ON THE STRUCTURE OF ABELIAN p-GROUPS

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ABSTRACT. A new kind of abelian p-group, called an A-group, is introduced. This class contains the totally projective groups and Warfield's S-groups as special cases. It also contains the N-groups recently classified by the author. These more general groups are classified by cardinal (numerical) invariants which include, but are not limited to, the Ulm-Kaplansky invariants. Thus the existing theory, as well as the classification, of certain abelian p-groups is once again generalized.

Having classified A-groups (by means of a uniqueness and corresponding existence theorem) we can successfully study their structure and special properties. Such a study is initiated in the last section of the paper.

1. Introduction. In this paper we determine the structure of a class of abelian p-groups more general than totally projective groups, S-groups, and N-groups, all of which have been classified earlier [5, 18, 6]. To introduce this new class of groups, we begin with the following definition. It is to be understood throughout that all groups are abelian.

DEFINITION. If μ is a limit ordinal, the class A_μ consists of those p-groups H for which there is a containing totally projective p-group G of length not exceeding μ that satisfies the following conditions.

(a) H is isotype in G.

(b) $p^\lambda(G/H) = \langle p^\lambda G, H \rangle/H$ whenever $\lambda < \mu$.

(c) G/H is the direct sum of a totally projective group and a divisible group.

The members of the class A_μ are called μ-elementary A-groups. An A-group is a direct sum of μ-elementary A-groups for various limit ordinals μ. If G is a totally projective p-group of length μ, the pair (H, G) is called an A_μ-pair whenever conditions (a)–(c) are satisfied. Thus, if H is an A-group, then $H = \Sigma \oplus H_i$, where (H_i, G_i) is an $A_{\mu(i)}$-pair for distinct limit limit ordinals $\mu(i)$.

Incidentally, if the terminology needs an explanation (and we hope not), one can interpret the letter "A" used here as an abbreviation for the word "accessible", which will prove to be a rather descriptive term. Certainly we follow tradition in using a single letter to represent this class of groups since their predecessors include T-groups, S-groups, and N-groups.

We are able to show that the structure of an A-group is completely determined by certain cardinal invariants including, but not limited to, the Ulm-Kaplansky invariants. The complete collection of invariants are called A-invariants (primarily because they suffice for A-groups). Observe that if we choose $H = G$ to form the A_μ-pair (H, G), we demonstrate that any totally projective group G is an A-group. Moreover, it follows from [18] that any S-group is an A-group and corresponds to the special case where G_i/H_i is divisible, for each i, in the representation $H = \Sigma \oplus H_i$, where (H_i, G_i) is an $A_{\mu(i)}$-pair (whereas $G_i/H_i = 0$ corresponds to a totally projective). But the new class also contains groups that are not S-groups. In particular, the N-group constructed by Megibben in [11] (and referred to by Warfield [18, p. 166] as an example of an isotype subgroup of a totally projective group that is not an S-group) is an A-group. A general existence theorem proved herein demonstrates the existence of many other A-groups.

Warfield's Problem 4 in [18] is the following. Is there a reasonable structure theory for a larger class of groups which are isotype subgroups of totally projective groups? The results of this paper establish a clear answer in the affirmative. Incidentally, Warfield once indicated that he thought the answer to this question was "no". Although this paper generalizes the main results of [18], our approach is somewhat different from that of Warfield. In particular, we do not require cotorsion completions. Otherwise, the techniques employed here in generalizing the theory of S-groups to a larger class of p-groups are similar to those of [18]. In fact, both are based on my original classification paper [5].

The reader who is familiar with Warfield's paper [18] knows the historical development of the classification of abelian p-groups, at least up through the classification of S-groups. Major steps in this development include: (1) countable groups (Ulm, 1933 [16]), (2) direct sums of countable groups (Kolettis, 1960 [10]), (3) totally projective groups (Hill, 1967 [5]), and (4) S-groups (Warfield, 1975 [18]). Other significant contributions to the classification problem include Zippin [20], Nunke [13], Hill and Megibben [8], Parker and Walker [15], and Crawley and Hales [1]; we do not claim to be exhaustive, but our intent is only to provide a reasonable bibliography. Moreover, our interest here is restricted to p-groups. For the classification of (certain classes of) mixed groups, see [14, 17, 19], or other sources.

Finally, we mention that the tendency upon reaching each of the milestones referenced above has been at first to perceive the situation as being as far as one can go. This is evidenced, for example, by the fact that a quarter of a century elapsed between Ulm's result for countable groups and Kolettis' extension to direct sums of such groups. Also, recall that Warfield initially thought that Problem 4 had a negative answer. My opinion is that no time soon will all p-groups be satisfactorily classified with numerical invariants, but it is inevitable that larger and larger classes will be reached with and encompassed in such a classification theory.

2. The reduction to cofinality greater than ω_0.

THEOREM 1. *Let μ denote an arbitrary limit ordinal. The class A_μ consists exclusively of totally projective groups if and only if μ is cofinal with ω_0.*

PROOF. If μ is not cofinal with ω_0, it is well known that there exists totally projective p-groups G of length μ with proper subgroups H that are isotype and dense in the sense that $\langle p^\lambda G, H \rangle = G$ whenever $\lambda < \mu$. Moreover, it is known that any such subgroup H cannot be totally projective. However, such an H belongs to the class A_μ. In other words, there is an S-group belonging to A_μ that is not totally projective [18].

It remains to show that if $\text{cof}(\mu) = \omega_0$ then H is totally projective for any A_μ-pair (H, G). In order to prove that such an H is totally projective, let $\mathscr{C}$ be a collection of nice subgroups of G that satisfy the third axiom of countability. Likewise let $\overline{\mathscr{C}}$ be a collection of nice subgroups of $(G/H)p^\mu(G/H)$ that satisfy the third axiom of countability. If N is a subgroup of G, let $\overline{N}$ denote the image of N under the natural map $G \twoheadrightarrow (G/H)/p^\mu(G/H)$.

As in earlier papers, we say that two subgroups A and B of G are compatible, in which case we write $A\|B$, if for every pair $(a, b) \in A \times B$ there exists c in $A \cap B$ such that $h_G(a + c) \geq h_G(a + b)$. We want to consider those subgroups N of G that satisfy the following conditions:

   (i) $N \in \mathscr{C}$.

   (ii) $\overline{N} \in \overline{\mathscr{C}}$.

   (iii) $N\|H$.

Claim. If N satisfies (i)–(iii) and C is any countable subgroup of G, there exists a countable subgroup K of G such that $\langle N, K \rangle \supseteq \langle N, C \rangle$ and such that the countable extension $N' = \langle N, K \rangle$ of N also satisfies conditions (i)–(iii).

For the moment, assume that the claim is valid. Since $\mathscr{C}$ and $\overline{\mathscr{C}}$ both satisfy the third axiom of countability and (iii) is an inductive property, we conclude from this that there is an ascending chain

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \cdots \subseteq G \qquad (\alpha < \sigma)$$

of subgroups N_α of G satisfying conditions (i)–(iii) with the property that $G = \bigcup_{\alpha < \sigma} N_\alpha$, $N_\beta = \bigcup_{\alpha < \beta} N_\alpha$ if β is a limit less than σ, and $N_{\alpha+1}/N_\alpha$ is countable for each α. It follows quickly that if we let $M_\alpha = N_\alpha \cap H$, then

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\alpha \subseteq \cdots \qquad (\alpha < \sigma)$$

is a chain of nice subgroups of H. Thus H satisfies the third axiom of countability [4], and the theorem is proved if we can validate the claim.

In order to verify the claim, it is enough to show only that there exists a countable group $B \supseteq C$ such that for each triple $(n, c, h) \in N \times C \times H$ there exists $x \in \langle N, B \rangle \cap H$ for which $h_G(n + c + x) \geq h_G(n + c + h)$. For if we can establish the above, then upon setting $C_0 = C$ and $C_1 = B$ and inductively replacing C_i by C_{i+1}, we can obtain an ascending sequence of countable subgroups $C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n \subseteq \cdots$ with the property that if $(n, c_n, h) \in N \times C_n \times H$ there exists $x \in \langle N, C_{n+1} \rangle \cap H$ such that $h_G(n + c_n + x) \geq h_G(n + c_n + h)$. If we let $K = \bigcup C_n$, then for each $(n, k, h) \in N \times K \times H$ there exists $x \in \langle N, K \rangle \cap H$ such that $h_G(n + k + x) \geq h_G(n + k + h)$, which shows that $\langle N, K \rangle\|H$. Since the subgroups $C_n, n \geq 1$, can be chosen so that $C_n \in \mathscr{C}$ and $\overline{C}_n \in \overline{\mathscr{C}}$, we can construct $K = \bigcup C_n$ so that $K \in \mathscr{C}$ and

$\overline{K} \in \overline{\mathscr{C}}$. Consequently, $\langle N, K \rangle \in \mathscr{C}$ and $\langle \overline{N}, \overline{K} \rangle \in \overline{\mathscr{C}}$. Therefore, $N' = \langle N, K \rangle$ satisfies (i)–(iii). Thus the claim will be validated (and the theorem proved) if we can merely show that for each countable subgroup C there is a countable subgroup $B \supseteq C$ such that, for each triple $(n, c, h) \in N \times C \times H$, there exists $x \in \langle N, B \rangle \cap H$ such that $h_G(n + c + x) \geqslant h_G(n + c + h)$. We remark that it would be easy to produce such a subgroup B if N were also countable, so the uncountability of N is what makes the problem nontrivial. Nevertheless, we shall produce such a countable subgroup B. List the elements of C (using the positive integers for indices) as $c_1, c_2, \ldots, c_i, \ldots$. First, we deal with each c_i individually. To each c_i we associate at most a countable number of elements $h_{i,k}$ (or h_i in case only a single element is required) in H as follows:

Case 1. $\bar{c}_i = c_i + H + p^\mu(G/H) \in \overline{N}$. Choose $m_i \in N$ so that

$$c_i + H + p^\mu(G/H) = m_i + H + p^\mu(G/H).$$

Since μ is cofinal with ω_0, there exists an ascending sequence $\mu(k)$ of ordinals less than μ with $\sup\{\mu(k)\} = \mu$. Since $p^{\mu(k)}(G/H) = \langle p^{\mu(k)}G, H \rangle/H$ for each $k \geqslant 1$, there exists $h_{i,k} \in H$ such that

$$m_i - c_i + h_{i,k} \in p^{\mu(k)}G.$$

The countable collection of elements $h_{i,k} \in H$ are associated with c_i.

