MONGE-AMPÈRE MEASURES ASSOCIATED
TO EXTREMAL PLURISUBHARMONIC FUNCTIONS IN $C^n$

BY

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Abstract. We consider the extremal plurisubharmonic functions $L_E^*$ and $U_E^*$ associated to a nonpluripolar compact subset $E$ of the unit ball $B \subset C^n$ and show that the corresponding Monge-Ampère measures $(ddcL_E^*)^n$ and $(ddcU_E^*)^n$ are mutually absolutely continuous. We then discuss the polynomial growth condition $(L^*)$, a generalization of Leja's polynomial condition in the plane, and study the relationship between the asymptotic behavior of the orthogonal polynomials associated to a measure on $E$ and the $(L^*)$ condition.

1. Introduction. Two natural extremal plurisubharmonic functions can be associated to a compact subset $E$ of the unit ball $B$ in $C^n$. One is the extremal function $L_E^*$ [S1], which is the Green function for the unbounded component of $C - E$ with pole at infinity when $n = 1$. The other is the relative extremal function $U_{E,B}^* = U_E^*$ introduced by Siciak and studied by Bedford and Taylor [BT], which is an analogue of the one-variable harmonic measure of $E$ relative to the unit disc. Both of these functions satisfy the complex Monge-Ampère equation $(ddc^*u)^n = 0$ outside of $E$ and hence both functions give rise to nonnegative measures supported in $E$.

In the case where the compact set $E$ is regular, i.e., $L_E^*$ and $U_E^*$ are continuous, Nguyen Thanh Van and Zeriahi [NZ] have shown that the set $E$, together with the measure $\mu_E = (ddcU_E^*)^n$, satisfy the polynomial condition $(L^*)$ which is a generalized version of Leja's polynomial condition in the complex plane. This result was used by Zeriahi [Ze] to show that certain classes of orthogonal polynomials associated with the measure $\mu_E$ exhibit asymptotic behavior similar to that of the extremal function $L_E^*$.

The main result of this paper shows that if $E$ is a nonpluripolar compact subset of $B$, then the Monge-Ampère measures $\mu_E = (ddcU_E^*)^n$ and $\bar{\mu}_E = (ddcL_E^*)^n$ are mutually absolutely continuous. As a corollary to this result, it follows that the pair $(E, \bar{\mu}_E)$ satisfies the $(L^*)$ condition for a regular compact set $E$. In addition, we show that the $(L^*)$ condition can be formulated entirely in terms of properties of the carriers of the measure. This result gives the extension to $C^n$ of a theorem of Ullman [U2] in the one-variable case.

2. Comparison of the Monge-Ampère measures. Given a domain $\Omega$ in $C^n$, we let $P(\Omega)$ denote the class of plurisubharmonic functions on $\Omega$. Let

$$L = \{ u \in P(C^n) : u(z) \leq \log|z| + O(1) \text{ as } |z| \to \infty \}$$

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and for \( E \subset B \) define

\[
L_E(z) = \sup\{ u(z) : u \in L, u \leq 0 \text{ on } E \}.
\]

The uppersemicontinuous regularization \( L_E^*(z) = \lim_{\xi \to z} L_E(\xi) \) is called the extremal function of \( E \). By a result of Siciak [S1], either \( L_E^* = +\infty \), in which case \( E \) is pluripolar, or \( L_E^* \in L \). In the latter case (see [BT]), then \( L_E^*(z) = 0 \) for \( z \in E - Z \), where \( Z \) is a pluripolar set. If \( E \) is compact, then \( E \) is regular when \( Z = \emptyset \).

Furthermore, \( (dd^cL_E^*)^n = 0 \) in \( C^n - E \). Here, \( d = \partial + \bar{\partial} \) and \( d' = i(\bar{\partial} - \partial) \), so that \( dd^cL_E^* = 2i\partial\bar{\partial}L_E^* \) and \( (dd^cL_E^*)^n = dd^cL_E^* \wedge \cdots \wedge dd^cL_E^* \) (n times) where the last two formulas are to be interpreted as currents on \( C^n \). We refer the reader to [Le] for a definition of currents and to [BT] for a more complete discussion of the complex Monge-Ampère operator \( (dd^c)^n \).

If \( \Omega \) is a strictly pseudoconvex domain containing \( E \), we define

\[
U_E(\Omega, z) = \sup\{ w(z) : w \in \mathcal{P}(\Omega), w \leq 0 \text{ on } \Omega, u \leq -1 \text{ on } E \}
\]

and call \( U_E^*(\Omega, z) = \lim_{\xi \to z} U_E(\Omega, \xi) \) the relative extremal function of \( E \) (relative to \( \Omega \)). It is easily seen that either \( U_E^*(\Omega, \cdot) = 0 \), in which case \( E \) is pluripolar, or \( U_E^*(\Omega, \cdot) \) is a nontrivial plurisubharmonic function in \( \Omega \) taking values between \(-1\) and \(0\). In the latter case, \( U_E^*(\Omega, z) \to 0 \) on \( \partial \Omega \) and \( U_E^*(\Omega, z) = -1 \) on \( E - Z \), where \( Z \) is pluripolar; \( Z = \emptyset \) when \( E \) is regular. In addition, \( (dd^cU_E^*(\Omega, \cdot))^n = 0 \) in \( \Omega - E \) if \( E \) is compact. Our first result shows that if \( E \) is not pluripolar and \( \Omega_1, \Omega_2 \) are two strictly pseudoconvex domains with \( E \subset \subset \Omega_1 \subset \Omega_2 \), and if \( \Omega_1 \) is a Runge domain relative to \( \Omega_2 \) (see [Ho, Theorem 4.3.3]), then \( (dd^cU_{E,1}^*(\Omega_1, \cdot))^n, i = 1, 2, \) are mutually absolutely continuous. This result is well known from classical potential theory in the case where \( n = 1 \) (see [Ne, Chapter 5]). For \( n \geq 2 \), the main ingredient in the proof is a comparison theorem for Monge-Ampère measures supported on \( E \).

