SUBSPACES OF BMO(R^n)

BY

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Abstract. We consider subspaces of BMO(R^n) generated by one singular integral transform. We show that the averages along x_i-lines of the jth Riesz transform of g ∈ BMO ∩ L^2(R^n) or g ∈ L^∞(R^n) satisfy a certain strong regularity property. One consequence of this result is that such functions satisfy a uniform doubling condition on a.e. x_i-line. We give an example to show, however, that the restrictions to x_i-lines of the Riesz transform of g ∈ BMO ∩ L^2(R^n) do not necessarily have uniformly bounded BMO norm. Also, for a Calderón-Zygmund singular integral operator K with real and odd kernel, we show that K(BMO) ⊆ L^∞ + K(L^p), where L^∞ and BMO are the spaces of L^∞ or BMO functions of compact support, respectively, and the closure is taken in BMO norm.

1. Introduction. Let BMO(R^n) be the set of complex-valued functions f ∈ L^1_{loc}(R^n) such that

\[ \|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f - f_Q| < \infty, \]

where the sup is taken over all cubes Q in R^n with sides parallel to the coordinate axes, f_Q = (1/|Q|)\int_Q f, and |Q| is the Lebesgue measure of Q. For us, a singular integral operator K with kernel Ω is an operator of the form

\[ Kf(x) = \lim_{\epsilon \to 0} \int_{|x-y| > \epsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y) \, dy, \]

where Ω is continuous, Ω(rx) = Ω(x) for r > 0, and \( \int_{S^{n-1}} \Omega(x) \, d\sigma(x) = 0 \) for surface measure d\sigma on \( S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\} \). For f ∈ \( L^p \) for 1 < p < \infty, Kf(x) exists a.e. (dx), and K is a bounded operator on L^p. Associated with K is the Fourier multiplier m so that for f ∈ L^2(R^n), (Kf)(\xi) = m(\xi)\hat{f}(\xi), where the Fourier transform is defined by \( \hat{\xi}(x) = \int_{\mathbb{R}^n} f(x) e^{2\pi i x \cdot \xi} \, dx \); m satisfies m(r\xi) = m(\xi) for r > 0 and \( \xi \in \mathbb{R}^n \). For \( f \in L^\infty(R^n) \), we define

\[ \tilde{K}f(x) = \lim_{\epsilon \to 0} \int_{|x-y| > \epsilon} \left( \frac{\Omega(x-y)}{|x-y|^n} - \frac{\Omega(y)}{|y|^n} \chi_{\{y : |y| > 1\}}(y) \right) f(y) \, dy, \]

where \( \chi_E \) denotes the characteristic function of the set E. For \( f \in L^p \cap L^\infty \), 1 ≤ p < \infty, Kf and \( \tilde{K}f \) differ only by a constant. The operator \( \tilde{K} \) maps \( L^\infty \) into
BMO boundedly. The $j$th Riesz transform $R_j, j \in \{1, \ldots, n\}$, is the singular integral operator with kernel $\Omega_j(x) = C_n x_j / |x|$ and multiplier $m_j(\xi) = i \xi_j / |\xi|$ [7, p. 58].

Let $H^1(\mathbb{R}^n)$ be the set of $f \in L^1(\mathbb{R}^n)$ such that
\[ \|f\|_{H^1} = \|f\|_{L^1} + \sum_{j=1}^n \|R_j f\|_{L^1} < \infty. \]

In [1], Fefferman and Stein proved that $(H^1)^* = BMO$ (in an appropriate sense), and obtained by duality as a corollary that $BMO = L^\infty + \sum_{j=1}^n \tilde{R}_j L^\infty$, with
\[ (1.1) \quad \|f\|_{BMO} \approx \inf \left\{ \sum_{j=0}^n \|g_j\|_{L^\infty}, f = g_0 + \sum_{j=1}^n \tilde{R}_j g_j, \text{modulo constants} \right\}, \]
where $\approx$ means that the norms are equivalent. In [8], Uchiyama obtained (1.1) constructively for $f$ of compact support, and proved that a certain condition on the multipliers of a set of singular integral operators $K_1, \ldots, K_n$ implies that $BMO = \sum_{j=1}^n \tilde{R}_j L^\infty$ with an appropriate equivalence of norms. This condition had been proven necessary by Janson [5].

Using in part techniques developed by Uchiyama in [8], we consider subspaces of BMO generated by one singular integral operator. The Riesz transforms are the most natural to consider in this context since one may expect that functions of the form $\tilde{R}_j g$ for $g \in L^\infty$, or $R_j g$ for $g \in BMO \cap L^2$, when restricted to lines in the $x_j$ direction, have regularity properties that functions in $BMO(\mathbb{R}^n)$ do not necessarily have. Results of this type are proved in §3, Theorem 3.2 and §4, Corollary 4.3. (In §2, we state Uchiyama’s results from [8] that are needed later.) Theorem 3.2 is obtained from Lemma 3.1, which characterizes $R_j(BMO \cap L^2)$ in terms of a decomposition closely related to Uchiyama’s decomposition of $BMO \cap L^2$ [8, p. 224]. One direction of the characterization follows easily from Uchiyama’s work; the other direction is obtained by solving the equation $R_j \varphi = b$ for $\varphi$ under appropriate assumptions on $b$.

Corollaries of Theorem 3.2, proved in §4, state that if $f = R_j g, g \in BMO \cap L^2$, or $f = \tilde{R}_j g, g \in L^\infty$, then $f$ has a uniform doubling property on $x_j$-lines; namely
\[ (1.2) \quad \left| \frac{1}{|I|} \int_I f(x_1, \ldots, x_n) \, dx_j - \frac{1}{|J|} \int_J f(x_1, \ldots, x_n) \, dx_j \right| \leq c \|g\|_{BMO} \]
for almost all $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in \mathbb{R}^{n-1}$, whenever $I$ and $J$ are adjacent intervals in $\mathbb{R}$ of the same length. An example of Kahane [6] from another context shows that the uniform doubling property for a function $f$ on $x_j$-lines is weaker than the uniform BMO property on $x_j$-lines. Both conditions restrict the singularities of $f$ along $x_j$-lines to be at most logarithmic, but the BMO condition further controls the packing of singularities. Kahane’s example is sketched in §4; it is then modified to show that $\exists G \in BMO \cap L^2(\mathbb{R}^2)$ such that $\text{ess sup}_{x_1} \|R_1 G(x_1, x_2)\|_{BMO(x_1)} = \infty$. Hence (1.2) cannot be improved to obtain a uniform BMO condition on lines for the class $R_j(BMO \cap L^2)$.

In §5 we show that if $K$ is a singular integral operator with a real and odd kernel (in particular if $K$ is a Riesz transform), then
\[ (1.3) \quad K(BMO_+) \subseteq L^\infty + K(L^\infty), \]
where $L^\infty_c$ and $\text{BMO}_c$ are the spaces of functions of compact support in $L^\infty$ and \text{BMO}, respectively, and the closure is taken in \text{BMO} norm. The proof is an adaptation of Uchiyama's main construction in \cite{8}. This suggests that the action of $K$ on \text{BMO} is in some sense close to the action of $K$ on $L^\infty$.

Finally, in §6 we make some remarks about related problems which are not yet solved.

**Notation.** The letter $c$ denotes various constants depending possibly on $n$ and, in §5, on $K$. All cubes $Q$, $I$, or $J$ are assumed to be open and to have sides parallel to the coordinate axes; $x_Q$ and $l(Q)$ denote the center and side length of $Q$, respectively. For $r > 0$, $rQ$ is the cube with sides parallel to the axes having center $x_Q$ and side length $rl(Q)$. For $j \in \{1, \ldots, n\}$, $D_j$ is the derivative with respect to $x_j$, and for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, $D^\alpha$ is the differential operator $\partial^{|\alpha|}/\partial x_{\alpha_1} \cdots \partial x_{\alpha_n}$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. For $f$ continuous on $\mathbb{R}^n$, let

$$\|f\|_{\text{Lip}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|},$$

and let $C^1(\mathbb{R}^n)$ be the space of continuously differentiable functions on $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, $x \cdot y = \sum_{j=1}^{n} x_j y_j$.

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**2. Background.** We state several lemmas from Uchiyama \cite{8} that will be used later. In fact, the following Lemmas U1–U6 are Lemmas 3.1–3.6, respectively, in \cite{8}.

**Lemma U1.** Suppose $f \in \text{BMO} \cap L^2(\mathbb{R}^n)$ and $\|f\|_{\text{BMO}} \leq 1$. Then there exist functions $\{b_I(x)\}_I$ and $\{\lambda_I\}_I$, $\lambda_I \geq 0$, where $I$ is taken over all dyadic cubes in $\mathbb{R}^n$, such that

\begin{equation}
(2.1) \quad f(x) = \lim_{k \to \infty \text{ in } L^2} \sum_{2^{-k} \leq |I| \leq 2^k} \lambda_I b_I(x),
\end{equation}

\begin{equation}
(2.2) \quad \text{supp } b_I \subseteq 3I, \quad \int b_I(x) \, dx = 0, \quad \|D^\alpha b_I\|_{L^\infty} \leq C_{|\alpha|/I}^{-|\alpha|},
\end{equation}

\begin{equation}
(2.3) \quad \sum_{I \subseteq J} \lambda_I^2 |I| \leq c|J| \quad \text{for any cube } J
\end{equation}

and

\begin{equation}
(2.4) \quad \sum_{I : |I| \leq 1} \lambda_I^2 |I| = c\|f\|^2_{L^2}.
\end{equation}

The last equality is not stated explicitly in \cite{8}, but follows from the definition of the $\{\lambda_I\}_I$ and Plancherel's Theorem quite easily.

**Lemma U2.** For $\{\lambda_I\}_I$ satisfying (2.3), $\lambda_I \geq 0 \quad \forall I$, set

$$\eta_k(x) = \sum_{I : |I| = 2^{-k}} \lambda_I (1 + 2^k |x - x_I|)^{-n-1}.$$
and
\[ \epsilon_k(x) = \sum_{j=0}^{\infty} \left( \frac{2}{3} \right)^j \eta_{k-j}(x). \]

Then \( \forall x, y, \)
\[ \eta_k(x) \leq \epsilon_k(x) \leq c, \]
\[ \eta_k(x) \leq c(2^k|x-y| + 1)^{n+1} \eta_k(y), \]
\[ \epsilon_k(x) \leq c(2^k|x-y| + 1)^{n+1} \epsilon_k(y), \]
and if \( |I| = 2^{-p}, \) then
\[ \int_I \sum_{k=p}^{\infty} \epsilon_k(x) \eta_k(x) \, dx \leq c|I|. \]

**Lemma U3.** Let \( j \) be a positive integer. Suppose \( \{b_j(x)\}_{d,dyadic} \) satisfy
\[ \sup \beta_j \leq 2^j, \int b_j(x) \, dx = 0, \quad \|b_j\|_{\text{Lip}_1} \leq c(2^j|I|)^{-1}. \]
If \( \lambda_I \geq 0 \ \forall I \) and \( \beta > \alpha > 0, \) then
\[ \left\| \sum_{l: \alpha < l(I) < \beta} \lambda_l b_l(x) \right\|_2 \leq c2^{jn} \left( \sum_l \lambda_l^2 |l| \right)^{1/2}. \]

**Lemma U4.** Suppose \( j, \{b_j\}, \) and \( \{\lambda_j\}, \) are as in Lemma U3 and (2.3) holds. For \( \alpha > 0, \) let \( f(x) = \sum_{l(I) < \alpha} \lambda_l b_l(x). \) Then
\[ \|f\|_{\text{BMO}} \leq C2^j. \]

**Lemma U5.** Let \( I \) and \( p_I(x) \in C^1(\mathbb{R}^n) \) be such that
\[ \int p_I(x) \, dx = 0, \]
\[ |p_I(x)| \leq \left( 1 + l(I)^{-1}|x - x_I| \right)^{-n-1} \]
and
\[ |\nabla p_I(x)| \leq l(I)^{-1} \left( 1 + l(I)^{-1}|x - x_I| \right)^{-n-2}. \]
Then \( \exists \{\beta_{I,j}(x)\}_{j=0}^{\infty} \) such that
\[ \|\beta_{I,j}\|_{\text{Lip}_1} \leq c(2^j|I|)^{-1}, \quad \sup \beta_{I,j} \subseteq 2^j I, \quad \int \beta_{I,j}(x) \, dx = 0 \]
and
\[ p_I(x) = \sum_{j=0}^{\infty} 2^{-j(n+1)} \beta_{I,j}(x). \]
If \( \tilde{p}_I = (p_{I,0}, \ldots, p_{I,m}) \) and each component of \( \tilde{p}_I \) is real valued and satisfies the conditions (2.12)–(2.14), and \( \tilde{a} = (a_0, \ldots, a_m) \in \mathbb{R}^{m+1} \) with \( |\tilde{a}| = 1 \) satisfies \( \tilde{p}_I(x) \cdot \tilde{a} = 0 \ \forall x, \) then we can take \( \tilde{\beta}_{I,j} = (\beta_{I,j}^0, \ldots, \beta_{I,j}^m) \) such that (2.15) is satisfied for each component of \( \tilde{\beta}_{I,j}, \tilde{p}_I(x) = \sum_{j=0}^{\infty} 2^{-j(n+1)} \tilde{\beta}_{I,j}(x) \) and \( \tilde{\beta}_{I,j}(x) \cdot \tilde{a} = 0 \ \forall x, j. \)
Lemma U6. Suppose $K$ is a singular integral operator with multiplier $m$, and $b_I(x)$ satisfies (2.2) for some cube $I$. Then $p_I(x) = K b_I(x)$ satisfies (2.12),

\begin{equation}
|p_I(x)| \leq C(K) \left(1 + l(I)^{-1}|x - x_I|\right)^{-n-1}
\end{equation}

and

\begin{equation}
|\nabla p_I(x)| \leq C(K) l(I)^{-1} \left(1 + l(I)^{-1}|x - x_I|\right)^{-n-2},
\end{equation}

where $|C(K)| \leq c \sup \{|D^a m(\xi)|: |\xi| = 1, |a| \leq C(n)\}$. (In fact, $C(n) = n + 4$ is sufficient.) Also it is enough to assume $\|D_x b_I\|_{Lip1} \leq c(I)^{-2}$ for $j = 1, \ldots, n$ instead of $\|D^a b_I\|_{L^\infty} \leq c_{|a|} (I)^{-|a|}$.