Case 2. $\bar{c}_j \notin \overline{N}$. Since $\overline{N}$ is nice in $(G/H)/p^\mu(G/H)$, there exist $m_j \in N$ so that the element $\bar{c}_j + \overline{m}_j$ is proper, that is, has maximal height in $(G/H)/p^\mu(G/H)$ among the elements of the coset $\bar{c}_j + \overline{N}$. Moreover, since $\bar{c}_j \notin \overline{N}$, the height of $\bar{c}_j + \overline{m}_j$ in $(G/H)/p^\mu(G/H)$ must be less than μ, say λ. Since

$$p^\lambda((G/H)/p^\mu(G/H)) = (p^\lambda(G/H))/p^\mu(G/H)$$

and

$$p^\lambda(G/H) = \langle p^\lambda G, H \rangle/H,$$

it follows for some $h_j \in H$ that $c_j + m_j + h_j \in p^\lambda G$. The single element $h_j \in H$ is associated with c_j.

Define $B = \langle C, h_j, h_{i,k} \rangle$, where i and j range over the appropriate subsets of the positive integers and $k \geqslant 1$. In order to show that B has the desired property, suppose $(n, c, h) \in N \times C \times H$ and $h_G(n + c + h) = \alpha$. If $\alpha \geqslant \mu$, then $n + c + h = 0$ since $p^\mu G = 0$. In this case we can take $x = h$, and we have $h_G(n + c + x) \geqslant h_G(n + c + h)$ with $x \in \langle N, B \rangle \cap H$, since $h \in \langle N, C \rangle \subseteq \langle N, B \rangle$. Therefore, we may assume that $\alpha < \mu$. If $c = c_i$, where Case 1 holds for c, we know that $\mu(k) > \alpha$ if k is chosen sufficiently large. Whence

$$m_i - c_i + h_{i,k} \in p^\alpha G,$$

and, consequently, $n + m_i + h + h_{i,k} \in p^\alpha G$. This implies that $n + m_i + x \in p^\alpha G$ for some $x \in N \cap H$ since $N \| H$. However, the latter implies, in turn, that

$$n + c_i + (x - h_{i,k}) \in p^\alpha G.$$

Since $c = c_i$ and $x - h_{i,k} \in \langle N, B \rangle \cap H$, the desired result holds.

If $c = c_j$ where Case 2 holds for c, the argument is similar but is based on the relation

$$c_j + m_j + h_j \in p^\lambda G$$

instead of $m_i - c_i + h_{i,k} \in p^\alpha G$. In the present case, however, we need to observe that $\lambda \geqslant \alpha$. But $h_G(n + c + h) = \alpha$ certainly implies that $\bar{c} + \bar{N}$ has height α, so $\lambda \geqslant \alpha$. This completes the proof of the theorem.

3. A-groups and A-invariants. Let H be an A-group, and let $H = \Sigma \oplus H_i$ and $G = \Sigma \oplus G_i$, where (H_i, G_i) is an $A_{\mu(i)}$-pair for distinct limit ordinals $\mu(i)$ that are not cofinal with ω_0. We define what we call the A-numbers ($=$ A-invariants) of H as follows. Actually we define a sequence of ordinal-to-cardinal functions F_μ. Although the values of these functions should be the A-numbers, to simplify the terminology and to follow tradition, we often refer to the functions F_μ themselves as the A-numbers or A-invariants. As a preliminary step toward defining the functions F_μ, set

$$E_\mu = \bigcap_{\substack{\lambda < \mu \\ \lambda + \sigma = \mu}} p^\sigma \big(\langle p^\lambda G, H \rangle / H \big) / \langle p^\mu G, H \rangle / H$$

whenever μ is a limit ordinal not cofinal with ω_0. Now, define $F_\mu = F_\mu^H$, in case $\mu = 0$, or μ is a limit ordinal not cofinal with ω_0, by

$$F_\mu(\alpha) = \begin{cases} \dim\big(p^\alpha H[p]/p^{\alpha+1}H[p] \big) & \text{if } \mu = 0 \text{ and } \alpha < \infty, \\ \dim\big(p^\alpha E_\mu[p]/p^{\alpha+1}E_\mu[p] \big) & \text{if } \mu \neq 0 \text{ and } \alpha < \infty, \\ \dim\big(p^\alpha E_\mu[p] \big) & \text{if } \mu \neq 0 \text{ and } \alpha = \infty. \end{cases}$$

We remark that there is no loss of generality in assuming, as we have done, that H and G are reduced. Hence, $p^\infty H = 0$, where $p^\infty A$ denotes the divisible part of a p-primary abelian group A, and there is no need to make a provision for $\alpha = \infty$ in case $\mu = 0$. We also mention the obvious: $F_0(\alpha)$ is just the αth Ulm-Kaplansky invariant of H. Further, it is not difficult to show, for a limit μ, that p^σ can be deleted from the definition of E_μ without altering its meaning. However, aside from its symmetry, the above definition appears to make E_μ smaller (a desirable feature) while yet retaining a measure of how far H is from being nice at the ordinal μ. More importantly, we will discover (in Lemma B) a more intrinsic characterization of E_μ, which can (and later will) be used as an alternate definition.

LEMMA A. *Let H be an A-group, and let $H = \Sigma \oplus H_i$ and $G = \Sigma \oplus G_i$, where (H_i, G_i) is an $A_{\mu(i)}$-pair for distinct limit ordinals $\mu(i)$ that are not cofinal with ω_0. If $F_\mu(\alpha)$ is as defined above and $\mu \neq 0$, then $F_\mu(\alpha) = 0$ unless $\mu = \mu(i)$ for some i. In case $\mu = \mu(i)$ for some i, then $F_\mu(\alpha)$ is the αth Ulm-Kaplansky invariant of $p^\mu(G_i/H_i)$ when $\alpha < \infty$, and $F_\mu(\infty)$ is the rank of the divisible part of this group.*

PROOF. Suppose $\mu \neq 0$. Let μ denote any limit ordinal not cofinal with ω_0. According to the definition of $F_\mu(\alpha)$, if $\alpha < \infty$,

$$F_\mu(\alpha) = \dim\big(p^\alpha E_\mu[p]/p^{\alpha+1}E_\mu[p] \big).$$

Since $E_\mu = \bigcap_{\lambda < \mu, \; \lambda + \sigma = \mu} p^\sigma(\langle p^\lambda G, H \rangle / H) / \langle p^\mu G, H \rangle / H$ and since $G = \Sigma \oplus G_i$ with the length of G_i not exceeding $\mu(i)$, it is clear that those summands G_i with $\mu(i) < \mu$ make no contributions to E_μ (and therefore make no contribution to $F_\mu(\alpha)$). Moreover, if $\mu(i) > \mu$ and $\lambda + \sigma = \mu$, then

$$p^\sigma\big(\langle p^\lambda G_i, H_i \rangle / H_i\big) = p^\sigma p^\lambda (G_i / H_i) = p^\mu (G_i / H_i) = \langle p^\mu G_i, H_i \rangle / H_i.$$

Thus, G_i again makes no contribution to E_μ, so $E_\mu = 0$ unless $\mu = \mu(i)$ for some i. If $\mu = \mu(i)$, then

$$E_\mu = \bigcap_{\substack{\lambda < \mu \\ \lambda + \sigma = \mu}} p^\sigma\big(\langle p^\lambda G_i, H_i \rangle / H_i\big) / \langle p^\mu G_i, H_i \rangle / H_i = p^\mu (G_i / H_i).$$

This completes the proof of Lemma A.

LEMMA B. *Assume the same notation and hypothesis as in Lemma* A, *and let μ be a limit ordinal not cofinal with ω_0. Let $H_\mu = H / p^\mu H$ and let $\overline{H}_\mu$ denote the completion of H_μ in the p^μ-topology (having $p^\lambda H / p^\mu H$, $\lambda < \mu$, as a base for the neighborhoods of zero). Then $E_\mu = \overline{H}_\mu / H_\mu$. In particular, the A-numbers of H are invariants of H.*

PROOF. Since G is totally projective and μ is a limit ordinal not cofinal with ω_0, $G / p^\mu G = \Sigma \oplus G_i / p^\mu G_i$ is Hausdorff and complete in its p^μ-topology. Since $H / p^\mu H = \Sigma \oplus H_i / p^\mu H_i$ is isotype in $G / p^\mu G$, the completion of $H / p^\mu H$ is simply its closure in $G / p^\mu G$. But from what has gone before, it follows that $H / p^\mu H$ is closed in $G / p^\mu G$ unless $\mu = \mu(i)$ for some i. Moreover, in case $\mu = \mu(i)$, we have $\overline{H}_\mu / H_\mu = p^\mu (G_i / H_i) = E_\mu$, with $\overline{H}_\mu$ being the closure of $H_\mu = H / p^\mu H$ in $G / p^\mu G$.

