CHAOS, PERIODICITY, AND SNAKELIKE CONTINUA

BY

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Abstract. The results of this paper relate the dynamics of a continuous map $f$ of the interval and the topology of the inverse limit space with bonding map $f$. These inverse limit spaces have been studied by many authors, and are examples of what Bing has called "snakelike continua". Roughly speaking, we show that when the dynamics of $f$ are complicated, the inverse limit space contains indecomposable subcontinua. We also establish a partial converse.

Introduction. Let $I$ be a closed interval, and let $f: I \rightarrow I$ be a continuous function. Associated with $f$ is the inverse limit space $(I, f) = \{(x_0, x_1, \ldots) | f(x_{n+1}) = x_n\}$. With a natural topology, $(I, f)$ is a compact, connected, metric space, and is an example of what Bing [Bi] has called a snakelike continuum. In this paper we will investigate the relationship between behavior of the orbits $\{f^n(x)|n \geq 0\}$ of points of $I$ under $f$, and the topological properties of the space $(I, f)$. These examples suggest some of the ideas which we will explore.

Example 1. Let $I = [0, 1]$, and define $f: I \rightarrow I$ by

$$f(t) = \begin{cases} \frac{3}{2}t & \text{if } 0 \leq t \leq \frac{2}{3}, \\ \frac{1}{3} - t & \text{if } \frac{2}{3} \leq t \leq 1. \end{cases}$$

It can be verified that (i) if $x \in I$, then $\{f^n(x)|n \geq 0\}$ is not dense in $I$, and (ii) $f$ has points of period 1 and 2, but no points of period $n \geq 3$. Now let $X$ be the subspace of the plane defined by

$$X = \{(x, y)|0 < x \leq 1 \text{ and } y = \sin(1/x)\} \cup \{(x, y)|x = 0 \text{ and } -1 \leq y \leq 1\}.$$

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It can be verified that $X$ is homeomorphic with $(I, f)$ and has the following property: if $C$ is a nondegenerate subcontinuum of $X$, then $C$ is the union of two of its proper subcontinua.

**Example 2.** Let $I = [0, 1]$, and let $f: I \to I$ be defined by

$$f(t) = \begin{cases} 2t & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 2 - 2t & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

It can be verified that (i) there is a point $x \in I$ such that \{f\(^n\)(x)|n \geq 0\} is dense in $I$, and (ii) for each positive integer $k$, $f$ has points of period $k$. Now let $X$ be the subspace of the plane described as follows: let $C$ be the standard Cantor set on the $x$-axis. Let $K_0$ be the union of all semicircles in the upper half-plane with endpoints on $C$, which are symmetric with the line $x = \frac{1}{3}$. For each positive integer $i$, let $K_i$ be the union of all semicircles in the lower half-plane with endpoints on $C$, which are symmetric about the line $x = 5/3 \cdot 2$. Then $X = \bigcup_{i=0}^{\infty} K_i$.

$$X \sim (I, f)$$

It can be verified that $X$ is homeomorphic with $(I, f)$ and has the following property: $X$ is not the union of two of its proper subcontinua.

**Definitions and terminology.** If $a$ and $b$ are distinct real numbers we will let $[a, b]$ denote the smallest closed interval containing both $a$ and $b$, and let $(a, b)$ denote the associated open interval. We will generically let $I$ be a closed interval and will be considering continuous functions $f: I \to I$. All of the functions which we will consider are continuous.

If $f: I \to I$ and $x \in I$, then the orbit of $x$ under $f$ is \{y\} for some integer $n$, $n \geq 0$, $y = f^n(x)$. We will be interested in functions $f: I \to I$ for which there is a point $x$ whose orbit is dense in $I$. (In [AY] a function $f$ is defined to be chaotic if there is a point whose orbit is dense and if every point is unstable. As a corollary to Lemma 2 we show that, for functions on the interval, the existence of a dense orbit implies that every point is unstable.)

If $f: I \to I$ and $x \in I$, the statement that $x$ has period $n$, means that $n$ is a positive integer, $f^n(x) = x$, and if $k$ is an integer, $1 \leq k < n$, then $f^k(x) \neq x$.

Associated with $f: I \to I$ is the compact, connected metric space $(I, f) = \{(x_0, x_1, \ldots)|f(x_i) = x_{i-1}\}$ with metric

$$d(((x_0, x_1, \ldots), (y_0, y_1, \ldots))) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^i}.$$
(I, f) is an example of what Bing [Bi] has called a snakelike continuum. The reason for this terminology is that for each \( \epsilon > 0 \), there is a finite open covering \( \{ g_1, g_2, \ldots, g_n \} \) of \((I, f)\) such that (i) \( \text{diam } g_i < \epsilon \), and (ii) \( g_i \cap g_j \neq \emptyset \) iff \( j = i - 1, i, \) or \( i + 1 \). We will denote elements of \((I, f)\) by subbarred letters, as \( \bar{x} = (x_0, x_1, \ldots) \).

The projection maps \( \pi_n \) of \((I, f)\) onto \( I \) given by \( \pi_n(x) = x_n \) are continuous. If \( H \) is a subcontinuum (compact, connected subspace) of \((I, f)\) we will let \( H_n \) denote \( \pi_n(H) \).

Note that \( H_n \) is a closed interval or point, and that \( f(H_{n+1}) = H_n \).

