times_\alpha G$ by the ideal $K \times_\alpha G$. Hence, if we write $B = GC(\Omega, A)^\alpha$ and $F = C_0(\Omega, A) \times_\alpha G$, then we want to construct an $F/(K \times_\alpha G)$-B/I imprimitivity bimodule. Rieffel has shown [**21**, Theorem 3.1] that an F-B imprimitivity bimodule X induces a lattice isomorphism between the (closed two-sided) ideals $\mathcal{I}(F)$ in F and $\mathcal{I}(B)$. Further, if the ideals $L \subset F$ and $M \subset B$ correspond under this bijection, then F/L is strongly Morita equivalent to B/M via an imprimitivity bimodule which is a quotient of X [**21**, Corollary 3.2]. Now $E = C_c(G \times \Omega, A)$ is dense in F, so the completion $\overline{X}$ of the E-B imprimitivity bimodule X constructed in Theorem 2.2 is an F-B imprimitivity bimodule, and it is enough for us to check that the ideals $K \times_\alpha G$ and I correspond under the bijection given by $\overline{X}$.

Rieffel's construction [**21**, §3] shows that the ideal I^F of F corresponding to I is the closed linear span of the set $\langle X.I, X.I\rangle_F$. A straightforward approximation argument using the Cauchy-Schwartz inequality [**20**, Proposition 2.9] shows that $\langle X.I, X.I\rangle_E$ still generates I^F. However, it is easy to check that if $b, c \in I$ and $q(\ker\pi) = p(x)$, then

$$\pi(\langle\phi b, \psi c\rangle_E(s, x)) = 0,$$

so that $\langle\phi b, \psi c\rangle_E \in C_c(G, K)$. Thus $I^F \subset K \times_\alpha G$.

To prove the reverse containment $K \times G \subset I^F$ we shall use a different description of I^F. We note that in terms of the representations $M(x, \pi)$ of Lemma 2.6 we have

$$I = \bigcap\{\ker M(x, \pi)\colon \pi \in \hat{A},\ x \in \Omega,\ q(\ker\pi) = p(x)\}.$$

Rieffel's bijection maps the primitive ideal $\ker\rho$ of B to the kernel of the representation X^ρ of F induced from ρ via X [**21**, Proposition 3.3] and it preserves intersections, so

$$I^F = \bigcap\{\ker X^{M(x,\pi)} \colon \pi \in \hat{A},\ x \in \Omega,\ q(\ker\pi) = p(x)\}.$$

We claim that $X^{M(x,\pi)}$ is equivalent to the representation $\operatorname{Ind}_{\{e\}}^G N(x,\pi)$ of F induced in the sense of Takesaki [**23**] from the representation $N(x,\pi)\colon z \to \pi(z(x))$ of $C_0(\Omega, A)$.

For we can realise $\operatorname{Ind} N(x,\pi)$ in $L^2(G, H_\pi)$ by the formula

$$[\operatorname{Ind}_{\{e\}}^G N(x,\pi)(f)\xi](s) = \int_G \pi(\beta_s^{-1}(f(t,sx)))\xi(t^{-1}s)\,dt. \tag{2.4}$$

On the other hand, $X^{M(x,\pi)}$ acts in the completion of $X \odot_B H$ with respect to the inner product

$$(\phi\otimes v, \psi\otimes w) = (M(x,\pi)(\langle\psi,\phi\rangle_B)v|w)_{H_\pi}$$

according to the formula

$$X^{M(x,\pi)}(f)(\phi\otimes v) = (f.\phi)\otimes v.$$

Define $U\colon X\odot_B H_\pi \to L^2(G, H_\pi)$ by

$$U(\phi\otimes v)(s) = \pi(\alpha_s^{-1}(\phi)(x))v.$$

A calculation then shows that U intertwines $X^{M(x,\pi)}$ and $\operatorname{Ind} N(x,\pi)$, so the claim is now justified.

We therefore have

$$I^F = \bigcap\{\ker \operatorname{Ind}_{\{e\}}^G N(x,\pi) \colon \pi \in \hat{A},\ x \in \Omega,\ q(\ker\pi) = p(x)\}.$$

However, it is easy to see using (2.4) that if $q(\ker\pi) = p(x)$ then $\operatorname{Ind} N(x,\pi)$ annihilates $C_c(G,K)$, so $I^F \supset K\times_\alpha G$. Thus I and $K\times_\alpha G$ correspond under $\overline{X}$, and the result follows. □

3. The structure of the fixed-point algebras. We now study the algebras $GC(\Omega,A)^\alpha$ of Theorem 2.2 and $GC(\Omega,A)^\alpha/I$ of Theorem 2.5. We begin by describing their spectra.