THEOREM 2. *Let H and H' be A-groups, and let $H = \Sigma \oplus H_i$, $H' = \Sigma \oplus H_i'$, $G = \Sigma \oplus G_i$, and $G' = \Sigma \oplus G_i'$, where (H_i, G_i) and (H_i', G_i') are $H_{\mu(i)}$-pairs for distinct limit ordinals $\mu(i)$ not cofinal with ω_0. If H and H' have the same A-numbers, there exists an isomorphism from $p^{\mu(i)}(G_i / H_i)$ onto $p^{\mu(i)}(G_i' / H_i')$ for each i.*

PROOF. We know that $E_{\mu(i)}$ and $E_{\mu(i)}'$ have the same Ulm-Kaplansky invariants and their divisible parts are isomorphic. By Lemma A, $E_{\mu(i)} = p^{\mu(i)}(G_i / H_i)$ and $E_{\mu(i)}' = p^{\mu(i)}(G_i' / H_i')$. Thus, the reduced parts of E_μ and E_μ' are totally projective. Therefore, since they have the same Ulm-Kaplansky invariants, E_μ and $E_{\mu(i)}'$ must be isomorphic, and the theorem is proved.

LEMMA C. *Suppose that (H_i, G_i) and (H_i', G_i') are $A_{\mu(i)}$-pairs for distinct ordinals $\mu(i)$. Further, suppose that $\phi_i \colon G_i / H_i \twoheadrightarrow G_i' / H_i'$ is an isomorphism between corresponding quotients. Let $G = \Sigma \oplus G_i$, $G' = \Sigma \oplus G_i'$, $H = \Sigma \oplus H_i$, and $H' = \Sigma \oplus H_i'$. Finally, let $\phi = \Sigma \phi_i$ denote the natural isomorphism from G / H onto G' / H' determined by the ϕ_i. If $g \in G$ but $g \notin H$ and $\phi(g + H) = g' + H'$, there exists $h' \in H'$ such that $h_{G'}(g' + h') \geqslant h_G(g)$. Moreover, if $h_{G / H}(g + H) > h_G(g)$, there exists $h' \in H'$ such that $h_{G'}(g' + h') > h_G(g)$.*

PROOF. Since $h_G(g) = \min\{h_{G_i}(g_i)\}$ if $g = \Sigma g_i$, with $g_i \in G_i$, it suffices to prove the lemma for the ith component. However, for this special case the lemma is almost an immediate consequence of (H_i, G_i) and (H_i', G_i') being $A_{\mu(i)}$-pairs. Specifically, if

$h_{G_i}(g_i) = \alpha$ then $\alpha < \mu(i)$ provided that $g_i \in G_i \setminus H_i$. Therefore,

$$g_i' + H' \in p^\alpha(G_i'/H_i') = \left\langle p^\alpha G_i', H_i' \right\rangle / H_i'$$

and

$$g_i' + H_i' \in p^{\alpha+1}(G_i'/H_i') = \left\langle p^{\alpha+1}G_i', H_i' \right\rangle / H_i'$$

if $g_i + H_i \in p^{\alpha+1}(G_i/H_i)$. This completes the proof of the Lemma.

4. Uniqueness theorem.

THEOREM 3. *If H and H' are A-groups with the same A-numbers, then H and H' are isomorphic.*

PROOF. For distinct ordinals $\lambda(i)$ not cofinal with ω_0 let (H_i, G_i) and (H_i', G_i') be $A_{\lambda(i)}$-pairs, where $H = \Sigma \oplus H_i$ and $H' = \Sigma \oplus H_i$. There is no loss of generality in assuming the index set is the same for both since, for example, we can add, if necessary, $A_{\lambda(i)}$-pairs of the type $(0, G_i)$ or $(0, G_i')$ without changing H or H'. Since H and H' have the same A-numbers, $p^{\lambda(i)}(G_i/H_i) \cong p^{\lambda(i)}(G_i'/H_i')$ according to Theorem 2. There is no loss of generality in assuming that $G_i = G_i'$ and $G_i/H_i \cong G_i'/H_i'$, for we can replace both G_i and G_i' by

$$G_i \oplus G_i' \oplus \sum_{\aleph_0} \oplus \left\{ G_i \oplus G_i' \oplus \left\{ (G_i/H_i)/p^{\lambda(i)}(G_i/H_i) \oplus (G_i'/H_i')/p^{\lambda(i)}(G_i'/H_i') \right\} \right\}.$$

Thus, to simplify notation we shall make this assumption. Indeed, we may assume (without loss of generality) that $G_i = G_i'$ and $G_i/H_i \cong G_i'/H_i'$. Let $\phi_i: G_i/H_i \twoheadrightarrow G_i/H_i'$ be an isomorphism, and set $\phi = \Sigma\phi_1$. Since $G_i = G_i'$, $G = \Sigma \oplus G_i = \Sigma \oplus G_i' = G'$. Moreover, ϕ is an isomorphism from G/H onto G/H' that maps the ith component G_i/H_i onto the corresponding ith component G_i/H_i'. Thus, Lemma C is applicable here since its hypotheses are satisfied.

We claim that there is an automorphism of G that maps H onto H'. In the original draft of this paper we included a detailed proof of this fact. However, since there are now two other proofs that are available we have suppressed these details. Our original proof was virtually the same as the proof of the uniqueness theorem for N-groups in [6] with the essential difference between the two proofs being reconciled by Lemma C. Hence, we simply offer this reference as one proof. A second (and much shorter) proof can be obtained as follows. Subsequent to the original proof, C. Megibben and I have proved the following

THEOREM (HILL AND MEGIBBEN [9]). *If H and H' are isotype subgroups of a totally projective group G, then H and H' are isomorphic provided they have the same Ulm-Kaplansky invariants and G/H and G/H' are isomorphic as valuated groups endowed with the coset valuation.*

The coset valuation of G/H is defined by

$$|g + H| = \sup\{|g + h| + 1: h \in H\}.$$

It can be quickly verified that the isomorphism $\phi_i: G_i/H_i \twoheadrightarrow G_i H_i'$ defined above preserves the coset valuation. Therefore, so does $\phi = \Sigma\phi_i$, and the theorem that

$H \cong H'$ is a consequence, at this point of the argument, of the theorem cited by Hill and Megibben.

5. Existence theorem. Although not called by that name, isolated examples of A-groups (even before the advent of N-groups) can be found in the literature that are not totally projective nor S-groups. One of the most notable cases is an example studied by Megibben in [**11**]. We suspect, in fact, that Megibben came close at that time to discovering the class of N-groups, if not A-groups, and their significance. As we have mentioned before, N-groups are a special case of A-groups. They were classified in [**6**], where a general existence theorem, as well as a uniqueness theorem, was established for them. Our purpose in this section is to establish the corresponding existence theorem for A-groups. Toward that effort we first take care of some preliminaries. As we shall see, the situation is much more complicated for A-groups.

Let f be an ordinal-to-cardinal function that eventually vanishes (for sufficiently large ordinals). If one prefers, the function can be restricted to an initial segment of the ordinals. We frequently adjoin ∞ to the ordinals and permit it to be in the domain of f with $f(\infty)$ *not* required to be zero (although $\infty > \alpha$ if α is an ordinal). If $\alpha < \beta \leqslant \infty$, define

$$\int_\alpha^\beta f(\gamma) = \sum_{\alpha \leqslant \gamma < \beta} f(\gamma).$$

Note that we conventionally integrate (or sum if one prefers) over a half-open interval. However, when it is desirable to integrate over a closed interval, we set

$$\int_\alpha^{\beta+} f(\gamma) = \sum_{\alpha \leqslant \gamma \leqslant \beta} f(\gamma) = \left(\int_\alpha^\beta f(\gamma) \right) + f(\beta);$$

in particular, $\int_\alpha^{\infty+} f(\gamma) = (\int_\alpha^\infty f(\gamma)) + f(\infty)$.

The function f is said to be *admissible* if

$$\int_\alpha^{\alpha+\omega} f(\gamma) = \int_\alpha^\infty f(\gamma)$$

for every ordinal α; compare with [**2**, p. 70]. A standard example of an admissible function f is the Ulm-Kaplansky function associated with a totally projective group G, defined (for any G) by

$$f(\gamma) = \dim(p^\gamma G[p]/p^{\gamma+1}G[p]).$$

If there is a need for a distinction, we call the one-point extension of f given by $f(\infty) = p^\infty G[p]$ the *extended* Ulm-Kaplansky function of G, which is relevant only when G is not reduced.

Throughout this section, it is understood that all functions considered, unless otherwise specified, are ordinal-to-cardinal functions. The length of such a function f is the smallest ordinal λ for which $f(\gamma) = 0$ whenever $\gamma \geqslant \lambda$ and $\gamma \neq \infty$. For example, the length of the extended Ulm-Kaplansky function of any p-group G is

the length of the reduced part of G. Another concept that will be used is the following

DEFINITION. If f and g are functions and f is of length $\mu > 0$, we say that f *totally dominates* g if

$$\int_\lambda^\infty f(\gamma) \geqslant \int_0^{\infty^+} g(\gamma)$$

whenever $\lambda < \mu$.

Before we are prepared to prove the general existence theorem for A-groups, certain combinatorial results are needed. These are provided in the next two lemmas. Since ω_0 appears frequently in this section, we let $\omega = \omega_0$. Likewise, $\mathrm{cof}(\mu)$ denotes the cofinality of a limit ordinal μ, and a cardinal is identified with the smallest ordinal of the given cardinality.