**Lemma 2.1.** Let \( E \) be a compact set in \( C^n \) with smooth boundary \( \partial E \) and nonempty interior \( E^0 \) such that \( E \) is the closure of \( E^0 \). Let \( \omega \subset C^n \) be a domain with \( E \subset \subset \omega \). Suppose \( u_1, u_2 \in P(\omega) \cap C(\omega) \) satisfy:

(i) \( u_1 = u_2 = 0 \) on \( E \);
(ii) \( u_1 \geq u_2 \) on \( \omega \);
(iii) \( (dd^cu_1)^n = (dd^cu_2)^n = 0 \) (as measures) on \( \omega - E \); and
(iv) there exists \( \eta > 0 \) with \( u_1 \geq u_2 + \eta \) on \( \partial \omega \).

Then \( \int_\omega \phi(dd^cu_1)^n \geq \int_\omega \phi(dd^cu_2)^n \) for all \( \phi \in C_0^\infty(\omega) \) with \( \phi \geq 0 \).

**Proof.** Without loss of generality, we assume \( E \) is connected. Given \( \epsilon > 0 \), let \( E_\epsilon = \{ z \in \omega : u_1(z) \leq u_2(z) + \epsilon \} \). From (i) and the continuity of \( u_i \) on \( \overline{\omega} \), \( E \subset \subset E_\epsilon \). By (iv), we also have that \( E_\epsilon \subset \subset \omega \) for \( \epsilon \) sufficiently small. For any family of smoothings \( \{ u_i^\delta \} \) with \( u_i^\delta \wedge u_i \) in a neighborhood of \( \omega \), we have \( E \subset \subset E_{\epsilon,\delta} \subset \subset \omega \) for sufficiently small \( \delta \), where

\[ E_{\epsilon,\delta} = \{ z \in \omega : u_1^\delta(z) \leq u_2^\delta(z) + \epsilon \}. \]
Also, since \( u_i \in C(\omega) \), there exists \( M > 0 \) such that
\[
(2.1) \quad u_i^\delta \leq M \quad (i = 1, 2) \quad \text{for } \delta \text{ sufficiently small.}
\]

Given \( \phi \in C^\infty_0(\omega) \) with \( \phi \geq 0 \),
\[
\int_\omega \phi \left[ (dd^c u_1^\delta)^n - (dd^c u_2^\delta)^n \right] = \int_\omega \phi \ dd^c (u_1^\delta - u_2^\delta) \wedge T_\delta,
\]
where \( T_\delta = \sum_{k=0}^{n-1} (dd^c u_1^\delta)^{n-k-1} \wedge (dd^c u_2^\delta)^k \) is a positive, closed \((n-1, n-1)\)-form with smooth coefficients. From two applications of Stokes' theorem, we have
\[
(2.2) \quad \int_\omega \phi \ dd^c (u_1^\delta - u_2^\delta) \wedge T_\delta = \int_\omega \left( u_1^\delta - u_2^\delta \right) dd^c \phi \wedge T_\delta
\]
\[
= \int_{E_{\epsilon,\delta}} \left( u_1^\delta - u_2^\delta \right) dd^c \phi \wedge T_\delta + \int_{\omega - E_{\epsilon,\delta}} \left( u_1^\delta - u_2^\delta \right) dd^c \phi \wedge T_\delta.
\]

We first consider the integral over \( E_{\epsilon,\delta} \). Since \( \phi \in C^\infty_0(\omega) \),
\[
\left| \int_{E_{\epsilon,\delta}} \left( u_1^\delta - u_2^\delta \right) dd^c \phi \wedge T_\delta \right| \leq c(\phi) \left\| u_1^\delta - u_2^\delta \right\|_{E_{\epsilon,\delta}} \int_{E_{\epsilon,\delta}} \beta \wedge T_\delta,
\]
where \( \left\| u_1^\delta - u_2^\delta \right\|_{E_{\epsilon,\delta}} = \sup_{z \in E_{\epsilon,\delta}} |u_1^\delta(z) - u_2^\delta(z)|, \beta = dd^c |z|^2, \) and \( c(\phi) \) is a constant depending only on \( \phi \) and \( \omega \). Choose a compact set \( K \) such that \( E_{\epsilon,\delta} \subseteq K \subseteq \omega \) for \( \epsilon, \delta \) sufficiently small. Then, since \( T_\delta \) is positive, \( \beta \wedge T_\delta \) is a positive multiple of the volume form \( \beta^n = \beta \wedge \cdots \wedge \beta \), and we have
\[
\left| \int_{E_{\epsilon,\delta}} \left( u_1^\delta - u_2^\delta \right) dd^c \phi \wedge T_\delta \right| \leq c(\phi) \left\| u_1^\delta - u_2^\delta \right\|_{E_{\epsilon,\delta}} \int_K \beta \wedge T_\delta
\]
\[
\leq c(\phi) \left\| u_1^\delta - u_2^\delta \right\|_{E_{\epsilon,\delta}} c'M^{n-1},
\]
where \( c' \) is a constant depending only on \( K \) and \( \omega \). This last inequality follows from (2.1) and the Chern-Levine-Nirenberg estimates on \((dd^c)^n\) for bounded plurisubharmonic functions (see [CLN, or BT, §2]). From the construction of the set \( E_{\epsilon,\delta} \), it follows that
\[
\lim_{\epsilon \to 0} \left\{ \lim_{\delta \to 0} \left\| u_1^\delta - u_2^\delta \right\|_{E_{\epsilon,\delta}} \right\} = 0
\]
so that
\[
\lim_{\epsilon \to 0} \left\{ \lim_{\delta \to 0} \left| \int_{E_{\epsilon,\delta}} \left( u_1^\delta - u_2^\delta \right) dd^c \phi \wedge T_\delta \right| \right\} = 0.
\]