PROPOSITION 3.1. *Let $A, G, \Omega, p, T, \beta, \alpha, q$ and I be as in Theorem 2.5, and for $(x,\pi)\in\Omega\times\hat{A}$ let $M(x,\pi)$ denote the representation $b\to\pi(b(x))$ of $GC(\Omega,A)^\alpha$. Let $\Omega\times\hat{A}$ carry the product action. Then the map M induces a homeomorphism of $(\Omega\times\hat{A})/G$ onto $(GC(\Omega,A)^\alpha)^\wedge$, and a homeomorphism of*

$$\Delta = \{G.(x,\pi)\in(\Omega\times\hat{A})/G\colon p(x) = q(\ker\pi)\}$$

onto $(GC(\Omega,A)^\alpha/I)^\wedge$. In particular, if β fixes $\hat{A}$, $(GC(\Omega,A)^\alpha/I)^\wedge$ is homeomorphic to $\hat{A}$.

PROOF. The map M is continuous from $\Omega\times\hat{A}$ to the spectrum of $B = GC(\Omega,A)^\alpha$, constant on orbits and hence induces a continuous map on $(\Omega\times\hat{A})/G$. This induced map is a bijection by Lemma 2.6, so for the first part it only remains to prove it is open. So suppose that $M(x_i,\pi_i)$ converges to $M(x,\pi)$ in $\hat{B}$; it will suffice to show that there is a subnet of $\{G.(x_i, \pi_i)\}$ which converges to $G.(x, \pi)$ in $(\Omega \times \hat{A})/G$. Extending these representations to $M(B)$ and then restricting to $C_0(\Omega/G) \subset ZM(B)$ gives another convergent net: it follows that $G.x_i \to G.x$ in Ω/G. If K is a compact neighborhood of x, then, as p is open, $p(K)$ is a compact neighbourhood of $G.x$ and eventually $(G.x_i) \cap K \neq \varnothing$. For each i we choose s_i with $s_i.x \in K$. By passing to a subnet we may suppose that $s_i.x_i$ converges to some $t.x$; then with $t_i = t^{-1}s_i$ we have $t_i.x_i \to x$. If we can prove that $t_i.\pi \to \pi$ in $\hat{A}$ then $G.(t_i, \pi_i) \to G.(x, \pi)$ and we are done.

A typical basic open neighbourhood of π in $\hat{A}$ is given by

$$M_1 = \{\rho \in \hat{A}\colon \exists \eta \in H_\rho,\ \|\eta\| = 1 \text{ with } |(\rho(a)\eta|\eta) - (\pi(a)\xi|\xi)| < \varepsilon\},$$

where $\varepsilon > 0$, $a \in A$ and $\xi \in H_\pi$ is a unit vector. Now pick $b \in B$ such that $b(x) = a$, and fix i_0 such that

$$\|b(t_i.x) - b(x)\| < \varepsilon/2 \quad \text{for } i \geq i_0.$$

Then

$$M_2 = \{\rho \in \hat{B}\colon \exists \eta \in H_\rho,\ \|\eta\| = 1 \text{ with } |(\rho(b)\eta|\eta) - (M(x,\pi)(b)\xi|\xi)| < \varepsilon/2\}$$

is an open neighborhood of $M(x, \pi)$ in $\hat{B}$, so we can find i_1 such that

$$M(t_i.x_i, t_i.\pi_i) = M(x_i, \pi_i) \in M_2 \quad \text{for } i \geq i_1.$$

Let $i_2 \geq i_0, i_1$ and suppose $i \geq i_2$. Then there is a unit vector $\eta \in H_{M(x_i,\pi_i)} = H_{\pi_i}$ such that

$$|(M(t_i.x_i, t_i.\pi_i)(b)\eta|\eta) - (M(x,\pi)(b)\xi|\xi)| < \varepsilon/2.$$

The Cauchy-Schwartz inequality gives

$$|(t_i.\pi_i(a)\eta|\eta) - (\pi(a)\xi|\xi)| < \varepsilon;$$

thus $t_i.\pi_i \in M_1$. This proves the first part, and the rest is straightforward. □

COROLLARY 3.2. *Let $A, G, \Omega, p, q, \beta$ be as in Theorem 2.5 and suppose β acts trivially on $\hat{A}$. Then $(p^*A \times_{p^*\beta} G)^\frown$ is homeomorphic to $\hat{A}$.*

PROOF. This follows immediately from the proposition, Theorem 2.5 and the fact that an imprimitivity bimodule induces a homeomorphism of spectra [**20**, Corollary 6.27]. □

This proposition suggests an obvious question: when is $GC(\Omega, A)^\alpha/I$ isomorphic to A, or at least strongly Morita equivalent to A? Our next proposition lists some cases where this does happen, but in general the relationship is more complicated, as we shall see later.