LEMMA D. *Let f and g_μ, $\mu \in \mathcal{M}$, be nonzero admissible functions, where $\mathcal{M}$ is a collection of limit ordinals not cofinal with ω. If*

$$(++) \qquad\qquad \int_\lambda^\infty f(\alpha) \geqslant \sum_{\mu > \lambda} \int_0^{\infty^+} g_\mu(\alpha)$$

for each ordinal λ (that satisfies $\mu > \lambda$ for some $\mu \in \mathcal{M}$), there exist functions f_μ, $\mu \in \mathcal{M}$, that satisfy the following conditions:

(a) *f_μ is admissible of length μ.*
(b) *f_μ totally dominates g_μ (that is, $\int_\lambda^\mu f_\mu(\alpha) \geqslant \int_0^{\infty^+} g_\mu(\alpha)$ when $\lambda < \mu$).*
(c) *$f(\alpha) = f(\alpha) + \sum_{\mu \in \mathcal{M}} f_\mu(\alpha)$ for each α.*

PROOF. For each limit ordinal λ, let $\mathcal{M}_\lambda = \{\mu \in \mathcal{M} : \mu > \lambda\}$. We define f_μ, for all $\mu \in \mathcal{M}$, on the half-open interval $[\lambda, \lambda + \omega)$ as follows. If $\mu \leqslant \lambda$ or, equivalently, if $\mu \notin \mathcal{M}_\lambda$, then $f_\mu(\alpha) = 0$ for every α in $[\lambda, \lambda + \omega)$; in particular, if $\mathcal{M}_\lambda$ is vacuous then $f_\mu(\alpha) = 0$ when $\alpha \geqslant \lambda$ for each $\mu \in \mathcal{M}$. We now deal with those μ in $\mathcal{M}_\lambda$. Assume that $\mathcal{M}_\lambda$ is nonempty. Two cases are distinguished: in both cases it is understood always that $\mu \in \mathcal{M}_\lambda$ and $\alpha \in [\lambda, \lambda + \omega)$.

Case 1. $f(\alpha) \geqslant |\mathcal{M}_\lambda|$ for infinitely many α (between λ and $\lambda + \omega$). Define

$$f_\mu(\alpha) = \begin{cases} f(\alpha) & \text{if } f(\alpha) \geqslant |\mathcal{M}_\lambda| \aleph_0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\int_{\lambda+n}^{\lambda+\omega} f(\alpha) = \int_{\lambda+n}^\infty f(\alpha) \geqslant \mathrm{cof}(\mu)$ when $\mu \in \mathcal{M}_\lambda$ and since $\mathrm{cof}(\mu) > \aleph_0$, we immediately see that Case 1 yields $f_\mu(\alpha) = f(\alpha)$ for infinitely many α; in other words, the condition $f(\alpha) \geqslant |\mathcal{M}_\lambda|$ for infinitely many α is equivalent to the condition $f(\alpha) \geqslant |\mathcal{M}_\lambda| \aleph_0$ for infinitely many α. Moreover, it is easy to deduce that

$$\int_{\lambda+n}^{\lambda+\omega} f_\mu(\alpha) = \int_{\lambda+n}^{\lambda+\omega} f(\alpha)$$

whenever $n < \omega$.

Case 2. $f(\alpha) < |\mathcal{M}_\lambda|$ for all but a finite number of α. Note that this precludes the countability of $\mathcal{M}_\lambda$. Since $\int_{\lambda+n}^{\lambda+\omega} f(\alpha) \geqslant |\mathcal{M}_\lambda|$, for each $n < \omega$, it is clear that $|\mathcal{M}_\lambda|$ must be cofinal with ω. Thus $\mathcal{M}_\lambda$ is the union of a countable number of disjoint infinite subsets S_n smaller than $\mathcal{M}_\lambda$; assume, without loss of generality, that

$|S_i| < |S_2| < \cdots < |S_n| < \cdots$. This partition of $\mathcal{M}_\lambda$ is arbitrary (but fixed), and its sole purpose is to contribute toward the definition of $f_\mu(\alpha)$ on the interval $[\lambda, \lambda + \omega)$. Indeed, we define $f_\mu(\alpha)$, for $\mu \in S_n$, by

$$f_\mu(\alpha) = \begin{cases} f(\alpha) & \text{if } f(\alpha) \geqslant |S_n|, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that, as in Case 1, $f_\mu(\alpha) = f(\alpha)$ for infinitely many α. Moreover, $\int_{\lambda+k}^{\lambda+\omega} f_\mu(\alpha)$ $= \int_{\lambda+k}^{\lambda+\omega} f(\alpha)$ whenever $k < \omega$ (because $f(\alpha)$ and $f_\mu(\alpha)$ disagree only if $f_\mu(\alpha) = 0$, in which case $f(\alpha)$ is relatively small).

The two cases no longer need distinguishing. It is a quick deduction that

$$f(\alpha) = f(\alpha) + \sum_{\mu \in \mathcal{M}_\lambda} f_\mu(\alpha)$$

for each α in the interval $[\lambda, \lambda + \omega)$. Since $f_\mu(\alpha) = 0$ if $\mu \notin \mathcal{M}_\lambda$, certainly condition (c) holds, first on the interval $[\lambda, \lambda + \omega)$ and then on the whole domain of f. For $\lambda < \mu$, the inequalities

$$\int_{\lambda+n}^{\infty} f_\mu(\alpha) \geqslant \int_{\lambda+n}^{\lambda+\omega} f_\mu(\alpha) = \int_{\lambda+n}^{\lambda+\omega} f(x) = \int_{\lambda+n}^{\infty} f(\alpha)$$

$$\geqslant \sum_{\nu > \lambda} \int_0^{\infty +} g_\nu(\alpha) \geqslant \int_0^{\infty +} g_\mu(\alpha)$$

demonstrate that f_μ totally dominates g_μ. Thus, condition (b) is satisfied. Similarly, the comparison

$$\int_{\lambda+n}^{\lambda+\omega} f_\mu(\alpha) = \int_{\lambda+n}^{\infty} f(\alpha) \geqslant \int_{\lambda+n}^{\infty} f_\mu(\alpha)$$

verifies that f_μ is admissible. It is clear from its definition that f_μ has length not exceeding μ, but the above inequalities for $\lambda < \mu$ imply that the length of f_μ is at least μ. Therefore, condition (a) is satisfied and the lemma is proved.

Another result similar to the preceding lemma that will be useful in the proof of the existence theorem is the following

LEMMA E. *For an arbitrary limit ordinal μ, let f be an admissible function of length μ. Let m be a fixed cardinal not cofinal with ω and suppose that $\int_\lambda^\mu f(\alpha) \geqslant m$ whenever $\lambda < \mu$. If $\mu(\tau)$ $(\tau < m)$ is a collection of not more than m limit ordinals less than μ, then there exist functions f_τ $(\tau < m)$ that satisfy the following conditions.*

(1) *f_τ is admissible of length $\mu(\tau)$.*
(2) *$\int_\lambda^{\lambda+\omega} f_\tau(\alpha) = \int_\lambda^{\lambda+\omega} f(\alpha)$ whenever $\lambda < \mu(\tau)$.*
(3) *$f_\tau(\alpha) = mf_\tau(\alpha) = \sum_{\rho \leqslant \tau} f_\rho(\alpha)$ for $\alpha < \mu(\tau)$.*
(4) *$f(\alpha) = f(\alpha) + \sum_{\tau < m} f_\tau(\alpha)$ for each α.*

PROOF. Define the function f_τ as follows:

$$f_\tau(\alpha) = \begin{cases} f(\alpha) & \text{if } \alpha < \mu(\tau) \text{ and } f(\alpha) \geqslant m, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that if $\lambda < \mu$ then $f(\alpha) \geqslant m$ for infinitely many α between λ and $\lambda + \omega$ because

$$\int_{\lambda+n}^{\lambda+\omega} f(\alpha) = \int_{\lambda+n}^{\infty} f(\alpha) \geqslant m$$

and m is not cofinal with ω. This leads at once to $\int_\lambda^{\lambda+\omega} f_\tau(\alpha) = \int_\lambda^{\lambda+\omega} f(\alpha)$ whenever $\lambda < \mu(\tau)$. The verification of conditions (1), (3), and (4) is now straightforward, and (2) is already settled.

THEOREM 4. *Let f and g_μ, $\mu \in \mathcal{M}$, be nonzero admissible functions, where $\mathcal{M}$ is a collection of limit ordinals not cofinal with ω. A necessary and sufficient condition for the existence of an A-group with A-invariants*

$$F_0(\alpha) = f(\alpha),$$

$$F_\mu(\alpha) = g_\mu(\alpha) \quad \textit{if } \mu \in \mathcal{M},$$

$$F_\nu(\alpha) = 0 \quad \textit{if } \nu \neq 0 \textit{ and } \nu \notin \mathcal{M},$$

is, for each λ (less than some $\mu \in \mathcal{M}$), the inequality

$$(++) \qquad\qquad \int_\lambda^{\lambda+\omega} f(\alpha) \geqslant \sum_{\mu > \lambda} \int_0^{\infty+} g_\mu(\alpha).$$

PROOF. *Sufficiency*. Suppose $(++)$ holds. If $\mathcal{M}$ is empty, all that is required is a totally projective group H whose Ulm-Kaplansky function is f. We know such an H exists [5 or 2]. If $\mathcal{M}$ is nonempty, Lemma D reduces the proof to the case where $\mathcal{M}$ contains a single element μ and f has length μ. Indeed we can write, according to Lemma D, $f = f + \sum_{\mu \in \mathcal{M}} f_\mu$, where f_μ is admissible of length μ and f_μ totally dominates g_μ. As we have mentioned, there is certainly a totally projective group whose Ulm-Kaplansky function if f. Thus, it suffices to prove the existence of an A-group H_μ with A-invariants

$$F_0(\alpha) = f_\mu(\alpha), \qquad F_\mu(\alpha) = g_\mu(\alpha), \qquad F_\nu(\alpha) = 0 \quad \text{if } \nu \neq 0 \text{ and } \nu \neq \mu.$$

To simplify notation let $f = f_\mu$ and $g = g_\mu$. Recall that f totally dominates g, that is,

$$(+) \qquad\qquad \int_\lambda^\mu f(\alpha) \geqslant \int_0^{\infty+} g(\alpha)$$

for each $\lambda < \mu$.