If \( f: I \to I \), then \( f \) induces a homeomorphism \( \hat{f}: (I, f) \to (I, f) \) by \( \hat{f}((x_0, x_1, \ldots)) = (f(x_0), x_0, x_1, \ldots) \). Notice that \( f \circ \pi_n = \pi_n \circ \hat{f}, \pi_n = \pi_{n+1} \circ \hat{f}, \) and \( f \circ \pi_{n+1} = \pi_n \).

Here are important facts about snakelike continua which we will utilize. Suppose \( S \) is snakelike; then (i) the intersection of any collection of subcontinua of \( S \) is a subcontinuum of \( S \), and (ii) if \( H \) is a subcontinuum of \( S \), then \( S - H \) has at most two components (see [Bi]).

If \( S \) is a continuum, the statement that \( S \) is indecomposable means that \( S \) is not the union of two of its proper subcontinua. Here are two conditions each of which is equivalent to the indecomposability of \( S \) (see [HY, pp. 139–141]).

1. If \( H \) is a subcontinuum of \( S \), then \( H \) contains no open set in \( S \).
2. \( S \) contains three distinct points \( x, y \) and \( z \) such that \( S \) is irreducible between each pair of these points. (Irreducibility between \( x \) and \( y \) means that no proper subcontinuum of \( S \) contains both \( x \) and \( y \).)

We will utilize the following important construction due to Bing [Bi]. Suppose that \((I, f)\) contains no indecomposable subcontinuum with interior. For each \( \bar{x} \in (I, f) \) let \( g_x \) be the intersection of all subcontinua of \((I, f)\) that contain interiorly a subcontinuum that contains \( \bar{x} \) in its interior. Then \( g_x \) is a subcontinuum of \((I, f)\). Furthermore, the sets \( g_x \) partition \((I, f)\) and if we let \( G = \{ g_x | \bar{x} \in (I, f) \} \) with the quotient topology, then \( G \) is an arc (i.e. homeomorphic with \( I \)). Moreover, \( f \) induces a homeomorphism \( \hat{f} \) of \( G \) onto \( G \) given by \( \hat{f}(g_x) = g_{f(x)} \). Bing also shows that, for each \( \bar{x} \in (I, f) \), \( g_x \) does not have interior.

If \( A \) is a set, we will let both \( \overline{A} \) and \( \text{cl}(A) \) denote the closure of \( A \) and let \( \text{int} A \) denote the interior of \( A \).

**Theorem 1.** Suppose that \( k \) and \( n \) are integers, \( k \geq 0, n \geq 1 \), and that \( f \) has a point of period \( 2^k(2n + 1) \); i.e., of period not a power of 2. Then \((I, f)\) has an indecomposable subcontinuum that is invariant under \( f^{2^k+1} \).

**Proof.** If follows from Sarkovskii's Theorem [S, N] and the hypothesis that \( f \) has a point of period \( (2^k+1)(3) \). Then \( f^{2^k+1} \) has a point of period 3. Now, for each positive integer \( j \), the spaces \((I, f)\) and \((I, f^j)\) are homeomorphic. One such homeomorphism is \( (x_0, x_1, \ldots, x_j, \ldots) \to (x_0, x_j, x_{2j}, \ldots) \). Using this fact, we need only show that if \( g: I \to I \) is continuous and \( g \) has a point of period 3, then \((I, g)\) has an indecomposable subcontinuum that is invariant under \( g \). Let \( g \) be such a function and let \( x \) be a point of period 3. Then we have the point \( \bar{x} = (x, g^2(x), g(x), x, \ldots) \) of \((I, g)\) with \( \hat{g}^3(\bar{x}) = \bar{x} \) and \( \hat{g}(\bar{x}) \neq \bar{x} \).
Now let $S$ be the intersection of all subcontinua of $(I, g)$ which contain \{x, \hat{g}(x), \hat{g}^2(x)\}. Since \{x, \hat{g}(x), \hat{g}^2(x)\} is contained in both \hat{g}(S) and \hat{g}^{-1}(S), it follows that $S = g(S)$. So $S$ is a subcontinuum of $(I, g)$ which is invariant under $\hat{g}$.

We will next show that $S$ is irreducible between each pair of $x$, $\hat{g}(x)$ and $\hat{g}^2(x)$. It will follow that $S$ is indecomposable. Suppose, for example, that $S$ is reducible between $\hat{g}(x)$ and $\hat{g}^2(x)$. Then there is a proper subcontinuum $H$ of $S$ which contains $\hat{g}(x)$ and $\hat{g}^2(x)$. Now, because $H$ is proper, $x \not\in H$, and it follows that there is an integer $N$ such that if $n > N$, then $\pi_n(x) \in H$. However, for every third $n$, $\pi_n(x)$ is between $\pi_n(\hat{g}(x))$ and $\pi_n(\hat{g}^2(x))$, which are in the closed interval $H_n$. This contradiction establishes Theorem 1.

In what follows, if $x \in I$, and $s$ and $k$ are integers, $s \geq 1$, $k \geq 0$, we let $A_{s,k}(x) = \{f^{in+k}(x) | n \geq 0\}$.

**Lemma 2.** Suppose that $f$ has a dense orbit, and that $x$ is a point whose orbit is dense, i.e., $A_{1,0}$ is dense in $I$. Then one of the following occurs.