3. Properties of $R_1(BMO \cap L^2)$. We characterize $R_1(BMO \cap L^2)$ in terms of a decomposition similar to the decomposition of $BMO \cap L^2$ in Lemma U1. For convenience we take $j = 1$ and write $x \in \mathbb{R}^n$ as $x = (x_1, x')$ for $x' \in \mathbb{R}^{n-1}$.

Lemma 3.1. Suppose $f = R_1 g$ for some $g \in BMO \cap L^2$ with $\|g\|_{BMO} \leq 1$. Then there exist functions $\{a_I(x)\}_I$ and $\{\lambda_I\}_I$, $\lambda_I > 0$, for $I$ taken over all dyadic cubes, such that

\begin{equation}
f(x) = \lim_{k \to \infty} \sum_{2^{-k} \leq l(I) \leq 2^k} \lambda_I a_I(x),
\end{equation}

\begin{equation}|a_I(x)| \leq c \left(1 + l(I)^{-1}|x - x_I|\right)^{-n-1},
\end{equation}

\begin{equation}|D^a a_I(x)| \leq c_{|a|} l(I)^{-|a|} \left(1 + l(I)^{-1}|x - x_I|\right)^{-n-1-|a|},
\end{equation}

\begin{equation}\int a_I(x_1, x') \, dx_1 = 0 \quad \forall x' \in \mathbb{R}^{n-1},
\end{equation}

\begin{equation}\sum_{I \subseteq J} \lambda_I^2 |I| \leq c |J| \quad \text{for any cube } J
\end{equation}

and

\begin{equation}\sum_{I \subseteq J} \lambda_I^2 |I| < \infty.
\end{equation}

Conversely, if $f$ satisfies (3.1)-(3.6) for some $\{a_I\}_I$ and $\{\lambda_I\}_I$, then $f = R_1 g$ for some $g \in BMO \cap L^2$ with $\|g\|_{BMO} \leq C$.

Proof. The forward direction follows easily from Uchiyama’s lemmas. By Lemma U1,

\begin{equation}g = \lim_{k \to \infty} \sum_{2^{-k} \leq l(I) \leq 2^k} \lambda_I b_I \end{equation}

with each $b_I$ satisfying (2.2), and (3.5) and (3.6) hold. Since $R_1$ is continuous on $L^2$, we obtain (3.1) by setting $a_I = R_1 b_I$. Then (3.2), and (3.3) for $|a| = 1$, follow by Lemma U6; for $|a| > 1$, $D^a a_I$ is estimated by taking the derivatives inside the integral in the expression for $a_I$ exactly as in the proof of Lemma U6. The cancellation property (3.4) holds because $\dot{a}_I(\xi) = i \xi |\xi|^{-1} \dot{b}_I(\xi)$. 

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The idea of the converse is to use (3.2)–(3.4) to write \( a_\ell = R_\ell \varphi_\ell \) with appropriate control of \( \varphi_\ell \). Let \( h_\ell(x_1, x') = \int_{-\infty}^{\infty} a_\ell(t, x') \, dt \). We claim

\[
|h_\ell(x)| \leq c(I) \left( 1 + l(I)^{-1} |x - x_1| \right)^n
\]

and

\[
|D^n h_\ell(x)| \leq c_a l(I)^{1-|a|} \left( 1 + l(I)^{-1} |x - x_1| \right)^{-n-|a|}.
\]

To prove (3.7), we may assume \( x_1 = 0 \). For \( x \) such that \( x_1 < 0 \), let \( k \) and \( m \) be positive integers such that \(- (k + 1) l(I) < x_1 < - k l(I)\) and \( ml(I) < |x'| < (m + 1) l(I) \). By (3.2),

\[
|h_\ell(x)| \leq \int_{-\infty}^{x_1} |a_\ell(t, x')| \, dt \leq c \sum_{p=-k}^{\infty} \frac{l(I)}{\left( 1 + \sqrt{p^2 + m^2} \right)^{n+1}}
\]

\[
\leq \frac{c l(I)}{(1 + \sqrt{k^2 + m^2})^n} \leq c l(I) \left( 1 + l(I)^{-1} |x - x_1| \right)^{-n}.
\]

If \( x_1 > 0, h_\ell(x) = - \int_{x_1}^{\infty} a_\ell(t, x') \, dt \) by (3.4), so the same estimates hold. Then (3.8) follows from (3.3) and (3.4) similarly. Note that (3.7) and (3.8) imply that \( h_\ell \in L^2 \) and \( |\nabla h_\ell| \in L^1 \cap L^2 \).

Define \( \varphi_\ell = - \sum_{j=-1}^{\infty} R_j D_{x_1} h_\ell \). Then \( \varphi_\ell \in L^2 \) since \( |\nabla h_\ell| \in L^2 \). Note that \( \hat{\varphi}_\ell(\xi) = -2\pi |\xi| \hat{h}_\ell(\xi) \); therefore \( (R_j \varphi_\ell)(\xi) = -2\pi i \xi_j \hat{h}_\ell(\xi) = \hat{a}_\ell(\xi), \) since \( D_{x_1} h_\ell = a_\ell \). Hence \( R_j \varphi_\ell = a_\ell \). If we can show that \( \sum \lambda_\ell \varphi_\ell = \sum \lambda_\ell R_j a_\ell = f \), then we are done since \( \lambda_\ell \varphi_\ell \in BMO \cap L^2 \), by Lemma U5.

To see that \( \sum \lambda_\ell \varphi_\ell \in BMO \cap L^2 \), we show that \( D_{x_1} h_\ell \) satisfies the assumptions of Lemma U5 for \( l = 1, \ldots, n \). We have \( \int D_{x_1} h_\ell(x) \, dx = 0 \) since \( |\nabla h_\ell| \in L^1 \) and \( \lim_{|x| \to \infty} h_\ell(x) = 0 \) by (3.7); the estimates (2.13) and (2.14) follow from (3.8). Hence for each \( l \), \( \exists (\beta_{l,j}(x))_{j=0}^{\infty} \) satisfying (2.15) and \( D_{x_1} h_\ell(x) = \sum_{j=0}^{\infty} 2^{-j(n+1)} \beta_{l,j}(x) \). Therefore, using Lemma U3,

\[
\left\| \sum \lambda_\ell D_{x_1} h_\ell \right\|_{L^2} \leq \sum_{j=0}^{\infty} 2^{-j(n+1)} \left\| \sum \lambda_\ell \beta_{l,j} \right\|_{L^2} \leq c \sum_{j=0}^{\infty} 2^{-j(n+1)} \left( \sum \lambda_\ell |I| \right)^{1/2} < \infty,
\]

by (3.6). Similarly, by Lemma U4,

\[
\left\| \sum \lambda_\ell D_{x_1} h_\ell \right\|_{BMO} \leq \sum_{j=0}^{\infty} 2^{-j(n+1)} \left\| \sum \lambda_\ell \beta_{l,j} \right\|_{BMO} \leq c.
\]

Therefore \( \sum \lambda_\ell D_{x_1} h_\ell \in BMO \cap L^2 \) for each \( l \); hence

\[
\sum \lambda_\ell \varphi_\ell = - \sum \lambda_\ell \sum_{j=1}^{n} R_j D_{x_1} h_\ell = - \sum_{j=1}^{n} R \sum \lambda_\ell D_{x_1} h_\ell \in BMO \cap L^2.
\]

The cancellation property of the Riesz transforms, reflected in condition (3.4), forces the \( x_1 \)-averages of \( f \in R_1(BMO \cap L^2) \) to have a certain regularity. Let \( Q \) be a
cube in $\mathbb{R}^n$, and write $Q = J \times L$ for $J \subseteq \mathbb{R}^1$ and $L \subseteq \mathbb{R}^{n-1}$. By Fubini’s Theorem, the average $f_{x,y} = (1/|J|) \int_J f(t, y) \, dt$ is defined for almost $y \in L$.

**Theorem 3.2.** Suppose $f = R_1 g$ for some $g \in \text{BMO} \cap L^2(\mathbb{R}^n)$ satisfying $\|g\|_{\text{BMO}} \leq 1$. For $Q, J,$ and $L$ as above, $\exists c > 0$ and a set $E \subseteq L$ of $(n-1)$-dimensional Lebesgue measure 0 such that

\[ |f_{x,y} - f_{x,y'}| \leq c \quad \forall y, y' \in L - E \quad \text{(3.9)} \]

and

\[ |f_{x,y} - f_{x,y'}| \leq c \frac{|y - y'|}{l(Q)} \log \frac{l(Q)}{|y - y'|} \quad \text{if } y, y' \in L - E \quad \text{(3.10)} \]

and $|y - y'| < \frac{1}{c} l(Q)$.

**Remarks.** The number $\frac{1}{c}$ is chosen because the function $t \mapsto t \log(\frac{1}{t})$ is increasing on $(0, \frac{1}{c})$. The proof uses only (3.1), (3.2), (3.3) for $\alpha = 1$, (3.4), and the estimate $\lambda_i \leq c$, which follows trivially from (3.5). An example of a function in $\text{BMO}(\mathbb{R}^n)$ which does not satisfy (3.10) is $f(x) = \log|x|$.

**Proof of Theorem 3.2.** We may assume $Q = [-1,1]^n$, so $J = [-1,1]$ and $L = [-1,1]^{n-1}$. For $y, y' \in L$ fixed, let $e = |y - y'|$. It is sufficient to obtain $|f_{x,y} - f_{x,y'}| < ce \log \frac{1}{e}$ for $e < \frac{1}{c}$, since (3.10) implies (3.9). By Lemma 3.1, $f = \sum \lambda_i a_i$ with (3.2)–(3.6). We also may assume that $\exists N$ so that $\lambda_i = 0$ whenever $l(I) > 2^N$ or $l(I) < 2^{-N}$. To see this, let $f_k = \sum_{2^{-k} \leq l(I) < 2^k} \lambda_i a_i$; then $\lim_{k \to \infty} f_k = f$ in $L^2$ and, for each $k, f_k$ is continuous and the sum converges absolutely. Let

\[ R_{y}^\delta = J \times \prod_{i=2}^n \left( y_i - \delta, y_i + \delta \right), \]

where $\delta > 0$ and $y = (y_1, \ldots, y_n)$, and similarly define $R_{y'}^\delta$. The result for $f_k$ implies that the averages $(f_k)_{R_{y}^\delta}$ and $(f_k)_{R_{y'}^\delta}$ of $f$ over $R_{y}^\delta$ and $R_{y'}^\delta$, respectively, satisfy

\[ |(f_k)_{R_{y}^\delta} - (f_k)_{R_{y'}^\delta}| \leq c e \log \frac{1}{e} \]

with $c$ independent of $k$ or $\delta$. Since $\lim_{k \to \infty} f_k = f$ in $L^2$, $|(f_k)_{R_{y}^\delta} - (f_k)_{R_{y'}^\delta}| \leq ce \log \frac{1}{e}$ with $c$ independent of $\delta$. By Lebesgue’s Theorem, $\lim_{\delta \to 0} f_{R_{y}^\delta} = f_{x,y}$ unless $y$ is taken from a set $E \subseteq L$ of $(n-1)$-dimensional Lebesgue measure 0, and similarly for $f_{x,y'}$; hence (3.10) follows.