Let the cofinality of μ be $\sigma > \omega$, and let $\mu(\tau)$, $\tau < \sigma$, be a strictly increasing sequence of ordinals with limit μ. Without loss of generality we may assume that $\mu(\tau)$ is a limit ordinal of the form $\gamma + \omega$ for every $\tau < \sigma$. In fact, we may assume that $\mu(\tau) = \lambda(\tau) + \omega$, where $\lambda(\tau)$ itself is of the form $\gamma + \omega$. Thus, choose $\mu(\tau)$ and $\lambda(\tau)$ in this manner. For convenience and agreement in notation with a previous lemma, set $m = |\sigma| = |\mathrm{cof}(\mu)|$ and note that $m \geqslant \aleph_1$. Denote by E the direct sum of a totally projective group and a divisible group that has g for its extended Ulm-Kaplansky function. If we consider the function f, then clearly, $\int_\lambda^\mu f(\alpha) \geqslant m$ for any $\lambda < \mu$ since f is admissible of length μ. Observe, in fact, that condition $(+)$ implies

$$\int_\lambda^{\lambda+\omega} f(\alpha) = \int_\lambda^\mu f(\alpha) \geqslant m|E|$$

in view of the inequality $|E| \leqslant \aleph_0 \int_0^{\infty+} g(\alpha)$ for any (nonreduced) totally projective group E with (extended) Ulm-Kaplansky function g. We now employ Lemma E and write

$$f(\alpha) = f(\alpha) + \sum_{\tau < \sigma} f_\tau(\alpha),$$

where f_τ is an admissible function of length $\mu(\tau)$, $\int_\lambda^{\lambda+\omega} f_\tau(\alpha) = \int_\lambda^{\lambda+\omega} f(\alpha)$ whenever $\lambda < \mu(\tau)$, and $f_\tau(\alpha) = mf_\tau(\alpha) = \sum_{\rho \leqslant \tau} f_\rho(\alpha)$. By our previous conclusion, observe that

$$\int_\lambda^{\lambda+\omega} f_\tau(\alpha) = \int_\lambda^{\lambda+\omega} f(\alpha) \geqslant m|E|$$

whenever $\lambda < \mu(\tau)$.

Let $B_0 \rightarrowtail B \twoheadrightarrow E$ be a pure-exact sequence where B is a direct sum of cyclic groups and $|B| = |E|$. Denote by C_τ the totally projective group of length $\mu(\tau) = \lambda(\tau) + \omega$ that has f_τ for its Ulm-Kaplansky function, and let C_τ' be the totally projective group (of length at most $\mu(\tau)$ and at least $\lambda(\tau)$) that satisfies $p^{\lambda(\tau)}C_\tau' = B$ and $C_\tau'/p^{\lambda(\tau)}C_\tau' = C_\tau/p^{\lambda(\tau)}C_\tau$. With C_τ' so defined for each $\tau < \sigma$, we inductively define an ascending chain of totally projective groups G_τ, for $\tau < \sigma$, as follows. Let $G_0 = B$. If G_α has been defined for $\alpha \leqslant \tau < \sigma$ so that G_α is a totally projective group with $p^\lambda G_\alpha = B$ for some $\lambda \leqslant \lambda(\alpha)$ (and more particularly $p^{\lambda(\alpha)}G_{\alpha+1} = B$ whenever $\alpha < \tau$), define $G_{\tau+1}$ to be the pushout associated with $B \rightarrowtail G_\tau$ and $B \rightarrowtail C_\tau'$. Thus we have the commutative diagram (with natural maps)

$$
\begin{array}{ccc}
B & \rightarrowtail & C_\tau' \\
\downarrow & & \downarrow \\
G_\tau & \rightarrowtail & G_{\tau+1}
\end{array}
$$

Alternately, we could say that $G_{\tau+1}$ is the amalgamated sum of G_τ and C_τ'. Observe that $p^{\lambda(\tau)}G_{\tau+1} = B = p^{\lambda(\tau)}C_\tau'$ since

$$G_{\tau+1}/B = G_\tau/B \oplus C_\tau'/B$$

and since $p^{\lambda(\tau)}G_\tau \subseteq B$ by the induction hypothesis. As expected, if τ is a limit less than σ and if G_α has been defined for $\alpha < \tau$, we let $G_\tau = \bigcup_{\alpha < \tau} G_\alpha$ (where $G_\alpha \subseteq G_\beta$, if $\alpha < \beta$, by virtue of the natural identification of G_α in $G_{\alpha+1}$). By induction, it is apparent that

$$G_\tau/B = \sum_{\gamma < \tau} \oplus\, C_\gamma'/B$$

and that $p^\lambda G_\tau = B$, where $\lambda = \sup_{\gamma < \tau}\{\lambda(\gamma)\} \leqslant \lambda(\tau)$. Thus, we have constructed a smooth ($G_\beta = \bigcup_{\alpha < \beta} G_\alpha$ when β is a limit), ascending chain of totally projective groups G_τ, $\tau < \sigma$, with $p^\lambda G_\tau = B$ where $\lambda = \sup_{\gamma < \tau}\{\lambda(\gamma)\}$, and $G_\tau/B = \sum_{\gamma < \tau} \oplus\, C_\gamma'/B$. We can extend this chain one additional link and retain the same properties simply by defining $G_\sigma = \bigcup_{\tau < \sigma} G_\tau$. Note, in particular, that $p^\mu G_\sigma = B$ since $\mu = \sup_{\tau < \sigma}\{\lambda(\tau)\}$. Indeed, if we set $A = G_\sigma/B_0$ then $p^\mu A = B/B_0 = E$ and $A/p^\mu A = G_\sigma/B = \sum_{\tau < \sigma} \oplus\, C_\tau'/B$. Therefore, $A/p^\mu A$ is the totally projective group whose Ulm-Kaplansky function is $\sum_{\tau < \sigma} f_\tau'$, where

$$f_\tau'(\alpha) = \begin{cases} f_\tau(\alpha) & \text{if } \alpha < \lambda(\tau), \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, f_τ' is the Ulm-Kaplansky function of $C_\tau'/B = C_\tau'/p^{\lambda(\tau)}C_\tau'$. Observe that

$$G_{\tau+1}/B = \sum_{\gamma \leqslant \tau} \oplus\, C_\gamma'/B \cong C_\tau'/B = C_\tau/p^{\lambda(\tau)}C_\tau$$

because all are totally projective and have the same Ulm-Kaplansky invariants; recall that $mf_\tau(\alpha) = f_\tau(\alpha) = \Sigma_{\rho \leqslant \tau} f_\rho(\alpha)$.

Since $\int_{\lambda(\tau)+n}^{\mu(\tau)} f_\tau(\alpha) \geqslant m|E|$ and $\rho^{\lambda(\tau)} C_\tau$ is a direct sum of cyclic groups, there exists an epimorphism

$$p^{\lambda(\tau)} C_\tau \twoheadrightarrow B = p^{\lambda(\tau)} G_{\tau+1}.$$

Let K_τ denote its kernel. Due to the total projectivity of $C_\tau/K_\tau \cong G_{\tau+1}$, the isomorphism

$$p^{\lambda(\tau)}(C_\tau/K_\tau) = p^{\lambda(\tau)} C_\tau/K_\tau \rightarrowtail\!\!\!\!\rightarrow B = p^{\lambda(\tau)} G_{\tau+1}$$

can be extended to an isomorphism between C_τ/K_τ and $G_{\tau+1}$ (by virtue of Zippin's theorem for totally projectives). Denote by ε_τ the resulting epimorphism from C_τ to $G_{\tau+1}$,

$$\varepsilon_\tau: C_\tau \twoheadrightarrow C_\tau/K_\tau \rightarrowtail\!\!\!\!\rightarrow G_{\tau+1},$$

and let

$$\phi_\tau: C_\tau \twoheadrightarrow G_{\tau+1} \twoheadrightarrow G_{\tau+1}/B_0 \subseteq A$$

be the composition of ε_τ and the projection of $G_{\tau+1}$ onto $G_{\tau+1}/B_0$, which is a subgroup of A. Observe that if $\lambda < \mu$ and $a \in p^\lambda A = \langle p^\lambda G_\sigma, B_0 \rangle/B_0$, then $a = y + B_0$ with $y \in p^\lambda G_\sigma$ implies $y \in p^\lambda G_{\tau+1}$ if τ is sufficiently large (namely, if $y \in G_{\tau+1}$ and $\lambda(\tau) > \lambda$) because $B = p^{\lambda(\tau)} G_{\tau+1}$ and $G_{\tau+1}/B$ is a direct summand of G_σ/B. Since ε_τ preserves heights less than $\lambda(\tau)$ (computed in C_τ and $G_{\tau+1}$), there exists x_τ in $p^\lambda C_\tau$ such that $\varepsilon_\tau(x_\tau) = y$ and $\phi_\tau(x_\tau) = y + B_0 = a$. Consequently, the mapping $\Sigma_{\tau < \sigma} \phi_\tau$ maps $p^\lambda(\Sigma_{\tau < \sigma} \oplus C_\tau)$ onto $p^\lambda A$ for each $\lambda < \mu$. This will prove to be an important feature in obtaining an almost balanced resolution of A.