(i) $A_{2,0}$ is dense in $I$, in which case $A_{s,k}$ is dense in $I$ for each $s \geq 1$, $k \geq 0$, or

(ii) $A_{2,0}$ is not dense in $I$, in which case $I = \bar{A}_{2,0} \cup \bar{A}_{2,1}$. $\bar{A}_{2,0}$ and $\bar{A}_{2,1}$ are closed intervals which intersect in a point, and $f(A_{2,0}) = A_{2,1}$, $f(A_{2,1}) = \bar{A}_{2,0}$. Moreover, for each $k \geq 1$, $A_{2k,0}$ is dense in $\bar{A}_{2,0}$ and $A_{2k,1}$ is dense in $\bar{A}_{2,1}$.

**Proof.** Let $s$ be an integer, $s \geq 1$, and for each integer $r$, $0 \leq r \leq s - 1$, let $B_r = A_{s,r}$. Then since $\cup_{0 \leq r \leq s - 1} A_{s,r} = A_{1,0}$, it follows that $\cup_{0 \leq r \leq s - 1} B_r = I$. From this we see that there is an $r$, $0 \leq r \leq s - 1$, so that $B_r$ has nonempty interior.

Next, notice that if $J$ is a closed subinterval of $I$, then $f(J)$ is a closed interval, because if $f(J)$ is a point, then for some integer $n$, $f^n(x)$ is periodic. From this remark, and the fact that $f(B_r) \subset B_{r+1}$ (mod $s$) it follows that, for each integer $i$, $0 \leq i \leq r - 1$, $B_r$ has nonempty interior.

We next show that if the interiors of $B_r$ and $B_j$ intersect, then these interiors are identical. For, if $B_i \cap \text{int} B_j \neq \emptyset$, then there is a positive integer $n$ so that $f^{2n+i}(x) \in \text{int} B_i \cap \text{int} B_j$, and there is a sequence $n_1$, $n_2$, $n_3$, ... of positive integers such that $f^{n_1+i}(x) \to f^{n_2+i}(x)$. Then for every integer $l > 0$ we have $f^{n_1+i+l}(x) \to f^{n_2+i+l}(x)$. From this it follows that $\text{cl}\{f^{in+i}(x), f^{in+i+l}(x), ...\} \subset B_j$ and hence that

$$B_i \subset B_j \cup \{f^{in+i}(x), f^{in+i+l}(x), ..., f^{(n-1)s+i}(x)\}.$$ 

From this we see that $\text{int} B_i \subset \text{int} B_j$. A similar argument shows that $\text{int} B_j \subset \text{int} B_i$, and hence $\text{int} B_i = \text{int} B_j$.

Now let $G = \{g_r\}$ for some $r$, $0 \leq r \leq s - 1$, $g$ is a component of $\text{int} B_i$. Notice that $G$ is a collection of disjoint open intervals whose union is dense in $I$. Since $G$ is countable we list $G$ as $\{g_1, g_2, ...\}$. Now for each $g_r \in G$ let $C_r = \bar{g}_r$. Then $C_r$ is a closed interval, $f(C_r)$ is a closed interval by an earlier remark, and there is an $r$, $0 \leq r \leq s - 1$, such that $f(C_r) \subset B_r$. Then $f(C_i) \subset \text{int} B_i$, and there is an integer $k$ so that $f(C_i) \subset C_k$. Because $x$ has a dense orbit, we see that if $i$ and $k$ are integers which are subscripts of elements of $G$, then there is a positive integer $j$ so that $f^j(C_i) \subset C_k$. Since we have this transitivity, it follows that $G$ is finite. Thus we may
list $G$ as $\{g_1, g_2, \ldots, g_n\}$ and their closures as $C_1, C_2, \ldots, C_n$. Notice that because of the above transitivity condition, the set $\{C_1, C_2, \ldots, C_n\}$ is permuted by $f$.

We next show that $n \leq 2$. Let $y$ be a fixed point of $f$. Now, if $y \in \text{int } C_i$, then $f(C_i) = C_i$, which is impossible unless $n = 1$. Similarly, if $y$ is an endpoint of $I$, then $n = 1$. If $y$ is a common endpoint of $C_i$ and $C_j$, then $f(C_i) = C_j$ and $f(C_j) = C_i$ which is impossible unless $n = 2$. Notice that the integer $n$ depends on $s$. In what follows we will refer to $n$ as $n(s)$.

We now verify the conclusion. First, assume that $A_{2,0}$ is dense in $I$. Let $s$ be an integer, $s \geq 1$, and suppose that $n(s) = 2$. Then, there are closed intervals $C_1$ and $C_2$ with $C_1 \cup C_2 = I$, $C_1 \cap C_2 = \{pt\}$, $f(C_1) = C_2$ and $f(C_2) = C_1$. Assuming that $x \in C_2$ we see that, for each $j$, $f^{2j}(x) \in C_2$ and hence $A_{2,0} \cap \text{int } C_1 \neq \emptyset$. This contradicts the fact that $A_{2,0}$ is dense in $I$, and hence $n(s) = 1$. Then, for each $r$, $0 \leq r \leq s - 1$, $B_r = I$. Then $A_{s,r} = I$, and so $A_{s,r}$ is dense in $I$. From this, we see that for any integer $k \geq 0$, $A_{s,k}$ is dense in $I$.