For the integral over \( \omega - E_{\epsilon,\delta} \) in (2.2), note that for each \( \delta > 0 \), the set \( E_{\epsilon,\delta} \) has smooth boundary for almost all \( \epsilon \) by Sard's theorem. Thus we can find a sequence of values of \( \delta \) tending to 0 and a sequence of values of \( \epsilon \) tending to 0 such that for each
pair \((\epsilon, \delta)\), the set \(E_{\epsilon, \delta}\) has smooth boundary. We can then apply Stokes’ theorem to obtain

\[
\int_{\omega - E_{\epsilon, \delta}} (u_1^\delta - u_2^\delta) \, dd^c \phi \wedge T_\delta = -\int_{\omega - E_{\epsilon, \delta}} d(u_1^\delta - u_2^\delta) \wedge d^c \phi \wedge T_\delta
\]

\[
- \int_{\partial E_{\epsilon, \delta}} (u_1^\delta - u_2^\delta) \, d^c \phi \wedge T_\delta
\]

\[
= -\int_{\omega - E_{\epsilon, \delta}} d\phi \wedge d^c(u_1^\delta - u_2^\delta) \wedge T_\delta - \epsilon \int_{\partial E_{\epsilon, \delta}} d^c \phi \wedge T_\delta
\]

\[
= \int_{\omega - E_{\epsilon, \delta}} \phi \left[ (dd^c u_1^\delta)^n - (dd^c u_2^\delta)^n \right]
\]

\[
+ \int_{\partial E_{\epsilon, \delta}} \phi d^c(u_1^\delta - u_2^\delta) \wedge T_\delta - \epsilon \int_{\partial E_{\epsilon, \delta}} d^c \phi \wedge T_\delta.
\]

Similar to before, we have

\[
\left| \epsilon \int_{\partial E_{\epsilon, \delta}} d^c \phi \wedge T_\delta \right| \leq \epsilon c(\phi) \int_K \beta \wedge T_\delta \leq \epsilon c(\phi) c'M^\delta_{n-1}
\]

which goes to zero as \(\epsilon \searrow 0\). We claim that the second term in (2.3) is nonnegative. This follows because \(\phi \geq 0\), \(T_\delta\) is a positive, closed \((n - 1, n - 1)\)-form, and \(u_1^\delta - u_2^\delta - \epsilon\) is a defining function for the interior of \(E_{\epsilon, \delta}\) so that \(d^c(u_1^\delta - u_2^\delta - \epsilon) \wedge T_\delta = d^c(u_1^\delta - u_2^\delta) \wedge T_\delta \geq 0\) on \(\partial E_{\epsilon, \delta}\) [Le, p. 68]. For the first term in (2.3), since \(E_{\epsilon} \supset E\) as \(\epsilon \searrow 0\), if we fix \(\epsilon > 0\), we can find \(\delta_0 > 0\) such that \(E_{\epsilon/2} \subset E_{\epsilon, \delta}\) for \(\delta < \delta_0\). As \(\delta \searrow 0\), the functions \(u_i^\delta\) decrease to \(u_i\) on \(\bar{\omega}\); hence the measures \((dd^c u_i^\delta)^n\) converge weakly to \((dd^c u_i)^n\), \(i = 1, 2\) [BT, Theorem 2.1]. Thus

\[
0 \leq \lim_{\delta \to 0} \int_{\omega - E_{\epsilon, \delta}} \phi (dd^c u_i^\delta)^n \leq \lim_{\delta \to 0} \int_{\omega - E_{\epsilon/2}} \phi (dd^c u_i^\delta)^n
\]

\[
= \int_{\omega - E_{\epsilon/2}} \phi (dd^c u_i)^n = 0
\]

by (iii) and the fact that \(E \subset \subset E_{\epsilon/2}\). Hence the sum of the integrals in (2.3) tends to a nonnegative number as \(\delta\) and then \(\epsilon\) decrease to zero so that from (2.2) we have

\[
\lim_{\delta \to 0} \int_\omega \phi \left[ (dd^c u_1^\delta)^n - (dd^c u_2^\delta)^n \right] = \int_\omega \phi \left[ (dd^c u_1)^n - (dd^c u_2)^n \right] \geq 0
\]

and the lemma is proved.

We will also make use of a domination principle of Bedford and Taylor for bounded plurisubharmonic functions [BT, Corollary 4.5].