We now construct another map from C_τ into A. This time we start with an epimorphism $p^{\lambda(\tau)} C_\tau[p] \twoheadrightarrow B[p]$. Since

$$B = p^{\lambda(\tau+1)} G_{\tau+2} \subseteq p^{\lambda(\tau)+\omega} G_{\tau+2} = p^\omega \big(p^{\lambda(\tau)} G_{\tau+2}\big)$$

and $p^{\lambda(\tau)} C_\tau$ is a direct sum of cyclic groups, the epimorphism $p^{\lambda(\tau)} C_\tau[p] \twoheadrightarrow B[p]$ can certainly be extended to a mapping (not necessarily epic) from $p^{\lambda(\tau)} C_\tau$ into $p^{\lambda(\tau)} G_{\tau+2}$. Since the extended map still does not decrease heights (computed in C_τ and $G_{\tau+2}$), it can be extended to a mapping from C_τ to $G_{\tau+2}$; let $\delta_\tau: C_\tau \to G_{\tau+2}$ with $\delta_\tau(p^{\lambda(\tau)} C_\tau[p]) = B[p]$. Denote by π_τ the composition of δ_τ and the projection of $G_{\tau+2}$ onto $G_{\tau+2}/B_0$. Thus, we have $\pi_\tau: C_\tau \to G_{\tau+2} \to G_{\tau+2}/B_0 \subseteq A$.

Some additional maps from totally projectives into A are still required in order for us to obtain the desired resolution of A. Let H_τ be a $p^{\lambda(\tau)}$-high subgroup of $G_{\tau+1}/B_0$. Since $\lambda(\tau)$ is cofinal with ω and since H_τ is isotype and $p^{\lambda(\tau)}$-dense in $G_{\tau+1}/p^{\lambda(\tau)} G_{\tau+1}$, it is totally projective [18, Theorem A]; in fact, $H_\tau \cong G_{\tau+1}/p^{\lambda(\tau)} G_{\tau+1}$ (and therefore has f_τ' for its Ulm-Kaplansky function). Let $i_\tau: H_\tau \subseteq G_{\tau+1}/B_0 \subseteq A$ denote the inclusion map of H_τ into A. Define

$$C = \sum_{\tau < \sigma} \oplus (C_\tau \oplus C_\tau \oplus H_\tau)$$

and let K be the kernel of the mapping $\Sigma_{\tau < \sigma}(\phi_\tau + \pi_\tau + i_\tau)$, which maps C onto A. In fact, $\Sigma \phi_\tau$ is epic. We want to show that (K, C) is an A_μ-pair. Since C_τ and H_τ are totally projective so is C, and the length of C is μ. Therefore, to demonstrate that (K, C) is an A_μ-pair, it suffices to prove that K is almost balanced in C in the sense that

  (i) $p^\lambda A = p^\lambda(C/K) = \langle p^\lambda C, K \rangle / K$ when $\lambda < \mu$, and
  (ii) K is isotype in C.

Condition (i) is a direct consequence of the fact that $\Sigma_{\tau < \sigma} \phi_\tau$ maps $p^\lambda(\Sigma_{\tau < \sigma} \oplus C_\tau)$ onto $p^\lambda A$ if $\lambda < \mu$. The proof that K is isotype in C is not quite as apparent. In order to show that K is isotype in C, we use the following criterion. If C is a p-group of limit length μ and K is a subgroup of G, then K is isotype in G provided that

$$(p) \qquad\qquad p^\lambda(C/K)[p] \subseteq \langle p^\lambda C[p], K \rangle / K$$

for each $\lambda < \mu$. It is easy to show by induction on α that this inclusion implies that $p^\alpha C \cap K \subseteq p^\alpha K$. Now, to show that condition (p) is, in fact, satisfied let $a \in p^\lambda A[p] = p^\lambda(C/K)[p]$. From previous considerations we know that $a = y + B_0$, where $y \in p^\lambda G_{\tau+1}$ for a sufficiently large τ. Since H_τ is $p^{\lambda(\tau)}$-high in $G_{\tau+1}/B_0$, it follows that

$$(G_{\tau+1}/B_0)[p] = H_\tau[p] + p^{\lambda(\tau)}(G_{\tau+1}/B_0)[p].$$

Therefore, if $\lambda < \lambda(\tau)$, we can write

$$y + B_0 = h_\tau + (b + B_0),$$

where $h_\tau \in p^\lambda H_\tau[p]$ and $b \in B = p^{\lambda(\tau)}G_{\tau+1}$. Since $pb \in B_0$ and B_0 is pure, we can choose $b \in B[p]$. Consequently, there exists $x \in p^{\lambda(\tau)}C_\tau[p]$ such that $\delta_\tau(x) = b$ and $\pi_\tau(x) = b + B_0$. Thus, we have produced an element $h_\tau + x$ in $p^\lambda C[p]$ that maps onto the given element a in A (under the mapping $\Sigma_{\tau < \sigma}(\phi_\tau + \pi_\tau + i_\tau)$). This completes the proof that (K, C) is an A_μ-pair.

   Let L be a totally projective group whose Ulm function is f and set $H = K \oplus L$. Since $(H, C \oplus L)$ is an A_μ-pair, H is certainly an A-group. To complete the proof of the sufficiency of $(++)$, it remains only to show that H has the desired invariants, namely,

$$F_0(\alpha) = f(\alpha), \qquad F_\mu(\alpha) = g(\alpha), \qquad F_\nu(\alpha) = 0 \quad \text{if } \nu \neq 0, \mu.$$

The fact that $(H, C \oplus L)$ is an A_μ-pair implies that $F_\nu(\alpha) = 0$ unless ν is zero or μ. Moreover, it is immediate from our construction that

$$p^\mu((C \oplus L)/H) = p^\mu(C/K) = p^\mu A = B/B_0 = E.$$

Since the Ulm-Kaplansky function of E is g, it must be that $F_\mu(\alpha) = g(\alpha)$. Finally, the αth Ulm-Kaplansky invariant of C is either zero or else $f(\alpha) = mf(\alpha)$ due to the properties of f_τ. Since K is isotype in C, it quickly follows that $H = K \oplus L$ has f for its Ulm-Kaplansky function since L does. This completes the proof of the sufficiency of $(++)$.

   Necessity. Suppose now that H is an A-group. Then, by definition, $H = \Sigma_{i \in I} \oplus H_i$, where (H_i, G_i) is an $A_{\mu(i)}$-pair for distinct limit ordinals $\mu(i)$ not cofinal with ω. As usual, let the A-invariants of H be denoted by $F_0(\alpha)$ and $F_\mu(\alpha)$, where $\mu \in \mathcal{M} = \{\mu(i) : i \in I\}$.

Set $f(\alpha) = F_0(\alpha)$ and $g_\mu(\alpha) = F_\mu(\alpha)$, $\mu \in \mathcal{M}$. We need to show that the condition

$$(+\,+) \qquad\qquad \int_\lambda^{\lambda+\omega} f(\alpha) \geqslant \sum_{\lambda < \mu} \int_0^{\infty\,+} g_\mu(\alpha)$$

is satisfied. Since the A-invariants are additive, it suffices to establish $(+\,+)$ for a μ-elementary A-group H. However, in this case $(+\,+)$ simplifies and takes the form

$$(+) \qquad\qquad \int_\lambda^{\lambda+\omega} f(\alpha) \geqslant \int_0^{\infty\,+} g_\mu(\alpha)$$

for $\lambda < \mu$, where (H, G) is an A_μ-pair. Furthermore, it is enough to show that $\int_\lambda^\mu f(\alpha) \geqslant \int_0^{\infty\,+} g_\mu(\alpha)$ is satisfied because f is admissible, since it is the Ulm-Kaplansky function of H. Recall that g_μ is the extended Ulm-Kaplansky function of $E_\mu = p^\mu(G/H)$. Let $p^\alpha H[p] = S_\alpha \oplus p^{\alpha+1}H[p]$ and set $S = \Sigma_{\lambda \leqslant \alpha < \mu} \oplus S_\alpha$. Choose T so that $S \subseteq T \subseteq p^\lambda G[p]$ and $\langle T, H \rangle/H = E_\mu[p]$; this is possible since H is isotype in G and since $E_\mu = p^\mu(G/H) = \cap_{\lambda < \mu} \langle p^\lambda G, H \rangle/H$. Consider S and T as valuated vector spaces (whose values are heights computed in G) and observe that S is dense in T, that is, if $t \neq 0$ belongs to T then $|t - s| > |t|$ for some $s \in S$. However, since G is totally projective the valuated vector space $G[p]$ is contained in a free space F. Therefore, we conclude that T is no larger (in cardinality) than S since no subspace of a free space can have a smaller dense subspace. Hence,

$$\int_\lambda^\mu f(\alpha) = |S| = |T| \geqslant |E_\mu[p]| \geqslant \int_0^{\infty\,+} g_\mu(\alpha)$$

for each $\lambda < \mu$, and the theorem is proved.

