FUNCTIONS OF UNIFORMLY BOUNDED CHARACTERISTIC ON RIEMANN SURFACES

BY

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ABSTRACT. A characteristic function $T(D, w, f)$ of Shimizu and Ahlfors type for a function $f$ meromorphic in a Riemann surface $R$ is defined, where $D$ is a regular subdomain of $R$ containing a reference point $w \in R$. Next we suppose that $R$ has the Green functions. Letting $T(w, f) = \lim_{R} T(D, w, f)$, we define $f$ to be of uniformly bounded characteristic on $R$, $f \in \text{UBC}(R)$ in notation, if $\sup_{w \in R} T(w, f) < \infty$. We shall propose, among other results, some criteria for $f$ to be in $\text{UBC}(R)$ in various terms, namely, Green's potentials, harmonic majorants, and counting functions. They reveal that $\text{UBC}(\Delta)$ for the unit disk $\Delta$ coincides precisely with that introduced in our former work. Many known facts on $\text{UBC}(\Delta)$ are extended to $\text{UBC}(R)$ by various methods. New proofs even for $R = \Delta$ are found. Some new facts, even for $\Delta$, are added.

0. Introduction. We shall extend the notion of UBC and UBC$_0$ from the unit disk $\Delta = \{ |z| < 1 \}$ (see [Y1 and Y2]) to hyperbolic Riemann surfaces, prove some results analogous to those in $\Delta$, and add some facts, new, even for $\Delta$. A hyperbolic Riemann surface $S$ is one possessing Green functions; thus, its universal covering surface $S^\infty$ must be conformally equivalent to $\Delta$, so that $S^\infty$ and $\Delta$ are identified.

Our study begins with how to define the Shimizu-Ahlfors characteristic function $T(D, w, f)$ on "good" subdomains $D$ containing a point $w$ of a Riemann surface $R$, hyperbolic or not, on which $f$ is meromorphic. Each point of $R$ is identified with its local-parametric image in the complex plane $C = \{ |z| < \infty \}$. By $D$ we always mean a relatively compact subdomain of $R$, whose boundary $\partial D$ consists of a finite number of mutually disjoint, analytic, simple and closed curves on $R$. If we refer to a pair $D$ and $w \in R$ we always assume that $w \in D$. The radius $r = r(D, w) > 0$ of $D$ is defined by

$$\log r = \lim (g_D(z, w) + \log |z - w|)$$

as $z \to w$ within the parametric disk of center $w$, where $g_D(z, w)$ is the Green function of $D$ with pole at $w$. We now set

$$T(D, w, f) = \pi^{-1} \int_0^r t^{-1} \left[ \int_{D_t} f^\#(z)^2 \, dx \, dy \right] \, dt,$$

where $D_t = \{ z \in D; g_D(z, w) \geq \log(r/t) \}$, $0 < t < r$, and

$$f^\#(z) = \begin{cases} \frac{|f'(z)|}{(1 + |f(z)|^2)}, & \text{if } f(z) \neq \infty, \\ (1/f)^\#(z), & \text{if } f(z) = \infty, \end{cases} \quad (0.1)$$

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is not a function on $R$, yet the second-order differential $f^*(z)^2 \, dx \, dy$, $z = x + iy \in R$, is well defined on $R$. The Green-potential expression

$$T(D, w, f) = \pi^{-1} \int \int_D f^*(z)^2 g_D(z, w) \, dx \, dy, \quad w \in R,$$

will be proved later. The nomenclature of $T$ is justified because for $R = \{|z| < \rho\}$, $D = \{|z| < r\}$ with $0 < r < \rho \leq \infty$, and $w = 0$, we have the usual one because $g_D(z, 0) = \log |r/z|$.

Henceforth we always assume that $R$ is hyperbolic and we set

$$T(w, f) = T(R, w, f) = \lim_{D \uparrow R} T(D, w, f) < \infty.$$

This means that given $\varepsilon > 0$ we can find a compact set $K$, $w \in K \subset R$, such that $|T - T(D)| < \varepsilon$ for all $D \supset K$, with the obvious change in case $T = \infty$. Lebesgue's convergence theorem applied to (0.2) yields

$$T(w, f) = \pi^{-1} \int \int_R f^*(z)^2 g(z, w) \, dx \, dy,$$

where $g = g_R$ is the Green function on $R$; (0.2) can be regarded as the case $R = D$.

A meromorphic $f$ on $R$ is said to be of uniformly bounded characteristic, $f \in \text{UBC} \equiv \text{UBC}(R)$ in notation, if the function $T(w, f)$ is bounded on $R$, while, $f \in \text{UBC}_0 \equiv \text{UBC}_0(R)$ if $\lim_{w \to \partial R} T(w, f) = 0$, that is, for $\varepsilon > 0$ there exists a compact $K \subset R$ such that $T(w, f) < \varepsilon$ in $R \setminus K$.

In §1 we extend our study from the family $M \equiv M(R)$ of meromorphic functions on $R$ to $M_e \equiv M_e(R)$ consisting of multiple-valued meromorphic functions with single-valued moduli on $R$. We can easily extend the definition of $\text{UBC}$ ($\text{UBC}_0$, respectively) for $f \in M$ to $\text{UBC}_e \equiv \text{UBC}_e(R)$ ($\text{UBC}_0 \equiv \text{UBC}_0(R)$, resp.) for $f \in M_e$.

In §2, (0.3) for $f \in M_e$ is proved. Thus, criteria are obtained in terms of the Green potentials (Corollary 2.2). The families $\text{BMOA}_e \equiv \text{BMOA}_e(R)$ and $\text{VMOA}_e \equiv \text{VMOA}_e(R)$ are defined for pole-free members of $M_e$; these are extensions of $\text{BMOA}$ and $\text{VMOA}$ in the disk. For the definition of $\text{BMOA}(R)$ see [M]; note that $\text{BMOA}(R) = M(R) \cap \text{BMOA}_e(R)$. The formulae $\text{BMOA}_e \subset \text{UBC}_e$ and $\text{VMOA}_e \subset \text{UBC}_0$ are now obvious. An expression of $T$ in terms of the limit $(D \uparrow R)$ of the mean of $\frac{1}{2} \log(1 + |f|^2)$ on $\partial D$ and the limit of

$$N(D, w, f) = \sum_{f(b) = \infty, b \in D} g_D(w, b)$$

will be of use to compare $T$ with L. Sario’s characteristic function $T_S$ (see [SN]). Sario’s class $M_e(B(R)$ coincides with that of $f \in M_e$ for which $T(w, f)$ is finite for a $w = w(f) \in R$.

§3 is devoted to the study of the least harmonic majorant $\varphi^\wedge$ of

$$\varphi = \frac{1}{2} \log(|f_1|^2 + |f_2|^2) \quad \text{for } f \in M_e,$$

where $|f| = |f_1|/|f_2|$, $f_1, f_2 \in M_e$, is an “admissible” decomposition. A new expression $T(w, f) = \varphi^\wedge(w) - \varphi(w)$, $w \in R$, is of use to obtain criteria for $f \in M_e$ to be of $\text{UBC}_e$ and of $\text{UBC}_0$ in terms of $\varphi^\wedge - \varphi$. Some remarks refer to strong parallels between $\text{BMOA}_e$ ($\text{VMOA}_e$, resp.) and $\text{UBC}_e$ ($\text{UBC}_0$, resp.).
In §4, criteria for $f \in M$ to be of UBC in terms of the supremum of the function $N(z) = \lim_{D \to R} N(D, w, 1/(f - z))$, $z \in C^* = C \cup \{\infty\}$, on the spherical circle of center $f(w)$ are obtained.