PROPOSITION 3.2. *Let $A, G, \Omega, p, q, \beta, \alpha$ be as in Theorem 2.5.*

(1) *Suppose that* (a) *each β_s is the identity, or* (b) *Ω is isomorphic to $G \times \Omega/G$ as a G-space. Then $B = GC(\Omega, A)^\alpha$ is isomorphic to $C_0(\Omega/G, A)$, and B/I is isomorphic to A.*

(2) *Suppose $\beta = \operatorname{Ad} u$ for some strictly continuous homomorphism $u\colon G \to M(A)$. Then $B = GC(\Omega, A)^\alpha$ is strongly Morita equivalent to $C_0(\Omega/G, A)$, and B/I to A.*

PROOF. (1) In case (a) it is immediate that functions in $GC(\Omega, A)^\alpha$ are constant on orbits, so $GC(\Omega, A)^\alpha = C_0(\Omega/G, A)$. The ideal I coincides with the ideal

$I_{C(\Omega/G)}$, so B/I is just the balanced tensor product $C_0(\Omega/G) \otimes_{C(\Omega/G)} A$, which is isomorphic to A by, for example, [**18**, Lemma 3.9]. In case (b) let $h\colon \Omega \to G \times \Omega/G$ be a homeomorphism of G-spaces and let $r\colon G\times\Omega/G \to G$ be the natural projection. Then it is easy to check that

$$\Phi(f)(x) = \beta_{r(h(x))}(f(G.x))$$

defines an isomorphism of $C_0(\Omega/G, A)$ onto B, which induces an isomorphism of $A \cong C_0(\Omega/G) \otimes_{C(\Omega/G)} A$ with B/I.

(2) An imprimitivity bimodule is given by

$$Y = \{f \in GC(\Omega, A)\colon f(tx) = u_t f(x) \text{ for } t \in G,\ x \in \Omega\}$$

with the module actions inherited from $GC(\Omega, A)$, and inner products given by

$$\langle f, g\rangle_B(x) = f(x)g(x)^*, \qquad \langle f, g\rangle_C(G.x) = f(x)^*g(x)$$

(we have written $C = C_0(\Omega/G, A)$ for convenience). It is easy to prove that this is an imprimitivity bimodule: only the density requirement on the inner products is not completely obvious. To see that $\langle\cdot,\cdot\rangle_C$ generates a dense ideal of C, it is enough to show that for each $x \in \Omega$, $\pi \in \hat{A}$ there exists $f \in Y$ with $\pi(\langle f, f\rangle_C(G.x)) \neq 0$. We choose $a \in A$ with $\pi(a) \neq 0$, and define $g\colon G.x \to A$ by $g(tx) = u_t a$. We fix a continuous function $\rho\colon \Omega \to [0,1]$ of compact support, extend $g|_{\Omega\cap \operatorname{supp}\rho}$ to a continuous function g_1 defined on all of Ω, and set

$$h(y) = \rho(y)g_1(y) \quad \text{for } y \in \Omega.$$

Since h has compact support, the wandering hypothesis shows that the function

$$f(y) = \int_G u_s h(s^{-1}y)\, ds$$

belongs to $GC(\Omega, A)$ (by the argument used to prove Lemma 2.3), and the invariance of Haar measure shows that $f \in Y$. It is easy to check that $\langle f, f\rangle_C(G.x)$ does not belong to $\ker\pi$. As the irreducible representations of B are also given by $b \to \pi(b(x))$ (Lemma 2.6), exactly the same reasoning shows that $\langle\cdot,\cdot\rangle_B$ generates a dense ideal in B. Thus Y is the required B-C imprimitivity bimodule. It is routine to check that the ideal I in B corresponds under Y to the ideal

$$J = \{f \in C_0(\Omega/G, A)\colon f(G.x) \in \ker\pi \text{ whenever } q(\ker\pi) = G.x\},$$

and hence B/I is strongly Morita equivalent to $C_0(\Omega/G, A)/J \cong A$ as in [**21**, §3]. □

We now turn to the construction of some examples where $GC(\Omega, A)^\alpha/I$ is not strongly Morita equivalent to A. We shall need the following lemma, which is essentially from [**16**, §2].

