ESSENTIAL DIMENSION LOWERING MAPPINGS HAVING DENSE DEFICIENCY SET

BY

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ABSTRACT. Two classes of surjective maps \( f : S^m \to S^n \) that are one-to-one over the image of a dense set are constructed. We show that for \( m, n \geq 3 \) there is a monotone surjection \( f : S^m \to S^n \) that is one-to-one over the image of a dense set; and for \( 3 \leq n \leq m \leq 2n - 3 \), each element of \( \pi_m(S^n) \) can be represented as a monotone surjection \( f : S^m \to S^n \) that is one-to-one over the image of a dense set.

1. Introduction. The present paper should be considered as a continuation of the study of surjective maps between spheres that are “one-to-one over the image of a dense set”. (A surjection \( f : X \to Y \) is one-to-one over the image of a dense set if there exists a dense set \( D \subseteq X \) such that for each \( y \in f(D), \# f^{-1}(y) = 1; \# = \text{cardinality}.\)

First inconceivable examples of such maps were constructed by J. J. Walsh [Wa 5]. Specifically, for any pair \( n \geq 3, d \geq 2 \) of integers, Walsh has built a monotone, surjective map \( f : S^n \to S^n \) of degree \( d \) that is one-to-one over the image of a dense set.

More recently, in [Be-Wa], it has been established that for any \( m, n \geq 2 \) there is a surjection \( f : S^m \to S^n \) that is one-to-one over the image of a dense set. By construction this map is not monotone and factors through a 1-dimensional compactum, and hence it is null-homotopic; even more, it has no stable values. (A point \( y \in Y \) is a stable value of a map \( f : X \to Y \) between metric spaces if there exists an open cover \( \mathcal{U} \) of \( Y \) so that for every \( \mathcal{U} \)-approximation \( f' \) to \( f \), \( y \) is in the image of \( f' \).)

In this paper we show:

(a) For \( m, n \geq 3 \) there is a monotone surjection \( f : S^m \to S^n \) that is one-to-one over the image of a dense set.

(b) For \( 3 \leq n \leq m \leq 2n - 3 \) each element of \( \pi_m(S^n) \) can be represented as a monotone surjection \( f : S^m \to S^n \) that is one-to-one over the image of a dense set. In particular, if \( 3 \leq n \leq m \leq 2n - 3 \) and \( \pi_m(S^n) \neq 0 \) (e.g., \( \pi_{n+1}(S^n) = Z_2 \)), there is a monotone, essential map \( f : S^m \to S^n \) that is one-to-one over the image of a dense set (and hence, all values of \( f \) are stable).

The techniques used in the paper stem from D. Wilson [Wi 1, Wi 2] and J. J. Walsh [Wa 1–Wa 5]. Mappings are constructed by making use of “defining sequences”. Although the necessary definitions are given, and in that respect the

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paper is self-contained, familiarity with [Wa 5] is desirable. We follow the notation developed in that paper.

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2. Preliminaries. For any family $P$ of subsets of a set $X$, and for any $A \subseteq X$, we define

$$St(A, P) = \bigcup \{ p \in P : p \cap A \neq \emptyset \}.$$

Following [Wa 5], by a (stratified) partition on a closed PL $n$-manifold $N$ we mean a collection $P = \{ p_1, \ldots, p_k \}$ of closed subsets of $N$ that cover $N$ with the following properties.

(P1) Each $p \in P$ is a PL $n$-submanifold (with boundary) of $N$.

(P2) If $p_{i(1)}, \ldots, p_{i(t)}$ are mutually distinct elements of $P$, then $p_{i(1)} \cap \cdots \cap p_{i(t)}$ is either empty or an $(n - t + 1)$-dimensional PL submanifold of the boundary of $p_{i(1)} \cap \cdots \cap p_{i(t-1)}$.

Observe that $p_{i(1)} \cap \cdots \cap p_{i(t)} \neq \emptyset$ has empty boundary if and only if $p \cap p_{i(1)} \cap \cdots \cap p_{i(t)} = \emptyset$ for all $p \in P - \{ p_{i(1)}, \ldots, p_{i(t)} \}$.

If $L$ is any triangulation of $N$, by $J_i^N$ denote the standard handlebody decomposition of $N$ associated with the $i$th barycentric subdivision $\beta^i L$ of $L$:

$$J_i^N = \{ St(v, \beta^{i+1} L) : v \text{ is a vertex of } \beta^i L \}.$$

It is easy to see that $J_i^N$ satisfies (P1) and (P2). For $i \geq 1$ and $j = St(v, \beta^{i+1} L) \in J_i^N$, define the index of $j$, $\text{Ind}(j)$, to be equal to $k$ if $v$ is the barycenter of a $k$-simplex in $\beta^i L$.

Let $M^m, N^n$ be PL manifolds, and $P, Q$ partitions on $M, N$ respectively. We say that a function $T : P \rightarrow Q$ is admissible, provided:

(A1) $T$ is a bijection;

(A2) for all $p_{i(1)}, \ldots, p_{i(t)} \in P$,

$$p_{i(1)} \cap \cdots \cap p_{i(t)} \neq \emptyset \Rightarrow T(p_{i(1)}) \cap \cdots \cap T(p_{i(t)}) \neq \emptyset;$$

(A3) for all $p, p' \in P$,

$$p \cap p' \neq \emptyset \Rightarrow T(p) \cap T(p') \neq \emptyset.$$

Let $L$ be any triangulation of $N$. If $T : P \rightarrow J$ is a triple satisfying (A2) and (A3), where $J = J_i^N$ is the handlebody decomposition of $N$ associated with $\beta^i L$, by an induced map we mean any map $h : M \rightarrow N$ with $h(p) \subseteq T(p)$ for all $p \in P$. We can define $h$ by the “backward induction” on $t$, requiring that $h(p_{i(1)} \cap \cdots \cap p_{i(t)}) \subseteq T(p_{i(1)}) \cap \cdots \cap T(p_{i(t)})$. Since each nonempty intersection of elements of $J$ is an absolute retract, the inductive step “goes through”. The same fact establishes that any two induced maps are homotopic (see [Wa 5]), which enables us to talk about the induced map.