**Lemma 2.2.** Let \(\Omega \subset \mathbb{C}^n\) be open and bounded and suppose that \(u, v \in P(\Omega) \cap L^\infty(\Omega)\) satisfy:

(i) \(\lim_{z \to \partial \Omega} (u(z) - v(z)) \geq 0\), and

(ii) \(\int_{\{u < v\}} (dd^c u)^n = 0\).

Then \(u \geq v\) in \(\Omega\).
**Theorem 2.1.** Let $E \subset \mathbb{C}^n$ be a nonpluripolar compact set and let $\Omega_1, \Omega_2$ be two strictly pseudoconvex domains with $E \subset \subset \Omega_1 \subset \Omega_2$. Then

$$
\mu_1 = (dd^cU^*_E(\Omega_1, \cdot))^n \geq \mu_2 = (dd^cU^*_E(\Omega_2, \cdot))^n.
$$

Furthermore, if $(\Omega_1, \Omega_2)$ is a Runge pair, then there exists a constant $c > 0$ such that $c\mu_1 \leq \mu_2$. In particular, in this latter case, $\mu_1$ and $\mu_2$ are mutually absolutely continuous.

**Proof.** For convenience, we work with the functions $u_i(z) = U^*_E(\Omega_i, z) + 1$, $i = 1, 2$. Given $\omega$ open with $E \subset \subset \omega \subset \Omega_1$, we will show that there exists a constant $c > 0$ such that

$$(2.4) \quad c \int_\omega \phi( dd^c u_1)^n \leq \int_\omega \phi( dd^c u_2)^n \leq \int_\omega \phi( dd^c u_1)^n$$

for all $\phi \in C_0^\infty(\omega)$ with $\phi \geq 0$. It will follow from the proof that only the first inequality will require that $(\Omega_1, \Omega_2)$ be a Runge pair. We first assume that $E$ satisfies the conditions in Lemma 2.1 so that $E$ is regular and $u_1, u_2$ are continuous. We then have that $u_1, u_2 \in C(\overline{\Omega}_i) \cap P(\Omega_i)$; also

(i) $u_1 = u_2 = 0$ on $E$

since $E$ is regular. Furthermore,

(ii) $u_1 \geq u_2$ on $\Omega_i$

by definition of the relative extremal functions; and

(iii) $(dd^c u_1)^n = (dd^c u_2)^n = 0$ on $\Omega_i - E$.

Since $u_1 = 1$ on $\partial \Omega_1$ and $\max_{z \in \partial \Omega_1} u_2(z) = \alpha < 1$ by the maximum principle for plurisubharmonic functions, we have

(iv) $u_1 \geq u_2 + (1 - \alpha)$ on $\partial \Omega_1$.

Thus we can apply Lemma 2.1 to conclude that for any open set $\omega \subset \Omega_1$ with $E \subset \subset \omega$,

$$
\int_\omega \phi( dd^c u_1)^n \geq \int_\omega \phi( dd^c u_2)^n
$$

for all $\phi \in C_0^\infty(\omega)$ satisfying $\phi \geq 0$. In the other direction, we first note that since $E \subset \subset \Omega_i$, $i = 1, 2$, it follows from the pseudoconvexity of $\Omega_i$ that $\hat{E}_{\Omega_i}^P \subset \subset \Omega_i$, where

$$
\hat{E}_{\Omega_i}^P = \left\{ z \in \Omega_i : u(z) \leq \sup_{z' \in E} u(z') \text{ for all } u \in P(\Omega_i) \right\}.
$$

Since $\Omega_1$ is a Runge domain relative to $\Omega_2$, $\hat{E}_{\Omega_1}^P = \hat{E}_{\Omega_2}^P$ [Ho, Theorems 4.3.3 and 4.3.4]. From the definitions of $\hat{E}_{\Omega_i}^P$ and $U_E(\Omega_i, z)$, we have that $U_E(\Omega_i, z) > -1$ for $z \in \Omega_i - \hat{E}_{\Omega_i}^P$. Thus if we let $\eta = \min_{z \in \partial \Omega_i} u_2(z)$, we see that $\eta > 0$, so that $\eta u_1 \leq u_2$ on $\partial \Omega_i$ and $\eta u_1 = u_2 = 0$ on $E$. By (iii), it follows that

$$
\int_{\{u_2 < \eta u_1\}} (dd^c u_2)^n = 0,
$$

so that by Lemma 2.2 we conclude that $\eta u_1 \leq u_2$ in $\Omega_1$. Given $0 < \varepsilon < \eta$, we then have that $(\eta - \varepsilon)u_1, u_2$ satisfy the hypotheses of Lemma 2.1 so that

$$
\int_\omega \phi[(dd^c(\eta - \varepsilon)u_1]^n \leq \int_\omega \phi(dd^c u_2)^n
$$
for all \( \phi \in C_0^\infty(\omega) \) with \( \phi \geq 0 \). Letting \( \varepsilon \searrow 0 \), we obtain

\[
\eta^n \int_\omega \phi \left( dd^c u_1 \right)^n \leq \int_\omega \phi \left( dd^c u_2 \right)^n,
\]

and (2.4) is proved with \( c = \eta^n \) for compact sets \( E \) satisfying the conditions in Lemma 2.1.