LEMMA 3.4. *Let H be an infinite-dimensional Hilbert space, and let X be a compact Hausdorff space. Given a continuous map $\phi\colon X \to \operatorname{Aut} K(H)$ we define an automorphism α_ϕ of $A = C(X, K(H))$ by $\alpha_\phi(f)(x) = \phi(x)(f(x))$. Then the map $\phi \to \alpha_\phi$ induces an isomorphism of the group $[X, \operatorname{Aut} K(H)]$ of homotopy classes onto the group $\operatorname{Aut}_{C(X)}A/\operatorname{Inn}A$ of outer $C(X)$-automorphisms of A. Both groups are isomorphic to the Čech cohomology group $H^2(X, \mathbf{Z})$.*

PROOF. It is easy to see that $\phi \to \alpha_\phi$ is a homomorphism into $\operatorname{Aut}_{C(X)}A$, and that it is surjective (see, for example, [**16**, Lemma 1.6]). A simple application of

[**16**, Proposition 2.6] shows that when ϕ is homotopic to the identity α_ϕ is inner, and the converse follows from the contractibility of $U(H)$ [**5**, 10.8.2], so the map is an isomorphism as claimed. The last statement follows from [**16**, Theorem 2.1]—the separability hypotheses of [**16**] are not necessary in case of the trivial field $E = X \times K(H)$. $\square$

PROPOSITION 3.5. *Let X be a compact Hausdorff space, let H be a Hilbert space, and let $A = C(S^1 \times X, K(H))$. Let $\phi\colon X \to \operatorname{Aut} K(H)$ be continuous and define $\beta\colon \mathbf{Z} \to \operatorname{Aut} A$ by $\beta_n(f)(z,x) = \phi(x)^n(f(z,x))$. Let $\mathbf{Z}$ act on $\mathbf{R}$, and hence also on $\Omega = \mathbf{R} \times X$, by translation, and let q be the obvious identification of $\Omega/\mathbf{Z}$ with $S^1 \times X = \hat{A}$.*

(1) *The algebra $GC(\mathbf{R} \times X, A)^\alpha/I$ is isomorphic to*

$$C_\phi = \{f \in C([0,1] \times X, K(H))\colon f(1,x) = \phi(x)(f(0,x))\}.$$

In particular, $GC(\Omega, A)^\alpha/I \cong A$ if and only if ϕ is homotopic to the identity.

(2) *Suppose H is infinite-dimensional. Then $GC(\Omega, A)^\alpha/I$ is isomorphic to A if and only if β_1 is an inner automorphism.*

PROOF. (2) follows from (1) and Lemma 3.4, so we concentrate on (1). It is routine to check that the map Ψ defined by

$$(\Psi f)(t,x) = f(t,x)(p(t,x))$$

is a homomorphism of $GC(\mathbf{R} \times X, A)^\alpha$ onto C_ϕ, and we have $\ker \Psi = 1$, so Ψ gives the required isomorphism. It is easy to construct an isomorphism of C_ϕ with A from a homotopy joining ϕ to id, so it remains to construct a homotopy from an isomorphism $\Phi\colon C_\phi \to C_{\mathrm{id}} = A$. By composing with a suitable automorphism of A we may assume that Φ induces the identity homeomorphism on $S^1 \times X = \hat{C}_\phi = \hat{A}$. Thus we can define isomorphisms $\Phi_{t,x}\colon K(H) \to K(H)$ by

$$\Phi_{t,x}(f(t,x)) = (\Phi f)(t,x) \quad \text{for } f \in C_\phi;$$

note that $t, x \to \Phi_{t,x}$ is continuous since Φ maps continuous functions to continuous functions. Then $t, x \to \Phi_{t,x}^{-1} \circ \Phi_{0,x}$ is a homotopy joining id to ϕ. $\square$

COROLLARY 3.6. *Let X be a compact metric space for which $H^2(X, \mathbf{Z}) \neq 0$, let $A = C(S^1 \times X, K(H))$ where H is infinite-dimensional and separable, and let $p\colon \mathbf{R} \times X \to S^1 \times X = \hat{A}$ be the usual quotient map. Then there is an automorphism group $\beta\colon \mathbf{Z} \to \operatorname{Aut}_{C(S^1 \times X)} A$ such that the crossed product $C_0(\mathbf{R} \times X, K(H)) \times_{p^*\beta} \mathbf{Z}$ is not strongly Morita equivalent to A.*

PROOF. By Lemma 3.4 there is a map $\phi\colon X \to \operatorname{Aut} K(H)$ which is not homotopic to a constant. If we define β as in the proposition then $GC(\Omega, A)^\alpha/I$ is not isomorphic to A. It is easy to see that both are separable continuous trace C^*-algebras given by locally trivial fields of elementary C^*-algebras, and it follows that their Dixmier-Douady classes are different [**5**, 10.8.4]. But this is not possible if the algebras are strongly Morita equivalent (see, for example, [**1**, §2.7]), so the result follows from Theorem 2.5. $\square$

Although Corollary 3.6 suggests that we cannot expect general results relating A to $GC(\Omega, A)^\alpha/I$, in fact they are always quite closely related, as the next proposition shows. Before stating it, we note that by Lemma 3.1 the map

$$r\colon \operatorname{Prim}(GC(\Omega, A)^\alpha) \to \Omega/G$$

defined by $r(\ker M(x,\pi)) = G.x$ is continuous, and we can therefore form the pull-backs $p^*(GC(\Omega,A)^\alpha)$ and $p^*(GC(\Omega,A)^\alpha/I)$. It is routine to check that the action of $C_b(\Omega/G)$ on $GC(\Omega/A)^\alpha$ defined by r and the Dauns-Hofmann theorem is the natural one given by $fb(x) = f(G.x)b(x)$.