For $f = \sum \lambda_i a_i$ satisfying $\lambda_i = 0$ for $l(I) > 2^N$ or $l(I) < 2^{-N}$, write $f = f_1 + f_2 + f_3 + f_4$, where

\[ f_1 = \sum_{l(I) \geq 1, l(I) < 2^1} \lambda_i a_i, \quad f_2 = \sum_{l(I) < \infty, l(I) \geq 3} \lambda_i a_i, \quad f_3 = \sum_{l(I) < \epsilon} \lambda_i a_i, \quad f_4 = \sum_{l(I) < \epsilon} \lambda_i a_i. \]

The functions $f_1$ and $f_2$ are estimated by Lipschitz estimates. By (3.3) and the fact that $\lambda_i \leq c$, if $t \in J$ we have

\[ |f_1(t, y) - f_1(t, y')| \leq c \sum_{k=0}^{\infty} \sum_{l(I) = 2^k} |a_i(t, y) - a_i(t, y')| \]

\[ \leq c \sum_{k=0}^{\infty} \sum_{l(I) = 2^k} \sup_{x \in Q} |\nabla a_i(x)| \epsilon \]

\[ \leq c e \sum_{k=0}^{\infty} \sum_{q \in Z^n} \frac{2^{-k}}{(1 + |q|)^{n+2}} \leq ce. \]
Therefore \( |(f_1)_J - (f_1)_{J_0}| \leq c \epsilon \). Similarly, if \( t \in J \),
\[
|f_2(t, y) - f_2(t, y')| \leq c \sum_{k=1}^{\infty} \sum_{l(I) = 2^{-k}} \sup_{l \cap 3Q = \emptyset} 2^k \left( 1 + 2^k|x - x_l| \right)^{n+2} \leq c \epsilon \sum_{k=1}^{\infty} 2^{-k} = c \epsilon.
\]

Therefore \( |(f_2)_J - (f_2)_{J_0}| \leq c \epsilon \).

Using (3.4), we will obtain \( |(f_3)_J| \leq c \epsilon \log \frac{1}{\epsilon} \) and \( |(f_3)_{J_0}| \leq c \epsilon \log \frac{1}{\epsilon} \). Let \( I \subseteq 3Q \) be such that \( l(I) = 2^{-k} \leq \epsilon \), with \( x_l = (x_l, x_l) \), \( x_l \in \mathbb{R} \), satisfying \( 0 < x_l < 1 \). Let \( m \in \{0, 1, 2, \ldots, 2^k - 1\} \) and \( p \in \mathbb{Z} \), \( 0 \leq p \leq 4n - 1 (2^k - 1) \), be such that \( m2^{-k} < 1 - x_l < (m + 1)2^{-k} \) and \( p2^{-k} < |x_l - y| \leq (p + 1)2^{-k} \). By (3.4), (3.2), and since \( |J| = 2 \),
\[
\left| \frac{1}{|J|} \int_J a_I(t, y) \, dt \right| \leq 2 \sum_{s=0}^{\infty} \left| a_I(t, y) \right| \, dt + 2 \int_{-\infty}^{\infty} \left| a_I(t, y) \right| \, dt 
\]
\[
\leq c \epsilon \sum_{s=0}^{\infty} 2^{-k} \left( 1 + \sqrt{p^2 + (m + s)^2} \right)^{n+1} \leq c \left( 1 + \frac{2^{-k}}{1 + \sqrt{p^2 + m^2}} \right)^n.
\]

If \( I \) and \( p \) are as above except that \(-1 < x_l < 0\), and \( m \in \{0, 1, \ldots, 2^k - 1\} \) is such that \( m2^{-k} < x_l + 1 \leq (m + 1)2^{-k} \), the same estimate is obtained by the same methods. If \( I \) and \( p \) are as above but \( 1 < |x_l| < 3 \), say \( m2^{-k} < x_l - 1 < (m + 1)2^{-k} \) if \( x_l > 1 \), or \( m2^{-k} < -1 - x_l < (m + 1)2^{-k} \) if \( x_l < -1 \) for some \( m \in \{0, 1, 2, \ldots, 2^k - 1\} \), then (3.4) is not used since (3.2) implies that
\[
\left| \frac{1}{|J|} \int_J a_I(t, y) \, dt \right| \leq c \sum_{s=0}^{\infty} \frac{2^{-k}}{\left( 1 + \sqrt{p^2 + (m + s)^2} \right)^{n+1}} \leq c \frac{2^{-k}}{\left( 1 + \sqrt{p^2 + m^2} \right)^n}.
\]

Summing over all \( I \subseteq 3Q \) with \( l(I) \leq \epsilon \), we obtain
\[
|(f_3)_J| = \sum_{k=\lfloor \log_2 1/\epsilon \rfloor}^{\infty} \frac{1}{|J|} \sum_{l(I) = 2^{-k}} \sup_{l \subseteq 3Q} \sum_{q \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{(1 + |q|)^n} \leq c \epsilon \log \frac{1}{\epsilon}.
\]
where we have used $\lambda_I \leq c$ again, and $\lceil x \rceil$ is the greatest integer in $x$. With the same estimate for $(f_3)_J$, we obtain $\|f_3\|_{J} - (f_3)_J,| \leq c\epsilon \log \frac{1}{\epsilon}.$

For $f_4$, (3.3) and (3.4) are both used. Suppose $\epsilon < l(I) = 2^{-k} < 1$ and $I \subseteq 3Q$. Let $y^* = \frac{1}{2}(y + y')$. Let $p \in \mathbb{Z}$, $0 \leq p < 4(\log - 1)(2^k - 1)$, be such that $p2^{-k} \leq |x' - y^*| \leq (p + 1)2^{-k}$. Let $m \in \{0, 1, \ldots, 2^{k-1} - 1\}$ be such that $m2^{-k} < x_{J \setminus I} - 1 < (m + 1)2^{-k}$ if $0 < x_{J \setminus I} < 1$, or such that $m2^{-k} < x_{J \setminus I} + 1 < (m + 1)2^{-k}$ if $-1 < x_{J \setminus I} < 0$. By (3.4),

$$
\left| \frac{1}{|J|} \int_J a_I(t, y') \ dt - \frac{1}{|J|} \int_J a_I(t, y) \ dt \right| = \left| \frac{1}{|J|} \int_{-\infty}^{-1} a_I(t, y') \ dt - \frac{1}{|J|} \int_{-\infty}^{-1} a_I(t, y) \ dt \right|
$$

$$
+ \frac{1}{|J|} \int_{1}^{\infty} a_I(t, y) \ dt - \frac{1}{|J|} \int_{1}^{\infty} a_I(t, y') \ dt \right|
$$

$$
\leq 2 \int_{-\infty}^{-1} |a_I(t, y) - a_I(t, y')| dt + 2 \int_{1}^{\infty} |a_I(t, y) - a_I(t, y')| dt.
$$

Therefore, by (3.3),

$$
|(f_4)_J - (f_4)_J,| \leq 2 \sum_{s=0}^{\infty} 2^{-k} \sup_{|x' - y^*| < \epsilon/2, (m + s)2^{-k} \leq |x_{J \setminus I} - x_{J \setminus \dot{I}}| \leq (m + s + 1)2^{-k}} |\nabla a_I(x)| \cdot \epsilon
$$

$$
\leq c2^{-k} \epsilon \sum_{s=0}^{\infty} \frac{2^k}{\left(1 + \sqrt{p^2 + (m + s)^2}\right)^{n+2}} \leq \frac{c\epsilon}{\left(1 + \sqrt{p^2 + m^2}\right)^{n+1}}.
$$

If $I$ and $p$ are as above but $1 < |x_{J \setminus I}| < 3$, and $m \in \{0, 1, \ldots, 2^{k+1} - 1\}$ is such that $m2^{-k} < x_{J \setminus I} - 1 < (m + 1)2^{-k}$ if $x_{J \setminus I} > 1$ or such that $m2^{-k} < -1 - x_{J \setminus I} < (m + 1)2^{-k}$ if $x_{J \setminus I} < -1$, then (3.3) leads directly to the same estimates without use of (3.4). Therefore, since $\lambda_{J \setminus I} \leq c$,

$$
\left| \frac{1}{|J|} \int_J \sum_{I \subseteq Q} \sum_{l(I) = 2^{-k}} \lambda_I a_I(t, y) \ dt - \frac{1}{|J|} \int_J \sum_{l(I) = 2^{-k}} \lambda_I a_I(t, y') \ dt \right|
$$

$$
\leq c\epsilon \sum_{q \in \mathbb{Z}^n} \frac{1}{(1 + |q|)^{n+1}} = c\epsilon.
$$

Therefore

$$
|(f_4)_J - (f_4)_J,| \leq c\epsilon \# \{k : \epsilon < 2^{-k} < 1\} = c\epsilon \log \frac{1}{\epsilon}.
$$

Theorem 3.2 follows from the estimates on $f_1$, $f_2$, $f_3$ and $f_4$.

4. Corollaries and examples. Theorem 3.2 implies that functions in $R_f(BMO \cap L^2)$ satisfy a doubling condition on almost every $x_J$-line. For convenience we take $J = 1$ and write $x \in \mathbb{R}^n$ as $(x_1, x')$ for $x_1 \in \mathbb{R}$, $x' \in \mathbb{R}^{n-1}$. 

Corollary 4.1. Suppose \( f = R_1g \) for some \( g \in \text{BMO} \cap L^2(\mathbb{R}^n) \). Then \( \exists c > 0 \) and a set \( E \subseteq \mathbb{R}^{n-1} \) of \((n-1)\)-dimensional Lebesgue measure 0 such that for \( x' \notin E \) and \( I_1 \) and \( I_2 \) adjacent intervals in \( \mathbb{R} \) with \( |I_1| = |I_2| \),

\[
\left| \frac{1}{|I_1|} \int_{I_1} f(x_1, x') \, dx_1 - \frac{1}{|I_2|} \int_{I_2} f(x_1, x') \, dx_1 \right| \leq c \|g\|_{\text{BMO}}.
\]

Proof. We can assume \( \|g\|_{\text{BMO}} = 1 \). Assume first that \( I_1 \) and \( I_2 \) are dyadic intervals. Let \( F \) be the union over all dyadic cubes of the exceptional sets obtained for \( f \) in Theorem 3.2. For \( x' \notin F \), let \( L \subseteq \mathbb{R}^{n-1} \) be a closed dyadic cube containing \( x' \) such that \( |I_1| = l(L) \), and write \( Q_1 = I_1 \times L \), \( Q_2 = I_2 \times L \). By Lemma 3.2, since \( x' \notin F \), \( |f_{I_1} - f_{Q_1}| \leq c \) for a.e. \( s \in \mathbb{R} - F \). Therefore \( |f_{I_1} - f_{Q_1}| \leq c \), and similarly \( |f_{Q_2} - f_{I_2}| \leq c \). Since \( Q_1 \) and \( Q_2 \) are adjacent cubes in \( \mathbb{R}^n \) and \( f \in \text{BMO}(\mathbb{R}^n) \), \( |f_{Q_1} - f_{Q_2}| \leq c |f|_{\text{BMO}} \leq c [3, p. 223] \). Therefore \( |f_{I_1} - f_{I_2}| \leq c \) for \( x' \notin F \). To obtain a single exceptional set \( E \) for \( I_1 \), \( I_2 \) not necessarily dyadic, apply this result to all dilations and translations of \( f \) by rational coordinates. \( \square \)

Example 4.2. There exists \( f \in \text{BMO} \cap L^2(\mathbb{R}^n) \) which does not satisfy (4.1). In \( \mathbb{R}^2 \), for example, for each dyadic square \( I \), let \( f_I \) be a function satisfying \( 0 \leq f_I(x) \leq 1 \), \( \text{supp} \ f_I \subseteq 3I \), \( \|f_I\|_{\text{Lip}} \leq c l(I)^{-1} \) and \( f_I(x) = 1 \) \( \forall x \in I \). Let \( A_k = \{ I \text{ dyadic}: \ l(I) = 2^{-k}, \ 0 < l_I < 1, \text{ and } I \cap \{ x: x_2 = 0 \} \neq \emptyset \} \). Let \( f = \sum_{k=0}^{\infty} \sum_{f \in A_k} f_I \). Then it is not difficult to show that \( f \in \text{BMO} \cap L^2(\mathbb{R}^2) \), but

\[
\lim_{x_2 \to 0} \left( \int_0^1 f(x_1, x_2) \, dx_1 - \int_0^2 f(x_1, x_2) \, dx_1 \right) = \infty.
\]

For details, see [2].