For the general case, we take a sequence \( \{ E_k \} \) of compact sets as above which decrease to \( E \). If we fix \( \omega \) open with \( E \subset \subset \omega \subset \subset \Omega_1 \), we have \( E_k \subset \subset \omega \) for \( k \) sufficiently large and we obtain

\[
(\eta_k)^n \int_\omega \phi \left( dd^c u_k^1 \right)^n \leq \int_\omega \phi \left( dd^c u_k^2 \right)^n \leq \int_\omega \phi \left( dd^c u_1^k \right)^n
\]

from (2.4), where \( u_k^i(z) \equiv U_k^i(\Omega_1, z) + 1 \), \( \eta_k = \min_{z \in \partial \Omega_1} u_k^i(z) \), and \( \phi \in C_0^\infty(\omega) \) satisfies \( \phi \geq 0 \). Since \( u_k^i \not\sim u \), a.e. [BT, Proposition 6.4], the measures \( \left( dd^c u_k^i \right)^n \) converge weakly to \( \left( dd^c u_i \right)^n \) [BT, Theorem 2.1], \( i = 1, 2 \). Thus for \( \phi \) as above,

\[
\int_\omega \phi \left( dd^c u_2 \right)^n = \lim_{k \to \infty} \int_\omega \phi \left( dd^c u_k^2 \right)^n \leq \lim_{k \to \infty} \int_\omega \phi \left( dd^c u_k^1 \right)^n = \int_\omega \phi \left( dd^c u_1 \right)^n.
\]

In addition, if we fix \( k_0, \eta_k \geq \eta_{k_0} \) for \( k \geq k_0 \) so that

\[
\int_\omega \phi \left( dd^c u_2 \right)^n = \lim_{k \to \infty} \int_\omega \phi \left( dd^c u_k^2 \right)^n \geq \lim_{k \to \infty} (\eta_k)^n \int_\omega \phi \left( dd^c u_k^1 \right)^n
\]

\[
\geq \lim_{k \to \infty} (\eta_{k_0})^n \int_\omega \phi \left( dd^c u_k^1 \right)^n = (\eta_{k_0})^n \int_\omega \phi \left( dd^c u_1 \right)^n,
\]

then (2.4) holds in the general case and the theorem is proved. We note that the constant in (2.4) can be taken as \( c = \left( \min_{z \in \partial \Omega} u_2(z) \right)^n \) when \( E \) is regular.

We now come to the main theorem of the paper.

**Theorem 2.2.** Let \( E \subset B \) be a nonpluripolar compact set. Then the measures \( \mu_E \) and \( \tilde{\mu}_E \) are mutually absolutely continuous.

**Proof.** We first assume that \( E \) satisfies the conditions of Lemma 2.1. Let \( a = \| L_E \|_B \). Then

\[
u_1(z) \equiv U_1(z) + 1 \geq (1/a) L_E(z) \equiv u_2(z)
\]

for \( z \in B \) by definition of \( U_E \). Furthermore, since \( \partial E \) is smooth, \( E \) is regular, so that \( u_1 = u_2 = 0 \) on \( E \). Also, \( \left( dd^c u_1 \right)^n = \left( dd^c u_2 \right)^n = 0 \) outside of \( E \). Thus, given \( 0 < \varepsilon < 1 \), the functions \( u_1 \) and \( (1 - \varepsilon)u_2 \) satisfy the conditions (i)-(iv) of Lemma 2.1 and we have

\[
\int_\omega \phi d\mu_E \geq \left( \frac{1 - \varepsilon}{a} \right)^n \int_\omega \phi \tilde{d}\tilde{\mu}_E
\]

for all \( \phi \in C_0^\infty(\omega) \) with \( \phi \geq 0 \) where \( E \subset \subset \omega \subset \subset B \) and \( \omega \) is open. Letting \( \varepsilon \searrow 0 \), we obtain

\[
\int_\omega \phi d\mu_E \geq \left( \frac{1}{a} \right)^n \int_\omega \phi \tilde{d}\tilde{\mu}_E.
\]

(2.5)
In the other direction, we first remark that by definition of the extremal function, if \( E_1 \subset E_2 \), then \( L_{E_1}^+(z) \geq L_{E_2}^+(z) \) for all \( z \in \mathbb{C}^n \). It is easy to show that the extremal function for a closed ball \( B(a, r) = \{ z \in \mathbb{C}^n : |z - a| \leq r \} \) is

\[
u(z) = \log^+ \left( \frac{|z - a|}{r} \right) = \max \left( \log \frac{|z - a|}{r}, 0 \right); \]

thus, since \( E \subset B \), \( L_E(z) \geq \log^+ |z| \) for all \( z \). In particular, \( \nu_1(z) = L_E(z)/\log 2 \geq 1 \) for \( |z| = 2 \).

If we let \( \nu_2(z) = U_E(B(0, 2), z) + 1 \), then \( \nu_1 = \nu_2 = 0 \) on \( E \); also, since \( (dd^c \nu_1)^n \) is supported in \( E \),

\[ \int_{\{\nu_1 < \nu_2 \}} (dd^c \nu_1)^n = 0. \]

Thus by Lemma 2.2, \( \nu_1 \geq \nu_2 \) on \( B(0, 2) \). Applying Lemma 2.1 to \( \nu_1 \) and \( (1 - e)\nu_2 \), we obtain

\[ \frac{1}{(\log 2)^n} \int_\omega \phi \, d\tilde{\mu}_E \geq (1 - e)^n \int_\omega \phi (dd^c \nu_2)^n \]

for \( \phi \in C_0^\infty(\omega) \) with \( \phi \geq 0 \). Letting \( e \downarrow 0 \), we obtain

\[ \int_\omega \phi \, d\tilde{\mu}_E \geq (\log 2)^n \int_\omega \phi (dd^c \nu_2)^n. \]