PROPOSITION 3.7. *Let $A, G, \Omega, p, q, \beta, \alpha$ be as in Theorem* 2.5.

(1) *The map $\Phi\colon C_0(\Omega) \odot GC(\Omega,A)^\alpha \to C_0(\Omega,A)$ defined on elementary tensors by $\Phi(\phi\otimes b)(x) = \phi(x)b(x)$ induces an isomorphism of $p^*(GC(\Omega,A)^\alpha)$ onto $C_0(\Omega,A)$.*

(2) *The algebra $p^*(GC(\Omega,A)^\alpha/I)$ is isomorphic to p^*A.*

PROOF. (1) The map Φ is the tensor product of the embeddings of $C_0(\Omega)$ and $GC(\Omega,A)^\alpha$ in $M(C_0(\Omega,A))$, and so extends to a well-defined homomorphism on the C^*-algebraic tensor product. In fact Φ kills the ideal $I_{C(\Omega/G)}$ and so defines a homomorphism Ψ of the quotient $p^*(GC(\Omega,A)^\alpha)$ into $C_0(\Omega,A)$. A standard partition of unity argument shows that we can approximate arbitrary elements of the form $\phi\otimes a \in C_0(\Omega)\otimes A$ by members of the range of Φ, so Ψ is surjective. If ε_x denotes evaluation at $x\in\Omega$, and $\pi\in\hat{A}$, then

$$\varepsilon_x\otimes\pi\left(\Phi\left(\sum_i \phi_i\otimes b_i\right)\right) = \varepsilon_x\otimes M(x,\pi)\left(\sum_i \phi_i\otimes b_i\right).$$

If $y = s.x$, then $\ker M(y,\pi) = \ker M(x, s^{-1}.\pi)$, so we have

$$\ker\Phi \subset \bigcap\{\ker(\varepsilon_x\otimes M(y,\pi))\colon x,y\in\Omega,\ \pi\in\hat{A} \text{ and } G.x = G.y\}.$$

But by Lemma 1.1 the right-hand side is just $I_{C(\Omega/G)}$, so Ψ is an isomorphism.

(2) Let $B = GC(\Omega,A)^\alpha$, let Ψ be the isomorphism above, and let J be the ideal in p^*B which is the quotient of $C_0(\Omega)\otimes I$. We shall prove that Ψ defines an isomorphism of p^*B/J onto p^*A, and then identify p^*B/J with $p^*(B/I)$.

Recall that $\operatorname{Prim} B$ is homeomorphic to $(\Omega\times\operatorname{Prim} A)/G$, and that by Lemma 1.1, $\operatorname{Prim} p^*B$ is homeomorphic to

$$\{(x, G.(y,\ker\pi)) \in \Omega\times(\Omega\times\operatorname{Prim} A)/G\colon G.x = G.y\}.$$

Since $(\varepsilon_x\otimes\pi)\circ\Psi = \varepsilon_x\otimes M(x,\pi)$, the induced homeomorphism $\hat{\Psi}$ of $\operatorname{Prim} p^*B$ onto $\Omega\times\operatorname{Prim} A$ is given by

$$\hat{\Psi}(x, G.(y,\ker\pi)) = (x, \ker(s^{-1}.\pi)) \quad \text{where } s.y = x.$$

On the other hand,

$$I = \bigcap\{\ker M(x,\pi)\colon q(\ker\pi) = G.x\},$$

and so the ideal $C_0(\Omega)\otimes I$ in $C_0(\Omega)\otimes B$ corresponds to the closed subset of $\Omega\times\operatorname{Prim} B$ given by

$$\{(x, G.(y,\ker\pi))\colon q(\ker\pi) = G.y\}.$$

The quotient J therefore corresponds to the closed subset

$$\{(x, G.(y,\ker\pi))\colon G.x = G.y = q(\ker\pi)\}$$

of $\operatorname{Prim} p^*B$. The image of this set under $\hat{\Psi}$ is

$$\{(x,\ker(s^{-1}.\pi))\colon s\in G,\ q(\ker\pi) = G.x\} = \{(x,\ker\pi)\colon q(\ker\pi) = G.x\}.$$

However, this closed subset defines the ideal K generated by

$$\{\phi f \otimes a - \phi \otimes fa \colon \phi \in C_0(\Omega),\ f \in C_0(\Omega/G),\ a \in A\},$$

so we deduce that Ψ maps J onto K and hence induces an isomorphism of p^*B/J onto p^*A. The identification of p^*B/J with $p^*(B/I)$ follows from a general lemma:

LEMMA 3.8. *Let B, C be C^*-algebras with C nuclear. Let X be a locally compact Hausdorff space and suppose there are continuous maps $p\colon \operatorname{Prim} C \to X$ and $r\colon \operatorname{Prim} B \to X$. Let I be an ideal in B and let J be the ideal in $C \otimes_{C(X)} B$ which is the image of $C \otimes I$. Then*

$$(C \otimes_{C(X)} B)/J \cong C \otimes_{C(X)} B/I.$$

In fact, J is isomorphic to $C \otimes_{C(X)} I$.

PROOF. Let F be the closed subset of $\operatorname{Prim} B$ corresponding to I, so that $\operatorname{Prim}(B/I)$ can be identified with F, and $C(X)$ acts on B/I by composition with $r|_F\colon F \to X$. Let L, K be the ideals such that

$$C \otimes_{C(X)} B = (C \otimes B)/L \quad \text{and} \quad C \otimes_{C(X)} B/I = [C \otimes B/I]/K.$$

Then $J = (C \otimes I)/[L \cap (C \otimes I)]$. Using [**5**, 1.8.4] we have

$$\begin{aligned} [(C \otimes B)/L]/J &\cong [(C \otimes B)/L]/[(C \otimes I + L)/L] \\ &\cong (C \otimes B)/(C \otimes I + L) \\ &\cong [(C \otimes B)/(C \otimes I)]/[(C \otimes I + L)/(C \otimes I)] \\ &\cong [(C \otimes B)/(C \otimes I)]/[L/L \cap (C \otimes I)]. \end{aligned}$$

Thus we need to show that the isomorphism of $C \otimes B/C \otimes I$ onto $C \otimes B/J$ carries $L/[L \cap (C \otimes I)]$ onto K. However, since C is nuclear, both $\operatorname{Prim}(C \otimes B/C \otimes I)$ and $\operatorname{Prim}(C \otimes B/I)$ can be naturally identified with $(\operatorname{Prim} C) \times F$ [**2**, Theorem 3.3]. Moreover, the isomorphism in question preserves these identifications, and hence it will be enough to check that the ideals correspond to the same closed subset of $(\operatorname{Prim} C) \times F$.

By Lemma 1.1, L corresponds to the subset

$$\Delta = \{(\ker \pi, \ker \rho)\colon \pi \in \hat{C},\ \rho \in \hat{B} \text{ and } p(\ker \pi) = r(\ker \rho)\}$$

of $\operatorname{Prim} C \times \operatorname{Prim} B$, so $L/[L \cap (C \otimes I)]$ is given by the intersection of Δ with $(\operatorname{Prim} C) \times F$. But as the action of $C(X)$ on B/I is given by the restriction of r to F, Lemma 1.1 shows that K corresponds to the same set. This establishes the main part of the lemma. To see that $J \cong C \otimes_{C(X)} I$, we need to show that if we regard $C \otimes I$ as an ideal in $C \otimes B$, then $L \cap (C \otimes I)$ is the ideal generated by elements of the form $cf \otimes b - c \otimes fb$ for $b \in I$. Again this follows from Lemma 1.1. □

This completes the proof of Proposition 3.7. □

COROLLARY 3.9. *Let G be a locally compact group, suppose G acts freely on a locally compact space Ω so that compact subsets are wandering, and let $p\colon \Omega \to \Omega/G$ be the orbit map. Let A be a continuous trace C^*-algebra with spectrum homeomorphic to Ω/G and suppose $\beta\colon G \to \operatorname{Aut} A$ consists of automorphisms which preserve the resulting action of $C_0(\Omega/G)$. Then $p^*A \times_{p^*\beta} G$ is a continuous trace C^*-algebra* *whose Dixmier-Douady class satisfies*

$$p^*(\delta(p^*A \times_{p^*\beta} G)) = p^*(\delta(A)).$$

It need not be the case that $\delta(p^*A \times G) = \delta(A)$.