Next, assume that $A_{2,0}$ is not dense in $I$. Let $s = 2$. Since $A_{2,0}$ is not dense, $B_0 \neq I$ and so $n(2) = 2$. Now let $j$ be an integer, $j \geq 1$. Then, for each integer $l$, $A_{2,2} \subset A_{2,2,l}$ and since $A_{2,2} \neq I$, we have $n(2j) = 2$. Now, notice that the intervals $C_1$ and $C_2$ which we construct for $s = 2$ are independent of $j$. This is because their common endpoint is the only fixed point for the function $f$. Then, assuming $x \in C_2$, we have $C_2 = A_{2,0}$, $C_1 = A_{2,1}$, and, for each integer $k \geq 1$, $A_{2,k} = C_2$ and $A_{2,k,1} = C_1$. This establishes Lemma 2.

The following result is known [N], but we include it for completeness.

**Corollary.** Suppose that $f$ has a dense orbit. Then the set of periodic points of $f$ is dense in $I$.

**Proof.** Let $V$ be an open interval in $I$. Let $x$ be a point of $V$ whose orbit is dense in $I$. If $\{f^{2n}(x)\}_{n \geq 0}$ is not dense in $I$, we may assume from Lemma 1 that $V \subset \text{cl}(\{f^{2n}(x)\}_{n \geq 0})$. Let $j$ be an integer such that $f^{2j}(x) \in V$. We may assume that $x < f^{2j}(x)$. Let $g : I \rightarrow I$ be the function $g = f^j$. From Lemma 2, it follows that $\{g^k(x)\}_{k \geq 0}$ is dense in $V$. Now let $l$ be the smallest positive integer such that $g^l(g(x)) \neq g(x)$. Then $g^l(x) = g^{l-1}(g(x)) > g(x) > x$ and $g^l(g(x)) < g(x)$. So $g^l(x) > x$ and $g^l(g(x)) < g(x)$. Consequently $g^l$ has a fixed point $y$, $x < y < g(x)$. Since $g^l(y) = y, f^{k_l}(y) = y$ and, since $y \in V$, $V$ contains a periodic point of $f$.

**Definition** If $y \in I$, the statement that $y$ is **topologically stable** means that if $\varepsilon > 0$, then there is a $\delta > 0$ such that if $z \in I$ and $|y - z| < \delta$ then for each positive integer $n$, $|f^n(y) - f^n(z)| < \varepsilon$. If $y$ is not topologically stable, then $y$ is called **topologically unstable**.

**Corollary.** Suppose that $f$ has a dense orbit. Then every point of $I$ is topologically unstable.

**Proof.** Suppose that $y \in I$ and that $y$ is topologically stable. Let $x$ be a point of $I$ whose orbit under $f$ is dense. We first show that the orbit of $y$ is dense. Suppose that $U$ is an open interval in $I$ and, for each $n$, $f^n(y) \notin U$. Let $V$ be the open interval which is the open middle third of $U$. Let $\varepsilon = \frac{1}{2} \text{diam } U$. Then, since $y$ is topologically
stable, there is a $\delta > 0$ such that if $|z - y| < \delta$ then, for each $n$, $|f^n(y) - f^n(z)| < \epsilon$. In particular, if $|z - y| < \delta$ then, for each $n$, $f^n(z) \notin V$. Now since $x$ has a dense orbit, there is a $j$ such that $|f^j(x) - y| < \delta$. Then there is an integer $k > j$ such that $f^k(x) \in V$. But then $f^{k-j}(f^j(x)) \in V$ and this is a contradiction. Therefore, the orbit of $y$ is dense.

Now it follows from Lemma 2 that there is a positive number $\epsilon$ and a subinterval $C$ of $I$ such that $\text{diam} \ C > 3\epsilon$, and, for each positive integer $n$, $\{f^{kn}(y)|k \geq 0\}$ is dense in $C$. Now choose $\delta$ such that if $|z - y| < \delta$, then for each $j$, $|f^j(z) - f^j(y)| < \epsilon$. Using the previous corollary, let $t$ be a periodic point such that $|t - y| < \delta$. Let $n$ be the period of $t$. Then, for each $k$, $|f^{kn}(t) - f^{kn}(y)| < \epsilon$, so $|t - f^{kn}(y)| < \epsilon$. But then $\{f^{kn}(y)|k \geq 0\}$ is not dense in $C$. This establishes the Corollary.

Example 3. Let $I = [0,1]$, and let $f: I \to I$ be defined by

\[
f(x) = \begin{cases} 
2x + \frac{1}{2}, & 0 \leq x \leq \frac{1}{4}, \\
-2x + \frac{1}{2}, & \frac{1}{4} \leq x \leq \frac{1}{2}, \\
2x - \frac{1}{2}, & \frac{1}{2} \leq x \leq 1.
\end{cases}
\]

Then $f$ has the following properties. (i) There is a number $x$ such that $\{f^n(x)|n \geq 0\}$ is dense in $I$, and (ii) if $y \in I$, then $\{f^2n(y)|n \geq 0\}$ is not dense in $I$.

It can be shown that $(I, f)$ is homeomorphic with Example 2 together with its reflection through the origin.

Theorem 3. Suppose that $x \in I$, and that $x$ has a dense orbit under $f$. Then one of the following occurs.