We can apply Theorem 2.1, since \( B \) is a Runge domain relative to \( B(0, 2) \), to get

\[ \int_\omega \phi (dd^c \nu_2)^n \geq \left( \min_{z \in \partial B} \nu_2(z) \right)^n \int_\omega \phi \, d\mu_E. \]

Thus from (2.5) and the above inequalities, we have

(2.6) \[ \left( \frac{1}{a} \right)^n \int_\omega \phi \, d\tilde{\mu}_E \leq \int_\omega \phi \, d\mu_E \leq \left[ (\log 2) \left( \min_{z \in \partial B} \nu_2(z) \right) \right]^n \int_\omega \phi \, d\tilde{\mu}_E \]

for all \( \phi \in C_0^\infty(\omega) \) with \( \phi \geq 0 \), and the theorem is proved for \( E \) satisfying the conditions of Lemma 2.1. The general case follows as in the proof of Theorem 2.1 by choosing a sequence \( \{E_k\} \) of compact sets as above which decrease to \( E \). Applying (2.6) to \( E_k, \tilde{\mu}_{E_k}, \mu_{E_k}, a_k = \|L_{E_k}\|_B \), and \( \nu_2^k(z) = U_{E_k}(B(0, 2), z) + 1 \), and then using the convergence theorems of Bedford and Taylor, the proof is complete.

3. **Condition \( (L^*) \) and orthogonal polynomials.** Given a compact set \( E \subset \mathbb{C}^n \) and a nonnegative measure \( \mu \) with support in \( E \), we say that the pair \( (E, \mu) \) satisfies condition \( (L^*) \) if for each family \( \Phi \) of polynomials with \( \sup_{p \in \Phi} |p(z)| < \infty \) \( \mu \)-a.e. on \( E \), and for each \( \lambda > 1 \), there exists a constant \( M \) and a neighborhood \( U \) of \( E \) such that

\[ \|p\|_U \leq M \lambda^{\deg(p)} \text{ for all } p \in \Phi. \]

The condition \( (L^*) \) is an important concept in discussing the regularity of a compact set; for example, if there exists a nonnegative measure \( \mu \) on \( E \) such that \( (E, \mu) \).
satisfies \((L^*)\), it follows that \(E\) is regular. On the other hand, Nguyen Thanh Van and Zeriahi have shown [NZ] that if \(E\) is regular, then the pair \((E, \mu_E)\) satisfies \((L^*)\). From Theorem 2.2 and the definition of \((L^*)\), it follows that, in addition, \((E, \tilde{\mu}_E)\) satisfies \((L^*)\). We now give an alternate characterization of the condition \((L^*)\).

**Theorem 3.1.** Let \(E \subset B\) be a regular compact set and let \(\mu\) be a measure on \(E\). Then \((E, \mu)\) satisfies \((L^*)\) if and only if for any Borel set \(F \subset E\) with \(\mu(F) = \mu(E)\), we have \(L^*_E = L_F\).

**Proof.** We first assume that \((E, \mu)\) satisfies \((L^*)\). Let \(F \subset E\) be a Borel set with \(\mu(F) = \mu(E)\). Since \(F \subset E\), \(L^*_F \geq L_E\); to prove the reverse inequality, it suffices to show that \(L^*_F \leq 0\) on \(E\). Suppose there exists a point \(z_0 \in E\) with \(L^*_E(z_0) > 0\). Let \(F_0 = \{z \in F : L^*_E(z) = 0\}\). Then \(F - F_0\) is a pluripolar set; thus, by a result of Siciak [S2, Theorem 1.6], there exists \(v \in L\) with \(v = -\infty\) on \(F - F_0\). Without loss of generality, we may assume that \(v \leq 0\) on \(E\).

We claim that the set \(N = \{z \in E : L^*_E(z) > L_E(z)\}\) is not pluripolar. For if it were, then by Proposition 3.11 of [S1], \(L^*_{E - N} = L_E\). However, since \(L^*_F \leq L_E = 0\) on \(E - N\), it follows that \(L^*_F \leq L_{E - N} = L_E\) in \(C^n\), contradicting our assumption that \(L^*_F(z_0) > 0\). Thus \(N\) is not pluripolar and we may assume that \(v(z_0) > -\infty\) and \(L^*_E(z_0) = 5\delta > 0\). We now choose \(\eta > 0\) sufficiently small so that:

\[
\begin{align*}
\text{(i)} & \quad 0 > \eta v(z_0) > -\delta, \\
\text{(ii)} & \quad (1 - \eta)L^*_E(z_0) > 4\delta.
\end{align*}
\]

Since \(L^*_E, v \in L\), it follows that \((1 - \eta)L^*_E + \eta v \in L\). For \(m \in \mathbb{Z}^+\) (positive integers), let \(u_m = [(1 - \eta)L^*_E + \eta v] x_{1/m}\), where \(x_{1/m}\) is a smooth, radially symmetric bump function supported in \(|z| \leq 1/m\). Then

\[
\begin{align*}
(3.2) & \quad u_m \in P(C^n) \cap C(C^n); \text{ furthermore, } u_m \in L [S1, Proposition 1.2]. \text{ By an approximation theorem [Fe, Proposition 2, Chapter VII], for each } m, \text{ there exists a sequence of positive integers } \{d_m\} \text{ and a family of polynomials } \{p_m\} \text{ with } \deg(p_m) = d_m \\
\text{such that } & \quad u_m(z) = \sup_k \frac{1}{d_m} \log |p_m(z)| \quad \text{for all } z \in E.
\end{align*}
\]

In particular, for each \(m\), we can find a polynomial \(p_m\) of degree \(d_m\) with

\[
\begin{align*}
\text{(i)} & \quad (1/d_m) \log |p_m(z)| \leq u_m(z) \text{ for all } z \in E, \text{ and} \\
\text{(ii)} & \quad (1/d_m) \log |p_m(z_0)| > u_m(z_0) - \delta.
\end{align*}
\]

Moreover, by taking powers of \(p_m\) if necessary, we may assume that \(d_m \not\to \infty\).