REMARK. Since it is obviously necessary for the functions $f = F_0$ and $g_\mu = F_\mu$ to be admissible in order to qualify as the A-invariants for an A-group H, an equivalent form of the existence theorem is the following. Let f and g_μ, $\mu \in \mathcal{M}$, be ordinal-to-cardinal functions, where $\mathcal{M}$ is a collection of limit ordinals not cofinal with ω. A necessary and sufficient condition for the existence of an A-group with A-invariants

$$F_0(\alpha) = f(\alpha), \qquad F_\mu(\alpha) = g_\mu(\alpha) \quad \text{if } \mu \in \mathcal{M}, F_\nu(\alpha) = 0 \quad \text{if } \nu \neq 0 \text{ and } \nu \notin \mathcal{M},$$

is that f and g_μ, for each $\mu \in \mathcal{M}$, be admissible and $(+\,+)$ be satisfied.

By combining the existence and uniqueness theorems, we obtain a complete classification of A-groups.

THEOREM 5. *The class of A-groups are in 1-1 correspondence, via their A-invariants, with the class of sequences of admissible functions $\{F_\mu\}$, $\mu = 0$ or a limit ordinal not cofinal with ω, that satisfy condition $(+\,+)$ upon setting $f = F_0$ and $g_\mu = F_\mu$ if $\mu \neq 0$. The given correspondence preserves finite and infinite sums.*

6. The structure and properties of A-groups.

In this section we formulate some of the most important properties of A-groups. In cases where these results are immediate consequences of the complete classification of A-groups provided by Theorem 5, proofs are either omitted entirely or only short proofs are included. However, because of their significance to the theory, we label these results (including direct corollaries) as theorems.

Our first result shows how to identify among A-groups the well-known special types simply by looking at the invariants.

THEOREM 6. *Let H be an A-group with A-invariants $F_\mu(\alpha)$, $\mu \geqslant 0$. Then H is totally projective if and only if $F_\mu(\alpha) = 0$ when $\mu > 0$ for all α (including $\alpha = \infty$). Moreover, H is an S-group if and only if $F_\mu(\alpha) = 0$ when $\mu > 0$ for all α different from ∞. Finally, H is an N-group if and only if $F_\mu(\alpha) = 0$ unless $\mu = 0$ or $\mu = \omega_1$ and $\alpha = 0$.*

The next result implies, among other things, that every A-group H has a totally projective summand T such that all the isotype subgroups of H containing T (including T and H) have exactly the same Ulm-Kaplansky invariants.

THEOREM 7. *If H is an A-group, then $H = T \oplus K$, where K is an A-group and T is totally projective and has the same Ulm-Kaplansky invariants as H.*

PROOF. Apply Lemma D and Theorem 5.

Recall that an A-group H is a μ-elementary A-group if $F_\nu(\alpha) = 0$, for every α, unless $\nu = 0$ or $\nu = \mu$. If, in addition, $F_\mu(\infty) = 0$, we say that H is an *adjusted* μ-elementary A-group.

THEOREM 8. *Any A-group H can be written as $H = T \oplus S \oplus A$, where T is totally projective and where $S = \Sigma \oplus S_\mu$ and $A = \Sigma \oplus A_\mu$ are direct sums of μ-elementary S-groups and adjusted μ-elementary A-groups, respectively.*

PROOF. Let H be an A-group with A-invariants $F_\mu(\alpha)$, $\mu = 0$ or a limit ordinal not cofinal with ω. It is understood that μ ranges over the appropriate ordinals. Set $f(\alpha) = F_0(\alpha)$ and $g_\mu(\alpha) = F_\mu(\alpha)$ if $\mu \neq 0$. By Lemma D we can decompose the function f as $f = f + \Sigma_{\mu \neq 0} f_\mu$ so that f_μ is admissible of length μ and totally dominates g_μ in the sense that $\int_\lambda^\mu f_\mu(\alpha) \geqslant \int_0^{\infty +} g_\mu(\alpha)$ when $\lambda < \mu$. Clearly, we can further decompose the function f_μ as $f_\mu = f_{S,\mu} + f_{A,\mu}$ into admissible functions $f_{S,\mu}$ and $f_{A,\mu}$ so that, when $\lambda < \mu$,

$$\int_\lambda^\mu f_{S,\mu}(\alpha) \geqslant g_\mu(\infty) \quad \text{and} \quad \int_\lambda^\mu f_{A,\mu}(\alpha) \geqslant \int_0^\infty g_\mu(\alpha).$$

According to the existence theorem, there are A-groups T, S_μ, and A_μ that have the following prescribed A-invariants:

$$T: F_0(\alpha) = f(\alpha) \quad \left(F_\mu(\alpha) = 0 \text{ if } \mu \neq 0 \right),$$

$$S_\mu: F_0(\alpha) = f_{S,\mu}(\alpha); \quad F_\mu(\alpha) = \begin{cases} 0 & \text{if } \alpha \neq \infty, \\ g_\mu(\infty) & \text{if } \alpha = \infty \end{cases}$$

$$\left(F_\nu(\alpha) = 0 \text{ if } \nu \neq 0 \text{ and } \nu \neq \mu \right),$$

$$A_\mu: F_0(\alpha) = f_{A,\mu}(\alpha); \quad F_\mu(\alpha) = \begin{cases} g_\mu(\alpha) & \text{if } \alpha \neq \infty, \\ 0 & \text{if } \alpha = \infty \end{cases}$$

$$\left(F_\nu(\alpha) = 0 \text{ if } \nu \neq 0 \text{ and } \nu \neq \mu \right).$$

Obviously, A_μ is an adjusted μ-elementary A-group, while S_μ can be decomposed into μ-elementary S-groups (with μ-fixed) if it is not already. Since the A-group $T \oplus \Sigma_{\mu \neq 0} S_\mu \oplus \Sigma_{\mu \neq 0} \oplus A_\mu$ has the same A-invariants as H, the theorem follows.

It perhaps should be noted in passing that H is an adjusted μ-elementary A-group if and only if H is an isotype subgroup of a totally projective group G of length μ

such that G/H is a reduced totally projective and $p^\lambda(G/H) = \langle p^\lambda G, H\rangle/H$ when $\lambda < \mu$. In particular, an N-group is always an adjusted μ-elementary A-group with $\mu = \omega_1$.

The next theorem generalizes the same result for totally projective groups (Nunke [13]) and S-groups (Warfield [18]).

THEOREM 9. *Let H be an arbitrary reduced p-group and α an arbitrary ordinal. The H is an A-group if and only if both $p^\alpha H$ and $H/p^\alpha H$ are A-groups.*

PROOF. The "only if" part is easy. First, it reduces quickly to the case where H is a μ-elementary A-group. Thus suppose that (H, G) is an A_μ-pair for some limit ordinal μ not cofinal with ω. If $\alpha < \mu$, it is routine to verify that $(p^\alpha H, p^\alpha G)$ is an A_λ-pair, where $\mu = \alpha + \lambda$. Thus $p^\alpha H$ is an A-group since $p^\alpha H = 0$ if $\alpha \geqslant \mu$. Likewise, if $\alpha < \mu$, $\langle H, p^\alpha G\rangle/p^\alpha G$ is balanced in $G/p^\alpha G$ and the quotient is totally projective. Hence, $H/p^\alpha H = \langle H, p^\alpha G\rangle/p^\alpha G$ is totally projective if $\alpha < \mu$, and in any event $H/p^\alpha H$ is an A-group.

To prove the converse, we first observe that H is an A-group if $p^n H$ is for a positive integer n. Suppose that $p^n H$ is an A-group. Let $p^n H = \Sigma \oplus K_i$, where K_i is a $\mu(i)$-elementary A-group for each i. We can lift the decomposition of $p^n H = \Sigma \oplus K_i$ to a decomposition of H (see, for example, Theorem 11 in [8]). Therefore, let $H = \Sigma \oplus H_i$, where $p^n H_i = K_i$. If (K_i, T_i) is an $A_{\mu(i)}$-pair, then (H_i, G_i) is also an $A_{\mu(i)}$-pair for a suitable totally projective group G_i with $p^n G_i = T_i$. Hence, $H = \Sigma \oplus H_i$ must be an A-group if $p^n H$ is an A-group.

Now, suppose α is arbitrary and both $p^\alpha H$ and $H/p^\alpha H$ are A-groups. Let $\alpha = \beta + n$, where β is a limit and $n < \omega$. From what has preceded, we conclude that $p^\beta H$ and $H/p^\beta H$ are A-groups. Our objective, of course, is to prove that H is an A-group, and we shall do this based on the fact that $p^\beta H$ and $H/p^\beta H$ are for a limit ordinal β. We may assume without loss of generality that $p^\beta H \neq 0$. It is rather immediate that H is at least an isotype subgroup of a totally projective group, for if $p^\beta H$ is isotype in $p^\beta G$ the identity map on $p^\beta H$ can be extended to a mapping from H to G since $H/p^\beta H$ is weakly p^β-projective [7, Theorem 2.2]. Under our hypothesis we may assume that G, as well as $p^\beta G$, is totally projective. Clearly, if we combine such a map from H to G with $H \twoheadrightarrow H/p^\beta H \twoheadrightarrow G'$, we can obtain the desired embedding of H in a totally projective group $G \oplus G'$. A benefit of this is that we now know that the Ulm-Kaplansky function of H is admissible [18, Theorem 4.7].