(a) $\{f^{2n}(x)|n \geq 0\}$ is dense in $I$, in which case $(I, f)$ is indecomposable, or

(b) $\{f^{2n}(x)|n \geq 0\}$ is not dense in $I$, in which case there are proper subcontinua $H$ and $K$ of $(I, f)$ such that (i) $H$ and $K$ are indecomposable, (ii) $H \cup K = (I, f)$, (iii) $H \cap K$ is a point, (iv) $\hat{f}(H) = K$, and (v) $\hat{f}(K) = H$.

Proof. First, assume that $\{f^{2n}(x)|n \geq 0\}$ is dense in $I$. Assume further that $(I, f)$ has no indecomposable subcontinuum with interior. Then Bing's construction [Bi] applies to yield a homeomorphism $\hat{f}: G \to G$ of the arc $G$ onto itself. Since $x$ has a dense orbit, the function $f$ is onto, and by choosing inverse images we may construct the point $x = (x, f^{-1}(x), \ldots)$ of $(I, f)$. It is clear that $\{f^n(x)|n \geq 0\}$ is dense in $(I, f)$. From this it follows that $\{\hat{f}^{n}(x)|n \geq 0\}$ is dense in $G$. This is impossible since $\hat{f}$ is a homeomorphism and $G$ is an arc. Therefore there is a subcontinuum $S$ of $(I, f)$ such that $S$ is indecomposable and has interior.
We will next show that \( S = (I, f) \). First, suppose that for each positive integer \( k \), \( \text{int}(f^k(S)) \cap \text{int} S = \emptyset \). As above, let \( \chi = (x, f^{-1}(x), f^{-2}(x), \ldots) \). The previous assumption makes it impossible for \( \{f^n(\chi)\}_{n \geq 0} \) to be dense in \( (I, f) \). It follows that there is a positive integer \( k \) such that \( \text{int}(f^k(S)) \cap \text{int} S = \emptyset \). Then \( f^k(S) \cap S \) is a subcontinuum of both \( f^k(S) \) and \( S \) which has interior, and, since \( S \) is indecomposable, \( f^k(S) = S \). Then, for each positive integer \( j \), \( f^j(S) = S \). Now let \( I \) be a positive integer such that \( f^I(\chi) \in S \). From Lemma 2 we have \( \{f^j(x)\}_{j \geq 1} \) in dense in \( I \) and hence \( \{f^j(x)\}_{j \geq 1} \) is dense in \( (I, f) \). Since \( f^j(x) \in S \) we see that \( S = (I, f) \).

Next, we consider the case where \( \{f^2(x)\}_{n \geq 0} \) is not dense in \( I \). By Lemma 2, there are closed subintervals \( C_1 \) and \( C_2 \) such that \( I = C_1 \cup C_2, C_1 \cap C_2 = \{p\} \), \( f(C_1) = C_2 \) and \( f(C_2) = C_1 \). Now let

\[
H = \{y \in (I, f), y_{2n} \in C_1 \text{ and } y_{2n+1} \in C_2 \text{ if } n \geq 0\}
\]

and

\[
K = \{y \in (I, f), y_{2n+1} \in C_1 \text{ and } y_{2n} \in C_2 \text{ if } n \geq 0\}.
\]

Then \( H \) and \( K \) are subcontinua of \( (I, f) \), \( H \cup K = (I, f), H \cap K = (p, p, p, \ldots), f(C_1) = C_2 \) and \( f(C_2) = C_1 \). In order to see that \( K \) is indecomposable, consider the function \( h = f^2: C_2 \to C_2 \). Then, assuming that \( x \in C_2 \), it follows from Lemma 2 that both \( \{h^n(x)\}_{n \geq 0} \) and \( \{h^2n(x)\}_{n \geq 0} \) are dense in \( C_2 \). By the first part of this theorem, \((C_2, h)\) is indecomposable. The correspondence

\[
(y, h^{-1}(y), h^{-2}(y), \ldots) \leftrightarrow (y, f(h^{-1}(y)), h^{-1}(y), f(h^{-2}(y)), h^{-2}(y), \ldots)
\]

is a homeomorphism between \((C_2, h)\) and \( K \). Therefore \( K \) is indecomposable and, as \( H = f(K), H \) is indecomposable. This establishes Theorem 3.

**Definition.** Suppose that \( y \) is a fixed point of \( f \). This statement that \( x \) is homoclinic to the fixed point \( y \) means that \( x \neq y \) and there is a choice of inverse images \( f^{-1}(x), f^{-2}(x), \ldots \) such that both \( f^n(x) \to y \) and \( f^{-n}(x) \to y \). If \( y \) is a periodic point of \( f \) with period \( s \), then the statement that \( x \) is homoclinic to \( y \) means that \( x \) is homoclinic to the fixed point \( y \) under \( f^s \).

The next result can be obtained from Theorem 1 and [BI]. We include a direct proof.

**Theorem 4.** If \( f \) has a point homoclinic to a periodic point, then \((I, f)\) contains an indecomposable subcontinuum.

**Proof.** Since, for each positive integer \( s \), \((I, f)\) is homeomorphic with \((I, f^s)\), we will assume that \( f \) has a point homoclinic to a fixed point. Let \( y \) be a fixed point and let \( x \neq y \), together with a choice of inverse images, be such that \( f^n(x) \to y \) and \( f^{-n}(x) \to y \). In \((I, f)\) let \( y = (y, y, y, \ldots) \) and \( \chi = (x, f^{-1}(x), f^{-2}(x), \ldots) \). Then \( \hat{f}(y) = y, \hat{f}^n(\chi) \to y \) and \( \hat{f}^{-n}(x) \to y \).