The projection $\pi: \Delta \to R$ is considered in §5, and the identity $T(R, \pi(\delta), f) = T(\Delta, \delta, f \circ \pi)$, $\delta \in \Delta$, for $f \in M(R)$ is proved. As applications we obtain: (1) If $f \in UBC(S)$ and $h: R \to S$ is an analytic map, then $f \circ h \in UBC(R)$. (2) If $h: R \to S$ is of type $Bl$ in the sense of M. Heins [H1], and if $f \circ h \in UBC(R)$, then $f \in UBC(S)$. Finally, a contribution is made to the classification of Riemann surfaces: $O_{UBC} \subsetneq O_{BMOA}$.

1. Families $UBC_e$ and $UBC_{e0}$. The functions on $R$, which we shall actually study, are, for the most part, the “generalized” meromorphic functions on $R$. Let $M_e = M_e(R)$ be the family of multiple-valued functions $f = \exp(u + iu^*)$ on $R$, where $u$ is a single-valued function harmonic on $R$ except for countably many logarithmic singularities $a_n$ clustering nowhere in $R$, such that $u(z) - k_n \log|z - a_n|$ is harmonic in the parameter disk of center $a_n$ with the integral coefficient $k_n$. The multiple-valuedness of $f$ arises from that of the conjugate function $u^*$ of $u$ on $R$. It is natural to regard the constant zero as a member of $M_e$.

The modulus $|f|$ of $f \in M_e$ is single-valued throughout $R$, and each branch of nonconstant $f$ in the parametric disk of each point $w \in R$ is single-valued there, and has the Laurent expansion

$$f(z) = c_\lambda(z - w)^\lambda + c_{\lambda+1}(z - w)^{\lambda+1} + \cdots,$$

where $\lambda$ is an integer with $c_\lambda \neq 0$; the branches differ by multiplicative constants of moduli one. Therefore $|c_\lambda|$ is definite and

$$|f(w)| \leq |c_\lambda| \text{ if } |f(w)| \neq \infty.$$

We call $w \in R$ a zero of $f$ of order $\lambda$ if $|f(w)| = 0$. Similarly, $w \in R$ is a pole (or, $\infty$-point) of $f$ of order $-\lambda$ if $|f(w)| = \infty$.

The family $M \equiv M(R)$ of single-valued members of $M_e$ consists of all the meromorphic functions on $R$. For $a \in C$ we call $w \in R$ an $a$-point of order $\lambda$ of $f \in M$ if $w$ is a zero of order $\lambda$ of $f - a$.

It is now easy to extend $T(D, w, f)$ and $T(w, f) \equiv T(R, w, f)$ to $f \in M_e$. Actually, $|f'(z)|$ for $|f(z)| \neq \infty$, as well as $f^\#(z)$ defined by (0.1), is definite, so that the differentials $|f'(z)|^2 dx dy$ for pole-free $f$ and $f^{\#}(z)^2 dx dy$ for arbitrary $f$ are defined on $R$. The definitions of $UBC_e \equiv UBC_e(R)$ and $UBC_{e0} \equiv UBC_{e0}(R)$ are thus clear; just extend those of $UBC \equiv UBC(R)$ and $UBC_0 \equiv UBC_0(R)$ in the introduction to $f \in M_e$.

2. The Shimizu-Ahlfors characteristic function. We begin with the Green-potential expression (0.3) for $f \in M_e$.

**Theorem 2.1.** The identity

$$T(w, f) = \pi^{-1} \int \int_R f^\#(z)^2 g(z, w)\, dx dy \quad (\leq \infty)$$

holds for each $f \in M_e(R)$ and each $w \in R$.

**Proof.** It suffices to establish

$$T(D, w, f) = \pi^{-1} \int \int_D f^\#(z)^2 g_D(z, w)\, dx dy$$
for each pair $D, w$. Set
\[ c_t(z) = \begin{cases} 1, & \text{if } z \in D_t, \\ 0, & \text{if } z \in D \setminus D_t. \end{cases} \]

Then the identity
\[ \int_0^r c_t(z)t^{-1} \, dt = g_D(z, w), \quad z \in D, \]

together with
\[ T(D, w, f) = \pi^{-1} \int_D \left( \int_0^r c_t(z)t^{-1} \, dt \right) f^\#(z)^2 \, dx \, dy \]

proves (2.2).

**REMARK.** Suppose that $w$ is in the parametric disk $U_{w'}$ of center $w' \in R$, and define $r' > 0$ by
\[ \log r' = \max(g_D(z, w) + \log|z - w|), \]

where, this time $z \to w$ within $U_{w'}$. The same proof as above then shows that
\[ T(D, w, f) = \pi^{-1} \int_0^{r'} t^{-1} \left( \int_{D'_t} f^\#(z)^2 \, dx \, dy \right) \, dt, \]

where $D'_t = \{ z \in D; g_D(z, w) \geq \log(r'/t) \}$, $0 < t < r'$.

Two corollaries follow from Theorem 2.1.

**COROLLARY 2.2.** For $f \in M_c(R)$ the following are valid.

(I) $f \in UBC_{c}(R)$ if and only if
\[ \sup_{w \in D} \int_R f^\#(z)^2 g(z, w) \, dx \, dy < \infty. \]

(II) $f \in UBC_{c0}(R)$ if and only if
\[ \lim_{w \to \partial R} \int_R f^\#(z)^2 g(z, w) \, dx \, dy = 0. \]

This corollary extends [Y1, Theorem 2.2, p. 352].

A pole-free $f \in M_c(R)$ is said to be of bounded (vanishing, resp.) mean oscillation on $R$, $f \in BMOA_c(R)$ ($f \in VMOA_c(R)$, resp.) in notation, if
\[ \sup_{w \in R} \int_R |f'(z)|^2 g(z, w) \, dx \, dy < \infty \]

\[ \left( \lim_{w \to \partial R} \int_R |f'(z)|^2 g(z, w) \, dx \, dy = 0, \text{ resp.} \right). \]

The family $BMOA(R) = BMOA_c(R) \cap M(R)$ is introduced by T. A. Metzger [M].

An immediate consequence of Corollary 2.2 is the following which extends [Y1, Theorem 7.1, p. 364].

**COROLLARY 2.3.** $BMOA_c(R) \subset UBC_{c}(R)$ and $VMOA_c(R) \subset UBC_{c0}(R)$.

Following Sario we shall define the proximity function $m_s(D, w, f)$, the counting function $N_s(D, w, f)$, and the characteristic function $T_s(D, w, f)$ for $f \in M_c$ and $w \in D$. They are extensions of M. Parreau's [P, p. 183 ff.] for $f \in M$. The reader is expected to be familiar with [SN, Chapter III] or with the papers [S1 and S2].
Let $\log^+ x = \max(\log x, 0)$ for $0 \leq x \leq \infty$, and set

$$m_\beta(D, w, f) = -(2\pi)^{-1} \int_{\partial D} \log^+ |f(z)| \, d\gamma_D(z, w),$$

where $\partial D$ is oriented positively with respect to $D$, and the star means the conjugate. The comparison

(2.5)  $m_\beta(D, w, f) \leq m(D, w, f) \leq \frac{1}{2} \log 2 + m_\beta(D, w, f)$

of $m_\beta(D, w, f)$ with

$$m(D, w, f) = -(4\pi)^{-1} \int_{\partial D} \log(1 + |f(z)|^2) \, d\gamma_D(z, w)$$

will be of use; (2.5) is a consequence of

(2.6)  $\log^+ x \leq \frac{1}{2} \log(1 + x^2) \leq \frac{1}{2} \log 2 + \log^+ x$  \  for  \  $0 \leq x \leq \infty$.

For $0 < t < r$, let $n(t, f)$ be the number of the poles of $f$, counted with the orders, in $D_t$. We define

$$N_\beta(D, w, f) = \int_0^r t^{-1} [n(t, f) - n(0, f)] \, dt + n(0, f) \log r,$$

where $n(0, f) = \lim_{t \to 0} n(t, f)$. Set $N(D, w, f) = \sum g_D(w, b)$, where the sum is extended over all the poles $b$ of $f$ in $D$, each counted with its order. If $f$ is pole-free in $D$, then $N = N_\beta = 0$. A routine procedure [SN, p. 76] yields that $N_\beta(D, w, f) = N(D, w, f)$ in case $|f(w)| \neq \infty$. Therefore

$$N(D, w, f) = \int_0^r t^{-1} n(t, f) \, dt$$

for all $w \in D$ because $N(D, w, f) = \infty$ if $|f(w)| = \infty$. The characteristic function for $f$ is now defined by

$$T_\beta(D, w, f) = m_\beta(D, w, f) + N_\beta(D, w, f).$$

For $X = m, N, m_\beta, N_\beta, \text{and } T_\beta$, we set $X(w, f) = \lim_{D \to R} X(D, w, f)$.

