GENERAL DEFECT RELATIONS OF HOLOMORPHIC CURVES

BY
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Dedicated to Professor Yusaku Komatu on his 70th birthday

ABSTRACT. Let \( x : \mathbb{C} \to \mathbb{C} \) be a holomorphic curve of finite lower order \( \mu \), and let \( A = \{ \alpha \} \) be an arbitrary finite family of holomorphic curves \( \alpha : \mathbb{C} \to (\mathbb{C}^*)^n \) satisfying \( T(r, \alpha) = o(T(r, x)) \) \( (r \to \infty) \). Suppose \( x \) is nondegenerate with respect to \( A \), and \( A \) is in general position. We show the following general defect relations: (1) \( x \) has at most \( n \) deficient curves in \( A \) if \( \mu = 0 \). (2) \( \sum_{\alpha \in A} \delta(\alpha) \leq n \) if \( 0 < \mu \leq 1/2 \). (3) \( \sum_{\alpha \in A} \delta(\alpha) \leq \lceil 2\mu \rceil + 1 \) if \( 1/2 < \mu < +\infty \).

1. Introduction. Let \( f \) be a transcendental meromorphic function on the complex plane \( \mathbb{C} \) and let \( a_j \) \((j = 1, 2, \ldots, q)\) be distinct meromorphic functions on \( \mathbb{C} \) satisfying

\[
T(r, a_j) = o(T(r, f)), \quad r \to \infty.
\]

Then the deficiency of \( a_j \) with respect to \( f \) is defined by

\[
\delta(a_j, f) = 1 - \limsup_{r \to \infty} \frac{N(r, 0, f - a_j)}{T(r, f)},
\]

and \( a_j \) is called a deficient function of \( f \) when \( \delta(a_j, f) > 0 \).

In 1929 Nevanlinna [10] proved that the defect relation

\[
(1.1) \quad \sum_{j=1}^{q} \delta(a_j, f) \leq 2
\]

is valid for \( q = 3 \) and asked if (1.1) is true for all positive integers \( q \). In 1939 Dufresnoy [4] showed that \( \sum_{j=1}^{q} \delta(p_j, f) \leq 2 + d \) for distinct polynomials \( p_j \) of degree \( \leq d \). In 1964 Chuang [3] proved (1.1) when \( \delta(\infty, f) = 1 \). Recently, Yang [17] gave the following:

**Theorem A.** Let \( \mu \) be the lower order of \( f \).

(1) If \( \mu = 0 \), \( f \) has at most one deficient function (cf. Mori [8, Theorem 1]).

(2) If \( \mu > 0 \) then

\[
\sum_{j=1}^{q} \delta(a_j, f) \begin{cases} 
\leq 1 & (0 < \mu \leq 1/2, \quad q = 1), \\
< 1 - \cos \pi \mu & (0 < \mu \leq 1/2, \quad q \geq 2), \\
\leq 2 - \sin \pi \mu & (1/2 < \mu \leq 1), \\
\leq \min\{[2\mu] + 1, (\sqrt{2}/2)\pi \mu\} & (\mu > 1).
\end{cases}
\]

On the other hand, Mori [9] regarded \( \delta(a_j, f) \) as a deficiency of the moving divisor and extended the above result of Nevanlinna to the case of holomorphic...
mappings of $\mathbb{C}^n$ into $P_m \mathbb{C}$. Further, under some conditions, Shiffman [12], Mori [9], and Stoll [13] obtained general defect relations for moving divisors with respect to meromorphic functions on $\mathbb{C}^n$, holomorphic mappings of $\mathbb{C}^n$ into $P_m \mathbb{C}$, and holomorphic mappings of $\mathbb{M}$ into $P_m \mathbb{C}$, respectively, where $\mathbb{M}$ is a parabolic complex manifold.

The purpose of this paper is to extend Theorem A to the case of holomorphic curves. We assume the reader is familiar with Nevanlinna theory of meromorphic functions and holomorphic curves (cf. [10, 15, 16]).

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2. Notations and main results. We denote complex projective $n$-space by $P_n \mathbb{C}$ and dual complex projective $n$-space by $(P_n \mathbb{C})^*$. Let $x: \mathbb{C} \to P_n \mathbb{C}$ be a holomorphic curve and $\tilde{x} = (x_0, x_1, \ldots, x_n): \mathbb{C} \to \mathbb{C}^{n+1} \setminus \{0\}$ its reduced representation. We define the characteristic (order) function $T(r, x)$ of $x$ by

$$T(r, x) = \frac{1}{2\pi} \int_0^{2\pi} \log |\tilde{x}(re^{i\theta})| d\theta - \log |\tilde{x}(0)|,$$

where $|\tilde{x}(z)| = (\sum_{j=0}^{n} |x_j(z)|^2)^{1/2}$.

Let $\alpha: \mathbb{C} \to (P_n \mathbb{C})^*$ be a holomorphic curve and $\tilde{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n): \mathbb{C} \to (\mathbb{C}^{n+1})^* \setminus \{0\}$ its reduced representation. We denote by $N(r, \alpha) \equiv N(r, \alpha; x)$ the counting function of zeros of the entire function

$$F(z) = (x(z), \tilde{\alpha}(z)) = \sum_{j=0}^{n} \alpha_j(z)x_j(z) \neq 0,$$

that is, $N(r, \alpha) = N(r, 0, F)$. We define the deficiency $\delta(\alpha) \equiv \delta(\alpha, x)$ of $\alpha$ with respect to $x$ by

$$\delta(\alpha) = 1 - \limsup_{r \to \infty} \frac{N(r, \alpha)}{T(r, x)}.$$

Since $|F(z)| \leq |\tilde{\alpha}||\tilde{x}|$, (2.1) and Jensen’s formula imply $0 \leq \delta(\alpha) \leq 1$. If $\delta(\alpha) > 0$ then we say that $\alpha$ is a deficient curve. We remark that if $\alpha$ satisfies