PROOF. By Proposition 3.1 the spectrum of $GC(\Omega, A)^\alpha$ is homeomorphic to $\Omega/G \times \Omega/G$ and, in particular, is Hausdorff. Let $M(x, \pi)$ be a fixed irreducible representation and choose $p \in A$ such that $\rho(p)$ is a rank one projection for ρ near π in $\hat{A}$. Let $b \in GC(\Omega, A)^\alpha$ satisfy $b^* = b$ and $b(x) = p$ and choose a neighbourhood N of x such that $\|b(y) - p\| < 1/4$ for $y \in N$. Let $f \in C_b(\mathbf{C})$ satisfy

$$f(z) = \begin{cases} 1 & \text{if } |z-1| < 1/4, \\ 0 & \text{if } |z| < 1/4. \end{cases}$$

By standard properties of the functional calculus we have

$$M(y,\rho)(f(b)) = \rho(f(b)(y)) = \rho(f(b(y))) = f(\rho(b(y))) = f(M(y,\rho)(b)),$$

so $M(y, \rho)(f(b))$ is a rank one projection for $y \in N$ and ρ near π. Thus $GC(\Omega, A)^\alpha$ and its quotient $GC(\Omega, A)^\alpha/I$ have continuous trace. By Theorem 2.5 $p^*A \times_{p^*\beta} G$ is strongly Morita equivalent to $GC(\Omega, A)^\alpha/I$, so it too has continuous trace (see, for example, [**25**, Theorem 2.15]) and its Dixmier-Douady class is equal to that of $GC(\Omega, A)^\alpha/I$ [**1**, §2.7]. It now follows from Propositions 1.4 and 3.7 that

$$p^*(\delta(A)) = \delta(p^*A) = \delta(p^*GC(\Omega, A)^\alpha/I) = p^*(\delta(p^*A \times_{p^*\beta} G)).$$

The last statement is illustrated by Corollary 3.6. □

This corollary suggests an intriguing problem: is it possible to compute $\delta(p^*A \times_{p^*\beta} G)$ in terms of topological data associated with p, A and β? At least for abelian G and locally unitary β there is a complete formula, which will be discussed in a forthcoming article by the first author and Jon Rosenberg [**19**].

4. Some twisted transformation group C^*-algebras. Let G be a separable locally compact group, $p\colon \Omega \to T$ a locally trivial principal G-bundle and $\omega\colon G \times G \to \mathbf{T}$ a multiplier. We shall compute the twisted crossed product $C^*(G, \Omega, \omega)$ up to strong Morita equivalence. To state our result we need to construct a pairing

$$\delta\colon H^2(G, \mathbf{T}) \times \check{H}^1(T, \mathcal{G}) \to \check{H}^3(T, \mathbf{Z});$$

here the first group is the (Moore cohomology) group of equivalence classes of multipliers [**14**], and the $\check{H}$'s are Čech cohomology groups.

We start with the natural pairing

$$\beta\colon H^1(G, PU) \times \check{H}^1(T, \mathcal{G}) \to \check{H}^1(T, \mathcal{PU}),$$

where $H^1(G, PU)$ is the collection of projective representations $\pi\colon G \to PU(H)$ for some Hilbert space H: a class $c \in \check{H}^1(T, \mathcal{G})$ can be realized as a cocycle $\lambda_{ij}\colon N_{ij} \to G$ relative to some open cover $\{N_i\}$ of T, and then $\beta(\pi, c)$ is the class of the cocycle $\pi \circ \lambda_{ij}\colon N_{ij} \to PU$. We then define $\delta(\pi, c)$ to be the image of $\beta(\pi, c)$ under the composition

$$\check{H}^1(T, \mathcal{PU}) \xrightarrow{\Delta_1} \check{H}^2(T, \mathcal{S}) \xrightarrow{\Delta_2} \check{H}^3(T, \mathbf{Z}),$$

where Δ_1, Δ_2 are the coboundary maps corresponding to the short exact sequences

$$0 \to \mathcal{S} \to \mathcal{U} \xrightarrow{\mathrm{Ad}} \mathcal{PU} \to 0, \qquad 0 \to \mathbf{Z} \to \mathcal{R} \xrightarrow{\exp 2\pi i(\cdot)} \mathcal{S} \to 0.$$

(Note that Δ_2 is an isomorphism since $\mathcal{R}$ is fine, and Δ_1 is a bijection if H is infinite dimensional [**7**, Lemme 22].) A well-known result of Mackey implies that for each $\pi \in H^1(G, PU)$ there is a multiplier ω on G and an ω-representation $V: G \to U(H)$ such that $\omega = \operatorname{Ad} V$, and ω, V are unique up to equivalence. Simple calculations show that $\delta(\pi, c)$ depends only on the class of ω in $H^2(G, \mathbf{T})$, and in fact is represented in $H^2(T, \mathcal{S})$ by the cocycle $\{N_i, \mu_{ijk}\}$, where $\mu_{ijk}(t) = \omega(\lambda_{ij}(t), \lambda_{jk}(t))$. It therefore makes sense to define our pairing by

$$\delta(\omega, c) = \delta(\operatorname{Ad} V, c) \quad \text{where } V: G \to U(H) \text{ is an } \omega\text{-representation.}$$

If Ω is a locally trivial principal G-bundle over T corresponding to the class $c \in \check{H}^1(T, \mathcal{G})$, we also write $\delta(\omega, \Omega)$ for $\delta(\omega, c)$. Routine calculations show that for fixed Ω the map $\omega \to \delta(\omega, \Omega)$ is a homomorphism of $H^2(G, \mathbf{T})$ into $\check{H}^3(T, \mathbf{Z})$.