We shall compare $T_\beta(w, f)$ with $T(w, f)$ in Corollary 2.5 below.

**Theorem 2.4.** If $|f(w)| \neq \infty$ for $f \in M_\beta(R)$, then

(2.7)  $T(w, f) = m(w, f) + N(w, f) - \frac{1}{2} \log(1 + |f(w)|^2).$

If $|f(w)| = \infty$, then

(2.8)  $T(w, f) = m(w, f) + N_\beta(w, f) - \log |c_\lambda|,$

where $c_\lambda$ is defined in (1.1).

**Remark.** If $f \in M_\beta(R)$ is bounded, $|f| \leq K$, then $T(w, f) \leq \frac{1}{2} \log(1 + K^2)$ by (2.7), so that $f \in \text{UBC}_\beta(R)$.

**Corollary 2.5.** If $f \in M_\beta(R)$ is nonconstant, then

(2.9)  $|T(w, f) - T_\beta(w, f)| \leq \frac{1}{2} \log 2 + \log |c_\lambda|.$

Read (2.9), in the specified case, as $T(w, f) = \infty$ if and only if $T_\beta(w, f) = \infty$.

Fix $w \in R$ and let $BC_\beta \equiv BC_\beta(R)$ be the family of $f \in M_\beta$ such that $T_\beta(w, f) < \infty$. As will be observed later in Remark (a) after Theorem 3.1, $BC_\beta$ does not depend on $w$. Note that $BC_\beta(R) = M_\beta B(R)$ in [SN, p. 78]; we let $BC(R) = BC_\beta(R) \cap M(R)$. An immediate consequence is the following.
COROLLARY 2.6. UBC\(_{\varepsilon}(R) \subset BC_{\varepsilon}(R)\).

Hereafter, mainly in the proofs, we shall frequently use the following abbreviations:

\[(2.10)\quad X(f) = X(D, w, f) \quad \text{for} \quad X = m, N, T, mS, NS, TS.\]

**Proof of Theorem 2.4.** It suffices to prove

\[(2.7')\quad T(f) = m(f) + N(f) - \frac{1}{2} \log(1 + |f(w)|^2), \quad \text{if} \quad |f(w)| \neq \infty;\]

\[(2.8')\quad T(f) = m(f) + N_S(f) - \log |c_{\lambda}|, \quad \text{if} \quad |f(w)| = \infty.\]

Suppose first that no pole of \(f\) lies on \(\partial D\), and let \(b\) be all the distinct poles of \(f\) with orders \(k(b)\) in \(D\). For sufficiently small \(\varepsilon > 0\), we let \(\gamma_w = \{|z - w| \leq \varepsilon\}\) and \(\gamma_b = \{|z - b| \leq \varepsilon\}\). Apply the Green formula to the function \(\psi = (1/2) \log(1 + |f|^2)\) on the domain \(D_\varepsilon = D \setminus (\gamma_w \cup \gamma_b)\). Since

\[\Delta \psi = \left(\frac{3}{2}\partial_x^2 + \frac{3}{2}\partial_y^2\right)\psi = 2f^2 \quad \text{in} \quad D_\varepsilon,\]

it follows that

\[(2.11)\quad \int \int_{D_\varepsilon} g_D(z, w) \Delta \psi(z) \, dx \, dy = \int_{\partial D_\varepsilon} g_D(z, w) \frac{\partial \psi(z)}{\partial n} \, |dz| + \int_{\partial D_\varepsilon} \psi(z) \frac{\partial g_D(z, w)}{\partial n} \, |dz|,
\]

where the normal derivatives \(\partial / \partial n\) are considered in the direction of the inner normal. As to the first integral in the right-hand side of (2.11), that on \(\partial D\) is zero, and those on \(\partial \gamma_w\) and \(\partial \gamma_b\) tend to zero and \(2\pi k(b) g_D(b, w)\) as \(\varepsilon \to 0\), respectively. Furthermore, as to the second, that on \(\partial D\) equals \(2\pi m(f)\), and those on \(\partial \gamma_w\) and \(\partial \gamma_b\) tend to \(-2\pi \psi(w)\) and \(0\) as \(\varepsilon \to 0\), respectively. The resulting identity divided by \(2\pi\), together with \(g_D(b, w) = g_D(w, b)\), yields

\[(2.12)\quad \pi^{-1} \int \int_D g_D(z, w) f^#(z) \, dz \, dy = m(f) - \psi(w) + \sum_b k(b) g_D(w, b).\]

In view of (2.2) we immediately observe that (2.7') is true.

Suppose now that \(\partial D\) contains at least one pole of \(f\). For \(t, 0 < t < r\), sufficiently near \(r\), we obtain (2.7') for \(D_t \setminus \partial D_t\) instead of \(D\). Observing that \(T, m,\) and hence \(N\), all are continuous in \(t\), one obtains (2.7') for \(D\) by letting \(t \uparrow r\).

For the proof of (2.8') we quote Jensen’s formula

\[(2.13)\quad \log |c_{\lambda}| = T_S(f) - T_S(1/f),\]

valid for \(f\) without any assumption on \(|f(w)|\) (see [SN, (7), p. 77]).

Now, for \(f\) with \(|f(w)| = \infty\), we first note that (2.7') for \(1/f\) yields

\[(2.14)\quad N_S(1/f) = N(1/f) = T(1/f) - m(1/f) = T(f) - m(1/f).\]

On the other hand, (2.13) for the present \(f\), together with \(\log |f| = \log^+ |f| - \log^+ |1/f|\), yields

\[\log |c_{\lambda}| = -(2\pi)^{-1} \int_{\partial D} \log |f(z)| \, dg_D(z, w) + N_S(f) - N_S(1/f),\]
so that, by (2.14),

\[-(2\pi)^{-1} \int_{\partial D} \log |f(z)| \, dg_D(z,w) + N_S(f) - T(f) + m(1/f) \]

\[= N_S(f) - T(f) + m(f), \]

which completes the proof of (2.8').

**Proof of Corollary 2.5.** We again use (2.10). It suffices to prove

\[(2.15) |T(f) - T_S(f)| < |\log 2 + |\log |c_\lambda||,\]

which yields (2.9) on letting \(D \uparrow R\). In the case \(|f(w)| \neq \infty\), it follows from \(N_S(f) = N(f)\), the inequalities (2.5) and (2.6) for \(x = |f(w)|\), and (2.7'), that

\[(2.16) |T(f) - T_S(1/f)| \leq |T(1/f) - T_S(1/f)| \leq \frac{1}{2} \log 2.\]

As is observed in (1.2), \(|f(w)| \leq |c_\lambda|\), so that (2.15) now follows from (2.16). In the case \(|f(w)| = \infty\), we apply (2.16) to \(1/f\) instead of \(f\), to obtain

\[|T(f) - T_S(1/f)| = |T(1/f) - T_S(1/f)| \leq \frac{1}{2} \log 2.\]

It then follows from (2.13) that

\[|T(f) - T_S(f) + \log |c_\lambda|| \leq \frac{1}{2} \log 2,\]

whence (2.15).

3. **An admissible and the canonical decompositions.** For \(f \in M_e(R)\) we call

\[(3.1) |f| = |f_1/f_2| \quad \text{on } R\]

an admissible decomposition if \(f_k \in M_e\) is pole-free \((k = 1,2)\) and furthermore if both have no common zero, or,

\[(3.2) |f_1|^2 + |f_2|^2 > 0 \quad \text{on } R.\]

This decomposition is not unique.