Corollary 4.3. If \( f = R_1g \) for some \( g \in L^\infty \) such that \( \|g\|_{L^\infty} \leq 1 \), then (3.9), (3.10) and (4.1) hold.

Proof. The estimates hold if \( f = R_1g_m \), where \( g_m(x) = g(x)(x:|x|<m) \) since \( g_m \in L^2 \cap L^\infty \) so that \( R_1g_m \) and \( \tilde{R}_1g_m \) differ by a constant. The result follows since, by a calculation in [2, p. 48], if \( K \) is any compact set in \( \mathbb{R}^n \),

\[
\lim_{m \to \infty} \left( \sup_{x \in K} |\tilde{R}_1(g - g_m)(x)| \right) = 0. \quad \square
\]

A proof of Corollary 4.3 based on duality with singular measures is included in [2].

Example 4.4. If \( f(x) = \log \log(1/|x'|) \) for \( |x'| < 1/e \) and \( f(x) = 0 \) otherwise, then \( f \in \text{BMO}(\mathbb{R}^n) \) but \( f \) does not satisfy (3.9). Although \( f \notin L^\infty \), \( f \) belongs to the BMO closure of \( L^\infty \). Therefore \( L^\infty + R_1L^\infty, L^\infty + R_1(\text{BMO} \cap L^2) \) and \( L^\infty + R_1(L^\infty) \) are not closed subspaces of BMO. This observation is relevant to §5 below. The space \( L^\infty + R_1L^\infty \) is not dense in BMO in BMO norm; a proof due to Paul Koosis is included in [2].

Example 4.5 (Kahane's Example). Kahane's example [6] can be used to show that there exists a sequence \( \{ f_n \}_{n=1}^\infty \) of functions on \((0,1)\) satisfying a uniform
doubling condition but such that \( \lim_{n \to \infty} \| f_n \|_{\text{BMO}(0, 1)} = \infty \). For \( m_1, m_2, \ldots, m_n \in \{1, 2, 3, 4\} \), define

\[
I_{m_1, \ldots, m_n} = \left( \frac{m_1 - 1}{4} + \frac{m_2 - 1}{4^2} + \cdots + \frac{m_n - 1}{4^n} \right),
\]

and \( Q_{m_1, \ldots, m_n} = I_{m_1, \ldots, m_n} \times [0, 4^{-n}] \). Let \( f_1(x_1, x_2) \) be defined by

\[
f_1 = x_{Q_1} - x_{Q_2} - x_{Q_3} + x_{Q_4}.
\]

With \( \text{sgn} \, t = t/|t| \) if \( t \neq 0 \) and \( \text{sgn} \, 0 = 0 \), define \( f_{n+1} \) inductively by

\[
f_{n+1} = f_n + \sum_{m_1, \ldots, m_n = 1}^{4} a_{m_1, \ldots, m_n},
\]

where

\[
a_{m_1, \ldots, m_n} = \text{sgn} \, f_n (-x_{Q_{m_1, \ldots, m_n}} + x_{Q_{m_1, \ldots, m_n}} + x_{Q_{m_1, \ldots, m_n}} - x_{Q_{m_1, \ldots, m_n}}).
\]

Regard \( f_n \) now as restricted to \((0, 1)\). Then clearly

\[
\| f_n \|_{L^1} = 1 \quad \forall n = 1, 2, 3, \ldots
\]

If \( \alpha_n = |\text{supp} \, f_n| \), then \( \alpha_n \) decreases to 0 as \( n \to \infty \). That \( \alpha_n \) decreases is clear; that \( \lim_{n \to \infty} \alpha_n = 0 \) can be seen by the fact that a simple random walk in one dimension is recurrent. Since \( f_0^1 f_n(x) \, dx = 0 \forall n \), and since

\[
\| f \|_{\text{BMO}} \approx \sup_Q \left( \frac{1}{|Q|} \int_Q |f - f_Q|^2 \right)^{1/2},
\]

we obtain \( \lim_{n \to \infty} \| f_n \|_{\text{BMO}(0, 1)} = \infty \) since

\[
(4.2) \quad \| f_n \|_{\text{BMO}(0, 1)} \geq c \| f_n \|_{L^2(0, 1)} \geq c \frac{\| f_n \|_{L^1(0, 1)}}{\sqrt{\alpha_n}} = \frac{c}{\sqrt{\alpha_n}}.
\]

If \( I_1 \) and \( I_2 \) are adjacent intervals of the same length in \( \mathbb{R} \), then \( \| f_n \|_{L^1(0, 1)} \leq c \), with \( c \) independent of \( I_1, I_2, \) or \( n \), by the main estimate in Kahane's example: see [6, p. 190].

**Example 4.6.** There exists \( G \in \text{BMO} \cap L^2(\mathbb{R}^2) \) such that

\[
(4.3) \quad \text{ess sup}_{x_2} \| R_1 G(x_1, x_2) \|_{\text{BMO}(x_1)} = \infty.
\]

We construct \( R_1 G \) by symmetrizing and smoothing the functions \( f_n(x_1, x_2) \) of Example 4.5 in the \( x_2 \) direction.

Regarding \( f_n \) above as a function of \( (x_1, x_2) \) again, let \( \tilde{f}_n(x_1, x_2) \) be the function which agrees with \( f_n(x_1, x_2) \) for \( x_2 \geq 0 \) and satisfies \( f_n(x_1, -x_2) = f_n(x_1, x_2) \) \( \forall (x_1, x_2) \). Define \( \tilde{a}_{m_1, \ldots, m_n} \) similarly related to \( a_{m_1, \ldots, m_n} \). Since

\[
\| \tilde{f}_{n+1} - \tilde{f}_n \|_{L^2(\mathbb{R}^2)} \leq c 2^{-n},
\]

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\[ \tilde{f} = \lim_{n \to \infty} \tilde{f}_n \] exists in \( L^2(\mathbb{R}^2) \) and a.e. The following sequence of steps can be carried out to show that \( \tilde{f} \in \text{BMO}(\mathbb{R}^2) \):

(a) Since the measure of the supports of \( \tilde{f}_{n+1} - \tilde{f}_n \) decrease rapidly,
\[
\frac{1}{|Q|} \int_Q |\tilde{f} - \tilde{f}_Q| \leq c
\]
for any 4-adic square \( Q \).

(b) If \( Q_1 \) and \( Q_2 \) are 4-adic squares of the same side length with \( \overline{Q}_1 \cap \overline{Q}_2 \neq \emptyset \), then \( |\tilde{f}_{Q_1} - \tilde{f}_{Q_2}| \leq c \). For \( Q_1 \) and \( Q_2 \) horizontally adjacent, this follows from the doubling condition for \( f_n \). For \( Q_1 \) and \( Q_2 \) vertically adjacent, this follows from the construction.

(c) An easy observation, noted by Varopoulous, shows that (a) and (b) imply \( \tilde{f} \in \text{BMO}(\mathbb{R}^2) \).

In this example, we wish to replace the functions \( \tilde{a}_{m_1 \ldots m_n} \) with versions which are smoothed out in the \( x_2 \) direction. Let \( \psi \in C^\infty(\mathbb{R}) \) be a fixed function satisfying
\[ 0 \leq \psi(x) \leq 1 \quad \forall x, \quad \text{supp} \psi \subseteq (-2,2), \quad \text{and} \quad \psi(x) = 1 \text{ if } x \in [-1,1]. \]
For \( m_1, \ldots, m_n \in \{1,2,3,4\} \), let
\[
\varphi_{m_1 \ldots m_n}(x_1, x_2) = \chi_{I_{m_1 \ldots m_n}}(x_1) \psi(4^n x_2).
\]
Let \( g_1 = \varphi_1 - \varphi_2 - \varphi_3 + \varphi_4 \), and define \( g_{n+1}(x_1, x_2) \) inductively by
\[
g_{n+1} = g_n + \sum_{m_1 \ldots m_n = 1}^4 b_{m_1 \ldots m_n},
\]
where
\[
b_{m_1 \ldots m_n} = \text{sgn} \; g_n(x_1, 0)(-\varphi_{m_1 \ldots m_n1} + \varphi_{m_1 \ldots m_n2} + \varphi_{m_1 \ldots m_n3} - \varphi_{m_1 \ldots m_n4}).
\]
Note that \( \text{sgn} \; g_n(x_1, 0) = \text{sgn} \; f_n(x_1, 0) \) for \( f_n \) in Example 4.5; in particular,
\[
\text{supp}(b_{m_1 \ldots m_n} - \tilde{a}_{m_1 \ldots m_n}) \subseteq I_{m_1 \ldots m_n} \times \{ x_2 : 4^{-n-1} \leq |x_2| \leq 2 \cdot 4^{-n-1} \}.
\]
Hence if \( g = \lim_{n \to \infty} g_n \), then \( g - \tilde{f} \in L^\infty(\mathbb{R}^2) \) by the disjointness of the supports of the different \( g_n - \tilde{f}_n \). Therefore \( g \in \text{BMO} \cap L^2(\mathbb{R}^2) \) since \( \tilde{f} \in \text{BMO} \cap L^2(\mathbb{R}^2) \).

We assert that \( g = R_1 G \) for some \( G \in \text{BMO} \cap L^2 \); we solve for \( G \) by the method of Lemma 3.1. Let
\[
h_0(x_1, x_2) = \int_{-\infty}^{x_1} g_1(t, x_2) \, dt.
\]
and
\[
h_{m_1 \ldots m_n}(x_1, x_2) = \int_{-\infty}^{x_1} b_{m_1 \ldots m_n}(t, x_2) \, dt.
\]
Note that \( h_{m_1 \ldots m_n}(x_1, x_2) = 0 \) if \( x_1 \notin I_{m_1 \ldots m_n} \)(set \( I_0 = (0,1) \)), or if \( |x_2| > 2 \cdot 4^{-n-1} \).

Further,
\[
|D_{x_2} h_{m_1 \ldots m_n}(x_1, x_2)| = |D_{x_2}(\psi(4^n x_2))\int_{-\infty}^{x_1}(\chi_{I_{m_1 \ldots m_n1}}(t) - \chi_{I_{m_1 \ldots m_n3}}(t) + \chi_{I_{m_1 \ldots m_n4}}(t)) \, dt| \leq \psi'(4^n x_2) \cdot 4^n \cdot 4^{-n-1} \leq C.
\]
Also
\[\text{supp } D_{x_2} h_{m_1 \ldots m_n} \subseteq I_{m_1 \ldots m_n} \times \{x_2: 4^{-n-1} \leq |x_2| \leq 2 \cdot 4^{-n-1}\},\]
since \(\psi(x_2) = 1\) if \(x_2 \in [-1, 1]\). Therefore
\[H = D_{x_2} h_0 + \sum_{n=1}^{\infty} \sum_{m_1 \ldots m_n=1}^{4} D_{x_2} h_{m_1 \ldots m_n} \in L^\infty_c.\]

By Fourier transform we obtain
\[g = g_1 + \sum_{n=1}^{\infty} \sum_{m_1 \ldots m_n=1}^{4} b_{m_1 \ldots m_n} = R_1(-R_1 g - R_2 H).\]

Therefore
\[g = g_1 + \sum_{n=1}^{\infty} \sum_{m_1 \ldots m_n=1}^{4} b_{m_1 \ldots m_n} \in R_1(-R_1 g - R_2 H).\]

Since \(g, H \in \text{BMO} \cap L^2\), \(G = -R_1 g - R_2 H \in \text{BMO} \cap L^2\) and \(R_1 G = g\). Note that
\[\tilde{f}(x_1, x_2) = f_n(x_1, x_2) \text{ for } 4^{-n-1} < x_2 < 4^{-n}, \text{ and hence by (4.2)}\]
\[\text{ess sup} \|\tilde{f}(x_1, x_2)\|_{\text{BMO}(x_1)} = \infty.\]

Since \(R_1 G - \tilde{f} \in L^\infty\), (4.3) follows. \(\square\)

5. A density theorem. Throughout this section \(K\) is a fixed singular integral operator with kernel \(\Omega\) which is real and odd; it follows [7, p. 39] that the multiplier \(m\) for \(K\) is odd and pure imaginary. We will show that
\[\text{K}(\text{BMO}_c) \subseteq L^\infty + \text{K}(L^\infty_c),\]
where the closure is taken in BMO norm. To fix notation for this section, \(I\) always denotes a dyadic cube, \(\tilde{f}\) denotes a vector-valued function with two components, \(\tilde{f} = (f_0, f_1)\), and we define
\[|\tilde{f}(x)| = (f_0^2(x) + f_1^2(x))^{1/2}, \quad \|\tilde{f}\| = \|f_0\| + \|f_1\|\]
for any function space norm \(\| \cdot \|\), and \(\tilde{K} \cdot \tilde{f} = f_0 + Kf_1\). Also define
\[|\nabla \tilde{f}(x)| = |\nabla f_0(x)| + |\nabla f_1(x)|\]
and
\[\int \tilde{f}(x) \, dx = \left(\int f_0(x) \, dx, \int f_1(x) \, dx\right).\]

Our method is based closely on Uchiyama’s construction in [8]. Uchiyama obtains uniformly bounded solutions, which satisfy a certain orthogonality condition, to a certain inversion problem. We are not able to obtain uniformly bounded solutions to the corresponding inversion problem in our case. Hence, some additional estimates are necessary.