A sequence of triples $\{ T_i : P_i \rightarrow J_i \}_{i=0}^{\infty}$ is a defining sequence provided, for all $i \geq 0$:

(DS1) $J_i = J_i^N$ is the handlebody decomposition of $N$ associated with $\beta^i L$;

(DS2) $P_i$ is a partition on $M$;

(DS3) $T_i$ is an admissible function;
(DS4) for all \( p \in P_i, p' \in P_{i+1} \),
\[
p \cap p' \neq \emptyset \iff \text{Int}(p \cap p') \neq \emptyset \iff T_i(p) \cap T_{i+1}(p') \neq \emptyset.
\]

The reader should find establishing the following result a useful exercise.

**Proposition 2.1 (see [Wa 5]).** Let \( \{T_i: P_i \to J_i\}_{i=0}^{\infty} \) be a defining sequence.

(i) Setting \( h^{-1}(y) = \bigcap_{i=0}^{\infty} \text{St}(p_i, P_i) \) for any choice of \( p_i \in P_i \) with \( T_i(p_i) \ni y \) defines a surjective map \( h: M \to N \). Moreover, \( \text{Int St}(p_i, P_i) \supseteq \text{St}(p_{i+1}, P_{i+1}) \) \( (i = 0, 1, 2, \ldots) \).

(ii) If \( h_i: M \to N \) is the map induced by \( T_i: P_i \to J_i \), then \( h = \lim_{i \to \infty} h_i \) and \( (i \geq 0, 1, 2, \ldots) \) is connected, then \( h \) is a monotone map.

(iii) If each \( p \in P_i, i \geq 0 \), is connected, then \( h \) is a monotone map.

(iv) If for each \( i \geq 1 \) and each \( j \in J_i \) with \( \text{Ind}(j) = n \) there exists a PL \( m \)-cell \( B \subseteq M \) with
\[
\text{St}(T_i^{-1}(j), P_i+1) \subseteq \text{Int}B \subseteq \text{St}(T_i^{-1}(j), P_i),
\]
then the points \( y \in N \) for which \( h^{-1}(y) \subseteq M \) is a cellular set form a dense subset of \( N \).

To construct interesting maps between manifolds using 2.1, we have to produce defining sequences. The major step consists of generating a triple \( T_{i+1}: P_{i+1} \to J_{i+1} \) from a triple \( T_i: P_i \to J_i \) previously constructed. To make the notation easier, the triple \( T_i: P_i \to J_i \) will be denoted by \( T: P \to J \), and the triple \( T_{i+1}: P_{i+1} \to J_{i+1} \) by \( \tilde{T}: \tilde{P} \to \tilde{J} \). Coherently, we will rename the subdivision \( \beta\mathcal{L} \) and again call it \( L \). Hence
\[
J = \{\text{St}(v, \beta\mathcal{L}): v \text{ is a vertex of } L\},
\]
\[
\tilde{J} = \{\text{St}(v, \beta^2\mathcal{L}): v \text{ is a vertex of } \beta\mathcal{L}\}.
\]

The construction of \( \tilde{P} \) is in two stages. We define an intermediate triple \( \tilde{T}: \tilde{P} \to \tilde{J} \). _Warning._ \( \tilde{P} \) will be a partition of \( M \), and \( \tilde{T} \) will satisfy (A2) and (A3), but not necessarily (A1).

The elements of \( \tilde{P} \) will be indexed by the set \( S \) of all collections \( \{p_{i(1)}(t), \ldots, p_{i(t)}(t)\} \subseteq P \) that have nonempty intersections. (These intersections, in Walsh's terminology, are called the _strata_ of \( P \).)

The collection \( \tilde{P} = \{p_s, s \in S\} \) will satisfy the following properties.

(H1) \( \tilde{P} \) is a partition on \( M \).
(H2) \( p_{s(1)}, \ldots, p_{s(t)}(t) \in \tilde{P} \) have nonempty intersection if and only if \( \{s(1), \ldots, s(t)\} \subseteq S \) is well-ordered with respect to inclusion.
(H3) For any \( p \in P \) and \( p_s \in \tilde{P} \),
\[
p \cap p_s \neq \emptyset \iff \text{Int}(p \cap p_s) \neq \emptyset \iff p \in s.
\]
(H4) For any \( p \in P \), \( p \) and \( p_{(p)} \) are homeomorphic.

Still following [Wa 5], we construct the elements \( p_s \in \tilde{P} \) as follows (see Figure 1). Let \( K \) be a triangulation of \( M \) so that each stratum \( \bigcap s, s \in S \), is a full subcomplex of \( K \). Define the _core_ of \( s \in S \) by
\[
c(s) = \bigcup\{\tau: \tau \text{ is a simplex of } \beta K \text{ contained in } \bigcap s - \partial(\bigcap s)\}.
\]
Finally, set
\[
p_s = \bigcup\{\text{St}(v, \beta^2K): v \in c(s) \text{ is a vertex of } \beta K\}.
\]
Observe that, by choosing a sufficiently small triangulation $K$:

(H5) Given neighborhoods $U(p)$ of $p \in P$, we can arrange that $St(p, \hat{P}) \subseteq U(p)$ for all $p \in P$.

We can also define the function $\hat{T}: \hat{P} \to \hat{J}$ by $\hat{T}(p_s) = St(v_s, \beta^2I) \in \hat{J}$, where $v$ is determined as follows. If $s = \{p_{i(1)}, \ldots, p_{i(t)}\}$ and $\hat{T}(p_{i(r)}) = St(v_r, \beta L)$, then $v$ is the barycenter of the simplex whose vertices are $v_1, \ldots, v_t$. Property (H2) implies that $\hat{T}$ satisfies (A2) and (A3). It is evident that $\hat{T}$ is a one-to-one function (but not necessarily a surjection).

We will “repair” the triple $\hat{T}: \hat{P} \to \hat{J}$ to get the triple $\bar{T}: \bar{P} \to \bar{J}$, but the repairation will depend on the desired properties of the function $h: M \to N$ determined by the defining sequence. The “repairation process”, as well as the construction of the triple $T_0: P_0 \to J_0$, is explained in detail in forthcoming sections.