**Lemma 3.0.** Each \(f \in M_e(R)\) has an admissible decomposition (3.1), where one of \(f_1\) and \(f_2\) is a member of \(M(R)\).

**Proof.** If \(f\) is pole-free, then set \(f_1 \equiv f\) and \(f_2 \equiv 1\). If \(f\) has one pole at least, then by H. Florack’s theorem [F, Satz II, pp. 3-4], there exists a pole-free \(f_2 \in M\) such that \(z \in R\) is a zero of order \(k\) of \(f_2\) if and only if \(z\) is a pole of order \(k\) of \(f\). We now obtain (3.1) on setting \(f_1 = ff_2\). We note that if \(f\) is zero-free, then set \(f_1 \equiv 1\) and \(f_2 = 1/f\). If \(f\) has one zero at least, then there exists a pole-free \(h_1 \in M\) such that \(z \in R\) is a zero of order \(k\) of \(h_1\) if and only if \(z\) is a zero of order \(k\) of \(f\). We then obtain another admissible decomposition \(|f| = |h_1/h_2|\) for \(h_2 = h_1/f\).

By a harmonic majorant of a subharmonic function \(v\) on \(R\) we mean a function \(u\) harmonic and \(v \leq u\) on \(R\). If \(v\) has a harmonic majorant on \(R\), then it has the least harmonic majorant \(v^\wedge = v^\wedge_R\) in the sense that \(v^\wedge\) is a harmonic majorant of \(v\) and \(v^\wedge \leq u\) for each harmonic majorant \(u\) of \(v\) on \(R\).

For an admissible decomposition (3.1) for \(f \in M_e(R)\) we denote

\[\varphi = \frac{1}{2} \log(|f_1|^2 + |f_2|^2) \quad \text{on } R.\]
This is a finite-valued subharmonic function on $R$ because (3.2) holds and

$$\Delta \varphi(z) \, dx \, dy = 2 f^\#(z)^2 \, dx \, dy \quad \text{on } R.$$  

Therefore, for $\varphi_1 = \frac{1}{2} \log(|F_1|^2 + |F_2|^2)$ for another admissible decomposition $|f| = |F_1/F_2|$, $\varphi - \varphi_1$ is a harmonic function on $R$; thus $\varphi$ is unique modulo harmonic functions on $R$.

**Theorem 3.1.** Suppose that $f \in M_c(R)$. Then $f \in BC_e(R)$ if and only if $\varphi$ has a harmonic majorant on $R$. If $f \in BC_e(R)$, then

$$T(w, f) = \varphi_R^+(w) - \varphi(w) \quad \text{for each } w \in R,$$

so that the function $\varphi_R^+ - \varphi$ is independent of a choice of decomposition (3.1).

**Remarks.** (a) It follows from Corollary 2.5, together with (3.5) that if $f \in BC_e(R)$, then $T_S(w, f) < \infty$ for every $w \in R$. Consequently, $BC_e$ does not depend on a choice of $w \in R$.

(b) If $f \in BC_e(R)$, then (3.5) shows that $T(z, f)$ is a $C^\infty$ function of real variables $x, y, z = x + iy$, together with

$$\Delta T(z, f) \, dx \, dy = -2 f^\#(z)^2 \, dx \, dy$$

by (3.4). This answers the question raised in [Y1, Remark (b), p. 361]. Note that

[Y1, Theorem 5.1, p. 360] is a specified case of Theorem 3.1 for pole-free $f \in M(\Delta)$.

As an immediate consequence we obtain

**Corollary 3.2.** Suppose that $f \in M_c(R)$. Then $f \in UBC_e(R)$ if and only if $\varphi$ has a harmonic majorant on $R$ and $\varphi_R^+ - \varphi$ is bounded there. Further, $f \in UBC_{e0}(R)$ if and only if $\varphi$ has a harmonic majorant on $R$ and

$$\lim_{w \to \partial R} [\varphi_R^+(w) - \varphi(w)] = 0.$$

The inclusion formula $UBC_{e0}(R) \subset UBC_e(R)$ is now obvious. The proof of [Y1, Lemma 2.1, p. 352] is thus facilitated. In fact, this is an immediate consequence of continuity of $T(w, f)$ observed in Remark (b) above.

For clarity we propose

**Corollary 3.3.** For $f \in M_c(R)$ to be in $UBC_e(R)$ it is necessary and sufficient that $\lim \sup_{w \to \partial R} T(w, f) < \infty$.

More precisely, there exists a compact set $K \subset R$ such that $T(w, f)$ is bounded in $R \setminus K$.

As the third corollary of Theorem 3.1 we claim that a compact set of capacity zero on $R$ is removable for functions of UBC. A set $E \subset R$ is said to be of capacity zero if the intersection of $E$ with the parameter disk at each point of $E$ is of capacity zero [T, p. 55] in $C$. If $E$ is closed further, then $E$ is totally disconnected, so that $R \setminus E$ is connected. The following is never obvious and needs a proof.

**Corollary 3.4.** Let $E$ be a compact set of capacity zero on $R$ and suppose that $f \in UBC(R \setminus E)$. Then there exists $F \in UBC(R)$ such that $F \equiv f$ in $R \setminus E$.

**Remark.** If $f \in UBC(R \setminus E)$ is pole-free, then, is $F$ pole-free? The answer is “no”. Just let $f(z) = 1/z$, $R = \Delta$, and $E = \{0\}$. 

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Proof of Theorem 3.1. Let \( w \) be the point in the definition of \( \text{BC}_e(R) \). We use the device (2.10). Since

\[
\varphi_D^\wedge(w) = -(2\pi)^{-1} \int_{\partial D} \varphi(z) d\varphi_D^\wedge(z, w),
\]

it follows from (2.2) and (3.4), together with the Green formula, that

\[
2\pi T(f) = \int_D \int_D (\Delta \varphi(z)) g_D(z, w) \, dx \, dy = -\int_{\partial D} \varphi(z) d\varphi_D^\wedge(z, w) - 2\pi \varphi(w),
\]

so that

\[
(3.6) \quad T(f) = \varphi_D^\wedge(w) - \varphi(w).
\]

Thus, \( \lim_{D \downarrow R} \varphi_D^\wedge(w) < \infty \) if and only if \( T_S(w, f) < \infty \), by (2.9). Now, if \( \varphi_R^\wedge \) exists, then (3.5) follows because the same calculation shows that (3.6) is true for each \( w \in R \).

Proof of Corollary 3.4. We may assume that \( f \) is nonconstant. First of all, \( \text{BC}(R) \) coincides with Parreau's class \((AM_0)\) in \( R \) [P, Definition 1, p. 180]; compare [P, Théorème 19, a), p. 181] with [SN, Theorem 3B, (33), p. 83]. By the case \( \alpha = 0 \) in [P, Théorème 20, p. 182] there exists \( F \in \text{BC}(R) \) \( (F = f_1 \) in Parreau's proof) with \( F \equiv f \) in \( R \setminus E \). An admissible decomposition \( F = f_1 / f_2 \) of \( F \) in \( R \) yields that of \( f \) in \( R \setminus E \) also. Thus, for \( \varphi \) of (3.3) for \( F \),

\[
\varphi_R^\wedge - \varphi = (\varphi_R^\wedge - \varphi_R^\wedge \setminus E) + (\varphi_R^\wedge \setminus E - \varphi) \quad \text{in} \ R \setminus E.
\]

Since \( \varphi_R^\wedge - \varphi \) is continuous on \( R \) and since the second term in the right-hand side is bounded in \( R \setminus E \), it suffices to show that \( \varphi_R^\wedge - \varphi_R^\wedge \setminus E \) is bounded in \( R \setminus E \). To prove this we remark that there exists a sequence of domains \( D = D_E \cup K_E \) in \( R \) such that (a) \( K_E \supset E \) is compact, (b) \( \partial K_E \) consists of piecewise analytic Jordan curves, (c) \( D_E = D \setminus K_E \uparrow R \setminus E \) as \( D \uparrow R \). Then,

\[
\varphi_D^\wedge - \varphi_D^\wedge_E = \begin{cases} 0 & \text{on } \partial D; \\ \varphi_D^\wedge - \varphi & \text{on } \partial K_E, \end{cases}
\]

so that

\[
\varphi_D^\wedge - \varphi_D^\wedge_E \leq \max_{K_E}(\varphi_R^\wedge - \varphi) \quad \text{in } D_E.
\]

Letting \( D \uparrow R \) we now obtain

\[
\varphi_R^\wedge - \varphi_R^\wedge \setminus E \leq \max_K(\varphi_R^\wedge - \varphi) \quad \text{in } R \setminus E,
\]

where \( K \) is a compact set whose interior contains \( E \).