First we state explicitly the assumptions we require regarding the inversion problem. We say that a collection of real-valued functions \(\{b_I(x)\}_I\) satisfies (\(\ast\)) if
there exist constants $c > 0$ and $l > 0$ such that for all $I$, 
(i) for any $\vec{a} = (a_0, a_1) \in \mathbb{R}^2$ with $a_1 \neq 0$, the equations

\begin{equation}
\vec{K} \cdot \vec{u}_I = b_I
\end{equation}

and

\begin{equation}
\vec{a} \cdot \vec{u}_I(x) = \vec{0} \quad \forall x
\end{equation}

have a solution $\vec{u}_I = (u_0, u_1), \vec{u}_I : \mathbb{R}^n \to \mathbb{R}^2$, satisfying

\begin{equation}
|\vec{u}_I(x)| \leq c \left( 1 + \left| \frac{a_0}{a_1} \right| \right) \left( 1 + l(I)^{-1}|x - x_I| \right)^{-n-1},
\end{equation}

\begin{equation}
|\nabla \vec{u}_I(x)| \leq c \left[ 1 + \left| \frac{a_0}{a_1} \right| \right] l(I)^{-1}(1 + l(I)^{-1}|x - x_I|)^{-n-2}
\end{equation}

and

\begin{equation}
\int \vec{u}_I(x) \, dx = \vec{0};
\end{equation}

and

(ii) the equation $K(u_{1I}) = b_I$ has a solution $u_{1I} : \mathbb{R}^n \to \mathbb{R}$ satisfying

\begin{equation}
|u_{1I}(x)| \leq c \left( 1 + l(I)^{-1}|x - x_I| \right)^{-n-1},
\end{equation}

\begin{equation}
|\nabla u_{1I}(x)| \leq c l(I)^{-1}(1 + l(I)^{-1}|x - x_I|)^{-n-2}
\end{equation}

and

\begin{equation}
\int u_{1I}(x) \, dx = 0.
\end{equation}

**Theorem 5.1.** Suppose $f(x) = \lim_{k \to \infty} \sum_{1 \leq |I| \leq 2^k} \lambda_I b_I(x)$ (convergence in $L^2$) with $\lambda_I \geq 0 \quad \forall I$, where the real valued functions $\{b_I(x)\}$, satisfy (*), and for some $c > 0$,

\begin{equation}
\sum_{I \subseteq J} \lambda_I^2 |I| \leq c |J| \quad \text{for all cubes } J,
\end{equation}

\begin{equation}
\sum_{I} \lambda_I^2 |I| < \infty.
\end{equation}

\begin{equation}
\lim_{M \to \infty} \left\| \sum_{|I| > 2^M} \lambda_I b_I \right\|_{\text{BMO}} = 0
\end{equation}

and

\begin{equation}
\exists \text{ a cube } S \text{ such that } \lambda_I = 0 \text{ if } 3I \cap S = \emptyset.
\end{equation}

Then $f \in L^\infty + K(L^\infty)$. 

Before proving Theorem 5.1, we mention some consequences.

**Corollary 5.2.** For $K$ as above, $K(\text{BMO}_c \subseteq L^\infty + K(L^\infty)$, where the closure is taken in BMO norm.
PROOF. Suppose \( g \in \text{BMO}_c \). We may assume that \( g \) is real valued. By Lemma U1, 
\( g = \sum \lambda_i a_i \) with (5.9) and (5.10). The functions \( a_i \) satisfy \( \text{supp } a_i \subseteq 3I \), \( \|D^\alpha a_i\|_{L^\infty} \leq c_{|\alpha|} l(I)^{-|\alpha|} \) and \( \int a_i(x) \, dx = 0 \) \( \forall I \). That (5.12) and the condition \( \lim_{M \to \infty} \|\sum_{|I| > 2^M} \lambda_i a_i\|_{\text{BMO}} = 0 \) are also obtained by the decomposition in Lemma U1 for functions of compact support is proved in [8, pp. 232 and 234]. Set \( b_i = K a_i \); since \( K \) is bounded on \( L^2 \) and \( \text{BMO} \), \( Kg = \sum \lambda_i Ka_i = \sum \lambda_i b_i \) in \( L^2 \). Each \( b_i \) is real valued since the \( \{a_i\} \), obtained by Lemma U1 are real valued (since \( g \) is real valued), and since \( K \) has a real-valued kernel.

We need only show that \( \{b_i\} \), satisfies (\( \ast \)). Suppose \( a_i \neq 0 \). Then (5.1) and (5.2) can be solved by setting

\[
(5.13) \quad u_{0i}(x) = \left[ \left( \frac{-a_1}{m(\xi)a_0 - a_1} \right) \hat{b}_i(\xi) \right]^{\ast}(x)
\]

and

\[
(5.14) \quad u_{1i}(x) = \left[ \left( \frac{a_0}{m(\xi)a_0 - a_1} \right) \hat{b}_i(\xi) \right]^{\ast}(x),
\]

where \( \ast \) is the inverse Fourier transform. (This is Uchiyama’s solution reduced to our case.) Since \( \hat{b}_i(\xi) = m(\xi)\hat{a}_i(\xi) \), the \( u_{0i} \) and \( u_{1i} \) are obtained from \( a_i \) by the Fourier multipliers

\[
m_0(\xi) = \frac{-a_1m(\xi)}{m(\xi)a_0 - a_1} \quad \text{and} \quad m_1(\xi) = \frac{-a_0m(\xi)}{m(\xi)a_0 - a_1},
\]

respectively, which are \( C^\infty \) on \( S^{n-1} \) (since \( m(\xi) \) is pure imaginary and \( a_1 \neq 0 \)), and homogeneous of degree 0 (since \( m \) is). In fact, since \( m(-\xi) = m(\xi) \), the same is true for \( m_0 \) and \( m_1 \) and hence \( u_{0i} \) and \( u_{1i} \) may be taken real valued. By [7, p. 75], \( \exists a_0, a_1 \in \mathbb{R} \) and singular integral operators \( T_0 \) and \( T_1 \) such that \( u_{1i} - \alpha_i a_i + T_{\alpha_i} a_i \), \( i = 0, 1 \). An easy calculation and the fact that \( m \) is pure imaginary show that \( |D^\alpha m_0(\xi)| \leq c_{|\alpha|}(1 + |a_0/a_1|^{|\alpha|}) \forall \xi \in S^{n-1} \). Also \( |a_i| \leq \|m_i\|_{L^\infty(S^{n-1})} \leq C \) for \( i = 0, 1 \). Hence by Lemma U6, the solutions \( \hat{u}_i \) of (5.1) and (5.2) satisfy (5.3), (5.4) and (5.5) for \( l = C(n) \). Solving \( K(u_{1i}) = b_i \) is trivial since \( b_i = Ka_i \), so that \( u_{1i} = a_i \) satisfies (5.6), (5.7) and (5.8). \( \Box \)

**Corollary 5.3.** If \( f(x) = \sum \lambda_i b_i(x) \) in \( L^2 \) with \( \lambda_i \geq 0 \) \( \forall I \), and if (5.9)–(5.12) hold and each \( b_i \) satisfies

\[
(5.15) \quad \text{supp } b_i \subseteq 3I, \quad \|D_{x_i} b_i\|_{L^1} \leq c l(I)^{-2}, \quad \text{and} \quad \int b_i(x', x'') \, dx' = 0 \quad \forall x'',
\]

where \( x' = (x_1, \ldots, x_k) \) and \( x'' = (x_{k+1}, \ldots, x_n) \), then

\[
f \in L^\infty + \sum_{j=1}^k R_j(L^\infty_c),
\]

where the closure is taken in \( \text{BMO} \) norm.

**Proof.** We may assume each \( b_i \) is real valued. If \( k = 1 \), we need only to show that (\( \ast \)) holds for \( K = R_1 \) and \( \{b_i\} \), under the assumption that \( \int b_i(x_1, x'') \, dx_1 = 0 \) \( \forall x'', I \). For \( a_i \neq 0 \), (5.1) and (5.2) can be solved by (5.13) and (5.14) with properties
(5.3), (5.4) and (5.5) as above. We solve $R_1 u_1 = b_1$ as in Lemma 3.1 by setting $h_1(x_1, x') = \int_{-\infty}^{x_1} b_1(t, x') \, dt$. Then by (5.15), $\text{supp } h_1 \subseteq 3I$. Set

$$u_{1t} = - \left( R_1 b_1 + \sum_{j=2}^{n} R_j D_s h_j \right);$$

then $R_1 u_1 = b_1$ as in Lemma 3.1. Since $\text{supp } D_s h_j \subseteq 3I$, $\int D_s h_j(x) \, dx = 0$ and $\| D_{s,p} h_j \|_{\text{Lip}1} \leq c(I)^{-2}$, $\forall j, p \in \{1, \ldots, n\}$, we obtain (5.6)–(5.8) by Lemma U6.

In general, it is sufficient to apply the case above once we note that if $b_1$ satisfies (5.15), then $b_j = \sum_{p=1}^{k} b_{1,p}$, where each $b_{1,p}$ satisfies $\text{supp } b_{1,p} \subseteq 3I$, $\| D_s b_{1,p} \|_{\text{Lip}1} \leq c(I)^{-2}$, and

$$\int b_{1,p}(x_1, \ldots, x_{p-1}, x_p, x_{p+1}, \ldots, x_n) \, dx_p = 0 \forall (x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_n).$$

To see this, suppose $I = \prod_{j=1}^{I} J_j$ and let $\alpha(x_1)$ be a function satisfying $\text{supp } \alpha(x_1) \subseteq 3I_1$, $(1/|I_1|) \alpha(x_1) \, dx_1 = 1$ and $\| \partial^p \alpha/\partial x_1 \|_{L^\infty} \leq c_p(I)^{-p}$ for $p = 1, 2, 3, \ldots$. Let $\bar{x} = (x_2, \ldots, x_n)$ and define $\beta(\bar{x}) = (1/|I_1|) \int b_1(x_1, \bar{x}) \, dx_1$. Set $b_1^*(x) = \alpha(x_1) \beta(\bar{x})$ and $b_{1,t} = b_t - b_1^*$. Then it follows from the definitions that $\int b_{1,t}(x_1, \bar{x}) \, dx_1 = 0 \forall \bar{x}$, $\text{supp } b_{1,t} \subseteq 3I$ and $\| D_s b_{1,t} \|_{\text{Lip}1} \leq c(I)^{-2}$ for $j = 1, \ldots, n$. Similarly $\text{supp } b_1^* \subseteq 3I$, $\| D_s b_1^* \|_{\text{Lip}1} \leq c(I)^{-2}$ for $j = 1, \ldots, n$, and

$$\int b_1^*(x_1, \ldots, x_n) \, dx_2 \cdots dx_k = \alpha(x_1) \int \beta(\bar{x}) \, dx_2 \cdots dx_k$$

Continuing similarly with $x_2$ and $b_1^*$, after $k - 1$ more steps we obtain $b_t = \sum_{p=1}^{k} b_{1,p}$ with the desired properties.

The proof of Theorem 5.1 is an adaptation of the proof of the Main Lemma in [8, pp. 231–238], we follow the organization and terminology of Uchiyama’s proof quite closely. Estimates which are essentially the same as Uchiyama’s will only be stated or sketched. More detail is given in [2].