REMARK. If the partition $P$ is the standard handlebody decomposition corresponding to a triangulation $K$ of the manifold $M$, then the partition $\hat{P}$ constructed above is (up to an ambient isotopy) the standard handlebody decomposition of $M$ corresponding to the barycentric subdivision $K'$ of $K$. This fact will be implicitly used in the sequel.

3. Essential maps. The purpose of this section is to establish the following

**Proposition 3.1.** Let $f: S^m \to S^n$ be any map, and let $3 \leq n \leq m \leq 2n - 3$. Then there exists a surjective monotone map $h: S^m \to S^n$ homotopic to $f$ such that the set $\{y \in S^n: h^{-1}(y) \text{ is cellular in } S^m\}$ is dense in $S^n$.

A routine consequence of 3.1 is the result announced in the Introduction.

**Theorem 3.2.** For any map $f: S^m \to S^n$, $3 \leq n \leq m \leq 2n - 3$, there exists a surjective monotone map $g: S^m \to S^n$ homotopic to $f$ that is one-to-one over the image of a dense set.

**Proof.** Let $h: S^m \to S^n$ be a map whose existence is promised by 3.1. We “carefully shrink countably many cellular fibers of $h$” in order to obtain the sought-after map $g: S^m \to S^n$. The shrinking process can be described as follows.

Let $U_1, U_2, \ldots$ be a countable basis of open sets of $S^m$. Choose a fiber $F$ of $h$ with $F \cap U_1 \neq \emptyset$, and pick a cellular fiber $C$ of $h$ in a “small” connected neighborhood $V$ of $F$ (here we use the fact that $h$ is a monotone map). Let $\lambda: S^m \to S^m$ be a surjection whose only nondegenerate point-preimage is $C$. We can arrange that $\lambda(C) \subseteq U_1$ and $\lambda = \text{identity off of } V$. Then $g_1 = h\lambda^{-1}$ is a monotone surjection “close” to $h$, and one of the fibers of $g_1$ is a point in $U_1$. In a similar fashion we produce monotone maps $g_2, g_3, \ldots$ such that $g_{i+1}$ is “close” to $g_i$, it agrees with $g_i$ off of a “small” neighborhood of a fiber of $g_i$, and $g_{i+1}$ has degenerate point-preimages in each of the sets $U_1, \ldots, U_{i+1}$.

Exercising sufficient control on all choices made, and carefully interpreting the quoted words in the preceding paragraph, we can arrange that the sequence $g_1, g_2, \ldots$ converges to a monotone map $g: S^m \to S^n$ homotopic to $h$ that is one-to-one over the image of a dense set.

Before giving a proof of Proposition 3.1, we state and prove an interesting corollary of Theorem 3.2.
In what follows, $E^r$ denotes Euclidean $r$-dimensional space. Observe that the homogeneity properties of $E^r$ establish that the set of all stable values of a surjection $f : X \to E^r$ is open in $E^r$.

**Corollary 3.3.** Let $m, n \geq 2$ be integers.

(i) If $\pi_{m-1}(S^{n-1}) = 0$, then any surjection $f : E^m \to E^n$ that is one-to-one over the image of a dense set has no stable values.

(ii) If $3 \leq n \leq m \leq 2n - 3$ and $\pi_{m-1}(S^{n-1}) \neq 0$, then there exists a proper monotone surjection $f : E^m \to E^n$ that is one-to-one over the image of a dense set and has all values stable.

**Proof.** (i) In view of the observation made before the statement of Corollary 3.3, it suffices to prove that if $\# f^{-1}(y) = 1$, then $y$ is not a stable value of $f$. Let $B$ be a “small” ball around $f^{-1}(y)$. The assumption about the homotopy group reveals that $f|\partial B$ is a null-homotopic map in a “small” deleted neighborhood of $y$. Redefine $f$ in Int $B$, using the homotopy, to get an approximation $f'$ to $f$ whose image misses $y$.

(ii) By Freudenthal’s Suspension Theorem (see [Sp, p. 458]), $\pi_{m-1}(S^{n-1}) \cong \pi_m(S^n)$. Application of 3.2 gives an essential monotone surjection $f : S^m \to S^n$ that is one-to-one over the image of a dense set. Pick $y \in S^n$ such that $\# f^{-1}(y) = 1$. Then $f : S^m - f^{-1}(y) \to S^n - y$ is a proper monotone surjection that is one-to-one over the image of a dense set. All values of $f$ are stable since the opposite would violate the fact that $f$ is essential.

**Proof of 3.1.** We construct a defining sequence $\{T_i : P_i \to J_i\}_{i=0}^\infty$ with the following additional properties.

(E1) For mutually distinct elements $p_i(1), \ldots, p_i(t) \in P_i$, $p_i(1) \cap \cdots \cap p_i(t) \neq \varnothing \Leftrightarrow t \leq n + 1$ and $T(p_i(1) \cap \cdots \cap T(p_i(t))) \neq \varnothing$ ($i = 0, 1, \ldots$).

(E2) Each element $p \in P_i$ is $(m - n)$-connected ($i = 0, 1, \ldots$).

(E3) If $j \in J_i$, $\text{Ind}(j) = n$, then $T_r^{-1}(j)$ is a PL $m$-ball ($i = 1, 2, \ldots$).

As announced in §2, we construct the defining sequence by induction. Suppressing indices, we start with an admissible function $T : P \to J$ satisfying (E1) and (E2) (produced following the inductive analysis). Let $K$ be a triangulation of $S^m$ such that all strata of $P$ are full subcomplexes of $K$. Let $\hat{T} : \hat{P} \to \hat{J}$ be the triple constructed in §2, $\hat{P} = \{p_s, s \in S\}$. Observe that (E1) implies that $\hat{T}$ satisfies (A1)–(A3). Also, (H2) implies that $\hat{T}$ satisfies (E1).