To get a contradiction to the \((L^*)\) condition, we take \(\Phi = \{e^{-\delta d_m}p_m\}\). We first show that

\[
\sup_m e^{-\delta d_m} |p_m(z)| < \infty \quad \text{for all } z \in F.
\]

For \(z \in F - F_0, v(z) = -\infty\); thus \(u_m(z) \leq 0\) for \(m \geq m(z)\) by (3.2) so that

\[
\sup_m e^{-\delta d_m} |p_m(z)| \leq e^{-\delta d_m} e^{d_m u_m(z)}
\]
(by (3.3)(i)) is finite. For \( z \in F_0 \),
\[
e^{-\delta m}|p_m(z)| \leq e^{-(\delta - \epsilon)} e^{d_m \sigma_m(z)}
\]
where \( \sigma_m = \sigma_m(z) \to 0 \) as \( m \to \infty \) by (3.2). Since \( v \leq 0 \) on \( E \) and \( L^*_F = 0 \) on \( F_0 \),
\[
(1 - \eta)L^*_F(z) + \eta v(z) \leq 0 \text{ if } z \in F_0 \text{ and we obtain}
\]
\[
e^{-\delta m}|p_m(z)| \leq e^{-(\delta - \epsilon)}.
\]
Since \( \sigma_m \to 0 \) as \( m \to \infty \), \( \delta - \sigma_m \) is eventually positive and thus
\[
\sup_{m} e^{-\delta m}|p_m(z)| < \infty \text{ for } z \in F_0.
\]

We show that \((L^*)\) is violated by proving that there is an \( \epsilon > 0 \) such that

\[
(3.4) \quad e^{-\delta m}|p_m(z_0)| \geq m(1 + \epsilon)^{d_m} \text{ for } m \text{ sufficiently large.}
\]

From (3.3)(ii), (3.2) and (3.1) we obtain
\[
e^{-\delta m}|p_m(z_0)| \geq e^{-\delta m} e^{d_m \sigma_m(z_0) - \delta}
\]
\[
\geq e^{-2\delta m} e^{d_m [(1 - \eta)L^*_F(z_0) + \eta v(z_0)]}
\]
\[
\geq e^{\delta m} > (1 + \delta)^{d_m} = (1 + \epsilon)^{d_m (1 + \delta)/(1 + \epsilon)} d_m.
\]
Thus by choosing \( \epsilon < \delta \), we obtain (3.4).

For the converse, we assume that if \( F \subset E \) is a Borel set with \( \mu(F) = \mu(E) \), then
\( L^*_F = L_E \). Let \( \Phi \) be a family of polynomials such that
\[
\sup_{p \in \Phi} |p(z)| = M(z) < \infty \text{ \( \mu \)-a.e. on } E.
\]
Let \( F = \{ z \in E: M(z) < \infty \} \). Then \( \mu(F) = \mu(E) \); furthermore, for each positive integer \( m \), if we let \( F_m = \{ z \in E: M(z) \leq m \} \), then \( F_m \nearrow F \), and since \( M(z) \) is lower semicontinuous, each \( F_m \) is compact. Thus \( F \) is a Borel set and hence \( L^*_F = L_E \).

Since it is clear that for a polynomial \( p \),
\[
\frac{1}{\deg(p)} \log|p(z)| \in L,
\]
it follows that for any \( p \in \Phi \),
\[
(3.5) \quad \frac{1}{\deg(p)} \log \frac{|p(z)|}{m} \leq L_{F_m}(z) \leq L^*_{F_m}(z) \text{ for all } z \in C^n.
\]
Also, since \( L_E \) is continuous, given \( \epsilon > 0 \),
\[
U_\epsilon = \{ z \in C^n: L_E(z) < \epsilon \}
\]
is an open neighborhood of \( E \). The extremal functions \( L^*_{F_m} \) decrease pointwise to \( L^*_F = L_E \) on \( U_\epsilon \); thus, by Dini’s theorem, the functions \( L^*_{F_m} \) converge uniformly to \( L_E \) on \( U_\epsilon \) and we can find \( m_0 = m_0(\epsilon) \) such that
\[
L^*_{F_m}(z) < 2\epsilon \text{ for } z \in \overline{U_\epsilon} \text{ if } m > m_0.
\]
For such \( m \),
\[
\|p\|_{U_\epsilon} = \sup_{z \in U_\epsilon} |p(z)| \leq m(e^{2\epsilon})^{\deg(p)} \text{ for all } p \in \Phi
\]
by (3.5). Thus \( (E, \mu) \) satisfies \((L^*)\).
In the complex plane, Ullman [U1] calls a measure whose support $E = E(\mu)$ is an infinite set, a determined measure if for any Borel subset $F \subset E$ with $\mu(F) = \mu(E)$, the logarithmic capacity $C(F)$ of $F$ equals $C(E)$. Since the function $L^*_E$ is the Green function in one variable, Theorem 3.1 shows that for regular compact sets $E \subset C$, $(E, \mu)$ satisfies $(L^*_*)$ precisely when $\mu$ is determined. The condition that $\mu$ has infinite support is a density condition which insures the linear independence of the monomials $\{z^k\}$ in the space $L^2(\mu)$ of square-integrable functions with respect to $\mu$ and allows one to construct the orthogonal polynomials $\{p_k\}$ with respect to $\mu$ via the Hilbert-Schmidt process. The polynomials $p_k(z) = z^k + \cdots$ are the unique monic orthogonal polynomials of degree $k$ ($k = 0, 1, 2, \ldots$) of minimal $L^2(\mu)$-norm among all monic polynomials of degree $k$. If we denote the $L^2(\mu)$-norm of $p_k$ by $N_k$, Ullman [U1] has shown that if $C(E) > 0$ and $\mu$ is a determined measure, then