Remarks. We pose here for some references to \( \text{BMOA}_e(R) \) and \( \text{VMOA}_e(R) \).

(a) For \( f \in M_e(R) \) pole-free, we have

\[
\Delta(|f(z)|^2) \, dx \, dy = 4|f'(z)|^2 \, dx \, dy, \quad z \in R.
\]

By the Green formula it is now easy to prove the following.

(a) \( f \) is of Hardy class \( H^2_0(R) \), that is, \(|f|^2 \) has a harmonic majorant on \( R \) if and only if for a point \( w \in R \),

\[
\int \int_R |f'(z)|^2 g(z, w) \, dx \, dy < \infty.
\]
(a) \( f \in \text{BMOA}_c(R) \) if and only if \( f \in H^2_e(R) \) and \((|f|^2)_R \) is bounded there, while, \( f \in \text{VMOA}_c(R) \) if and only if \( f \in H^2_e(R) \) and
\[
\lim_{w \to \partial R} \left( (|f|^2)_R(w) - |f(w)|^2 \right) = 0.
\]

The BMOA\(_c\) version of Corollary 3.3 is obvious.

(b) The analogue of Corollary 3.4 is valid.

If \( E \subset R \) is compact and of capacity zero, and if \( f \in \text{BMOA}(R \setminus E) \), then there exists \( F \in \text{BMOA}(R) \) with \( F \equiv f \) in \( R \setminus E \).

The proof is essentially the same as that of Corollary 3.4. The existence of a single-valued \( F \in H^2_e(R) \) with \( F \equiv f \) in \( R \setminus E \) follows from \( \alpha = 2 \) in [P, Théorème 20, p. 182].

Returning to \( f \in \text{BC}_e(R), f \neq 0 \), we now consider \( m_s(w, f) \) and \( N(w, f) \); see \( X(w, f) \) before Theorem 2.4. Then \( m_s(w, f) \) is a nonnegative harmonic function of \( w \) on \( R \) and \( N(w, f) = \sum g(w, b) \), the summation being extended over all the poles \( b \) of \( f \) in \( R \), each counted with its order, is harmonic in \( R \) minus the poles of \( f \). Set
\[
F(w, f) = \exp \left[-m_s(w, f) - im_s^*(w, f)\right],
\]
\[
\beta(w, f) = \exp \left[-N(w, f) - iN^*(w, f)\right]
\]
for \( w \in R \); both are bounded by one, and hence are in \( \text{UBC}_e(R) \) by the remark after Theorem 2.4. In particular, \( \beta \) is a Blaschke product if \( R = \Delta \).

The canonical decomposition, referred to in the title of the present section, is obtained by Sario [SN, Theorem 3B, p. 83 and Corollary 7, p. 86], which, for the clarity, we propose in the form of a lemma.

**Lemma 3.5.** For \( f \in \text{BC}_e(R), f \neq 0 \), we have
\[
\beta(w, f)f(w) \in \text{UBC}_e(R), \tag{3.7}
\]
\[
|f(w)| = \left| \frac{\beta(w, f)F(w, f)}{\beta(w, f)F(w, f)} \right| \text{ at each } w \in R.
\]

Note that \( 1/f \in \text{BC}_e(R) \). The following is an extension of [Y, Corollary 4.1, p. 359].

**Theorem 3.6.** If \( f \in \text{UBC}_e(R), f \neq 0 \), then \( F(w, 1/f)/F(w, f) \in \text{UBC}_e(R) \).

It is known that the converse is false for \( R = \Delta \); see [Y, p. 359]. Further it is known that \( \text{UBC}(\Delta) \) is not closed for multiplication and summation [Y, Theorem 4.2, p. 359].

**Proof of Theorem 3.6.** To prove first that
\[
\beta(w, f)f(w) \in \text{UBC}_e(R), \tag{3.8}
\]
we set
\[
\beta(w) = \beta(w, f), \quad h(w) = \beta(w)f(w), \quad w \in R.
\]
Then \( \beta \in \text{BC}_e \) and \( h \in \text{BC}_e \) is pole-free. First we consider those \( w \in R \) for which \( |f(w)| \neq \infty \). It follows from (2.7) of Theorem 2.4 that
\[
m(w, f) = T(w, f) + \log |\beta(w)| + \frac{1}{2} \log (1 + |f(w)|^2)
\]
\[
= T(w, f) + \frac{1}{2} \log (|\beta(w)|^2 + |h(w)|^2).
\]
Since
\[ m(w, h) \leq m(w, \beta) + m(w, f), \quad \text{and} \]
\[ A \equiv (|\beta(w)|^2 + |h(w)|^2)/(1 + |h(w)|^2) \leq 1, \]
it again follows from (2.7) that
\[ T(w, h) = m(w, h) - \frac{1}{2} \log(1 + |h(w)|^2) \]
\[ \leq m(w, \beta) + m(w, f) - \frac{1}{2} \log(1 + |h(w)|^2) \]
\[ \leq \frac{1}{2} \log 2 + T(w, f) + \frac{1}{2} \log A \]
\[ \leq \frac{1}{2} \log 2 + T(w, f). \]

Therefore,
\[ (3.9) \quad T(w, h) \leq \frac{1}{2} \log 2 + \sup_{\zeta \in \mathbb{R}} T(\zeta, f) \]
for \( w \in \mathbb{R} \) with \( |f(w)| \neq \infty \). Since \( T(w, h) \) is continuous on \( \mathbb{R} \), and since the poles of \( f \) are isolated, we observe that \( \sup T(w, h) \) for \( w \in \mathbb{R} \) does not exceed the right-hand side of (3.9), which completes the proof of (3.8).

Now, \( 1/h \in \text{UBC}_e(R) \) by (3.8). Apply (3.8) to \( 1/h \) instead of \( f \). Since \( |\beta(w, 1/f)| = |\beta(w, 1/h)| \), we conclude that \( F(w, f)/F(w, 1/f) \in \text{UBC}_e(R) \) because
\[ |F(w, f)/F(w, 1/f)| = |\beta(w, 1/h)(1/h(w))|. \]

Thus, \( F(w, 1/f)/F(w, f) \in \text{UBC}_e(R) \).

4. Counting function. For nonconstant \( f \in M(R), \ w \in D, \) and \( z \in \mathbb{C}^* \) we define
\[ N_s(D, w, z, f) = N_s(D, w, 1/(f - z)) \quad \text{and} \quad N(D, w, z, f) = N(D, w, 1/(f - z)). \]

Then
\[ N(D, w, z, f) = \sum g_D(w, \zeta), \]
where the sum is extended over all \( z \)-points \( \zeta \) of \( f \) in \( D \), each counted with its order. Further, set
\[ N_s(w, z, f) = \lim_{D \to R} N_s(D, w, z, f) \quad \text{and} \quad N(w, z, f) = \lim_{D \to R} N(D, w, z, f) = \sum g(w, \zeta), \]
where the sum is extended over all \( z \)-points \( \zeta \) of \( f \) in \( R \), each counted with its order. Note that \( N(w, z, f) = G(w, z, f) \) in [SN, p. 90]. Apparently, \( N(w, z, f) = \infty \) if \( f(w) = z \). Heins [H2] called \( f \in M(R) \) a Lindelöfian map from \( R \) into \( \mathbb{C}^* \), or \( f \) is Lindelöfian and meromorphic, if \( N(w, z, f) < \infty \) for each pair \( z \in \mathbb{C}^*, \ w \in R, \) with \( f(w) \neq z \). It is known that \( f \in \text{BC}(R) \) if and only if \( f \) is Lindelöfian and meromorphic [SN, Theorem 6E, p. 92].