Proof of Theorem 5.1. Let $\varepsilon > 0$ be given. We will obtain $g_0 \in L_\infty$, $g_1 \in L_\infty$ such that $\| f - g_0 - K g_1 \|_{\text{BMO}} < \varepsilon$. We may assume $S = [-1,1]^n$. Define $\{ \eta_k \}^{\infty}_{k=\infty}$ and $\{ \varepsilon_k \}^{\infty}_{k=\infty}$ as in Lemma U2 for the $(\lambda_I)_I$ given in Theorem 5.1. Let $l > 0$ be the number given by $(\ast)$. A sufficiently large positive integer $M$, and sufficiently large positive numbers $R$ and $p$ will be determined later, depending on $\varepsilon$. We inductively construct the functions $\{ \tilde{g}_k(x) \}^{\infty}_{k=-M}$, $\{ \tilde{\varphi}_k(x) \}^{\infty}_{k=-M}$, $\{ \tilde{\beta}_{l,j}(x) \}^{\infty}_{j=0, l(I) \leq 2^M}$, $\{ \tilde{\psi}_k(x) \}^{\infty}_{k=-M}$ and $\{ \tilde{\xi}_k(x) \}^{\infty}_{k=-M}$, satisfying

$$\text{(5.16) } \text{supp } \tilde{\beta}_{l,j} \subseteq 2^j I, \quad \| \nabla \tilde{\beta}_{l,j}(x) \| \leq c R^{1/p} (2^j l(I))^{-1}, \quad \int \tilde{\beta}_{l,j}(x) \, dx = 0,$$

$$\text{(5.17) } \tilde{K} \cdot \sum_{j=0}^{\infty} 2^{-j(n+1)} \tilde{\beta}_{l,j} = b_t \quad \forall I \text{ such that } l(I) \leq 2^M,$$

$$\text{(5.18) } \tilde{\varphi}_k = \tilde{\psi}_k + \tilde{\xi}_k \quad \text{for } -M \leq k < \infty,$$

$$\text{(5.19) } |\tilde{\varphi}_k(x) - \tilde{\psi}_k(y)| \leq c (2^{(n+2)M} R (2^{l/p-1} + R^{-1/p}) 2^k |x - y| \quad \text{if } |x - y| \leq 2^{-k},$$
\begin{align}
(5.20) \quad & |\tilde{\psi}_k(x)| \leq c2^{(n+2)MR^{(2l/p-1)}}\varepsilon_k(x)\eta_k(x), \\
(5.21) \quad & \text{supp } \tilde{\psi}_k \subseteq \{x: |x| \leq 2\sqrt{n} \max(2^{-k}, 1) \}, \\
(5.22) \quad & \text{supp } \tilde{s}_k \subseteq \{x: |x| \leq 2\sqrt{n} \max(2^{-k}, 1) \}, \\
(5.23) \quad & \tilde{g}_{-M-1} = (R, 0), \\
(5.24) \quad & |\tilde{g}_k(x)| = R \ \forall x, \ \forall k \geq -M - 1, \\
(5.25) \quad & \tilde{g}_k(x) - \tilde{g}_{k-1}(x) = \sum_{I(l) = 2^{-k}} \lambda_i \sum_{j=0}^{M} 2^{-j(n+1)}\beta_{I,j}(x) - \tilde{\psi}_k(x)
\end{align}

and

\begin{align}
(5.26) \quad & |\tilde{g}_k(x) - \tilde{g}_k(y)| \leq c_0 R^{l/p}\varepsilon_k(x) 2^k|x - y| \text{ if } |x - y| \leq 2^{-k}.
\end{align}

As a consequence of the induction, we will obtain

\begin{align}
(5.27) \quad & \sum_{j = -M}^{k} \tilde{\varepsilon}_j(x) \text{ converges in } L^2 \text{ as } k \to \infty
\end{align}

and

\begin{align}
(5.28) \quad & \left\| \sum_{j = -M}^{\infty} \tilde{\varepsilon}_j(x) \right\|_{\text{BMO}} \leq cR^{-1/p}.
\end{align}

First we assume the construction and complete the proof of the theorem. By (5.25) and (5.18), if \( k > -M \)

\begin{align}
(5.29) \quad & \tilde{g}_k(x) - \tilde{g}_{-M-1}(x) = \sum_{j=0}^{\infty} 2^{-j(n+1)} \sum_{I: 2^{-M} \gg \frac{l(I)}{2^k}} \lambda_i \beta_{I,j} \\
& \quad - \sum_{j=M+1}^{\infty} 2^{-j(n+1)} \sum_{I: 2^{-M} \gg \frac{l(I)}{2^k}} \lambda_i \beta_{I,j} \\
& \quad - \sum_{j=-M}^{k} \tilde{\psi}_j + \sum_{j=-M}^{k} \tilde{\varepsilon}_j.
\end{align}

By (5.10), (5.16) and Lemma U3, the first two terms on the right-hand side of (5.29) converge in \( L^2 \) as \( k \to \infty \). By (5.27) the last term in (5.29) converges in \( L^2 \) as \( k \to \infty \). By (5.20), (5.21), (5.9) and Lemma U2, \( \Sigma_{j=-M}^{k} \tilde{\psi}_j \) converges in \( L^1 \) as \( k \to \infty \). Since \( \|\tilde{g}_k - \tilde{g}_j\|_{L^\infty} \leq cR \ \forall k, j \geq -M - 1 \), \( \tilde{g}_k - \tilde{g}_{-M-1} \) converges in \( L^2 \) as \( k \to \infty \) by the argument in \([8, \text{p. 233}]\). Hence \( \Sigma_{j=-M}^{k} \tilde{\varepsilon}_j \) converges in \( L^2 \) as \( k \to \infty \) as well. Set

\begin{align}
\tilde{g}(x) = \tilde{g}_{-M-1}(x) + \lim_{k \to \infty} \left( \tilde{g}_k(x) - \tilde{g}_{-M-1}(x) \right).
\end{align}

Then \( |\tilde{g}(x)| = R \ \forall x \in \mathbb{R}^n \). By (5.12), (5.16), (5.21), (5.29) and (5.23),

\begin{align}
\text{supp}(\tilde{g} - (R, 0)) \subseteq \{x: |x| \leq 2\sqrt{n} 2^M \}.
\end{align}
Hence \( g_1 \in L^\infty \). By (5.29) and (5.17),

\[
\tilde{K} \cdot \tilde{g}_k = \sum_{2^M \geq l(I) \geq 2^k} \lambda_j b_I - \tilde{K} \cdot \left\{ \sum_{j = M + 1}^\infty 2^{-j(n+1)} \sum_{l: 2^M \geq l(I) \geq 2^k} \lambda_j \tilde{b}_{I,j} \right. \\
+ \sum_{j = -M}^k \tilde{\psi}_j + \sum_{j = -M}^k \tilde{\xi}_j \right\}.
\]

Therefore,

\[
\tilde{K} \cdot \tilde{g} = f - \sum_{l(I) > 2^M} \lambda_j b_I - \tilde{K} \cdot \left\{ \sum_{j = M + 1}^\infty 2^{-j(n+1)} \sum_{l: 2^M \geq l(I)} \lambda_j \tilde{b}_{I,j} \right. \\
+ \sum_{j = -M}^k \tilde{\psi}_j + \sum_{j = -M}^k \tilde{\xi}_j \right\}.
\]

By (5.11), \( \lim_{M \to \infty} \|\sum_{l(I) > 2^M} \lambda_j b_I \|_{\text{BMO}} = 0 \). By the boundedness of \( K \) on BMO, and by (5.16) and Lemma U4, we obtain, as in [8, p. 237],

\[
\left\| \tilde{K} \cdot \sum_{j = M + 1}^\infty 2^{-j(n+1)} \sum_{l: 2^M \geq l(I)} \lambda_j \tilde{b}_{I,j} \right\|_{\text{BMO}} \leq c_1 2^{-M} R^{1/p}.
\]

As in [8, p. 238], from (5.19) and (5.20) we obtain

\[
\left\| \tilde{K} \cdot \sum_{j = M}^\infty \tilde{\psi}_j \right\|_{\text{BMO}} \leq c_2 (2^{(n+2)} M R^{(2l/p - 1)} + R^{-1/p}).
\]

By (5.28), \( \| \tilde{K} \cdot \sum_{j = -M}^{j=-M} \tilde{\xi}_j \|_{\text{BMO}} \leq c_3 R^{1/p} \). Let \( c_4 = \max(c_1, c_2, c_3, 1) \). We will choose \( R \) sufficiently large and \( p \) so that \( R^{1/p} = 8\epsilon_1 c_4 \). Choose \( M \) so that \( \|\sum_{l(I) > 2^M} \lambda_j b_I \|_{\text{BMO}} < \epsilon/4 \), and so that \( c_1 2^{-M} R^{1/p} = c_1 2^{-M} (8\epsilon_1 c_4)^l < \epsilon/4 \). Now choose \( R \) sufficiently large so that

\[
c_2 2^{(n+2)} M R^{(2l/p - 1)} = c_2 2^{(n+2)} M (8\epsilon_1 c_4)^2 R^{-1} < \epsilon/8.
\]

Finally pick \( p \) so that \( R^{1/p} = 8\epsilon_1 c_4 \), then \( c_2 R^{1/p} < \epsilon/8 \) and \( c_3 R^{-1/p} < \epsilon/8 \). Then \( \| \tilde{K} \cdot \tilde{g} - f \|_{\text{BMO}} < \epsilon \), with \( g_0 \in L^\infty \), \( g_1 \in L^\infty \). For future reference, note that our choices allow us to assume \( R^{1/p} > 1 \) and \( R^{2l/p} < R \).

We begin the proof of the construction. Define \( \tilde{g}_{-M-1} \) by (5.23). Assume \( \{ \tilde{g}_p \}_{p = -M}^{-1-M-1} \), \( \{ \tilde{\psi}_p \}_{p = -M}^{-1-M-1} \), \( \{ \tilde{b}_{I,j} \}_{j = 0}^{k-1} \), \( \{ \tilde{b}_{I,j} \}_{j = 0}^{k-1} \), \( \{ \tilde{\xi}_p \}_{p = -M}^{-1-M-1} \), satisfying (5.16)–(5.26), have been constructed. Let \( I \) be a dyadic cube with \( l(I) = 2^{-k} \). Let \( a(I) = (a_0(I), a_1(I)) = \tilde{g}_{-k}(x) \). Let

\[
A_k = \{ I: l(I) = 2^{-k} \text{ and } |a_0(I)/a_1(I)| \leq R^{1/p} \}.
\]

Let \( B_k = \{ I: l(I) = 2^{-k} \text{ and either } a_1(I) = 0 \text{ or } |a_0(I)/a_1(I)| > R^{1/p} \} \).

For \( I \in A_k \), apply (i) to \( I \), \( b_I \), and \( \tilde{a} = a(I) \) to obtain \( \tilde{u}_I: \mathbb{R}^n \to \mathbb{R} \) satisfying (5.1)–(5.5). Since \(|a_0(I)/a_1(I)| \leq R^{1/p} \), by Lemma U5, \( \exists \{ \tilde{b}_{I,j}(x) \}_{j = 0}^{\infty} \) satisfying \( \text{supp} \tilde{b}_{I,j} \leq 2I, \| \tilde{b}_{I,j} \|_{\text{Lip}1} \leq c R^{l/p} (2l(I))^{-1} \), \( \int \tilde{b}_{I,j}(x) dx = 0, \tilde{a}(I) \cdot \tilde{b}_{I,j}(x) = 0 \forall x \), and \( \tilde{u}_I(x) = \sum_{j = 0}^{\infty} 2^{-j(n+1)} \tilde{b}_{I,j}(x) \). Note that therefore \(|\tilde{b}_{I,j}(x)| \leq c R^{l/p} \forall x \).
For $I \in B_k$, apply (**) to solve $Ku_{1I} = b_I$ with $u_{1I}$ satisfying (5.6)-(5.8). By Lemma U5, $\exists \{ \beta_{1,I,j}(x) \}_{j=0}^\infty$ satisfying $\operatorname{supp} \beta_{1,I,j} \subseteq 2^{j} I$, $\| \beta_{1,I,j} \|_{\operatorname{Lip}_1} \leq c(2^{j/l}(I))^{-1}$ and $\int_{R^n} \beta_{1,I,j}(x) \, dx = 0$, such that
\[
u_{1I} = \sum_{j=0}^\infty 2^{-j(n+1)} \beta_{1,I,j}(x).
\]
Define $\tilde{\beta}_{I,j}(x) = (0, \beta_{1,I,j})$. Note that $|\tilde{\beta}_{I,j}(x)| \leq c \forall x$, and $|a(I) \cdot \tilde{\beta}_{I,j}(x)| = |a_1(I) \beta_{1,I,j}(x)| \leq cR^{1-1/p}$, since $|a_1(I)| \leq |a_0(I)|R^{-1/p}$ and $I \in B_k$, and $|a_0(I)| \leq |g_{k-1}(x)| = R$ by (5.24).