$$T(r, \alpha) = o(T(r, x)), \quad r \to \infty,$$

then (2.2) is reduced to

$$\delta(\alpha) = 1 - \limsup_{r \to \infty} \frac{T(r, \alpha)}{T(r, x)}.$$

Now let $\alpha^{(k)}: \mathbb{C} \to (P_n \mathbb{C})^*$ ($k = 1, 2, \ldots, q$; $q > n + 1$) be holomorphic curves and $\tilde{\alpha}^{(k)} = (\alpha_0^{(k)}, \alpha_1^{(k)}, \ldots, \alpha_n^{(k)}): \mathbb{C} \to (\mathbb{C}^{n+1})^* \setminus \{0\}$ their reduced representation. If $\det\{(\alpha_1^{(k)}(z))_{1 \leq j \leq n+1, 0 \leq k \leq n}\} \neq 0$ for any $\{k_1, k_2, \ldots, k_{n+1}\} \subset \{1, 2, \ldots, q\}$, then we say that $\{\alpha_1^{(k)}\}_{k=1}^{q}$ is in general position. A curve $x: \mathbb{C} \to P_n \mathbb{C}$ is called nondegenerate with respect to $\{\alpha^{(k)}\}$ if

$$F_k(z) = \sum_{j=0}^{n} \alpha_j^{(k)}(z)x_j(z) \neq 0 \quad \text{for all } k.$$

Using Yang’s idea [17], we prove the following defect relations:
Theorem 1. Let \( x: C \to P_n C \) be a holomorphic curve of lower order \( \mu \) and \( A = \{ \alpha \} \) an arbitrary finite family of holomorphic curves \( \alpha: C \to (P_n C)^* \) satisfying
\[
T(r, \alpha) = o\left(T(r, x)\right), \quad r \to \infty.
\]
Suppose \( x \) is nondegenerate with respect to \( A \), and \( A \) is in general position. Then:

(I) If \( \mu = 0 \) there are at most \( n \) deficient curves in \( A \).

(II) Assume \( 0 < \mu \leq 1/2 \) and put \( B = \{ \alpha \in A; \delta(\alpha) > 1 - \cos \pi \mu \} \) and \( C = \{ \alpha \in A; \delta(\alpha) = 1 - \cos \pi \mu \} \). If there are \( n \) curves belonging to \( B \cup C \), then all the remaining deficiencies are equal to zero. If \( \#B + p < n \), then
\[
\sum_{\alpha \in A-B} \delta(\alpha) \leq (n-p)(1-\cos \pi \mu),
\]
where equality holds if and only if \( \#C = n-p \) and \( \delta(\alpha) = 0 \) for all \( \alpha \in A-(B \cup C) \).

(III) If \( 1/2 < \mu < +\infty \) we have
\[
\sum_{\alpha \in A} \delta(\alpha) \leq [2n\mu] + 1 - \cos(2n\mu - [2n\mu])(\pi/2).
\]

As an immediate consequence of (I) and (II) we have

**Corollary.** If \( 0 \leq \mu \leq 1/2 \), then \( \sum_{\alpha \in A} \delta(\alpha) \leq n \).

3. Spread relation. Let \( x: C \to P_n C \) be a holomorphic curve of lower order \( \mu \) (\( 0 < \mu < +\infty \)) and \( \hat{x} = (x_0, x_1, \ldots, x_n): C \to C^{n+1} - \{0\} \) its reduced representation. Suppose that a holomorphic curve \( \alpha: C \to (P_n C)^* \) and its reduced representation \( \hat{\alpha}: C \to (C^{n+1})^* - \{0\} \) satisfy
\[
T(r, \alpha) = o\left(T(r, x)\right), \quad r \to \infty,
\]
and \( (\hat{x}(z), \hat{\alpha}(z)) \neq 0 \). We put
\[
\|x(z), \alpha(z)\| = \|\hat{x}(z), \hat{\alpha}(z)\|/(\|\hat{x}(z)\|\|\hat{\alpha}(z)\|).
\]

A positive, increasing, unbounded sequence \( \{r_m\} \) is called a sequence of Pólya peaks of order \( \mu \) of \( T(r, x) \) if it is possible to find positive sequences \( \{r'_m\}, \{r''_m\} \), and \( \{\varepsilon_m\} \) such that, as \( m \to \infty \), then \( r_m \to \infty \), \( r_m/r'_m \to \infty \), \( r''_m/r'_m \to \infty \), \( \varepsilon_m \to 0 \), and
\[
T(t, x)/T(r_m, x) \leq (t/r_m)^\mu(1 + \varepsilon_m) \quad (r'_m < t < r''_m).
\]

We now define the spread \( \sigma(\alpha) \equiv \sigma(\alpha, x) \) of \( \alpha \) with respect to \( x \). Let \( \{r_m\} \) be a sequence of Pólya peaks of order \( \mu \) of \( T(r, x) \) and \( \Lambda(r) \) a positive function satisfying
\[
\Lambda(r) = o\left(T(r, x)\right), \quad r \to \infty.
\]

Put
\[
E_\Lambda(r, \alpha) \equiv E_\Lambda(r, \alpha; x) = \{\theta; \log \|x(re^{i\theta}), \alpha(re^{i\theta})\| < -\Lambda(r)\} \subset (-\pi, \pi)
\]
and let
\[
\sigma_\Lambda(\alpha) = \liminf_{m \to \infty} \text{meas} E_\Lambda(r_m, \alpha).
\]

Then we define
\[
\sigma(\alpha) = \inf_\Lambda \sigma_\Lambda(\alpha),
\]
where the infimum is taken over all functions \( \Lambda(r) \) satisfying (3.3).

We obtain the following spread relation, which is a generalization of Baernstein [1]—Yang [17] and the author [11]—Krytov [7].
Theorem 2 (Spread Relation). Let \( x: \mathbb{C} \to P_n \mathbb{C} \) be a holomorphic curve of lower order \( \mu \) \((0 < \mu < +\infty)\). Then

\[
\sigma(\alpha) \geq \min\{2\pi, (4/\mu) \sin^{-1}(\delta(\alpha)/2)^{1/2}\}
\]

for every holomorphic curve \( \alpha: \mathbb{C} \to (P_n \mathbb{C})^* \) satisfying (3.1) and \( \|x(z), \alpha(z)\| \neq 0 \).