As usual we consider the chordal distance
\[ \chi(a, b) = |a - b|/[(1 + |a|^2)(1 + |b|^2)]^{1/2}, \quad a, b \in \mathbb{C}^*, \]
with the obvious change for \( a = \infty \) or \( b = \infty \). The length of a curve on \( \mathbb{C}^* \) measured by \( dx(\zeta) = |d\zeta|/(1 + |\zeta|^2), \ \zeta \in \mathbb{C}^* \), is the same as its Euclidean length, considered as a curve on the Riemann sphere in the Euclidean space. Set
\[ \Gamma(a, \rho) = \{ z \in \mathbb{C}^*; \chi(z, a) = \rho \}, \quad a \in \mathbb{C}^*, \ 0 < \rho < 1, \]
and set, for \( w \in R, \ 0 < \rho < 1, \) and \( f \in M(R), \)
\[ C(w, \rho, f) = \sup_{z \in \Gamma(f(w), \rho)} N(w, z, f). \]

We begin with criteria for \( f \in M(R) \) to be of \( \text{BC}(R) \) in terms of \( C. \)
THEOREM 4.1. For $f \in M(R)$ the following are mutually equivalent.

(I) $f \in BC(R)$.

(II) There exists a pair $w, \rho$, with $w \in R$, $0 < \rho < 1$, such that $C(w, \rho, f) < \infty$.

(III) For each pair $w, \rho$ with $w \in R$, $0 < \rho < 1$, we have $C(w, \rho, f) < \infty$.

A weaker condition than (II) implies (I). Actually, if $N(w, z, f) < \infty$ for a set of $z \in C^*$ of positive capacity (see §5 for the definition of "capacity" of a set on $C^*$), then $f \in BC(R)$ by [P, Théorème 22, p. 190]; the set $\Gamma(f(w), \rho)$ is of positive capacity. A reason of proposing (II) is to compare it with (V) in

THEOREM 4.2. For $f \in M(R)$ the following are mutually equivalent.

(IV) $f \in UBC(R)$.

(V) There exists $\rho$, $0 < \rho < 1$, such that $\sup_{w \in R} C(w, \rho, f) < \infty$.

(VI) For each $\rho$, $0 < \rho < 1$, $\sup_{w \in R} C(w, \rho, f) < \infty$.

Postponing the proofs of Theorems 4.1 and 4.2 we first note that the length of $\Gamma(a, \rho)$ is independent of $a \in C^*$ and is

(4.1) \[ l(\rho) = 2\pi \rho(1 - \rho^2)^{1/2}, \quad 0 < \rho < 1; \]

for example, the length of the equator $\Gamma(0, 1/\sqrt{2}) = \Gamma(\infty, 1/\sqrt{2})$ is $l(1/\sqrt{2}) = \pi$. Set

(4.2) \[ c(\rho) = \max[\rho^{-1}(1 - \rho^2)^{1/2}, \rho(1 - \rho^2)^{-1/2}], \quad 0 < \rho < 1. \]

THEOREM 4.3. The following estimates hold for $f \in M(R)$, $0 < \rho < 1$, and $w \in R$.

(VII) $c(\rho)^2 T(w, f) \geq l(\rho)^{-1} \int_{\Gamma(f(w), \rho)} N(w, z, f) d\chi(z) - (1/2) \log 2$.

(VIII) $c(\rho)^{-2} T(w, f) \leq l(\rho)^{-1} \int_{\Gamma(f(w), \rho)} N(w, z, f) d\chi(z) + (1/2) \log 2$.

The estimates (VII) and (VIII) are motivated by the celebrated Cartan formula [SN, (56), p. 89]:

\[ T_S(D, w, f) = \log^+ |f(w)| + (2\pi)^{-1} \int_0^{2\pi} N_S(D, w, e^{it}, f) dt \]

for $f \in M(R)$, provided that $f(w) \neq \infty$. A merit of (VII) and (VIII) might be that the right-hand sides have no term like $\log^+ |f(w)|$; also no assumption on the value $f(w)$ is posed.

With the aid of (VIII) of Theorem 4.3, (II) $\Rightarrow$ (I) of Theorem 4.1 and (V) $\Rightarrow$ (IV) of Theorem 4.2 are immediately obtained.

PROOF OF THEOREM 4.3. We may suppose that $f$ is nonconstant. We shall use the following abbreviation like (2.10):

(4.3) \[ N_f(z) = N(D, w, z, f). \]

Set

(4.4) \[ \delta = \rho(1 - \rho^2)^{-1/2}, \]

so that $l(\rho) = 2\pi \delta(1 + \delta^2)^{-1}$ by (4.1) and $c(\rho) = \max(\delta, \delta^{-1})$ by (4.2). We consider $h \in M(R)$ defined by

(4.5) \[ h = (f - f(w))/(1 + |f(w)|) \quad (= 1/f \text{ if } f(w) = \infty). \]
First we observe that
\begin{equation}
(2\pi)^{-1} \int_0^{2\pi} N_h(\delta e^{it}) \, dt = l(\rho)^{-1} \int_{\Gamma(f(w),\rho)} N_f(z) \, d\chi(z).
\end{equation}
For the proof we note that the Möbius transformation
\[\varsigma = (z - f(w))/(1 + f(w)z)\]
maps the circle \(\Gamma(f(w),\rho)\) one-to-one onto the circle \(\{|\varsigma| = \delta\}\) and further,
\[|d\varsigma| = (1 + \delta^2) \, d\chi(z) \quad \text{for } z \in \Gamma(f(w),\rho).
\]
Now, the left-hand side of (4.6), denoted by \(A\), is expressed as
\[A = (2\pi\delta)^{-1} \int_{|\varsigma| = \delta} N_h(\varsigma)|d\varsigma|,
\]
which yields (4.6).

Next we claim that
\begin{equation}
|T(h/\delta) - A| \leq \frac{1}{2} \log 2.
\end{equation}
Since \(h(w) = 0\), it follows from (2.16) for \(h/\delta\) instead of \(f\) that
\begin{equation}
|T(h/\delta) - T_S(h/\delta)| \leq \frac{1}{2} \log 2.
\end{equation}
We now remember the identity
\begin{equation}
(2\pi)^{-1} \int_0^{2\pi} \log |b - \delta e^{it}| \, dt = \log \delta + \log^+ |b/\delta|
\end{equation}
for each \(b \in \mathbb{C}\); see [N, p. 178] for the calculation. Jensen’s formula [SN, (2), p. 76], applied to \(h - \delta e^{it}\) with \(h(w) - \delta e^{it} = \delta e^{it}\), yields
\begin{equation}
\log \delta = -(2\pi)^{-1} \int_{\partial D} \log |h(z) - \delta e^{it}| \, dg_D^*(z, w) + N(h) - N_h(\delta e^{it})
\end{equation}
for each \(t \in [0,2\pi)\). Calculating the integral means of both sides of (4.10) with respect to \(dt\) in \([0,2\pi)\), and observing (4.9), together with \(N(h) = N(h/\delta)\), we now obtain
\[\log \delta = \log \delta + m_S(h/\delta) + N(h/\delta) - A,
\]
whence \(T_S(h/\delta) = A\). The estimate (4.7) follows then from (4.8).

Finally for \(c = c(\rho)\), some computations yield \(c^{-1}h^\# \leq (h/\delta)^\# \leq ch^\#\). Since \(T(h) = T(f)\), it follows that \(c^{-2}T(f) \leq T(h/\delta) \leq c^2T(f)\).