We now “repair” the triple $\hat{T} : \hat{P} \to \hat{J}$ to get another triple $\hat{T} : \hat{P} \to \hat{J}$ which satisfies (E2) and (E3). We want to maintain all properties that $\hat{T} : \hat{P} \to \hat{J}$ already satisfies. For all $s \in S$ choose $p(s) \in s$; if possible, choose $p(s)$ so that $\text{Ind} T(p(s)) = n$. A quick remark: each $s \in S$ contains at most one $p$ with $\text{Ind} T(p) = n$. We interrupt the proof to introduce some notation.

For a compactum $X$, denote by $C(X) = X \times [0, 1]/(x_1, 1) \sim (x_2, 1)$ the cone over $X$. We identify $X = X \times \{0\} \subseteq C(X)$. Name $\frac{1}{2}C(X) = X \times [0, \frac{1}{2}] \subseteq C(X)$ the bottom half of the cone over $X$. If $A$ is a subcomplex of $K$, by $A^{(r)}$ we denote the $r$-skeleton of $A$ with respect to $K$. Finally, “$\sim$” means “PL homeomorphic”.

For all $s \in S$ with $\# s > 1$ choose a polyhedron $X_s \subseteq S^m$ containing $c(s)$ with the following properties.
(a) $X_s \subseteq p(s) \cap (p_s \cup \text{Int} p_s(p_s))$
(b) $X_s \cap \partial p(s) = c(s)$
(c) $(X_s, X_s \cap p_s, c(s)) \approx ((c(s)) \cup C((c(s))^{(m-n)}), (c(s) \cup \frac{1}{2}C((c(s))^{(m-n)}), c(s))$ and
(d) if $s_1 \neq s_2$, then $X_{s_1} \cap X_{s_2} = \emptyset$.

Sets $X_s$ exist, since $\dim(C(c(s))^{(m-n)}) \leq m - n + 1$, and $2(m - n + 1) < m$.
The same inequality, coupled with (e) and the fact that each $p_s$ with $\#s = 1$ is $(m - n)$-connected (see (H4)), testifies that $\{p_s\}_{s \in S, \#s > 1}$ is still $(m - n)$-connected for all $s \in S$.

Let $K'$ be a subdivision of $K$ such that all mentioned subsets of $S^m$ are full subcomplexes with respect to $K'$. Let $N_s$ be the second derived neighborhood of $X_s$ in $p(s)$ with respect to $K'$. Define

$$\tilde{p}_s = \begin{cases} p_s \cup N_s & \text{if } \#s > 1, \\ p_s \setminus \{\text{Int} N_{s'}, s' \in S, \#s' > 1\} & \text{if } \#s = 1. \end{cases}$$

The reader can easily verify that $\tilde{P} = \{\tilde{p}_s, s \in S\}$ is a partition on $S^m$, that $\tilde{T}: \tilde{P} \rightarrow \tilde{J}$ defined by $\tilde{T}(\tilde{p}_s) = \tilde{T}(p_s)$ is an admissible function, and that the triple $\tilde{T}: \tilde{P} \rightarrow \tilde{J}$ satisfies (E1) and (E2). (Note that $p_s$ collapses to $c(s)$; by adding the cone over $c(s)$, we “killed” first $(m - n)$ homotopy groups).

If $\#s = n + 1$, then $\tilde{p}_s$ is a regular neighborhood of $X_s \approx C(c(s))$ (since $\dim(c(s)) = m - n$), and hence $\tilde{p}_s$ is an $m$-ball. This establishes (E3). From (a) and (H3), it easily follows that $T$ and $\tilde{T}$ satisfy (DS4).

Observe that, by our choice of $p(s), s \in S$, we have $\text{St}(p, \tilde{P}) = \text{St}(p, \tilde{P})$, for all $p \in P$ with $\text{Ind} T(p) = n$. Thus, taking into account (H5), we get:

(E4) Given neighborhoods $U(p)$ of $p \in P$ with $\text{Ind} T(p) = n$, we can arrange that $\text{St}(p, \tilde{P}) \subseteq U(p)$ for all such $p$.

If $h: S^m \rightarrow S^n$ is the map associated with a defining sequence $\{T_i: P_i \rightarrow J_i\}_{i=0}^\infty$ satisfying (E1)–(E3), then, by 2.1(iii), $h$ is a monotone surjection, and using 2.1(iv), together with (E4), we see that we can arrange that the set $\{y \in S^n: h^{-1}(y) \text{ is cellular}\}$ is dense in $S^n$. Indeed, let $B_j$ be a regular neighborhood of $T_i^{-1}(j)$ contained in $\text{St}(T_i^{-1}(j), P_i)$. By (E3), $B_j$ is an $m$-ball. By (E4) we can arrange that $\text{St}(T_i^{-1}(j), P_i+1) \subseteq \text{Int} B_j$.

To finish the proof of 3.1, in view of 2.1(ii), we need to construct a triple $T_0: P_0 \rightarrow J_0$ satisfying (E1) and (E2) such that the induced map $h_0: S^m \rightarrow S^n$ is homotopic to $f$. By Freudenthal’s Suspension Theorem [Sp, p. 458], there is a map $f': S^{m-1} \rightarrow S^{n-1}$ whose suspension $\Sigma f': S^m \rightarrow S^n$ is homotopic to $f$. Without loss of generality, we may assume that $f'$ is a surjective simplicial map, with respect to some triangulations $K_0, L_0$ of $S^{m-1}, S^{n-1}$ respectively. Then the map $\Sigma f': \Sigma K_0 \rightarrow \Sigma L_0$ is simplicial. To suppress unnecessary symbols, rename it as $f: K \rightarrow L$. Let

$$p_v = \bigcup \{\text{St}(w, \beta K): f(w) = v, w \text{ is a vertex of } K\}$$

and set $P = \{p_v: v \text{ is a vertex of } L\}$. Then $P$ is a partition on $S^m$, and the function $T: P \rightarrow J_0$ given by $T(p_v) = \text{St}(v, \beta L)$ satisfies (A1)–(A3) and (E1).