Note that the $\tilde{\beta}_{I,j}$ defined in the above two cases satisfy (5.16) and (5.17). Define
\[
\tilde{h}_k(x) = \sum_{I: l(I) = 2^{-k}} \frac{\lambda_I}{\sum_{j=0}^M 2^{-j(n+1)}} \tilde{\beta}_{I,j}(x),
\]
\[
\tilde{g}_k(x) = \tilde{g}_{k-1}(x) + \tilde{h}_k(x), \quad \tilde{g}_k(x) = R \frac{\tilde{h}_k(x)}{||\tilde{h}_k(x)||}
\]
and
\[
\tilde{y}_k(x) = \tilde{g}_{k-1}(x) + \tilde{h}_k(x) - \tilde{g}_k(x)
\]
\[
= \left[ \frac{\tilde{g}_{k-1}(x) + \tilde{h}_k(x)}{||\tilde{g}_{k-1}(x) + \tilde{h}_k(x)||} \right] \left[ \frac{R}{||\tilde{g}_{k-1}(x) + \tilde{h}_k(x)||} - 1 \right] = \frac{\tilde{y}_k(x)}{||\tilde{y}_k(x)||} \left[ ||\tilde{y}_k(x)|| - R \right].
\]

Then (5.24) and (5.25) hold. Note that $\tilde{h}_k(x) = 0$ and hence, by (5.30) and (5.24), $\tilde{g}_k(x) = 0$, if $|x| > 2\sqrt{n} \max(2^{M-k}, 1)$, by (5.12) and the fact that $\operatorname{supp} \tilde{\beta}_{I,j} \subseteq 2^{j} I$.

From $|\tilde{\beta}_{I,j}(x)| \leq cR^{1/p}$ we obtain
\[
|\tilde{h}_k(x)| \leq cR^{1/p} \eta_k(x)
\]
as in [8, p. 235]. Similarly, from $|\nabla \tilde{\beta}_{I,j}(x)| \leq cR^{1/p}(2^{j/l}(I))^{-1}$, we obtain
\[
|\tilde{h}_k(x) - \tilde{h}_k(y)| \leq c_5 R^{1/p} \eta_k(x)|x - y| \quad \text{if } |x - y| \leq 2^{-k}.
\]

Therefore, if $|x - y| \leq 2^{-k}$, using (5.26) inductively, we obtain
\[
|\tilde{y}_k(x) - \tilde{y}_k(y)| \leq \left| \tilde{g}_{k-1}(x) - \tilde{g}_{k-1}(y) \right| + |\tilde{h}_k(x) - \tilde{h}_k(y)|
\]
\[
\leq \left( (c_0/2)e_{k-1}(x) + c_5 \eta_k(x) \right) R^{1/p} 2^k |x - y|
\]
\[
\leq \frac{c_0}{2} e_k(x) R^{1/p} 2^k |x - y|,
\]
if we choose $c_0 > \frac{4}{3} c_5$. Since $|\tilde{h}_k(x)| \leq cR^{1/p} \ll R = |\tilde{g}_{k-1}(x)|$, we obtain
\[
|\tilde{g}_k(x) - \tilde{g}_k(y)| \leq \frac{1}{4} |\tilde{y}_k(x) - \tilde{y}_k(y)| \leq c_0 e_k(x) R^{1/p} 2^k |x - y|
\]
for $|x - y| \leq 2^{-k}$. Hence (5.26) holds.
We consider now $\tilde{q}_k(x) = \tilde{y}_k(x) \frac{|\tilde{y}_k(x)| - R}{|\tilde{y}_k(x)|}$. Write

$$|\tilde{y}_k(x)| - R = |\tilde{g}_{k-1}(x) + \tilde{h}_k(x)| - R$$

$$= R \left( 1 + \frac{2 \tilde{g}_{k-1}(x) \cdot \tilde{h}_k(x)}{R^2} + \frac{|\tilde{h}_k(x)|^2}{R^2} \right)^{1/2} - R$$

$$= R \left( 1 + \frac{\tilde{g}_{k-1}(x) \cdot \tilde{h}_k(x)}{R^2} + \frac{|\tilde{h}_k(x)|^2}{2R^2} \right) - R + E_1(x)$$

$$= \frac{\tilde{g}_{k-1}(x) \cdot \tilde{h}_k(x)}{R} + E_2(x),$$

where the last two equalities define $E_1$ and $E_2$. Since $|\sqrt{1 + a} - 1 - \frac{1}{2}a| \leq a^2$ for $|a| \leq \frac{1}{2}$, using (5.24) we obtain (5.33)

$$|E_2(x)| \leq \frac{|\tilde{h}_k(x)|^2}{2R} + cR \left( \frac{|\tilde{g}_{k-1}(x) \cdot \tilde{h}_k(x)|}{R^2} + \frac{|\tilde{h}_k(x)|^2}{R^2} \right)^2 \leq c \frac{|\tilde{h}_k(x)|^2}{R}.$$

Hence

$$\tilde{q}_k(x) = \frac{\tilde{y}_k(x)}{|\tilde{y}_k(x)|} \frac{\tilde{g}_{k-1}(x) \cdot \tilde{h}_k(x)}{R} + \frac{\tilde{y}_k(x)}{|\tilde{y}_k(x)|} E_2 \equiv (I)_k + (II)_k.$$

We write

$$(I)_k = \left( \frac{\tilde{g}_{k-1}(x) + \tilde{h}_k(x)}{\tilde{g}_{k-1}(x) + \tilde{h}_k(x)} \frac{\tilde{g}_{k-1}(x) \cdot \tilde{h}_k(x)}{R} \right)$$

$$= \left( \frac{\tilde{g}_{k-1}(x) \cdot \tilde{h}_k(x)}{R} \frac{\tilde{h}_k(x)}{\tilde{g}_{k-1}(x) + \tilde{h}_k(x)} + \frac{\tilde{g}_{k-1}(x) \cdot \tilde{h}_k(x) \tilde{g}_{k-1}(x)}{R} \right)$$

$$+ \left( \frac{\tilde{g}_{k-1}(x) \cdot \tilde{h}_k(x)}{R} \right) \frac{\tilde{g}_{k-1}(x)}{\tilde{g}_{k-1}(x) + \tilde{h}_k(x)} \frac{1}{\tilde{g}_{k-1}(x) + \tilde{h}_k(x)} - \frac{1}{R}$$

$$\equiv (III)_k + (IV)_k + (V)_k.$$

Now

$$(IV)_k = \frac{\tilde{g}_{k-1}(x)}{R} \left( \frac{1}{R} \sum_{l : |l(t)| = 2^{-k}} \lambda_l \sum_{j=0}^M 2^{-j(n+1)} \tilde{g}_{k-1}(x) \cdot \tilde{b}_{l,j}(x) \right)$$

$$= \frac{\tilde{g}_{k-1}(x)}{R^2} \sum_{l : |l(t)| = 2^{-k}} \lambda_l \sum_{j=0}^M 2^{-j(n+1)} (\tilde{g}_{k-1}(x) - \tilde{g}_{k-1}(x_i)) \cdot \tilde{b}_{l,j}(x)$$

$$+ \frac{1}{R^2} \sum_{l : |l(t)| = 2^{-k}} \lambda_l \sum_{j=0}^M 2^{-j(n+1)} (\tilde{g}_{k-1}(x) - \tilde{g}_{k-1}(x_i)) \tilde{g}_{k-1}(x_i) \cdot \tilde{b}_{l,j}(x)$$

$$+ \frac{1}{R^2} \sum_{l : |l(t)| = 2^{-k}} \lambda_l \sum_{j=0}^M 2^{-j(n+1)} \tilde{g}_{k-1}(x_i) \tilde{g}_{k-1}(x_i) \cdot \tilde{b}_{l,j}(x)$$

$$\equiv (VI)_k + (VII)_k + (VIII)_k.$$
Define $\tilde{\nu}_k = (\text{II})_k + (\text{III})_k + (\text{V})_k + (\text{VI})_k + (\text{VII})_k$, and $\tilde{\nu}_k = (\text{VIII})_k$. Then (5.18) holds. By (5.12) and the fact that $\text{supp} \tilde{\beta}_{I,j} \subseteq 2^{-l}$, (5.22) holds. By (5.30), $\tilde{\nu}_k(x) = 0$ if $|x| > 2\sqrt{n} \max(2^{M-k}, 1)$, and so (5.21) holds also.

We now prove (5.20). By (5.33), (5.31) and (2.5),

$$|(\text{II})_k| \leq \frac{c}{R} |\tilde{h}_k(x)|^2 \leq cR^{(l/p-1)}e_k^2(x)$$

Similarly,

$$|(\text{III})_k| \leq \frac{c}{R} |\tilde{h}_k(x)|^2 \leq cR^{(l/p-1)}e_k^2(x)$$

and

$$|(\text{V})_k| \leq \frac{|\tilde{g}_{k-1}(x) \cdot \tilde{h}_k(x)|}{R^2} \leq \frac{|\tilde{g}_{k-1}(x) \cdot \tilde{h}_k(x)|}{R^2} + \frac{|\tilde{g}_{k-1}(x) \cdot \tilde{h}_k(x)|}{R^2} \leq \frac{c}{R} |\tilde{h}_k(x)|^2 \leq cR^{(l/p-1)}e_k^2(x)$$

by our above estimates on $|y_k(x)| - R$ and $E_2$. To estimate (VI)$_k$ and (VII)$_k$, we note, as in [8, p. 235], that if $l(I) = 2^{-k}$ and $\tilde{\beta}_{I,j}(x) \neq 0$, $0 \leq j \leq M$, then $|x - x_j| \leq c2^{M-k}$, and

$$|(\text{VI})_k| \leq \sum_{l: l(I) = 2^{-k}} \lambda_l \sum_{j=0}^{M} 2^{-j(n+1)R/l}2^{(n+2)M\epsilon_{k-1}(x)}|\tilde{\beta}_{I,j}(x)|$$

Therefore, for the same $x$ and $I$,

$$|(\text{VII})_k| \leq \sum_{l: l(I) = 2^{-k}} \lambda_l \sum_{j=0}^{M} 2^{-j(n+1)R/l}2^{(n+2)M\epsilon_{k-1}(x)}|\tilde{\beta}_{I,j}(x)|$$

since $|x - x_j| \leq c2^{M-k}$ for these $x$ and $I$. Therefore

$$|(\text{VI})_k| \leq \sum_{l: l(I) = 2^{-k}} \lambda_l \sum_{j=0}^{M} 2^{-j(n+1)R/l}2^{(n+2)M\epsilon_{k-1}(x)}|\tilde{\beta}_{I,j}(x)|$$

since $|\tilde{\beta}_{I,j}(x)| \leq cR^{l/p}$, by the method used to obtain (5.31). Similarly, using (5.34) instead of (5.35),

$$|(\text{VII})_k| \leq \sum_{l: l(I) = 2^{-k}} \lambda_l \sum_{j=0}^{M} 2^{-j(n+1)R/l}2^{(n+2)M\epsilon_{k-1}(x)}|\tilde{\beta}_{I,j}(x)|$$

as in (5.31). Hence (5.20) holds, since $\epsilon_{k-1} \leq \frac{3}{2} \epsilon_k$. 