Now let \( S \) be the intersection of all subcontinua of \((I, f)\) which contain \( \{y\} \cup \{f^n(\chi)\}_{-\infty < n < \infty} \). Since both \( \hat{f}(S) \) and \( \hat{f}^{-1}(S) \) contain \( \{y\} \cup \{f^n(\chi)\}_{-\infty < n < \infty} \), we see that \( \hat{f}(S) = S \).

Now, for each \( n \geq 0 \), \( \pi_n \circ \hat{f} = f \circ \pi_n \), and so \( f(\pi_n(S)) = \pi_n(\hat{f}(S)) = \pi_n(S) \). Thus \( \pi_n(S) \) is invariant under \( f \). Now let \( J = \pi_0(S) \). Then \( f(J) = J \) and \( S = (J, f) \).
Suppose now that \( S = (J, f) \) contains no indecomposable subcontinuum with interior. Then Bing's construction \([Bi]\) yields an arc \( G \), and \( \hat{f}: G \to G \) is a homeomorphism. Now \( \hat{f}^n(g_x) \to g_y \) and \( \hat{f}^{-m}(g_x) \to g_y \). Since \( G \) is an arc and \( \hat{f} \) is a homeomorphism, this is impossible unless, for each \( j \), \( \hat{f}^j(g_x) = g_y \). This implies that, for each \( j \), \( g_{f^j(x)} = g_y \). But then \( g_y \) contains \( \{y\} \cup \{f^j(x)|-\infty < j < \infty\} \) and hence \( g_y = S \). But then \( G \) is degenerate, and this is impossible.

Therefore \( S \) contains an indecomposable subcontinuum with interior, and \((I, f)\) contains an indecomposable subcontinuum.

**Definition.** If \( f: I \to I \) is continuous, then the statement that \( f \) is *organic* means that if \( x \in (I, f) \), \( y \in (I, f) \) and \((I, f)\) is irreducible from \( x \to y \), then there is a positive integer \( n \) such that \( f^n([x, y]) = I \). The statement that \( f \) is *inorganic* means that \( f \) is not organic.

**Example 4.** The accompanying figure is a sketch of a function which is inorganic.

In \([H]\), Henderson shows that \((I, f)\) is a pseudo-arc, a particular snakelike continuum which is hereditarily indecomposable. Notice that \( f \) has no points of period greater than one.

**Lemma 5.** Let \( I = [a, b] \) and suppose that \( f: I \to I \) is continuous and onto. Further, suppose that \((I, f)\) is irreducible between \( \bar{x} = (x_0, x_1, \ldots) \) and \( \bar{y} = (y_0, y_1, \ldots) \). Then if \( c \) and \( d \) are numbers, \( a < c < d < b \), then there is an integer \( N \) such that if \( n > N \), then \( [c, d] \subseteq f^n([x, y]) \).

**Proof.** Recall that \([x_n, y_n]\) is the smallest closed interval containing \( x_n \) and \( y_n \).

First, notice that if \( n_2 > n_1 \), then \( f^{n_2}([x_{n_1}, y_{n_1}]) \subseteq f^{n_1}([x_{n_2}, y_{n_2}]) \). This is because \( f^{n_2-n_1}(x_{n_2}) = x_{n_1} \) and \( f^{n_2-n_1}(y_{n_2}) = y_{n_1} \). Then we have \( [x_0, y_0] \subseteq f([x_1, y_1]) \subseteq f^2([x_2, y_2]) \subseteq \cdots \). Now if \( k \) is an integer, \( k > 0 \), let \( J_k \) be \( \text{cl}(\bigcup_{n \geq k} f^{n-k}([x_n, y_n])) \). Then for each \( k \), \( J_k \) is a closed subinterval of \( I \), and \( f(J_{k+1}) = J_k \).

Now let \( J \) be the subcontinuum of \((I, f)\) defined by \( J = \{(z_0, z_1, z_2, \ldots)|z_k \in J_k \) and \( f(z_{k+1}) = z_k \} \). Since \( \bar{x} \) and \( \bar{y} \) belong to \( J \), and \((I, f)\) is irreducible from \( \bar{x} \to \bar{y} \), it follows that \( J = (I, f) \). Now, since \( f \) is onto, \( J_0 = I \). Then \( I = \text{cl}(\bigcup_{n=0}^{\infty} f^n([x_n, y_n])) \), and since \( f^n([x_n, y_n]) \subseteq f^{n+1}([x_{n+1}, y_{n+1}]) \), the conclusion follows.

**Lemma 6.** Suppose that \( I = [a, b] \) and that \( f: I \to I \) is continuous. If there are numbers \( p \) and \( q \), \( a < p < b \), \( a < q < b \), and integers \( r \) and \( s \) such that \( f^r(p) = a \), \( f^s(q) = b \), then \( f \) is organic.

**Proof.** Suppose that \((I, f)\) is irreducible between \( \bar{x} = (x_0, x_1, \ldots) \) and \( \bar{y} = (y_0, y_1, \ldots) \). It follows from the argument given in Lemma 5 that there is an integer
CHAOS, PERIODICITY AND SNAKELIKE CONTINUA

363

Nr such that if \( n > Nr \), then \( p \in f^{-r}([x_n, y_n]) \), and an integer \( N_r \) such that if \( n > N_r \), then \( q \in f^{-r}([x_n, y_n]) \). Then, if \( n > Nr \), \( a \in f^n([x_n, y_n]) \), and if \( n > N_r \), then \( b \in f^n([x_n, y_n]) \).