Proof. We may assume that the reduced representations \( \tilde{x} \) and \( \tilde{\alpha} \) of \( x \) and \( \alpha \), respectively, satisfy

\[
|\tilde{x}(0)| = 1 \quad \text{and} \quad |\tilde{\alpha}(0)| = 1.
\]

We have

\[
E_\Lambda(r, \alpha) = \{ \theta; \log |\tilde{x}(re^{i\theta})| + \log |\tilde{\alpha}(re^{i\theta})| - \log |F(re^{i\theta})| > \Lambda(r) \},
\]

where

\[
F(z) = \langle \tilde{x}(z), \tilde{\alpha}(z) \rangle = \sum_{j=0}^{n} \alpha_j(z)x_j(z).
\]

Define the entire function \( h(z) \) by

\[
h(z) = cz^{-k}F(z),
\]

where \( k \) is a nonnegative integer, \( c \) is a nonzero constant, and

\[
h(0) = 1.
\]

Then

\[
N(r, \alpha) = N(r, 0, F) = N(r, 0, h) + k \log r.
\]

We put

\[
G(z) = \log |\tilde{x}(z)| + \log |\tilde{\alpha}(z)| - \log |h(z)|,
\]

\[
\Lambda_1(r) = \Lambda(r) + k \log r - \log |c|,
\]

and

\[
E_{\Lambda_1}(r, G) = \{ \theta; G(re^{i\theta}) > \Lambda_1(r) \}.
\]

Then we deduce that

\[
E_{\Lambda_1}(r, G) = E_{\Lambda}(r, \alpha),
\]

and

\[
\Lambda_1(r) = o(T(r, x)) \quad (r \to \infty).
\]

We next write \( T(r) = T(r, x) + T(r, \alpha) \). Then it follows from (3.1) that

\[
T(r) \sim T(r, x), \quad r \to \infty,
\]

that is, \( T(r)/T(r, x) \to 1 \) as \( r \to 1 \). Hence, \( T(r) \) and \( T(r, x) \) have the same sequence of Pólya peaks, and, combining (3.6) and (3.10) with the definition of deficiency \( \delta(\alpha) \), we have

\[
\delta(\alpha) = 1 - \limsup_{r \to \infty} \frac{N(r, 0, h)}{T(r)}.
\]
Therefore, in order to prove Theorem 2, from (3.8)–(3.11) it is sufficient to prove that

\[(3.12) \quad \liminf_{r \to \infty} \text{meas} E_{\Lambda_1}(r, G) \geq \min \left\{ 2\pi, \frac{4}{\mu} \sin^{-1} \left( \frac{\delta(\alpha)}{2} \right)^{1/2} \right\}.\]

Following Baernstein [1] we now define

\[T^*(z) = \sup_E \frac{1}{2\pi} \int_E G(re^{i\omega}) \, d\omega + N(r, 0, h) \quad (z = re^{i\theta}, \ 0 < r < +\infty, \ 0 < \theta \leq \pi),\]

where the supremum is taken over all measurable sets \(E \subset (-\pi, \pi]\) whose measure equals 2\(\pi\). Hence, we deduce that

\[(3.13) \quad T^*(0) = 0.\]

Then since \(\log |x(z)| + \log |x(z)|\) and \(\log |h(z)|\) are subharmonic, from (3.4), (3.5), (3.7), (3.13), and Theorem A' in Baernstein [2], it follows that \(T^*(z)\) is subharmonic in \(\{z; \text{ Im} z > 0\}\) and continuous on \(\{z; \text{ Im} z \geq 0\}\). By definition of \(T^*(z)\),

\[(3.14) \quad T^*(re^{i\pi}) = T^*(-r) = T(r), \quad T^*(r) = N(r, 0, h) \equiv N(r).\]

Since \(|h(z)| \leq |c| |z|^{-k} |x(z)| |x(z)|\), then \(G(z) + \log |c| - k \log |z| \geq 0\) for all \(z\). Hence, we deduce that

\[(3.15) \quad T^*(re^{i\theta}) \leq T^*(re^{i\pi}) + ((\pi - \theta)/\pi) \{\log |c| - k \log r\} < T(r) + K \quad (r > 1),\]

where \(K\) is a positive constant such that \(\log^+ |c| < K\).

Now we can apply the arguments of Baernstein [1, pp. 430–433] and the author [11, pp. 364–365]. For the sake of clarification we sketch them here.

If \(\delta(\alpha) = 0\) there is nothing to prove, so we may assume \(\delta(\alpha) > 0\). Put

\[\gamma = (1/2\pi) \min \{2\pi, (4/\mu) \sin^{-1}(\delta(\alpha)/2)^{1/2}\}.\]

Then

\[(3.16) \quad 0 < \gamma \leq 1, \quad 0 < \gamma \mu \leq 1/2 \quad \text{and} \quad 1 - \delta(\alpha) \leq \cos \pi \gamma \mu.\]

Define

\[v(z) = T^*(z^\gamma) \quad (z = re^{i\theta}, \ 0 < r < \infty, \ 0 \leq \theta \leq \pi).\]

Then \(v(z)\) is subharmonic in \(\{z; \text{ Im} z > 0\}\) and continuous in \(\{z; \text{ Im} z \geq 0\}\). Hence, when \(re^{i\theta} \in D_R = \{z = re^{i\theta}; \ 0 < r < R, \ 0 < \theta < \pi\},

\[(3.17) \quad v(re^{i\theta}) \leq \int_{-R}^{R} v(t) A(t, r, \theta, R) \, dt + \int_{0}^{\pi} v(Re^{i\psi}) B(\psi, r, \theta, R) \, d\psi,\]

where

\[A(t, r, \theta, R) = \frac{1}{\pi} \frac{r \sin \theta}{t^2 - 2tr \cos \theta + r^2} - \frac{1}{\pi} \frac{R^2 r \sin \theta}{R^4 - 2rt R^2 \cos \theta + r^2 t^2},\]

\[B(\psi, r, \theta, R) = \frac{2Rr \sin \theta}{\pi} \frac{(R^2 - r^2) \sin \psi}{|R^2 e^{2t \psi} - 2r Re^{i\psi} \cos \theta + r^2|^2}.\]