We now “repair” the triple $T: P \rightarrow J_0$ to get a new triple $T_0: P_0 \rightarrow J_0$ satisfying, in addition, (E2). The strategy is the same as for obtaining $\tilde{T}$ from $T$. Observe
that if $\sigma$ is a suspension vertex of $L$, then $p_\sigma$ is an $m$-ball. Moreover, $p_\sigma$ intersects all elements of $P$ except for $p_\tau \in P$, where $\tau$ is the other suspension vertex. If $p \in P - \{p_\sigma, p_\tau\}$, then $p$ collapses to $p \cap p_\sigma$. For each $p \in P - \{p_\sigma, p_\tau\}$ choose a polyhedron $A_p \subseteq p \cap p_\sigma$ such that $\dim A_p \leq m - n$ and the pair $(p \cap p_\sigma, A_p)$ is $(m - n)$-connected. We can arrange that $A_p(1) \cap A_p(2) = \emptyset$ if $p(1) \neq p(2)$. (We can take $A_p$ to be the $(m - n)$-skeleton of a shrunk copy of $p \cap p_\sigma$.) Next, embed the cones $C(A_p)$ into $p_\sigma$ to obtain polyhedra $Y(p)$, $p \in P - \{p_\sigma, p_\tau\}$. We can arrange that $Y(p) \cap \partial p_\sigma = A_p$ for all $p \in P - \{p_\sigma, p_\tau\}$ and, by general positioning, that $Y(p) \cap Y(p') = \emptyset$ for $p \neq p'$ (we are in the range of dimensions where $2(m - n + 1) < m$). Let $K'$ be a subdivision of $K$ such that all mentioned subsets of $S^m$ are full subcomplexes of $K'$. If $N_0$ is the second derived neighborhood of $Y(P_0)$ in $p_\sigma$, set

$$\bar{p}_v = \begin{cases} p_v \cup N_v, & \text{if } v \text{ is a vertex of } K_0, \\ p_\sigma - \bigcup \{\text{Int } p_{v'}, v' \text{ is a vertex of } K_0\}, & \text{if } v = \sigma, \\ p_\tau, & \text{if } v = \tau. \end{cases}$$

Then $P_0 = \{\bar{p}_v : v \text{ is a vertex of } K\}$ is a partition on $S^m$, and $T_0 : P_0 \to J_0$ defined by $T_0(\bar{p}_v) = T(p_v)$ satisfies (A1)-(A3), (E1) and (E2). If $h_0 : S^m \to S^n$ is a map induced by $T_0 : P_0 \to J_0$, then $h_0$ and $f$ are $\mathcal{U}$-close, where $\mathcal{U} = \{\text{St}(j, J_0), j \in J_0\}$ is a closed cover of $S^n$ such that each nonempty intersection of elements of $\mathcal{U}$ is an absolute retract (in fact, it is a PL ball). Hence (see [Wa 5]) $h_0$ and $f$ are homotopic maps.

This finishes the proof of 3.1.

4. Monotone maps. In §3 we have shown that, in certain range of dimensions, there exist essential, monotone maps $f : S^m \to S^n$ that are one-to-one over the image of a dense set. In this section we show how to construct monotone (inessential) surjections $f : S^m \to S^n$ that are one-to-one over the image of a dense set for any $m, n \geq 3$. If $m > n \geq 4$, the existence of such maps follows from 3.2. Indeed, let $f_i : S^i \to S^{i-1}$ be a monotone surjection that is one-to-one over the image of a dense set ($i = n + 1, n + 2, \ldots, m$). Let $g_i : S^i \to S^i$ be a homeomorphism intermingling the two pertinent (countable) dense subsets of $S^i$ ($i = n + 1, n + 2, \ldots, m - 1$). Then the composition $f_{n+1}g_{n+1} \cdots f_{m-1}g_{m-1}f_m : S^m \to S^n$ is a map with the desired properties. However, we want to present an independent proof that also works for $3 \leq m \leq n$ or $n = 3$.

**Theorem 4.1.** For any $m, n \geq 3$ there exists a monotone surjection $h : S^m \to S^n$ that is one-to-one over the image of a dense set.

In the proof we need

**Lemma 4.2.** Let $h : X \to Y$ be a surjective map between compact metric spaces. If each nonempty open set in $X$ contains a fiber of $h$, then $h$ is one-to-one over the image of a dense set.

**Proof.** Suppose not. Let $F_\varepsilon = \bigcup \{h^{-1}(y) : y \in Y, \text{diam } h^{-1}(y) \geq \varepsilon\}$. Then $F_\varepsilon$ is a closed set for any $\varepsilon > 0$, and $\bigcup \{F_\varepsilon, \varepsilon > 0\}$ has nonempty interior. By Baire's Category Theorem [Du, p. 250], there exists $\varepsilon > 0$ such that $F_\varepsilon$ has nonempty interior. Let $U \subseteq F_\varepsilon$ be a nonempty open set with $\text{diam } U < \varepsilon$. Then $U$ does not contain any fibers of $h$, contrary to the hypothesis.
PROOF OF 4.1. We construct a defining sequence \( \{T_i : P_i \to J_i\}_{i=0}^{\infty} \) with the following properties.

(M1) Each \( p \in P_i \) is connected, \( i = 0, 1, 2, \ldots \).

(M2) If \( p_{i(1)}, p_{i(2)}, p_{i(3)}, p_{i(4)} \in P_i \) are mutually distinct elements, then \( p_{i(1)} \cap p_{i(2)} \cap p_{i(3)} \cap p_{i(4)} = \emptyset \).

As in §3, we show first how to construct the triple \( T_{i+1} : P_{i+1} \to J_{i+1} \) from the triple \( T_i : P_i \to J_i \) already constructed. As before, we suppress indices, starting with an admissible function \( T : P \to J \) satisfying (M1) and (M2). Let \( \hat{T} : \hat{P} \to \hat{J} \) be the triple constructed in §2. Observe that, although it is one-to-one, \( T \) will never be onto (this is the whole point of the construction).

We have to create new elements of \( P \) that will correspond to elements of \( \hat{J} - \text{Im} \hat{T} \), as well as make all elements of \( \hat{P} \) connected. Observe that if \( j \in \hat{J} \) with \( \text{Ind}(j) \leq 1 \), then \( (\text{by (A3)}) j \in \text{Im} \hat{T} \); and if \( \text{Ind}(j) \geq 3 \), then \( (\text{by (M2)}) j \notin \text{Im} \hat{T} \).