THEOREM 4.1. *Let G be a separable locally compact group, $\omega \in H^2(G, \mathbf{T})$, and Ω a locally trivial principal G-bundle over a locally compact space T. Then $C^*(G, \Omega, \omega)$ is a continuous trace C^*-algebra with spectrum T and Dixmier-Douady class $\delta(\overline{\omega}, \Omega)$.*

For the proof we need a couple of lemmas. The first is Lemma 3 in Wassermann's thesis [**24**, p. 19].

LEMMA 4.2. *Let (G, A, α) be a C^*-dynamical system, $\omega \in H^2(G, \mathbf{T})$ and suppose $V: G \to U(H)$ is a nondegenerate $\overline{\omega}$-representation. Then*

$$C^*(G, A, \omega) \otimes K(H) \cong [A \otimes K(H)] \times_{\alpha \otimes \operatorname{Ad} V} G.$$

LEMMA 4.3. *Let $V: G \to U(H)$ be an $\overline{\omega}$-representation and let $\Omega \to T$ be a locally trivial principal G-bundle. Then*

$$B = \{f \in GC(\Omega, K(H)) \colon f(gx) = \operatorname{Ad} V(g)(f(x)) \text{ for } g \in G,\ x \in \Omega\}$$

is a continuous trace C^*-algebra with spectrum T and Dixmier-Douady class $\delta(\overline{\omega}, \Omega)$.

PROOF. Let Ω be given by the cocycle $\lambda_{ij}: N_{ij} \to G$, so we can regard Ω as the disjoint union $\bigcup N_i \times G$ modulo the equivalence relation which identifies $(t, g) \in N_i \times G$ with $(t, g\lambda_{ij}(t)) \in N_j \times G$. We write $[t, g]_i$ for the class of $(t, g) \in N_i \times G$ in Ω. Then $\delta(\overline{\omega}, \Omega)$ is represented in $\check{H}^1(T, \mathcal{PU})$ by $\{N_i, \operatorname{Ad} V \circ \lambda_{ij}\}$, so that

$$A = \left\{ \{f_i\} \in \prod_i C_b(N_i, K(H)) \colon f_i(t) = \operatorname{Ad} V(\lambda_{ij}(t)) f_j(t) \text{ for } t \in N_{ij} \right\}$$

is a continuous trace C^*-algebra with $\delta(A) = \delta(\overline{\omega}, \Omega)$. We now define $\Phi: B \to A$ by

$$(\Phi f)_i(t) = f([t, e]_i);$$

on N_{ij} we have

$$\begin{aligned}(\Phi f)_i(t) = f([t, e]_i) &= f([t, \lambda_{ij}(t)]_j) = \operatorname{Ad} V(\lambda_{ij}(t)) f([t, e]_j) \\ &= \operatorname{Ad} V(\lambda_{ij}(t))[(\Phi f)_j(t)],\end{aligned}$$

so this does map into A. We also define $\Psi: A \to B$ by

$$\Psi\{f_i\}([t, g]_i) = \operatorname{Ad} V(f)(f_i(t));$$

routine calculations like the one two lines above show that $\Psi\{f_i\}$ is a well-defined function on Ω satisfying the appropriate condition along G-orbits, and that

$$\|\Psi\{f_i\}([t,g]_i)\| = \|f_i(t)\| \quad \text{for all } g \in G,$$

so $\Psi\{f_i\}$ belongs to $GC(\Omega, K(H))$. As Ψ is an inverse for Φ the lemma is proved. $\square$

PROOF OF THEOREM 4.1. Let γ denote the action of G on $C_0(\Omega)$, and let V be a nondegenerate $\overline{\omega}$-representation of G on H. Then by Lemma 4.2

$$C^*(G,\Omega,\omega) \otimes K(H) \cong [C_0(\Omega) \otimes K(H)] \times_{\gamma\otimes \mathrm{Ad}\, V} G.$$

By Theorem 2.2 this crossed product is strongly Morita equivalent to $GC(\Omega, K(H))^{\gamma\otimes\mathrm{Ad}\,V}$, which by Lemma 4.3 has continuous trace and the right Dixmier-Douady class. $\square$

REMARK. Theorem 5 of [**24**] is this result for G a compact Lie group, together with the observation that in this case $\omega \to \delta(\omega,\Omega)$ can also be realised as the composition

$$H^2(G,\mathbf{T}) \to H^3(G,\mathbf{Z}) \to \check{H}^3(BG,\mathbf{Z}) \to \check{H}^3(T,\mathbf{Z}).$$

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