Now if \( n > Nr + N_r \), then \( I = f^n([x_n, y_n]) \), and so \( f \) is organic.

**Theorem 7.** If \( f: I \to I \) is organic, and \((I, f)\) is indecomposable, then \( f \) has a periodic point whose period is not a power of 2.

**Proof.** Since \((I, f)\) is indecomposable, there are three points \( x = (x_0, x_1, \ldots) \), \( y = (y_0, y_1, \ldots) \) and \( z = (z_0, z_1, \ldots) \) in \((I, f)\), such that \((I, f)\) is irreducible between any two of them. Because \( f \) is organic, there is a positive integer \( n \) such that \( f^n([x_n, y_n]) = f^n([x_n, z_n]) = f^n([y_n, z_n]) = I \). We will assume that the notation is chosen so that \( x_n < y_n < z_n \). Now since \([x_n, y_n] \subset f^n([y_n, z_n])\), there is a closed subinterval \( J_1 \) of \([y_n, z_n]\) such that \( f(J_1) = [x_n, y_n] \). Now there is a closed subinterval \( J_2 \) of \([y_n, z_n]\) such that \( f(J_2) = J_1 \). Notice that \( y_n \in J_1 \cap J_2 \). Finally, let \( J_3 \) be a closed subinterval of \([x_n, y_n]\) such that \( f(J_3) = J_2 \).

Now \( f^3(J_3) \), and so there is a point \( p \) of \( J_3 \) such that \( f^3(p) = p \).

Now suppose \( f^3(p) = p \). Then \( p = y_n \) and \( f^3(y_n) = f^2(y_n) = f(y_n) = y_n \). But then \( y_n \in J_1 \cap J_2 \), which is a contradiction. Thus the points \( p, f^3(p) \) and \( f^2(p) \) are distinct.

Let \( s \) be the period of \( p \). Then \( f^3(p) = p \) and it follows that 3 divides \( s \). Therefore, \( s \) is not a power of 2. This establishes Theorem 7.

**Lemma 8.** Suppose \( f: I \to I \) is continuous and onto. Suppose that \( J \) is a proper closed subinterval of \( I \) and, for each \( n \geq 1 \), \( f^{-n}(J) \) is an interval. Then \((I, f)\) is decomposable.

**Proof.** Let \( H = \{ x \mid x \in (I, f) \text{ and } \pi_0(x) \in J \} \). Since \( J \) is proper and \( f \) is onto, \( H \) is a proper subset of \((I, f)\). Since, for each \( n \), \( f^{-n}(J) \) is an interval, \( H_n = \pi_0(H) = f^{-n}(J) \), and so \( H \) is a subcontinuum of \((I, f)\). Now let \( U = \pi_0^{-1}\left( \text{int } J \right) \). Then \( U \) is open in \((I, f)\) and \( U \subset H \). Therefore \( H \) is a proper subcontinuum of \((I, f)\) with interior, and it follows that \((I, f)\) is decomposable.

The following lemma is well known.

**Lemma 9.** Let \( f: I \to I \) be continuous, \( h: I \to I \) be a homeomorphism, and \( f_1 = h \circ f \circ h^{-1} \). Then \((I, f)\) and \((I, f_1)\) are homeomorphic.

**Proof.** Define \( H: (I, f) \to (I, f_1) \) by \( H((x_0, x_1, \ldots)) = (h(x_0), h(x_1), \ldots) \). It is clear that \( H \) is a homeomorphism.

**Definition.** If \( f: I \to I \) is continuous, then the statement that \( f \) has finitely many turning points means that there is a finite set \( \{ a_0, a_1, \ldots, a_l \} \), \( a = a_0 < a_1 < \cdots < a_l = b \) in \( I = [a, b] \) such that \( f \) is monotone on \([a_{i-1}, a_i] \) for \( i = 1, 2, \ldots, l \).

**Theorem 10.** Suppose that \( f: I \to I \) is continuous, onto, and has finitely many turning points. Then if \((I, f)\) is indecomposable, \( f \) has a periodic point whose period is not a power of 2.
**Proof.** If $f$ is organic, then the conclusion follows from Theorem 7. We will show that if $f$ is inorganic, then $f$ has a point of period 3. Suppose that $f$ is inorganic. Then it follows from Lemma 6 that either

(i) $f^{-1}(\{a\}) = \{a\}$,  
(ii) $f^{-1}(\{b\}) = \{b\}$, or  
(iii) $f(a) = b, f(b) = a$ and $f^{-1}(\{a, b\}) = \{a, b\}$.

Now if (ii) holds, define $h: I \to I$ by $h(x) = (a + b) - x$, and let $f_1 = h \circ f \circ h^{-1}$. Then $f_1^{-1}(\{a\}) = \{a\}$. It follows from Lemma 8 that $(I, f) \sim (I, f_1)$, and so the hypotheses of the theorem hold for $f_1$.

If (iii) holds let $f_1 = f^2$. Then $f_1^{-1}(\{a\}) = \{a\}$ and, since $(I, f) \sim (I, f^2)$, the hypotheses of the theorem hold for $f_1$.