First “connect up” all components of elements of \( \hat{P} \). The only important property of \( \hat{P} \) we use here is that for each disconnected \( p, p \in \hat{P} \) there is a connected element \( \hat{p} \in \hat{P} \) such that each component of \( p \) intersects \( \hat{p} \). (If \( p \in s, \hat{p} = p(p) \) would do; see (H4).) Fixing a triangulation \( K' \) of \( S^m \) such that all pertinent subsets of \( S^m \) are (full) subcomplexes, for each disconnected \( p \in \hat{P} \) choose a PL arc \( \alpha_p \) lying in a connected element \( \hat{c}(p) \in \hat{P} \) such that \( \alpha_p \cap \partial \hat{c}(p) \) is a finite set intersecting each component of \( p \). We can also arrange that different arcs are disjoint, and lie in the complement of the \( (m-2) \)-skeleton of \( K' \) (here we use \( m \geq 3 \)). Choose a subdivision \( K'' \) of \( K' \) such that \( \alpha_p \)'s are subcomplexes with respect to \( K'' \), and let \( N_p \) be the second derived neighborhood of \( \alpha_p \) in \( c(p) \). Finally, set

\[
\begin{align*}
p^0 &= \begin{cases} 
p \cup N_p & \text{if } p \in \hat{P} \text{ is disconnected}, 
\quad p - \bigcup \{N_{p'} : p' \in \hat{P} \text{ is disconnected} \} & \text{if } p \in \hat{P} \text{ is connected}.
\end{cases}
\end{align*}
\]

Set \( \hat{P}^0 = \{p^0, p \in \hat{P} \} \). Then \( \hat{T}^0 : \hat{P}^0 \to \hat{J} \) defined by \( \hat{T}^0(p) = \hat{T}(p) \) is a triple satisfying (A2), (A3), (M1) and (M2).

Now, we create new elements so that \( \hat{T}^0 \) can be extended to a bijection.

Let \( \hat{J} - \text{Im} \hat{T}^0 = \{j_1, j_2, \ldots, j_r\} \). We can order this set so that \( k \leq l \) implies \( \text{Ind}(j_k) \leq \text{Ind}(j_l) \). We define a sequence \( \{\hat{T}^k : \hat{P}^k \to \hat{J} \}_{k=1}^{\infty} \) of triples satisfying (A2), (A3), (M1) and (M2) and

\[
\begin{align*}
(a_k) \quad & \hat{J} - \text{Im} \hat{T}^k = \{j_{k+1}, j_{k+2}, \ldots, j_r\}; \\
(b_k) \quad & \text{for any } p \in P, p' \in \hat{P}^k, \\
& p \cap p' \neq \emptyset \Leftrightarrow \text{Int}(p \cap p') \neq \emptyset \Leftrightarrow T(p) \cap \hat{T}^k(p') \neq \emptyset.
\end{align*}
\]

Clearly, \( \hat{T}^0 : \hat{P}^0 \to \hat{J} \) satisfies \( (a_0) \) and \( (b_0) \). Finally, we set \( \{T_r : \hat{P} \to \hat{J} \} = \{\hat{T}^r : \hat{P}^r \to \hat{J} \} \).

An argument for the inductive step is as follows. Assume the triple \( \hat{T}^{k-1} : \hat{P}^{k-1} \to \hat{J} \) satisfies (A2), (A3), (M1), (M2), \( (a_{k-1}) \) and \( (b_{k-1}) \). Define

\[
B = \bigcup \{p \in \hat{P}^{k-1} : \hat{T}^{k-1}(p) \cap j_k \neq \emptyset \}.
\]

Claim 1. \( B \) is connected.

Indeed, let \( p_1, p_2 \in \hat{P}^{k-1} \) with \( \hat{T}^{k-1}(p_i) \cap j_k \neq \emptyset, i = 1, 2 \). Let \( v_1, v_2 \) be the two vertices of \( \beta L \) such that \( \hat{T}^{k-1}(p_i) = \text{St}(v_i, \beta^2 L), i = 1, 2 \). Similarly, let \( v \) be
the vertex of $\beta L$, such that $j_k = \text{St}(v, \beta^2 L)$. Let $\sigma, \sigma_1, \sigma_2$ be the simplexes of $L$ whose barycenters are $v, v_1, v_2$ respectively. Since $\text{St}(v, \beta^2 L) \cap \text{St}(v_1, \beta^2 L) \neq \emptyset$, $\sigma$ and $\sigma_1$ are comparable simplexes, $i = 1, 2$ (i.e. one is a face of the other). Choose a sequence $w_1, w_2, \ldots, w_l$ of vertices of $\sigma$, such that $w_1$ is a vertex of $\sigma_1$, and $w_l$ is a vertex of $\sigma_2$. Say, $\sigma_1 = \langle w_1, a_1, \ldots, a_q \rangle$, $\sigma_2 = \langle w_1, b_1, \ldots, b_u \rangle$. In the sequence $\langle w_1, a_1, \ldots, a_{q-1}, a_q \rangle, \langle w_1, a_1, \ldots, a_{q-1} \rangle, \ldots, \langle w_1, a_1 \rangle, \langle w_1, w_2 \rangle, \langle w_2, w_3 \rangle, \langle w_3 \rangle, \ldots, \langle w_{l-1}, w_l \rangle, \langle w_l \rangle, \langle w_1, b_1 \rangle, \ldots, \langle w_1, b_1, \ldots, b_u \rangle$ any two consecutive simplexes of $L$ are comparable. Let $j^1, j^2, \ldots, j^x$ be the corresponding sequence of elements of $J$ (determined by the barycenters of the simplexes in the sequence). Note that

1. any two consecutive elements in this sequence intersect, (2) we can arrange the vertices of $\sigma_1$ and $\sigma_2$ so that each element in the sequence intersects $j_k$, and
2. by our choice of indexing elements of $J - \text{Im} \bar{T}^0$ and the fact that $\text{Ind}(j) \leq 1$ implies $j \in \text{Im} T^0$, each element in the sequence is in $\text{Im} \bar{T}^{k-1}$. Now by (A3) and (M1), $(\bar{T}^{k-1})^{-1}(j^1), (\bar{T}^{k-1})^{-1}(j^2), \ldots, (\bar{T}^{k-1})^{-1}(j^x)$ is a sequence of connected sets whose union contains $p_1, p_2$ and itself is contained in $B$ such that any two consecutive elements intersect. Hence, $B$ is connected.