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To establish (5.19), we prove corresponding estimates for \( \tilde{\varphi}_k \) and \( \tilde{\xi}_k \) and use (5.18). Following [8, p. 236], for \( |x - y| < 2^{-k} \), write

\[
\tilde{\varphi}_k(x) - \tilde{\varphi}_k(y) = |\tilde{\gamma}_k(x)|^{-1}(|\tilde{\gamma}_k(x)| - R)(\tilde{\gamma}_k(x) - \tilde{\gamma}_k(y))
+ \tilde{\gamma}_k(y) R \frac{|\tilde{\gamma}_k(x)| - |\tilde{\gamma}_k(y)|}{|\tilde{\gamma}_k(x)||\tilde{\gamma}_k(y)|}
= (\mathbf{I})_k + (\mathbf{II})_k.
\]

Since \( |\tilde{\gamma}_k(x)| - R| < c R^{1/p} \), \( |\tilde{\gamma}_k(x) - \tilde{\gamma}_k(y)| \leq c_0 \varepsilon_k(x) R^{1/p} 2^k |x - y| \) and \( |\tilde{\gamma}_k(x)| > R/2 \) (since \( |\tilde{h}_k(x)| \leq R \)), we obtain

\[
|\mathbf{I}_k| \leq c R^{(2l/p - 1)} \varepsilon_k(x) 2^k |x - y| \text{ for } |x - y| \leq 2^{-k}.
\]

As in [8, p. 237],

\[
|\mathbf{II}_k| \leq 2 \left\{ |\tilde{g}_{k-1}(x) + \tilde{h}_k(x)| - |\tilde{g}_{k-1}(y) + \tilde{h}_k(y)| \right\}
+ 2 \left\{ |\tilde{g}_{k-1}(y) + \tilde{h}_k(x)| \right\}
\leq (\mathbf{III})_k + (\mathbf{IV})_k.
\]

Now

\[
(\mathbf{III})_k = \left| R \left( 1 + \frac{2 \tilde{g}_{k-1}(x) \cdot \tilde{h}_k(x)}{R^2} + \frac{\tilde{h}_k(x)^2}{R^2} \right)^{1/2} 
- R \left( 1 + \frac{2 \tilde{g}_{k-1}(y) \cdot \tilde{h}_k(x)}{R^2} + \frac{\tilde{h}_k(x)^2}{R^2} \right)^{1/2} \right|
\leq \frac{c}{R} \left| (\tilde{g}_{k-1}(x) - \tilde{g}_{k-1}(y)) \cdot \tilde{h}_k(x) \right|
\leq c R^{(2l/p - 1)} \varepsilon_{k-1}(x) 2^k |x - y| \leq c R^{(2l/p - 1)} \varepsilon_k(x) 2^k |x - y|,
\]

by (5.26), (5.31) and a Lipschitz estimate for the function \( f(t) = (1 + t)^{1/2} \) for \( t \) sufficiently small. Similarly,

\[
(\mathbf{IV})_k = \left| R \left( 1 + \frac{2 \tilde{g}_{k-1}(y) \cdot \tilde{h}_k(x)}{R^2} + \frac{\tilde{h}_k(x)^2}{R^2} \right)^{1/2} 
- R \left( 1 + \frac{2 \tilde{g}_{k-1}(y) \cdot \tilde{h}_k(y)}{R^2} + \frac{\tilde{h}_k(y)^2}{R^2} \right)^{1/2} \right|
\leq \frac{c}{R} \left| \tilde{g}_{k-1}(y) \cdot (\tilde{h}_k(x) - \tilde{h}_k(y)) \right|
+ \frac{c}{R} \left| \tilde{h}_k(x)^2 - \tilde{h}_k(y)^2 \right|
= |(\mathbf{V})_k| + |(\mathbf{VI})_k|.
By (5.32) and (2.5) we have $$|\langle V \rangle_k| \leq c R^{(2l/p-1)2^k} |x - y|$$. Now

$$
|\langle V \rangle_k| \leq \frac{c}{R} \left| \sum_{I: |I| = 2^{-k}} \lambda_I \sum_{j=0}^M 2^{-j(n+1)} \left( \hat{g}_{k-1}(y) - \hat{g}_{k-1}(x_I) \right) \cdot \left( \vec{\beta}_{I,j}(x) - \vec{\beta}_{I,j}(y) \right) \right|
+ \frac{c}{R} \left| \sum_{I: |I| = 2^{-k}} \lambda_I \sum_{j=0}^M 2^{-j(n+1)} \hat{g}_{k-1}(x_I) \cdot \left( \vec{\beta}_{I,j}(x) - \vec{\beta}_{I,j}(y) \right) \right|
= (\langle \text{VII} \rangle)_k + (\langle \text{VIII} \rangle)_k.
$$

If $$\vec{\beta}_{I,j}(x)$$ and $$\vec{\beta}_{I,j}(y)$$ are not both $$\vec{0}$$, then $$|y - x_I| \leq c \cdot 2^{-k}$$, since $$|x - y| \leq 2^{-k}$$; so by applying (5.34) we obtain

$$
|\langle \text{VII} \rangle_k| \leq c R^{(l/p-1)2(n+2)} \sum_{I: |I| = 2^{-k}} \lambda_I \sum_{j=0}^M 2^{-j(n+1)} |\vec{\beta}_{I,j}(x) - \vec{\beta}_{I,j}(y)|
\leq c R^{(l/p-1)2(n+2)} \eta_k(x) 2^k |x - y|,
$$

by the calculation in estimate (5.32). Since $$\hat{g}_{k-1}(x_I) \cdot \vec{\beta}_{I,j}(x) = 0$$ $$\forall x$$ whenever $$I \in A_k$$, we have

$$
|\langle \text{VIII} \rangle_k| = \frac{c}{R} \left| \sum_{I \in B_k} \lambda_I \sum_{j=0}^M 2^{-j(n+1)} a_I(I) \left( \vec{\beta}_{I,j}(x) - \vec{\beta}_{I,j}(y) \right) \right|,
$$

since $$\vec{\beta}_{I,j} = (0, \beta_I^1)$$ for $$I \in B_k$$. Since $$|a_I(I)| \leq R^{1-1/p}$$ and $$\|\beta_I^1\|_{Lip} \leq c(2^j l(I))^{-1}$$ for $$I \in B_k$$, and by (2.5) and the proof of (5.32),

$$
|\langle \text{VIII} \rangle_k| \leq c R^{-1/p} \sum_{j=0}^M 2^{-j(n+1)} \sum_{I: \text{dist}(x,I) \leq 2^{-k-j}} \lambda_I |x - y|
\leq c R^{-1/p} \eta_k(x) 2^k |x - y| \leq c R^{-1/p} 2^k |x - y| \text{ for } |x - y| \leq 2^{-k}.
$$

Hence

$$
|\vec{\psi}_k(x) - \tilde{\psi}_k(y)| \leq c(2^{n+2} M R^{(2l/p-1)} + R^{-1/p}) 2^k |x - y|
$$

if $$|x - y| \leq 2^{-k}$$.

Whenever $$|x - y| \leq 2^{-k}$$, we have

$$
|\vec{\phi}_k(x) - \tilde{\phi}_k(y)| = \frac{1}{R^2} \left| \sum_{I \in B_k} \lambda_I \sum_{j=0}^M 2^{-j(n+1)} \tilde{g}_{k-1}(x_I) \tilde{g}_{k-1}(x_I) \cdot \left( \vec{\beta}_{I,j}(x) - \vec{\beta}_{I,j}(y) \right) \right|
\leq c R^{-1/p} \sum_{I \in B_k} \lambda_I \sum_{j=0}^M 2^{-j(n+1)} |\beta_{I,j}(x) - \beta_{I,j}(y)|
\leq c R^{-1/p} 2^k |x - y|,
$$

since $$\tilde{g}_{k-1}(x_I) \cdot \vec{\beta}_{I,j}(x) = 0$$ for $$I \in A_k$$, and for $$I \in B_k$$, $$\tilde{g}_{k-1}(x_I) = (a_0(I), a_I(I))$$, with $$|a_I(I)| \leq R^{1-1/p}$$ and $$\vec{\beta}_{I,j} = (0, \beta_I^1)$$. The estimates for $$\vec{\psi}_k$$ and $$\vec{\phi}_k$$ give (5.19), by (5.18).
We now verify (5.27) and (5.28). We have, for \( k > q > -M \),
\[
\left\| \sum_{s=q+1}^{k} \bar{\xi}_s \right\|_{L^2}^2 = \frac{1}{R^2} \left\| \sum_{l \in \cup_{s=-q+1}^{q} B_s} \lambda_l \sum_{j=0}^{M} 2^{-j(n+1)} \bar{g}_{s-1}(x_l) a_l(I) \beta_{l,j} \right\|_{L^2}^2 \\
\leq c R^{-1/p} \sum_{j=0}^{M} 2^{-j(n+1)} \left\| \sum_{l \in \cup_{s=-q+1}^{q} B_s} \bar{g}_{s-1}(x_l) a_l(I) \lambda_l \beta_{l,j} \right\|_{L^2}^2.
\]
Note that \(|\bar{g}_{s-1}(x_l) a_l(I) / R^{2-1/p}| \leq c \) for \( l \in \cup_{s=-q+1}^{k} B_s \), so by Lemma U3
\[
\left\| \sum_{s=q+1}^{k} \bar{\xi}_s \right\|_{L^2}^2 \leq c R^{-1/p} \left( \sum_{l \in \cup_{s=-q+1}^{q} B_s} \lambda_l^2 |I_l| \right)^{1/2}.
\]
By (5.10), then, (5.27) holds. Similarly, by Lemma U4,
\[
\sum_{s=-M}^{k} \bar{\xi}_s \right\|_{\text{BMO}} \leq c R^{-1/p} \sum_{j=0}^{M} 2^{-j(n+1)} \left\| \sum_{l \in \cup_{s=-M}^{q} B_s} \bar{g}_{s-1}(x_l) a_l(I) \lambda_l \beta_{l,j} \right\|_{\text{BMO}}^{1/2} \leq c R^{-1/p}.
\]
This establishes (5.28) and hence Theorem 5.1. \( \square \)

6. Problems and remarks. The work of Janson [5] and Uchiyama [8] gives the complete result that a collection \( \{K_j\}_{j=1}^{p} \) of singular integral operators satisfy \( \Sigma_{j=1}^{p} K_j L^\infty = \text{BMO} \) if and only if
\[
(6.1) \quad \text{rank} \begin{pmatrix} m_1(\xi) & \cdots & m_p(\xi) \\ m_1(-\xi) & \cdots & m_p(-\xi) \end{pmatrix} = 2 \quad \forall \xi \in S^{n-1}.
\]
However, little is known about the subspace \( \Sigma_{j=1}^{p} K_j L^\infty \) of BMO when (6.1) fails. For the Riesz transforms, Corollary 4.3 implies that any \( f \in \Sigma_{j=1}^{p} \tilde{R}_j L^\infty \), \( p < n \), satisfies a regularity condition on appropriate \( p \)-dimensional subspaces of \( \mathbb{R}^n \) that is not satisfied for all BMO functions. One would hope to obtain some similar interpretation, whenever (6.1) fails, of the fact that \( \Sigma_{j=1}^{p} K_j L^\infty \neq \text{BMO} \). The study of the Riesz transforms was facilitated by their cancellation property; to proceed in the general case, one would presumably require a geometrical interpretation of the failure of (6.1).

However, even for the Riesz transforms the results are far from complete. Our regularity conditions hold for \( f \in R_1(\text{BMO} \cap L^2) \) as well as \( \tilde{R}_1 L^\infty \); boundedness played a role in Corollary 4.3 only in removing the \( L^2 \) restriction. Further, only the condition \( \lambda_j \leq c \), rather than the full strength of the BMO packing condition (3.5), is used in proving Theorem 3.2. Hence our regularity conclusions hold for larger classes of functions and cannot characterize \( R_1(\text{BMO} \cap L^2) \) or \( \tilde{R}_1 L^\infty \). The decomposition result in Lemma 3.1 is a complete, but not explicitly geometrical, characterization of \( R_1(\text{BMO} \cap L^2) \); it is not clear whether a complete geometrical characterization is to be expected. Example 4.6 shows that we cannot expect much stronger regularity along lines.
Since our regularity results hold for $R_1(BMO \cap L^2)$ as well as $\tilde{R}_1L^\infty$, it is natural to ask if there are additional geometrical properties that hold for $\tilde{R}_1L^\infty$ but not for $R_1(BMO \cap L^2)$. Corollary 5.2 suggests that any such distinction may be subtle. Explicit examples of functions $f \in L^\infty + R_1(BMO \cap L^2)$ such that $f \not\in L^\infty + \tilde{R}_1L^\infty$ would be helpful, but are not apparent since our only criterion for excluding $f$ from $L^\infty + \tilde{R}_1L^\infty$ is the failure of our regularity condition, which holds for $L^\infty + R_1(BMO \cap L^2)$ as well. Our function $G \in BMO \cap L^2(\mathbb{R}^2)$ in Example 4.6 is relevant: if $R_1 G \in L^\infty + \tilde{R}_1L^\infty$, then a uniform BMO condition on $x_1$-lines does not hold for $L^\infty + \tilde{R}_1L^\infty$; if not, then $R_1 G$ distinguishes $R_1(L^2 \cap BMO)$ from $L^\infty + \tilde{R}_1L^\infty$. Either case will give information as soon as it is determined which case holds.

A further question regarding each subspace $Y \subseteq BMO$ under consideration is the determination of $\text{dist}(f, Y) = \inf\{\|f - g\|_{BMO} : g \in Y\}$ for $f \in BMO$. When $Y = L^\infty$, Garnett and Jones [4] have complete results. For $K$ as in §5, Corollary 5.2 is obviously equivalent to the statement that

$$\text{dist}(f, L^\infty + K(L^\infty)) = \text{dist}(f, L^\infty + K(BMO_0)).$$

For the Riesz transforms, our conditions on lines should play a role in this problem.

As noted above, some of our results must apply in greater generality than presented here. Since the method of the decomposition of BMO (Lemma U1) appears quite general, one suspects that further information about the action of specific operators on appropriate spaces can be obtained by similar methods.

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