It follows from the preceding discussion that we may assume that $f^{-1}(\{a\}) = \{a\}$. Let the turning points of $f$ be $a = a_0 < a_1 < \cdots < a_i = b$. We will consider two cases.

**Case 1.** $f^{-1}(\{b\}) = \{b\}$. Now suppose that there is a number $c$ in $(a, b)$ such that if $x \in [a, c]$, then $f(x) > x$. Since $f^{-1}(\{a\}) = \{a\}$, we may assume that $c$ is chosen so that $f([c, b]) \cap [a, c] = \emptyset$. Then, for each $n \geq 0$, $f^{-n}([a, c])$ is an interval, and it follows from Lemma 8 that $(I, f)$ is decomposable. Therefore for each $c \in (a, b)$ there is an $x \in (a, c)$ such that $f(x) \leq x$. Similarly, for each $c \in (a, b)$ there is an $x \in (c, b)$ such that $f(x) \geq x$. It follows that there are fixed points for $f$ in $(a, b)$.

Now let $\mathcal{E} = \{c | a < c < b$ and $f([a, c]) = [a, c]\}$. We will show that $\mathcal{E}$ is nonempty. Let $q$ be a point of $(a, b)$ such that $f(q) = q$. Then either $f([a, q]) = [a, q]$ or there is a turning point $a_i, a < a_i < q$, with $f(a_i) > a_i$. But in this case there is a point $q_1, a < q_1 < a_1$, with $f(q_1) = q_1$. Since there are only finitely many turning points there is a number $c_0, a < c_0 < b$, such that $f([c_0, a]) = f([a, c_0]) = [a, c_0]$. Now since $f([a, c_0]) = [a, c_0]$ and $(I, f)$ is indecomposable, it follows from Lemma 8 that $f([c_0, b]) \cap [a, c_0] \neq \emptyset$. Let $a_n$ be the smallest turning point such that $f(a_{n+1}) < c_0 < a_n$. Notice that $a_{n+1} \neq b$. Now, there is a $q, a_{n+1} < q < b$, such that $f(q) = q$. Let $c_1$ be the smallest such $q$. Now either $f([a, c_1]) = [a, c_1]$, or there is a number $x, c_0 < x < c_1$, with $f(x) > c_1$. If $x > a_{n+1}$, there is a $q, a_{n+1} < q < x$, such that $f(q) = q$, which contradicts the choice of $c_1$. Therefore $c_0 < x < a_{n+1}$. Again, there is a $q, x < q < a_{n+1}$, such that $f(q) = q$. Now, we have $[c_0, c_1] \subset f([c_0, q])$ and $[c_0, c_1] \subset f([q, c_1])$, and using the same argument as in the proof of Theorem 7, we have a point of period 3. Thus we have the conclusion of the theorem, or $c_1 \in \mathcal{E}$.

If $c_1 \in \mathcal{E}$ we repeat the argument, replacing $c_0$ with $c_1$. Continuing this way we get $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ with $a_{n_1} < a_{n_2} < a_{n_3} < \cdots$. Since there are only finitely many turning points, the process will end in a finite number of steps with a point of period 3.

**Case 2.** There is a point $p \in (a, b)$ with $f(p) = b$. We proceed as in Case 1. We have $\mathcal{E} \cap (a, p) \neq \emptyset$. Choose $c_0$ such that $f([c_0]) = c_0$ and $f([a, c_0]) = [a, c_0]$. Let $a_{n_0}$ be as in Case 1, except now it might be that $a_{n_0} = b$. If $a_{n_0} = b$, there is a $q, p < q < b$, such that $f(q) = q$. Then $[c_0, a_{n_0}] \subset f([c_0, q])$ and $[c_0, q] \subset f([q, a_{n_0}])$. As before, it follows that $f$ has a point of period 3. In fact, if $a_{n_0} > p$, the same result holds.
Thus we may assume that $a_{n_0} < p$. We may then find $c_1, a_{n_0} < c_1 < p$, and proceed as in Case 1. Thus $f$ has a point of period 3.

**Example 5.** Let $f$ be as sketched in the accompanying figure. Then it can be shown that $(I, f)$ is indecomposable and that $f$ is inorganic. It follows from the argument in Theorem 10 that $f$ has a point of period 3.

![Graph showing a snakelike continuum](image)

**Corollary 11.** Suppose that $f: I \to I$ is continuous and has finitely many turning points. Then, there is an integer $l \geq 0$ and an indecomposable subcontinuum of $(I, f)$ which is invariant under $f^{2^l}$ if and only if $f$ has a periodic point whose period is not a power of 2.

**Proof.** Theorem 1 shows that if $f$ has the required periodic point, then $(I, f)$ has the required indecomposable subcontinuum.

Suppose then that $S$ is an indecomposable subcontinuum of $(I, f)$ which is invariant under $f^{2^l}$. Let $g = f^{2^l}$. Then $(I, f)$ is homeomorphic with $(I, g)$. Let $S_1$ be the image of $S$ under the natural homeomorphism. Then $S_1$ is invariant under $g$. Let $J = \pi_0(S_1)$. Then $J$ is invariant under $g$, and $g$ has finitely many turning points in $J$. We now apply Theorem 10 to $g: J \to J$ and find that $g$ has a periodic point whose period is not a power of 2, and it follows that $f$ has a periodic point whose period is not a power of 2.

**References**


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