Claim 2. If $p \in P$ and if $T(p) \cap j_k \neq \emptyset$, then $\text{Int}(B \cap p') \neq \emptyset$.

Indeed, if $j_k = \text{St}(v, \beta^2 L)$, and if $v$ is a barycenter of a simplex $\sigma = \langle a_1, \ldots, a_l \rangle$ of $L$, then $T(p) = \text{St}(a_i, \beta L)$ for some $i$, $1 \leq i \leq l$. But then

$$p' = (\bar{T}^{k-1})^{-1}(\text{St}(a_i, \beta^2 L)) \subseteq B$$

and $\text{Int}(p' \cap p) \neq \emptyset$ (by (b$_{k-1}$)).

Following the well-established pattern, once again choose a triangulation $K_k$ of $S^m$ such that all relevant subsets of $S^m$ are (full) subcomplexes. Let $\alpha$ be a PL arc in $\text{Int} B$ that intersects all sets of the form $p$ or $\text{Int}(B \cap p')$ for some $p \in \hat{P}^{k-1}$ with $\bar{T}^{k-1}(p) \cap j_k \neq \emptyset$ or some $p' \in P$ with $T(p') \cap j_k \neq \emptyset$. By Claims 1 and 2 such an arc exists. We can also arrange it so that it misses the $(m - 2)$-skeleton of $K_k$. Let $K'_k$ be a subdivision of $K_k$ such that $\alpha$ is a subcomplex, and let $N$ be the second derived neighborhood of $\alpha$ (in $B$). Define $A(p) = p - \text{Int} N$ for all $p \in \hat{P}^{k-1}$. Setting $\hat{P}^k = \{ A(p): p \in \hat{P}^{k-1} \} \cup \{ N \}$ and $\hat{T}^k(A(p)) = \hat{T}^{k-1}(p)$, $\hat{T}^k(N) = j_k$ defines a triple $\hat{T}^k: \hat{P}^k \to \hat{J}$. The reader should observe that this triple satisfies (A2), (A3), (M1), (M2), (a$_k$), (b$_k$).

We now proceed with the description of a number of improvements on the construction of a triple $\bar{T}: \bar{P} \to \bar{J}$. For convenience, we use the following notation. If $p_{i(1)}, \ldots, p_{i(t)} \in P$ with $\bigcap_{i=1}^t T(p_{i(i)}) \neq \emptyset$, then by $A(p_{i(1)}, \ldots, p_{i(t)})$ we denote the element of $\hat{P}$ such that $T(A(p_{i(1)}, \ldots, p_{i(t)})) = \text{St}(v, \beta^2 L)$, where $v$ is the barycenter of the simplex of $L$ whose vertices are determined by “centers” of $T(p_{i(1)}, \ldots, T(p_{i(t)}))$.

(i) Given $p_{i(1)}, p_{i(2)}, p_{i(3)}, p_{i(4)} \in P$ with $T(p_{i(1)}) \cap T(p_{i(2)}) \cap T(p_{i(3)}) \cap T(p_{i(4)}) \neq \emptyset$, we can arrange that there exists a PL arc $\alpha \subset \text{Int} p_{i(1)}$ such that $\alpha$ intersects the interior of each of the following elements of $\hat{P}$, and no other elements of $\hat{P}$:

$A(p_{i(1)}), A(p_{i(2)}), A(p_{i(3)}), A(p_{i(1)}, p_{i(2)}, p_{i(3)}), A(p_{i(1)}, p_{i(2)}, p_{i(3)}, p_{i(4)}).$

The trick is first to specify an arc $\alpha \subset \text{Int} p_{i(1)}$ that meets “right” elements of $\hat{P}$. In the process of “connecting up”, we can choose arcs to miss $\alpha$, and (choosing
a small triangulation of $S^m$) we can arrange that the “connecting tubes” miss $\alpha$. Consequently, $\alpha$ hits the “right” elements of $\hat{P}^0$. In the inductive process of constructing partitions $\hat{P}^1, \ldots, \hat{P}^r = \hat{P}$, we can choose the relevant arcs either to hit or to miss $\alpha$ (according to the nature of the partition element that is about to be constructed).

(ii) Along with the hypotheses as in (i), assume that $U$ is an open set in $S^m$ and $U \cap p_i(1) \neq \emptyset$. Then we can arrange that $\alpha$ (which satisfies the conclusion of (i)) is contained in $U$.

Indeed, if $\alpha$ is any arc as in (i), we can find a PL homeomorphism $\psi: S^m \to S^m$ such that $\psi = \text{identity off of } \text{Int} p_i(1)$ and $\psi(\alpha) \subset U$. Then $\hat{P}' = \{ \psi(\tilde{p}) : \tilde{p} \in \tilde{P} \}$ and $\alpha' = \psi(\alpha)$ satisfy all conclusions of (ii).

(iii) Given $p_i(1), p_i(2), p_i(3), p_i(4) \in P$ as in (i) and an arc $\alpha \subset \bigcup_{t=1}^4 \text{Int} p_i(t)$ intersecting each $\text{Int} p_i(t)$, $t = 1, 2, 3, 4$, we can arrange that $A(p_i(1), p_i(2), p_i(3), p_i(4))$ is contained in a prechosen neighborhood $U$ of $\alpha$.

Using an argument of the same type as in (i), we can arrange that $\alpha$ has all properties as an arc serving as a guide for constructing $A(p_i(1), p_i(2), p_i(3), p_i(4))$. Then it remains to choose a small triangulation of $S^m$ to get the required containment (in the inductive process, already “born” elements cannot “grow”).

The improvements (ii) and (iii), applied to

$$A(p_i(1)), A(p_i(1), p_i(2)), A(p_i(1), p_i(2), p_i(3)), A(p_i(1), p_i(2), p_i(3), p_i(4)),$$

coupled together yield the following (here we use $n \geq 3$).