Letting f_G denote the Ulm-Kaplansky function of an arbitrary group G and letting $p^\beta H = H(\beta)$, we define the function $f'_{H(\beta)}$ by the rule

$$f'_{H(\beta)}(\alpha) = \begin{cases} 0 & \text{if } \alpha < \beta, \\ f_{H(\beta)}(\gamma) & \text{if } \alpha = \beta + \gamma. \end{cases}$$

Note that $f'_{H(\beta)}$ is not itself an Ulm-Kaplansky function, but rather a shift of the Ulm-Kaplansky function $f_{H(\beta)}$. Since β is a limit, $f_H = f_{H/p^\beta H} + f'_{p^\beta H}$. Denote the nonzero A-invariants of $H/p^\beta H$ as follows:

$$F_0 = f_{H/p^\beta H}, \qquad F_\mu = g_\mu \quad \text{for } 0 \neq \mu \in \mathscr{M}.$$

Observe that no $\mu \in \mathcal{M}$ can exceed β (since $H/p^{\beta}H$ has length β). Likewise, denote the nonzero A-invariants of $H(\beta) = p^{\beta}H$ as follows:

$$F_0 = f_{H(\beta)}, \quad F_{\mu} = g_{\beta + \mu} \quad \text{for } \beta < \beta + \mu \in \mathcal{M}'.$$

Note that $\mathcal{M}$ and $\mathcal{M}'$ are disjoint. Hereafter in this proof it is understood that μ ranges over $\mathcal{M} \cup \mathcal{M}'$. If $\lambda \leqslant \beta$,

$$\int_{\lambda}^{\lambda + \omega} f_H(\alpha) \geqslant \int_{\beta}^{\infty} f_H(\alpha) = \int_0^{\infty} f_{H(\beta)}(\alpha)$$

since f_H is admissible. Therefore, if λ is less than some μ,

$$\int_{\lambda}^{\lambda + \omega} f_H(\alpha) \geqslant \int_{\lambda}^{\lambda + \omega} f_{H/p^{\beta}H}(\alpha) + \int_0^{\infty} f_{H(\beta)}(\alpha) \geqslant \sum_{\mu > \lambda} \int_0^{\infty +} g_{\mu}(\alpha).$$

Consequently, there exists an A-group K whose A-invariants are closely related to H, namely,

$$F_0^K = f_H, \qquad F_{\mu}^K = g_{\mu} \quad \text{if } \mu \neq 0.$$

Recall that

$$g_{\mu}(\alpha) = \begin{cases} F_{\mu}^{H/p^{\beta}H}(\alpha) & \text{if } \mu \in \mathcal{M}, \\ F_{\nu}^{p^{\beta}H}(\alpha) & \text{if } \mu = \beta + \nu \in \mathcal{M}'. \end{cases}$$

Thus, $p^{\beta}K$ and $p^{\beta}H$ are A-groups with the same invariants, and therefore $p^{\beta}K \cong p^{\beta}H$. Likewise, $K/p^{\beta}K \cong H/p^{\beta}H$. Finally, $K \cong H$ by virtue of a recent result of Hill and Megibben [9] that implies that an A-group is totally Zippin, which means (by definition) that an isomorphism $p^{\beta}H \rightarrowtail p^{\beta}K$ can be extended to an isomorphism $H \rightarrowtail K$ provided the quotients $H/p^{\beta}H$ and $K/p^{\beta}K$ are isomorphic and totally Zippin. Thus, H is an A-group since K is an A-group and the theorem is proved.

The existence theorem is evidence enough that A-groups exist in abundance. However, we include the following examples to illustrate how naturally they occur.

EXAMPLE 1. let $\Omega = \omega_1$ and let G be a totally projective group of length $\Omega 2$. Let M be Megibben's example in [11, p. 109] identified in the d.s.c. group $p^{\Omega}G$. If H is maximal in G with respect to $H \cap p^{\Omega}G = M$, then H is an A-group that is not an S-group. Note that $H/p^{\Omega}H$ is an S-group.

EXAMPLE 2. Let W be an S-group of length Ω that is not totally projective and let T be totally projective of length greater than Ω. If W is isotype and p^{Ω}-dense in the d.s.c. group G, then the p^{Ω}-pure exact sequence $\mathrm{TOR}(W, T) \rightarrowtail \mathrm{TOR}(G, T) \twoheadrightarrow \Sigma \oplus T$ reveals that $\mathrm{TOR}(W, T)$ is an A-group (but not an S-group).

The following result is particularly striking since it was only in 1964 that a subgroup of a d.s.c. was first exhibited [12] that did not itself decompose into a d.s.c. Shortly after, it was shown that no isotype subgroup of a d.s.c. of countable length could fail to decompose into countable groups. Moreover, all isotype subgroups of d.s.c.'s that are S-groups must decompose into groups of cardinality at most $\aleph_1$.

THEOREM 10. *There exist arbitrarily large A-groups that are, in fact, isotype subgroups of d.s.c.'s that cannot be written as the direct sum of smaller groups. (Hence they are quasi-indecomposable.)*

PROOF. Let $\aleph \geqslant \aleph_1$ be a cardinal and let σ be the first ordinal of cardinality $\aleph$. Set $\mu = \omega_1$ and define

$$f(\alpha) = \begin{cases} \aleph & \text{if } \alpha < \mu, \\ 0 & \text{otherwise (if } \alpha \text{ is uncountable).} \end{cases}$$

Define

$$g_\mu(\alpha) = \begin{cases} \aleph & \text{if } \alpha \leqslant \sigma, \\ 0 & \text{otherwise (if } \alpha \text{ exceeds } \sigma). \end{cases}$$

Note that g_μ has length $\sigma + 1$, not σ. Obviously, f and g_μ are admissible and f totally dominates g_μ since

$$\int_\lambda^\mu f(\alpha) = \aleph = \int_0^{\infty +} g_\mu(\alpha).$$

Therefore, there is a μ-elementary A-group H that has for its A-invariants $F_0(\alpha) = f(\alpha)$ and $F_\mu(\alpha) = g_\mu(\alpha)$. Since $\mu = \omega_1$, there is a d.s.c. group G such that (H, G) is an A_μ-pair. The proof of the existence theorem reveals that H has cardinality $\aleph$. To demonstrate that H cannot be decomposed as a direct sum of smaller groups, assume that it can. Suppose that $H = \Sigma \oplus H_i$, where $|H_i| < \aleph$ for each i. Consequently, g_μ must decompose as $g_\mu = \Sigma g_{\mu,i}$, where $g_{\mu,i}$ is admissible and $\int_0^\infty g_{\mu,i}(\alpha) \leqslant |H_i| < \aleph$. But this is absurd, since for at least one i we must have $g_{\mu,i}(\sigma) \neq 0$, which implies that $\int_0^\infty g_{\mu,i}(\alpha) \geqslant \aleph$ since $g_{\mu,i}$ is admissible.

Our final result shows that there is no class of reduced p-groups that is closed with respect to direct sums properly containing the class of A-groups whose members are determined by their A-invariants. Here, in the more general setting, we *define E_μ* $= \overline{(H/p^\mu H)}/(H/p^\mu H)$.

THEOREM 11. *Let $\mathscr{C}$ be a class of reduced p-groups closed with respect to direct sums (and such that membership is independent of notation). Suppose that the A-invariants determine the structure of all the members of $\mathscr{C}$. If $\mathscr{C}$ contains the A-groups then it is exactly the class of A-groups.*

PROOF. Suppose that E belongs to $\mathscr{C}$. Since $E \oplus H$ and H have the same A-invariants for some A-group H and since both belong to $\mathscr{C}$, it is evident that $E \oplus H = H$ and E is a direct summand of an A-group. Thus we know already that the only members of $\mathscr{C}$ that are not A-groups, if any, must be summands of A-groups. We shall demonstrate, however, that summands of A-groups belonging to $\mathscr{C}$ are, in fact, A-groups. This will be accomplished if we can show that a summand of an A-group has the same invariants as some A-group.

Let H be an A-group and let $H = K \oplus L$. We prove that the A-invariants of K agree with the A-invariants of some A-group by induction on the length σ of K. This is certainly the case if σ is countable, so assume that σ is uncountable. By employing Theorem 9, we may assume that $\sigma = \beta + \sigma$ for each limit ordinal $\beta < \sigma$. For otherwise $p^\beta K$ and $K/p^\beta K$ have the same invariants as A-groups by the induction hypothesis, and therefore so does K. Observe that $E_\mu(H) = E_\mu(K) \oplus E_\mu(L)$ and, of course, $f_H = f_K + f_L$. Since f_K is admissible and $\int_\beta^{\beta+\omega} f_K(\alpha) \geqslant \int_\gamma^{\gamma+\omega} f_K(\alpha)$ when

$\beta < \gamma$, we may assume without loss of generality (for some fixed cardinal m) that

$$\int_\beta^{\beta+\omega} f_K(\alpha) = m = \int_\gamma^{\gamma+\omega} f_K(\alpha)$$

whenever β and γ are limit ordinals less that σ. Again, since f_K is admissible, it is clear that $m \geqslant |\sigma|$. By the same argument used in the proof of the necessity of condition $(++)$ in the proof of the existence theorem, we can obtain the inequality

$$\int_\lambda^\mu f_K(\alpha) \geqslant |E_\mu(K)[p]|$$

for each $\lambda < \mu$. Therefore, if $\lambda < \mu$,

$$m \geqslant |E_\mu(K)[p]| \geqslant \int_0^{\infty^+} g_\mu(\alpha),$$

where $g_\mu = F_\mu^K$ is the extended Ulm-Kaplansky function of $E_\mu(K)$. Since g_μ does not exist (or is zero) when $\mu > \sigma$, and since $m \geqslant |\sigma|$, the preceding inequality yields

$$(++) \qquad\qquad \int_\lambda^{\lambda+\omega} f_K(\alpha) \geqslant m \geqslant \sum_{\lambda<\mu} \int_0^{\infty^+} g_\mu(\alpha).$$

Therefore, according to the existing theorem, K has the same A-invariants as some A-group, and the theorem is proved.

There remains at least one significant open question about A-groups. Are they closed with respect to direct summands? We suspect (and there is some historical basis for this opinion) that the summand question is relatively difficult. At least its solution has not yet fallen to our particular attack.

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