The estimates (VII) and (VIII) for \(R = D\) now follow from the above, together with (4.6) and (4.7). Letting \(D \uparrow R\) we arrive at the requested conclusions.

For the remaining proofs of Theorems 4.1 and 4.2 we prove

**Lemma 4.4.** Let \(f \in M(R)\), \(0 < \rho < 1\), \(w \in R\), and \(1 < q < \infty\). Then
\begin{equation}
C(w, \rho, f) \leq q c(\rho)^2 T(w, f) + (q/2) \log 2 + \log[(q + 1)/(q - 1)].
\end{equation}
On setting \(q = 2\), say, we see that (I)⇒(III) of Theorem 4.1 and (IV)⇒(VI) of Theorem 4.2 follow from (IX). Since (III)⇒(II) and (VI)⇒(V) are trivial, this completes the proofs of Theorems 4.1 and 4.2.
PROOF OF LEMMA 4.4. Fix $D$, $w \in D$. As is noted by O. Lehto [L] the function $N_f(z)$ (see (4.3)) of $z \in \mathbb{C}^*$ is subharmonic in $\mathbb{C}^* \setminus \{f(w)\}$ and $N_f$ has the logarithmic singularity at $f(w)$ in the sense that
\[ N_f(z) + \log |z - f(w)| \quad (N_f(z) - \log |z| \text{ if } f(w) = \infty) \]
is subharmonic in $\mathbb{C}^* \setminus \{-1/f(w)\}$. Thus,
\begin{equation}
A' = \sup_{\chi(z, f(w)) \geq \rho} N_f(z) = \sup_{z \in \Gamma(f(w), \rho)} N_f(z).
\end{equation}
Our task is therefore to show that
\begin{equation}
A' \leq q\rho^2 T(f) + (q/2) \log 2 + \log [(q + 1)/(q - 1)].
\end{equation}
Since $T(w, f) \geq T(f)$ and since $N_f \uparrow N(w, f)$ as $D \uparrow R$, (IX) follows from (4.12).

We consider again $h$ of (4.5) for which $h(w) = 0$. Then $u(z) \equiv N_h(1/z) = N_1/h(z)$ is subharmonic in $\mathbb{C}$ and $u(z) - \log |z|$ is subharmonic in $\mathbb{C}^* \setminus \{0\}$. By (4.6) and (VII) of Theorem 4.3 we obtain
\begin{equation}
(2\pi)^{-1} \int_0^{2\pi} u(\delta^{-1}e^{it}) dt = (2\pi)^{-1} \int_0^{2\pi} u(\delta^{-1}e^{-it}) dt
\end{equation}
\begin{equation}
= (2\pi)^{-1} \int_0^{2\pi} N_h(\delta e^{it}) dt \leq c(\rho^2 T(f) + \frac{1}{2} \log 2 \equiv \alpha),
\end{equation}
where $\delta$ is defined in (4.4).

Our aim is to prove that
\begin{equation}
|z| \leq \delta^{-1} \Rightarrow u(z) \leq q\alpha + \log \lambda, \quad \lambda = (q + 1)/(q - 1).
\end{equation}
Then, the identities
\[ A' = \sup_{\delta \leq |z| \leq \infty} N_h(z) = \sup_{|z| \leq \delta^{-1}} u(z), \]
together with (4.14), show (4.12).

Let $u_\delta^\wedge$ be the least harmonic majorant of $u$ in the disk $|z| < \delta^{-1}$, which can be expressed by the Poisson formula,
\[ u_\delta^\wedge(z) = (2\pi)^{-1} \int_0^{2\pi} \frac{\delta^{-2} - |z|^2}{|\delta^{-1}e^{it} - z|^2} u(\delta^{-1}e^{it}) dt, \quad |z| < \delta^{-1}. \]
Set $\Lambda = \lambda^{-1}\delta^{-1}$. Then, on the circle $|z| = \Lambda$ we have the estimate
\[ u_\delta^\wedge(z) \leq \frac{\Lambda^{-1} + \delta}{\Lambda^{-1} - \delta} (2\pi)^{-1} \int_0^{2\pi} u(\delta^{-1}e^{it}) dt, \]
whence, by (4.13), we obtain
\begin{equation}
\max_{|z| = \Lambda} u_\delta^\wedge(z) \leq q\alpha.
\end{equation}
The maximum principle now yields
\begin{equation}
|z| \leq \Lambda \Rightarrow u(z) \leq u_\delta^\wedge(z) \leq q\alpha.
\end{equation}
On the other hand, by the maximum principle for $u(z) - \log |z|$, with (4.15), we have
\[ \Lambda < |z| < \infty \Rightarrow u(z) - \log |z| \leq \sup_{|z| = \Lambda} (u(\zeta) - \log |\zeta|) \leq q\alpha - \log \Lambda. \]
Therefore,
\[ \Lambda < |z| \leq \delta^{-1} \Rightarrow u(z) \leq q\alpha - \log \Lambda + \log \delta^{-1} \]
(4.17)
\[ = q\alpha + \log \lambda. \]
The estimate (4.14) now follows from (4.16) and (4.17).

5. Uniformization. For a projection map \( \pi: \Delta \to R \), there exists the group \( \mathcal{G} \) of conformal homeomorphisms from \( \Delta \) onto \( \Delta \), called the covering transformation group, such that \( \pi \circ \gamma = \pi \) for each \( \gamma \in \mathcal{G} \). Thus, for \( f \in M(R) \), the function \( f \circ \pi \) is \( \mathcal{G} \)-automorphic in the sense that \( (f \circ \pi) \circ \gamma = f \circ \pi \) for all \( \gamma \in \mathcal{G} \). We begin with

**Theorem 5.1.** For each \( f \in M(R) \) and each \( \delta \in \Delta \), we have
\[ T(R, \pi(\delta), f) = T(\Delta, \delta, f \circ \pi). \]
(5.1)

Set
\[ \|f\|_{UBC(R)} = \sup_{w \in R} T(R, w, f) \]
for \( f \in M(R) \). Besides the obvious consequence of Theorem 5.1 that \( f \in BC(R) \) if and only if \( f \circ \pi \in BC(\Delta) \), we have

**Corollary 5.2.** For \( f \in M(R) \), the equality
\[ \|f\|_{UBC(R)} = \|f \circ \pi\|_{UBC(\Delta)} \]
holds. Thus, \( f \in UBC(R) \) if and only if \( f \circ \pi \in UBC(\Delta) \).

A subset \( A \) of \( C^* \) is said to be of positive capacity if \( A \) contains a closed subset of \( C^* \) of positive elliptic capacity [T, p. 90], or equivalently, \( A \setminus \{\infty\} \) contains a bounded (in \( C \)) and closed set of positive capacity [T, p. 55].

**Corollary 5.3.** Suppose that for \( f \in M(R) \), the exceptional set \( C^* \setminus f(R) \) is of positive capacity. Then \( f \in UBC(R) \).

Applying the known theorem [Y2, Theorem 1] to \( f \circ \pi \) we observe that \( f \circ \pi \in UBC(\Delta) \), whence \( f \in UBC(R) \).

**Corollary 5.4.** Suppose that a subdomain \( G \) of \( C^* \) has the projection \( \pi: \Delta \to G \). Then, \( \pi \in UBC(\Delta) \) if and only if \( C^* \setminus G \) is of positive capacity.

The “only if” part is a consequence of [N, Satz 1, p. 213] because \( \pi \in BC(\Delta) \). The “if” part is a consequence of Corollary 5.3.