It follows from (3.14) and (3.15) that

\[(3.18) \quad v(t) = T^*(t^\gamma) = N(t^\gamma), \quad v(-t) \leq T(r) + K, \quad v(Re^{i\psi}) \leq T(r) + K.\]
We deduce that

$$0 < A(t, r, \theta, R) \leq \frac{r \sin \theta}{\pi t^2 - 2tr \cos \theta + r^2} \equiv P(t, r, \pi - \theta) \quad (0 < r < R, 0 < \theta < \pi),$$

(3.19)

$$0 < B(\psi, r, \theta, R) \leq 32r/\pi R \quad (0 < \theta < \pi, \ 0 < \psi < \pi, \ 0 < r < R/2),$$

and

$$\int_0^\infty P(t, r, \theta) \, d\theta = \frac{\theta}{\pi} < 1 \quad (0 < \theta < \pi, \ r > 0).$$

(3.21)

Using (3.18)-(3.21) in (3.17) we obtain

$$v(re^{i\theta}) \leq \int_0^R N(t') P(t, r, \pi - \theta) \, dt + \int_0^R T(t') P(t, r, \theta) \, dt$$

$$+ 32(r/R)(T(R') + K) + K \quad (0 < \theta < \pi, \ 0 < r < R/2).$$

(3.22)

Let \{r_m\} be any sequence of Pólya peaks of order \(\mu\) of \(T(r, \alpha)\) and put \(s_m = r_m^{1/\gamma}\). \{r_m\} is also a sequence of Pólya peaks of order \(\mu\) of \(T(r)\). Hence, it follows from the discussion of Baernstein [1, pp. 332–333] that

$$v(s_me^{i\theta}) \leq T(r_m) \{\cos(\pi - \theta)\gamma \mu + \alpha_m\} \quad (m = 1, 2, \ldots; \ 0 < \theta < \pi),$$

(3.23)

where \{\alpha_m\} is a sequence tending to zero. Let

$$\sigma_m = \text{meas } E_{\Lambda_1}(r_m, G) \equiv \text{meas } E(r_m).$$

Then (3.12) is equivalent to the inequality

$$\liminf_{m \to \infty} \sigma_m \geq 2\pi \gamma.$$ 

(3.24)

It follows from (3.4) and (3.5) that

$$T(r_m) = \frac{1}{2\pi} \int_0^\pi (\log |\hat{x}(r_m e^{i\omega})| + \log |\hat{\alpha}(r_m e^{i\omega})|) \, d\omega$$

$$= \frac{1}{2\pi} \int_0^\pi G(r_m e^{i\omega}) \, d\omega + N(r_m)$$

$$\leq \frac{1}{2\pi} \int_{E(r_m)} G(r_m e^{i\omega}) \, d\omega + N(r_m) + \Lambda_1(r_m)$$

$$\leq T^*(r_m e^{i\sigma_m/2}) + \Lambda_1(r_m).$$

Dividing by \(T(r_m)\) and remembering (3.15), we find that

$$\lim_{m \to \infty} \frac{T^*(r_m e^{i\sigma_m/2})}{T(r_m)} = 1.$$ 

(3.25)

Let \(M = \{m; \sigma_m < 2\pi \gamma\}\). If \(M\) is a finite set, then (3.24) holds and we have finished, so we assume that \(M\) is infinite. The point \((r_m e^{i\sigma_m/2})^{1/\gamma} = s_m e^{i\sigma_m/2\gamma}\) belongs to the domain of \(v(z)\), i.e. the upper half-plane, if and only if \(m \in M\), in which case we have

$$T^*(r_m e^{i\sigma_m/2}) = v(s_m e^{i\sigma_m/2\gamma}) \quad (m \in M).$$

Using this in (3.25), comparing with (3.23), and remembering (3.16), we deduce that \(\lim_{m \to \infty, m \in M} \sigma_m/2\gamma = \pi\), which shows that (3.24) holds in this case also.

Thus the proof of Theorem 2 is complete.
4. Lemmas. In order to prove Theorem 1 we need several lemmas.

**Lemma 1.** Let \( x: C \to P_nC \) be a holomorphic curve, \( \alpha^{(k)}: C \to (P_nC)^* \) \((k = 1, 2, \ldots, n + 1)\) holomorphic curves in general position, and \( \Lambda(r) \) a positive function satisfying (3.2). Suppose \( x \) is nondegenerate with respect to \( \{\alpha^{(k)}\}_{k=1}^{n+1} \). Then

\[
\text{meas} \left\{ \bigcap_{k=1}^{n+1} E_{\Lambda}(r, \alpha^{(k)}, x) \right\} \leq \frac{2\pi}{\Lambda(r)} \left\{ \sum_{k=1}^{n+1} T(r, \alpha^{(k)}) + K \right\},
\]

where \( K \) is a constant independent of \( \Lambda \).

**Proof.** Let \( \tilde{x} = (x_0, x_1, \ldots, x_n): C \to C^{n+1} - \{0\} \) and \( \tilde{\alpha}^{(k)} = (\alpha_0^{(k)}, \alpha_1^{(k)}, \ldots, \alpha_n^{(k)}): C \to (C^{n+1})^* - \{0\} \) be reduced representations of \( x \) and \( \alpha^{(k)} \), respectively. By the linear equations

\[
\sum_{j=0}^{n} \alpha_j^{(k)}(z)x_j(z) = F_k(z), \quad k = 1, 2, \ldots, n + 1,
\]

we have \( x_j = D_j/D \), where \( D \) is the coefficient determinant of (4.1) and

\[
D_j = \sum_{k=1}^{n+1} (-1)^{j+k+1} D_{kj} F_k,
\]

where \( D_{kj} \) is the minor of \( D \) obtained by omitting the \( k \)th row and \((j + 1)\)th column from \( D \). The Hadamard inequality yields

\[
|D_{kj}| \leq \prod_{l=1, l \neq k}^{n+1} |\tilde{\alpha}^{(l)}|,
\]

so

\[
|x_j| \leq \frac{1}{|D|} \sum_{k=1}^{n+1} \left( \prod_{l=1, l \neq k}^{n+1} |\tilde{\alpha}^{(l)}| \right) |F_k|.
\]