$$\lim_{k \to \infty} \left[ \frac{|p_k(z)|}{N_k} \right]^{\frac{1}{k}} = e^{L^*_E(z)},$$

for $z$ in the unbounded component of $C - E$ except for a set of Lebesgue measure zero in the convex hull of $E$.

In $C^n$, $n > 1$, let $E \subset B$ be compact and let $\mu$ be a measure on $E$. As a replacement for the monic normalization, we first put an ordering on the monomials $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Let $\alpha: Z^+ \to (Z^+ \cup \{0\})^n$ be a bijective map on the positive integers such that $|\alpha(k)| \leq |\alpha(k + 1)|$ for all $k \in Z^+$, where $|\alpha(k)| = \alpha_1 + \cdots + \alpha_n$ if $\alpha(k) = (\alpha_1, \ldots, \alpha_n)$. Under certain conditions on the measure $\mu$ and the set $E$, the sequence of monomials $\{e_k(z) = z^{\alpha(k)}\}$ is linearly independent in the space $L^2(\mu)$ and therefore we can construct orthogonal polynomials $\{A_k\}$ of the form

$$A_k(z) = e_k(z) + \sum_{j=1}^{k-1} a_j e_j(z), \quad a_j \in C,$$

via the Hilbert-Schmidt process. Recall that if $E \subset C^n$ is compact, the Silov boundary of $E$ with respect to the uniform Banach algebra generated by the restrictions to $E$ of the analytic polynomials in $C^n$, denoted $S_E$, is the smallest closed subset of $E$ such that for any polynomial $p$, there exists $z_0 \in S_E$ with $||p||_E = ||p(z_0)||$.

**PROPOSITION 3.1.** Let $E \subset C^n$ be compact and suppose that $E$ is not pluripolar. If $\mu$ is a measure on $E$ such that $S_E \subset$ support of $\mu$, then the monomials $\{e_k\}$ are linearly independent in $L^2(\mu)$.

**PROOF.** By a result of Siciak [S1, Proposition 4.3], since $E$ is not pluripolar, $E$ is unisolvent, i.e., if $p$ is a polynomial satisfying $p(z) = 0$ for $z \in E$, then $p \equiv 0$ on $C^n$.

Suppose there exists a polynomial $p(z) = \sum_{j=1}^{k-1} c_j e_j(z), p \not\equiv 0$, with $\int_E |p|^2 \, d\mu = 0$. Then $||p||_E = M > 0$ and we can find $z_0 \in S_E$ with $|p(z_0)| = M$. By continuity, we can find $r > 0$ such that $|p(z)| > M/2$ for $z \in B(z_0, r)$. Since $S_E \subset$ support of $\mu$, $\mu(E \cap B(z_0, r)) > 0$ so that

$$0 = \int_E |p|^2 \, d\mu \geq \int_{E \cap B(z_0, r)} |p|^2 \, d\mu > \left( \frac{M}{2} \right)^2 \mu(E \cap B(z_0, r)) > 0,$$
which gives a contradiction. Thus $p \equiv 0$ on $\mathbb{C}^n$, i.e., $c_1 = \cdots = c_k = 0$, and the proposition is proved.

From the work of Nguyen Thanh Van and Zeriahi [NZ], if $(E, \mu)$ satisfies $(L^*)$, then $(E, \mu)$ satisfies the density condition in Proposition 3.1 and we may construct the orthogonal polynomials $\{A_k\}$ associated with $\mu$. Set $v_k \equiv \left( \int_E |A_k|^2 \, d\mu \right)^{1/2}$. By following the argument used by Zeriahi [Ze], we can now state a $\mathbb{C}^n$-version of Ullman's theorem.

**Theorem 3.2.** Let $E \subset B$ be a regular compact set and let $\mu$ be a finite measure on $E$ such that $(E, \mu)$ satisfies $(L^*)$. If $\{A_k\}$ is the sequence of orthogonal polynomials associated to $\mu$ and $\{v_k\}$ is the corresponding sequence of $L^2(\mu)$-norms, then:

1. $\lim_{k \to \infty} \left( \frac{\|A_k\|_E}{v_k} \right)^{1/\alpha(k)} = 1$, and
2. $\lim_{k \to \infty} \left( \frac{|A_k(z)|}{v_k} \right)^{1/\alpha(k)} = e^L_{\alpha}(z)$ for all $z \in \mathbb{C}^n - \hat{E}$, where $\hat{E}$ is the polynomial hull of $E$.

**References**


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