**Proof of Theorem 5.1.** We note that, by the construction of \( f_1 \) and \( f_2 \) in Lemma 3.0, both are members of \( M \). Thus,
\[ f \circ \pi = F_1/F_2, \quad F_k = f_k \circ \pi, \quad k = 1, 2, \]
yields a canonical decomposition of \( f \circ \pi \) in \( \Delta \). For \( \varphi \) of (3.3) we set
\[ \Phi = \varphi \circ \pi = \frac{1}{2} \log(|F_1|^2 + |F_2|^2). \]
(5.3)
We shall show that
\[ \Phi^\Delta - \Phi = (\varphi_R^\Delta - \varphi) \circ \pi; \]
(5.4)
that \( \Phi^\Delta \) exists if and only if \( \varphi_R^\Delta \) exists is clear in the following context. The identity (5.1) now follows from (3.5) with (5.4).
For the proof of (5.4) it suffices to observe that $\Phi^\Delta = \Phi^R \circ \pi$. Since $\varphi \leq \varphi^R$, it follows that $\Phi \leq \varphi^R \circ \pi$, whence $\Phi^\Delta \leq \varphi^R \circ \pi$. To prove the inverse we remark that $\Phi^\Delta$ is $g$-automorphic by (5.3). Then the function $\psi$ on $R$ well-defined by

$$
\psi(z) = \Phi^\Delta(\zeta) \quad \text{for } z \in R \text{ with } \pi(\zeta) = z, \ \zeta \in \Delta,
$$

is harmonic on $R$. Therefore, it follows from $\Phi^\Delta \geq \Phi = \varphi \circ \pi$ that $\psi \geq \varphi$, whence $\psi \geq \varphi^R$, so that $\Phi^\Delta \geq \varphi^R \circ \pi$ in $\Delta$.

We now extend [Y2, Theorem 3]. Let $S$ be a hyperbolic Riemann surface with the Green functions $g_s(z, w)$. An analytic mapping $h: \Delta \rightarrow S$ is said to be of type $Bl$ (see [H1]) if for each $w \in S$, the function $g_s(h(z), w) - \sum g_{R}(z, \zeta)$ of $z \in R$ is singular, that is, it does not dominate any strictly positive and bounded harmonic function on $R$, where the summation is taken over all roots $\zeta \in R$ of the equation $h = w$, counted with their multiplicities.

**Theorem 5.5.** The following hold for $f \in M(S)$.

(I) If $f \in UBC(S)$, then for each analytic map $h: \Delta \rightarrow S$, we have

$$
\|f \circ h\|_{UBC(R)} \leq \|f\|_{UBC(S)},
$$

so that $f \circ h \in UBC(R)$.

(II) If $h: \Delta \rightarrow S$ is of type $Bl$, and if $f \circ h \in UBC(R)$, then

$$
\|f \circ h\|_{UBC(R)} = \|f\|_{UBC(S)},
$$

so that $f \in UBC(S)$.

**Proof.** Let $\pi_R: \Delta \rightarrow R$ and $\pi_S: \Delta \rightarrow S$ be the projection maps. Then a branch of $\pi_S^{-1} \circ h \circ \pi_R$ is single-valued in $\Delta$, which we denote simply, $H = \pi_S^{-1} \circ h \circ \pi_R$. Set $F = f \circ \pi_S$, so that $f \circ h \circ \pi_R = F \circ H$. If $f \in UBC(S)$, then $F \in UBC(\Delta)$ with

$$
\|F\|_{UBC(\Delta)} = \|f\|_{UBC(S)}
$$

by Corollary 5.2. On the other hand, it follows from [Y2, Theorem 3, (I)] that

$$
\|F \circ H\|_{UBC(\Delta)} \leq \|F\|_{UBC(\Delta)}.
$$

Corollary 5.2 again shows that

$$
\|F \circ H\|_{UBC(\Delta)} = \|F\|_{UBC(\Delta)},
$$

so that (5.5) follows from (5.7), (5.8), and (5.9).

Now, if $h$ is of type $Bl$, then $H$ is of type $Bl$ by [H1, Corollary, p. 472] because $S$ is hyperbolic. Thus, if $f \circ h \in UBC(R)$, then by (5.9), $F \circ H \in UBC(\Delta)$, so that, by [Y2, Theorem 3, (II)] we have $\|F \circ H\|_{UBC(\Delta)} = \|F\|_{UBC(\Delta)}$, which, together with (5.7) and (5.9), yields (5.6).

**Remarks.** (a) Is it true that $f \in UBC_0(R)$ if and only if $f \circ \pi \in UBC_0(\Delta)$? This is open, as far as the author knows. The similar question for VMOA is also open.

(b) Let $N(R)$ denote the family of normal meromorphic functions on $R$; namely, $f \in N(R)$ if and only if there is a constant $c > 0$ such that $f^\#(z) \leq cR(z)$, $z \in R$, where $\sigma_R(z) \, |dz|$ is the hyperbolic metric on $R$; note that $\sigma_\Delta(z) = (1 - |2|^2)^{-1}$. It is easy to observe that $f \in N(R)$ if and only if $f \circ \pi \in N(\Delta)$. It then follows from the known inclusion formula $UBC(\Delta) \subset N(\Delta)$ [Y1, Theorem 3.1] that $UBC(R) \subset N(R)$.
(c) By the remarks after the proof of Corollary 3.4, we can prove the analogue of (5.1) for pole-free $f \in M(R)$ by the similar method. Namely,

$$\int_{R} \int_{\Delta} |f'(z)|^2 g(z, \pi(\delta)) \, dx \, dy = \int_{\Delta} |(f \circ \pi)'(z)|^2 g_{\Delta}(z, \delta) \, dx \, dy.$$ 

Metzger [M] obtained this by analyzing the Green functions.

(d) Define the BMOA norm for pole-free $f \in M(R)$ by

$$||f||_{BMOA(R)} = \sup_{w \in R} \int_{R} |f'(z)|^2 g(z, w) \, dx \, dy.$$ 

The BMOA analogue of (5.2) is then $||f||_{BMOA(R)} = ||f \circ \pi||_{BMOA(\Delta)}$.

(e) A BMOA analogue of Theorem 5.5 is valid. Instead of [Y2, Theorem 3] we use the corresponding one found by K. Stephenson [St, Theorem 3, p. 572, in particular]. The results are:

If $f \in BMOA(S)$, then

$$||f \circ h||_{BMOA(R)} \leq ||f||_{BMOA(S)}$$

for each analytic map $h$: $R \to S$.

If $h$: $R \to S$ is of type Bl, and if $f \circ h \in BMOA(R)$, then $||f \circ h||_{BMOA(R)} = ||f||_{BMOA(S)}$, so that $f \in BMOA(S)$.

(f) Metzger [M] introduced the family $BMOA$ of Riemann surfaces on which $BMOA$ consists only of constants. By an obvious reason we include Riemann surfaces of $G$ in $BMOA$. Let $UBC$ be the family of Riemann surfaces which are either of $G$ or admit no nonconstant UBC functions. We shall prove the strict inclusion formula

$$UBC \subset BMOA.$$ 

Our work should be finding $R \in BMOA \setminus UBC$. Let $E$ be a compact set of linear measure zero, yet of positive capacity lying on the real axis. Then $R = C^* \setminus E$ is the desired. Since the function $z$ is of $UBC(R)$ by Corollary 5.3, it follows that $R \notin UBC$. On the other hand, each $f \in BMOA(R)$ is of class $H^2(R)$, that is, $|f|^2$ has a harmonic majorant on $R$. It is familiar (see, for example, [Y3, p. 334]) that $f$ can be extended holomorphically to $C^*$, so that, $f$ must be a constant. Thus $R \in BMOA$.

(g) Let $UBCA(\Delta)$ be the set of all pole-free members of $UBC(\Delta)$. It is apparent that $BMOA(\Delta) \subset UBCA(\Delta)$. On observing [B, Corollary 2, p. 15], one might suspect that $BMOA(\Delta) = UBCA(\Delta)$. This is not the case. Let

$$f(z) = \frac{(1 + z)}{(1 - z)}.$$ 

By Corollary 5.3, $f \in UBCA(\Delta)$, yet $f \notin BMOA(\Delta)$ because $f$ is not Bloch in the sense that $(1 - |z|^2)|f'(z)|$ is unbounded in $\Delta$.

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