Hence,

\[
\log |x_j| \leq \log \left\{ \prod_{k=1}^{n+1} \left( \sum_{l=1, l \neq k}^{n+1} |\tilde{\alpha}^{(l)}| \right) |F_k| \right\} - \log |D| + \frac{1}{2} \log(n + 1).
\]

For \( z = re^{i\theta} \), with \( \theta \in E(r) \equiv \bigcap_{k=1}^{n+1} E_{\Lambda}(r, \alpha^{(k)}) \), we have

\[
|F_k(z)|/(|\tilde{z}(z)| |\tilde{\alpha}^{(k)}(z)|) < \exp(-\Lambda(r)), \quad k = 1, 2, \ldots, n + 1,
\]

so

\[
\left\{ \prod_{k=1}^{n+1} \left( \sum_{l=1, l \neq k}^{n+1} |\tilde{\alpha}^{(l)}| \right) |F_k| \right\} / \left( |\tilde{z}| \prod_{k=1}^{n+1} |\alpha^{(k)}| \right) < (n + 1) \exp(-\Lambda(r)).
\]

Hence, combining this with (4.2) we obtain

\[
\sum_{k=1}^{n+1} \log |\tilde{\alpha}^{(k)}(z)| - \log |D(z)| > \Lambda(r) - \frac{3}{2} \log(n + 1) \text{ for } z = re^{i\theta}, \ \theta \in E(r).
\]
The Hadamard inequality implies $|D(z)| \leq \prod_{k=1}^{n+1} |\tilde{a}(k)(z)|$, so
\begin{equation}
\sum_{k=1}^{n+1} \log |\tilde{a}(k)(z)| - \log |D(z)| \geq 0
\end{equation}
for all $z$. Hence, it follows from (4.3) and (4.4) that
\begin{align*}
&\frac{1}{2\pi} \left( \Lambda(r) - \frac{3}{2} \log(n + 1) \right) \text{meas } E(r) \\
&\leq \frac{1}{2\pi} \int_{E(r)} \left( \sum_{k=1}^{n+1} \log |\tilde{a}(k)(re^{i\theta})| - \log |D(re^{i\theta})| \right) d\theta \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{k=1}^{n+1} \log |\tilde{a}(k)(re^{i\theta})| - \log |D(re^{i\theta})| \right) d\theta \\
&= \sum_{k=1}^{n+1} \left( T(r, \alpha(k)) + \log |\tilde{a}(k)(0)| \right) - (N(r, 0, D) + \log |c_\lambda|),
\end{align*}
where
\begin{equation}
D(z) = c_\lambda z^\lambda + c_{\lambda+1} z^{\lambda+1} + \cdots \quad (c_\lambda \neq 0).
\end{equation}
Hence, choosing a constant $K$ such that
\begin{equation}
\sum_{k=1}^{n+1} \log |\tilde{a}(k)(0)| - \log |c_\lambda| + \frac{3}{2} \log(n + 1) < K,
\end{equation}
we obtain
\begin{equation}
\text{meas } E(r) \leq \frac{2\pi}{\Lambda(r)} \left\{ \sum_{k=1}^{n+1} T(r, \alpha(k)) + K \right\},
\end{equation}
which proves Lemma 1.

Now let $A = \{ \alpha(k) \}_{k=1}^{q} \quad (q \geq n + 1)$ be a finite family of holomorphic curves in general position which satisfy
\begin{equation}
T(r, \alpha(k)) = o(T(r, x)), \quad r \to \infty.
\end{equation}
Further suppose that $x$ is nondegenerate with respect to $A$. We define $I_A(r, \alpha(k)) \equiv I_A(r, \alpha^{(k)}, A)$ by
\begin{equation}
I_A(r, \alpha(k)) = \bigcup_{(l_1, \ldots, l_n)} \left\{ E_A(r, \alpha(k)) \cap \left( \bigcap_{j=1}^{n} E_A(r, \alpha(l_j)) \right) \right\},
\end{equation}
where the union is taken over all possible $n$-tuples $(l_1, \ldots, l_n) \subset (1, \ldots, k-1, k+1, \ldots, q)$, and
\begin{equation}
\sigma'_A(\alpha(k)) = \liminf_{m \to \infty} \text{meas } I_A(r_m, \alpha(k)).
\end{equation}
Then we can deduce

**Lemma 2.** For any \((k_1, \ldots, k_{n+1}) \subset (1, 2, \ldots, q)\) we have

\[
\bigcap_{j=1}^{n+1} I_A(r, \alpha^{(k_j)}) = \emptyset.
\]

We now prove

**Lemma 3.** There is a sequence of positive functions \(\Lambda_\nu(r)\) satisfying (3.3) such that

\[
\liminf_{\nu \to \infty} \sigma'_{\Lambda_\nu}(\alpha^{(k)}) \geq \min \left\{ 2\pi, \frac{4}{\mu} \sin^{-1} \left( \frac{\delta(\alpha^{(k)})}{2} \right)^{1/2} \right\}.
\]

**Proof.** We can choose a constant \(K\) such that Lemma 1 is valid for all \((n+1)\)-tuples \((k_1, \ldots, k_{n+1}) \subset (1, \ldots, q)\). Hence, it follows that

\[
\sum_{(l_1, \ldots, l_n)} \text{meas} \left\{ E_A(r, \alpha^{(k)}) \cap \left( \bigcap_{j=1}^{n} E_A(r, \alpha^{(l_j)}) \right) \right\}
\]

\[
\leq \sum_{(l_1, \ldots, l_n)} \frac{2\pi}{\Lambda(r)} \left\{ T(r, \alpha^{(k)}) + \sum_{j=1}^{n} T(r, \alpha^{(l_j)}) + K \right\}
\]

\[
\leq \frac{2\pi}{\Lambda(r)} \left( q - 1 \right) \left( \sum_{l=1}^{q} T(r, \alpha^{(l)}) + K \right).
\]