(iv) For any nonempty open set $U \subset S^m$, in the construction of the defining sequence $\{ T_i : P_i \to J_i \}_{i=0}^\infty$, whenever $P_i$ is given, we can arrange (by carefully choosing $P_{i+1}$ and $P_{i+2}$) that $P_{i+2}$ contains an element contained in $U$.

(v) Given $p \in P$ and an open set $U \subset S^m$ with $p \subset U$, we can arrange that $\text{St}(A(p), \tilde{P}) \subset U$.

Indeed, using (H5), we can arrange that $\text{St}(p, \tilde{P}) \subset U$. In the “connecting up” process, we choose $c(p')$ to be $p'' \in P$. In this way we get $\text{St}(p', \tilde{P}^0) \subset U$ where $\tilde{p} = p(p)$. Since $\tilde{p}$ is connected, we have $\tilde{p}^0 \subset \tilde{p}$ and hence $\text{St}(p^0, \tilde{P}^0) \subset U$. Inductively assume that $\text{St}(p^{k-1}, \tilde{p}^{k-1}) \subset U$, where $\tilde{p} \in \tilde{P}^{k-1}$ “comes” from $\tilde{p} \in \tilde{P}$. If $j_k \cap \tilde{T}^{k-1}(\tilde{p}^{k-1}) = \emptyset$, we have $\text{St}(\tilde{p}^k, \tilde{P}^k) \subset \text{St}(\tilde{p}^{k-1}, \tilde{P}^{k-1}) \subset U$. So assume that $j_k \cap \tilde{T}^{k-1}(\tilde{p}^{k-1}) \neq \emptyset$. The corresponding set $B$ defined along the inductive argument can be written as $B = B_1 \cup B_2$, where $B_1 = \bigcup \{ p' \in \tilde{P}^{k-1} : p' \subset B, p' \cap \tilde{p}^{k-1} = \emptyset \}$, and $B_2 = \bigcup \{ p' \in \tilde{P}^{k-1} : p' \cap \tilde{p}^{k-1} = \emptyset \}$. Then $B_1 \subset U$ and each $p' \subset B_2$ hits $B_1$. Consequently, if we replace the set $B$ in the inductive argument by the set $B' = B_1 \cup \{ \text{the collection of all components of } B_2 \cap U \text{ that hit } B_1 \}$, the constructed element $N$ will be contained in $U$, and hence $\text{St}(\tilde{p}^k, \tilde{P}^k) \subset U$.

Improvements (iv) and (v) give the following:

Let $\{ U_1, U_2, U_3, \ldots \}$ be a countable basis of open sets for the topology on $S^m$. Then we can arrange that, for each $i$, there exists $p(i) \in P_{3i}$ with $\text{St}(p(i), P_{3i}) \subset U_i$.

Since by 2.1(i) each of the sets of the form $\text{St}(p, P)$, $p \in P$, contains a fiber of the map $h: S^m \to S^n$ determined by such a defining sequence, we conclude that $h$ satisfies the hypotheses of 4.2, and hence it is one-to-one over the image of a dense set. (M1) implies that $h$ is monotone.
To finish the proof of 4.1, we need to construct a triple $T_0: P_0 \to J_0$ satisfying (A1)-(A3) and (M1), (M2). As in §3, we take advantage of the fact that spheres are suspensions.

Let $L'$ be any triangulation of $S^{m-1}$. Then there exists a partition $P$ of $S^{m-1}$ and an admissible map $T: P \to J$, $J = \{St(v, \beta L'): v$ is a vertex of $L'\}$ such that if $p_{i(1)}, p_{i(2)}, p_{i(3)} \in P$ are distinct, then $p_{i(1)} \cap p_{i(2)} \cap p_{i(3)} = \emptyset$. We now "suspend" the triple $T: P \to J$. Let $J_0$ be the standard handlebody decomposition of $S^n$ corresponding to the triangulation $L = \Sigma L'$. $P$ is a partition of $S^{m-1} \subset S^m$. Let $P'_0$ be the partition of $S^m$ consisting of slightly "thickened" copies of $p \in P$ together with two $m$-balls corresponding to the suspension points. Defining $T'_0: P'_0 \to J_0$ in the obvious way, the reader should realize that this triple satisfies (A1)-(A3) and (M2). It remains to "connect up" elements of $P'_0$. Observing that both $m$-balls in $P'_0$, corresponding to two suspension points, intersect all components of all disconnected elements of $P'_0$, the author leaves this as an exercise.

This completes the proof of 4.1.

**Remark 4.3.** In [Be-Wa] it is shown that a map $f: S^m \to S^2$ that is one-to-one over the image of a dense set is far from being monotone. Using the technique of this section, one can construct a map $f: S^m \to S^2$ that is one-to-one over the image of a dense set, thus giving an alternative proof of the result in [Be-Wa]. One
finds a defining sequence \( \{T_i : P_i \to J_i\}_{i=0}^{\infty} \) with the additional property:

(U) If \( p_i(1), p_i(2), p_i(3) \in P \) are mutually distinct, then \( p_i(1) \cap p_i(2) \cap p_i(3) = \emptyset \).

The argument is slightly easier than the one given in the case of monotone maps, since one does not have to worry about "connecting up" various components.

**Remark 4.4.** Maps \( h : S^m \to S^n \) constructed in 4.1 and 4.3, are totally unstable (i.e. they have no stable values). In fact \( h : S^m \to S^n \) can be approximated by a map \( h_1 : S^m \to S^n \) induced by the admissible function \( T_i : P_i \to J_i \), which, in turn, can be approximated by a map \( h_1' : S^m \to S^n \) induced by the triple \( T_i : \hat{P}_i \to \hat{J}_i \). If \( T_i : P_i \to J_i \) satisfies (M2), then \( \text{Im} h_1' \) is contained in the union of all elements \( j \) of \( J_i \) with \( \text{Ind}(j) \leq 2 \), and hence it is contained in a regular neighborhood of the 2-skeleton of \( \beta^2 L \). Consequently, \( h : S^m \to S^n \) constructed as in 4.1 can be approximated by maps that factor through 2-dimensional polyhedra.

**References**


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