Choose a sequence of positive numbers \(\varepsilon_\nu\) such that \(\lim_{\nu \to \infty} \varepsilon_\nu = 0\). Put

\[
\Lambda_\nu(r) = \frac{2\pi}{\varepsilon_\nu} \left( q - 1 \right) \left( \sum_{l=1}^{q} T(r, \alpha^{(l)}) + K \right).
\]

Then it follows from (4.5) that for every fixed \(\nu\), \(\Lambda_\nu(r)\) satisfies (3.3). From the definition of \(\sigma_\Lambda(\alpha)\) and (4.6)–(4.9) we deduce that, for every fixed \(\nu\),

\[
\text{meas} I_{\Lambda_\nu}(r_m, \alpha^{(k)}) \geq \text{meas} E_{\Lambda_\nu}(r_m, \alpha^{(k)})
\]

\[
- \sum_{(l_1, \ldots, l_n)} \text{meas} \left\{ E_{\Lambda_\nu}(r_m, \alpha^{(k)}) \cap \left( \bigcap_{j=1}^{n} E_{\Lambda_\nu}(r_m, \alpha^{(l_j)}) \right) \right\}
\]

\[
\geq \text{meas} E_{\Lambda_\nu}(r_m, \alpha^{(k)}) - \varepsilon_\nu,
\]

so

\[
\sigma'_{\Lambda_\nu}(\alpha^{(k)}) \geq \sigma_{\Lambda_\nu}(\alpha^{(k)}) - \varepsilon_\nu.
\]

Therefore, our spread relation implies

\[
\sigma'_{\Lambda_\nu}(\alpha^{(k)}) \geq \min \left\{ 2\pi, \frac{4}{\mu} \sin^{-1} \left( \frac{\delta(\alpha^{(k)})}{2} \right)^{1/2} \right\} - \varepsilon_\nu
\]

for every fixed \(\nu\). Here let \(\nu \to \infty\). Then we obtain Lemma 3.
We deduce the following obvious

**Lemma 4.** Let \( E_1, \ldots, E_q \) be measurable subsets of \( \{|z| = 1\} \). Define \( E = E_1 \cup \cdots \cup E_q \). Take an integer \( p \) with \( 1 \leq p \leq q \). Assume that for any selection of integers \( 1 \leq i_1 < i_2 < \cdots < i_p \leq q \), we have \( E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_p} = \emptyset \). Then

\[
\sum_{j=1}^{q} \text{meas } E_j \leq (p - 1) \text{meas } E.
\]

We finally prove

**Lemma 5.** Let \( x : \mathbb{C} \to \mathbb{P}_n \mathbb{C} \) be a holomorphic curve of lower order \( \mu \) (\( 0 < \mu < +\infty \)) and \( \alpha^{(k)} : \mathbb{C} \to (\mathbb{P}_n \mathbb{C})^* \), \( k = 1, \ldots, q \) (\( q \geq n + 1 \)) holomorphic curves satisfying (4.5). Suppose that \( x \) is nondegenerate with respect to \( \{\alpha^{(k)}\} \), and \( \{\alpha^{(k)}\} \) is in general position. Then

\[
\sum_{k=1}^{q} \min \left\{ 2\pi, \frac{4}{\mu} \sin^{-1} \left( \frac{\delta(\alpha^{(k)})}{2} \right)^{1/2} \right\} \leq 2n\pi.
\]

**Proof.** Lemmas 2 and 4 imply

\[
\sum_{k=1}^{q} \text{meas } I_A(r, \alpha^{(k)}) \leq 2n\pi,
\]

and, consequently,

\[
\sum_{k=1}^{q} \sigma_A'(\alpha^{(k)}) \leq 2n\pi.
\]

Hence, we deduce from Lemma 3 that

\[
\sum_{k=1}^{q} \min \left\{ 2\pi, \frac{4}{\mu} \sin^{-1} \left( \frac{\delta(\alpha^{(k)})}{2} \right)^{1/2} \right\} \leq 2n\pi,
\]

which proves Lemma 5.

5. **Proof of Theorem 1(1).** Theorem 1(1) is an immediate consequence of the Corollary to Theorem 3, which is a generalization of results of Edrei-Fuchs [6], Toda [14], and Mori [8] (cf. Yang [17]).

**Theorem 3.** Let \( x : \mathbb{C} \to \mathbb{P}_n \mathbb{C} \) be a holomorphic curve of lower order \( \mu \) and \( \alpha^{(k)} : \mathbb{C} \to (\mathbb{P}_n \mathbb{C})^* \) \( k = 1, \ldots, n + 1 \) holomorphic curves satisfying \( T(r, \alpha^{(k)}) = o(T(r, x)) \) \( r \to \infty \). Assume that \( x \) is nondegenerate with respect to \( \{\alpha^{(k)}\}_k^{n+1} \), \( \{\alpha^{(k)}\}_k^{n+1} \) is in general position, and \( \alpha^{(k)} \) \( k = 1, \ldots, n + 1 \) are deficient curves with respect to \( x \). If \( \gamma = \max_{1 \leq k \leq n+1} \{1 - \delta(\alpha^{(k)})\} \), then

\[
\mu \geq \begin{cases} 
\log(1/\gamma(2-\gamma)) & (\gamma \neq 0), \\
\log(1+4/\gamma(1-\gamma)) & (\gamma = 0).
\end{cases}
\]

**Corollary.** Holomorphic curves with more than \( n \) deficient curves in general position have a positive lower order.

**Proof of Theorem 3.** Let \( \tilde{x} = (x_0, x_1, \ldots, x_n) : \mathbb{C} \to \mathbb{C}^{n+1} - \{0\} \) and \( \tilde{\alpha}^{(k)} = (\alpha_0^{(k)}, \alpha_1^{(k)}, \ldots, \alpha_n^{(k)}) : \mathbb{C} \to (\mathbb{C}^{n+1})^* - \{0\} \) be reduced representations of \( x \)
and $\alpha^{(k)}$, respectively. Then we can choose a nonzero complex number $\beta$ such that a reduced representation of a constant curve $\alpha^{(n+2)}: C \rightarrow (P_n C)^*$ is $\bar{\alpha}^{(n+2)} = (1, \beta, \beta^2, \ldots, \beta^n)$, $\{\alpha^{(k)}\}_{k=1}^{n+2}$ is in general position, and $x$ is nondegenerate with respect to $\{\alpha^{(k)}\}_{k=1}^{n+2}$.

We now consider the holomorphic curve $y: C \rightarrow P_n C$ which Mori $[9]$ constructed from $x$ and $\{\alpha^{(k)}\}_{k=1}^{n+2}$. We put

$$F_k(z) = \sum_{j=0}^{n} \alpha^{(k)}_j(z) x_j(z).$$

Let $G = \det \{\alpha^{(k)}_j\}_{0 \leq j \leq n, 1 \leq k \leq n+1}$, be the determinant of the matrix replaced the $k$th column vector $t(\alpha^{(k)}_0, \ldots, \alpha^{(k)}_n)$ of the matrix $\{\alpha^{(k)}_j\}_{0 \leq j \leq n, 1 \leq k \leq n+1}$ by the column vector $t(\alpha^{(n+2)}_0, \ldots, \alpha^{(n+2)}_n)$, and $g_k = G_k/G$. We denote by $h(z)$ a common factor among $g_1(z)F_1(z), \ldots, g_n(z)F_n(z)$ and $g_{n+1}(z)F_{n+1}(z)$ up to non-vanishing entire functions such that $y_{k-1}(z) = g_k(z)F_k(z)/h(z)$ ($k = 1, \ldots, n+1$) are entire. Then, by the reasoning of Mori $[9]$, proof of Theorem 3.4, the holomorphic curve $y: C \rightarrow P_n C$ induced by $\bar{y} = (y_0, y_1, \ldots, y_n): C \rightarrow C^{n+1} - \{0\}$ has the properties

$$(5.2) \quad T(r, x) \sim T(r, y) \quad (r \rightarrow \infty),$$

and, for $k = 1, 2, \ldots, n+1$,

$$(5.3) \quad N(r, 0, y_{k-1}) = N(r, 0, F_k) + o(T(r, x)) \quad (r \rightarrow \infty).$$

Now we prove (5.2) and (5.3). Hadamard’s inequality implies

$$|G_k| \leq B \prod_{j=1, \neq k}^{n+1} |\alpha^{(j)}|,$$

where $B = |\bar{\alpha}^{(n+2)}|$ is a constant. Using $|F_k| \leq |\bar{z}| |\alpha^{(k)}|$, together with (5.4), we have

$$|y_{k-1}| = \left| \frac{G_kF_k}{hG} \right| \leq B|\bar{z}| \prod_{j=1}^{n+1} \left| \frac{\alpha^{(j)}}{|hG|} \right|,$$

so

$$|\bar{y}| \leq (n+1)^{1/2} B|\bar{z}| \prod_{j=1}^{n+1} \left| \frac{\alpha^{(j)}}{|hG|} \right|.$$

Hence, we have, with a suitable constant $K_1$,

$$T(r, y) \leq T(r, x) + \sum_{j=1}^{n+1} T(r, \alpha^{(j)}) - N(r, 0, G) - N(r, 0, h) + N(r, \infty, h) + K_1.$$

Since $N(r, \infty, h) \leq N(r, 0, G)$, we obtain

$$(5.5) \quad T(r, y) \leq T(r, x) + \sum_{j=1}^{n+1} T(r, \alpha^{(j)}) + K_1.$$

There is a mistake in Mori’s proof of his Theorem 3.4 $[9]$, but he provides a correct proof in the “Correction to $[9]$”. Unfortunately, his correction will not appear in the published paper, so we prove (5.2) and (5.3) along the lines of his correct proof.
On the other hand, it follows from
\[ G_k \sum_{j=0}^{n} \alpha_j^{(k)} x_j = G y_{k-1} \quad (k = 1, \ldots, n + 1) \]
that \( x_j = G H_j / H \), where \( H = \det \{ G_k \alpha_j^{(k)} \}_{1 \leq k \leq n + 1, 0 \leq j \leq n} \) and \( H_j \) is the determinant of the matrix obtained by replacing the \( j \)th column vector
\[ t(G_1 \alpha_1^{(1)}, \ldots, G_n+1 \alpha_{n+1}^{(n+1)}) \]
of the matrix \( \{ G_k \alpha_j^{(k)} \}_{1 \leq k \leq n + 1, 0 \leq j \leq n} \) by the column vector \( t(y_0, \ldots, y_n) \). We put
\[ \tilde{\eta}_k = t(G_1 \alpha_1^{(1)}, \ldots, G_n+1 \alpha_{n+1}^{(n+1)}) \]
Then using Hadamard’s inequality together with (5.4), we have
\[ |H_j| \leq |\tilde{y}| \prod_{k=0, \neq j}^{n} |\tilde{\eta}_k|, \]
and
\[ |\tilde{\eta}_k|^2 = \sum_{l=1}^{n+1} |G_l \alpha_l^{(l)}|^2 \leq \sum_{l=1}^{n+1} \left( B \prod_{j=1, \neq l}^{n+1} |\tilde{\alpha}^{(j)}| \right)^2 |\tilde{\alpha}^{(l)}|^2 = (n + 1) B^2 \prod_{j=1}^{n+1} |\tilde{\alpha}^{(j)}|^2, \]
so
\[ |H_j| \leq |\tilde{y}| \left( (n + 1)^{1/2} B \prod_{k=1}^{n+1} |\tilde{\alpha}^{(k)}| \right)^n. \]
By Hadamard’s inequality we also have
\[ (5.6) \quad |G| \leq \prod_{k=1}^{n+1} |\tilde{\alpha}^{(k)}|. \]
Hence,
\[ \log |\tilde{x}| = \log \left( \sum_{j=0}^{n} \left| \frac{h G H_j}{H} \right|^2 \right)^{1/2} \]
\[ = \log |G| + \log \left( \sum_{j=0}^{n} |H_j|^2 \right)^{1/2} + \log |h| - \log |H| \]
\[ \leq \log |\tilde{y}| + (n + 1) \sum_{k=1}^{n+1} \log |\tilde{\alpha}^{(k)}| + \log |h| - \log |H| + \log B^n (n + 1)^{(n+1)/2}, \]
so, with a suitable constant \( K_2 \),
\[ T(r, x) \leq T(r, y) + (n + 1) \sum_{k=1}^{n+1} T(r, \alpha^{(k)}) + N(r, 0, h) \]
\[ - N(r, \infty, h) - N(r, 0, H) + K_2. \]
Since \( N(r, 0, h) \leq N(r, 0, H) \), we obtain
\[ (5.7) \quad T(r, x) \leq T(r, y) + (n + 1) \sum_{k=1}^{n+1} T(r, \alpha^{(k)}) + K_2. \]
Next (5.4) and (5.6) imply

\begin{equation}
N(r, 0, G_k) \leq \sum_{j=0, \neq k}^{n} T(r, \alpha^{(j)}) + K_3
\end{equation}

and

\begin{equation}
N(r, 0, G) \leq \sum_{j=0}^{n} T(r, \alpha^{(j)}) + K_4,
\end{equation}

where $K_3$ and $K_4$ are constants. Hence, using (5.8) we have

\begin{equation}
N(r, 0, y_{k-1}) \leq N(r, 0, F_k) + N(r, 0, G_k)
\end{equation}

\begin{equation}
N(r, 0, y_k) \leq N(r, 0, F_k) + \sum_{j=0, \neq k}^{n} T(r, \alpha^{(j)}) + K_4.
\end{equation}

Since $|H| \leq (\prod_{j=1}^{n+1} |G_j|)|G|$, we have

\begin{equation}
N(r, 0, H) \leq N(r, 0, G) + \sum_{j=1}^{n+1} N(r, 0, G_k) + K_5,
\end{equation}

where $K_5$ is a constant. Hence, using $N(r, 0, h) \leq N(r, 0, H)$, together with (5.8), (5.9), and (5.11), we have

\begin{equation}
N(r, 0, F_k) \leq N(r, 0, y_{k-1}) + N(r, 0, G) + N(r, 0, h)
\end{equation}

\begin{equation}
\leq N(r, 0, y_{k-1}) + 2N(r, 0, G) + \sum_{j=1}^{n+1} N(r, 0, G_j) + K_5
\end{equation}

\begin{equation}
\leq N(r, 0, y_{k-1}) + (n + 2) \sum_{j=1}^{n+1} T(r, \alpha^{(j)}) + K_6,
\end{equation}

where $K_6$ is a suitable constant. Therefore using (5.1) in (5.5), (5.7), (5.10), and (5.12), we obtain (5.2) and (5.3).

Now it follows from (5.2) and (5.3) that

\[ \delta(\alpha^{(k)}, x) = 1 - \limsup_{r \to \infty} \frac{N(r, 0, y_{k-1})}{T(r, y)} = \delta(\alpha^{(k)}, y), \]

where $\alpha^{(k)} \in (P_n C)^*$ are induced by $(0, \ldots, 0, 1, 0, \ldots, 0)$ ($(k - 1)$th spot). Therefore, Theorem 3 follows Theorem 2 in Toda [14].

6. Proof of Theorem 1(II). We may assume that $\delta(\alpha^{(k)}) \geq 1 - \cos \pi \mu, k = 1, 2, \ldots, n$. Then

\[ (4/\mu) \sin^{-1}(\delta(\alpha^{(k)})/2)^{1/2} \geq 2\pi, \]

so

\[ \min\{2\pi, (4/\mu) \sin^{-1}(\delta(\alpha^{(k)})/2)^{1/2}\} = 2\pi, \quad k = 1, \ldots, n. \]

Hence, it follows from Lemma 5 that

\[ \sum_{k=n+1}^{q} \min\left\{ 2\pi, \frac{4}{\mu} \sin^{-1}\left( \frac{\delta(\alpha^{(k)})}{2} \right)^{1/2} \right\} = 0 \]
and, consequently, 
\[ \delta(\alpha^{(k)}) = 0, \quad k = n + 1, \ldots, q. \]

We next assume that \( \delta(\alpha^{(k)}) > 1 - \cos \pi \mu \) \((k = 1, \ldots, p; \ 0 \leq p < n)\) and \( \delta(\alpha^{(k)}) \leq 1 - \cos \pi \mu \) \((k = p + 1, \ldots, q)\). Then it follows from Lemma 5 that
\[
\sum_{k=p+1}^{q} \frac{4}{\mu} \sin^{-1} \left( \frac{\delta(\alpha^{(k)})}{2} \right)^{1/2} \leq 2(n-p)\pi.
\]

Hence, from the reasoning of §3 of Edrei [5] we can deduce that
\[
(6.1) \quad \sum_{k=p+1}^{q} \delta(\alpha^{(k)}) \leq (n-p)(1-\cos \pi \mu).
\]

It follows from Lemma 1 of Edrei [5] that equality in (6.1) is possible if and only if exactly \(n-p\) of \(\{\delta(\alpha^{(k)})\}_{k=p+1}^{q}\) are equal to \(1 - \cos \pi \mu\) and all other \(\delta(\alpha^{(k)})\) are equal to zero.

Thus the proof of Theorem 1(II) is complete.

\textbf{7. Proof of Theorem 1(III).} Our assumption \(\mu > 1/2\) implies
\[
(4/\mu) \sin^{-1}(\delta(\alpha^{(k)})/2)^{1/2} < 2\pi,
\]
so it follows from Lemma 5 that
\[
\sum_{k=1}^{q} \frac{4}{\mu} \sin^{-1} \left( \frac{\delta(\alpha^{(k)})}{2} \right)^{1/2} \leq 2n\pi.
\]

Hence, from the reasoning of §2 of Edrei [5] we can deduce that
\[
\sum_{k=1}^{q} \delta(\alpha^{(k)}) \leq [2n\mu + 1 - \cos(2n\mu - [2n\mu])(\pi/2),
\]

which proves Theorem 1(III).

\textbf{REFERENCES}


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