

MINIMAL SURFACES OF CONSTANT CURVATURE IN S^n

BY

ROBERT L. BRYANT¹

ABSTRACT. In this note, we study an overdetermined system of partial differential equations whose solutions determine the minimal surfaces in S^n of constant Gaussian curvature. If the Gaussian curvature is positive, the solution to the global problem was found by [Calabi], while the solution to the local problem was found by [Wallach]. The case of nonpositive Gaussian curvature is more subtle and has remained open. We prove that there are no minimal surfaces in S^n of constant negative Gaussian curvature (even locally). We also find all of the flat minimal surfaces in S^n and give necessary and sufficient conditions that a given two-torus may be immersed minimally, conformally, and flatly into S^n .

0. Introduction. In this paper, we classify the connected minimal surfaces of constant Gaussian curvature in the unit n -sphere $S^n \subset \mathbf{E}^{n+1}$ for all n . It is a classical fact (and follows easily from the structure equations) that the only examples up to rigid motion in S^3 are the open subsets of the geodesic spheres (with $K \equiv 1$) and the Clifford torus (with $K \equiv 0$). In an early paper Boruvka constructed a linearly full immersion $S^2 \subseteq S^{2m}$ for each $m > 0$, where the induced metric had $K \equiv 2/m(m+1)$. Later [Calabi] showed that, up to rigid motion, these Boruvka spheres were the only compact minimal surfaces with $K \equiv K_0 > 0$ in S^n for any n . [Wallach] proved that any connected piece of minimal surface with K a positive constant in S^n was a subset of a Boruvka sphere. [Kenmotsu, 1976] found all of the flat minimal surfaces ($K \equiv 0$) in S^n and quite recently, [Kenmotsu, 1983] showed that $K \equiv K_0 < 0$ is impossible for minimal surfaces in S^4 . The techniques used in the above proofs range from harmonic analysis to rather involved calculations with the moving frame.

On the other hand, in this paper, we take a somewhat different point of view. If (M^2, ds^2) is a surface of constant Gaussian curvature K , then the minimal surfaces in $S^n \subseteq \mathbf{E}^{n+1}$ of constant Gaussian curvature K are given locally by smooth maps $f: M \rightarrow \mathbf{E}^{n+1}$ which satisfy three conditions: first, $\langle f, f \rangle \equiv 1$; second, $\Delta f = -2f$ (Δ is the Laplace-Beltrami operator of ds^2); and third, that f should be an isometry, $\langle df, df \rangle = ds^2$. We study the more general class of maps $f: M \rightarrow \mathbf{E}^{n+1}$ which only satisfy the first two conditions. Note that this set of equations is already overdetermined (by one equation).

Received by the editors August 14, 1984.

1980 *Mathematics Subject Classification*. Primary 53A10; Secondary 53B25 53C40.

Key words and phrases. Minimal surfaces, Gauss curvature, n -sphere.

¹ This work was done while the author was an Alfred P. Sloan Fellow.

In §1, we define the fundamental operators on the space of (vector-valued) smooth functions on M . These operators $X, Y,$ and H form a three-dimensional Lie algebra of differential operators. We prove some basic identities and use these to prove our first result: If $K > 0$, then there is no $f: M \rightarrow S^n$ satisfying $\Delta f = -2f$ (let alone the isometry condition) unless $K = \binom{m}{2}^{-1}$ for some $m \geq 2$. We then show that if $K = \binom{m}{2}^{-1}$ for some $m \geq 2$, then any $f: M \rightarrow S^n$ satisfying $\Delta f = -2f$ must necessarily be an isometric immersion. At this point, it would be possible to quote [Wallach] to classify the minimal surfaces of constant positive Gauss curvature. However, we find it simpler to prove the result directly. Of course, we get the same result: Each such surface is an open subset of a Boruvka sphere.

In §2, we consider the case $K \leq 0$ and derive a further set of identities. We then restrict to the case $K < 0$ and prove that the identities lead to a contradiction. Thus, we conclude that *there are no minimal surfaces of constant negative Gaussian curvature in any S^n .*

In §3, we consider the case $K \equiv 0$. This case has been treated already by [Kenmotsu, 1976] and we include this mainly for the sake of completeness. On the other hand, our proof is considerably shorter, so there is some merit to its inclusion in this paper.

Finally, in §4 we consider the minimal surfaces of constant Gaussian curvature in E^n and H^n . We prove that these minimal surfaces are necessarily totally geodesic surfaces in E^n or H^n .

1. The fundamental structure equations and the case $K > 0$. Let (M^2, ds^2) be an oriented, connected surface with a smooth Riemannian metric ds^2 . We do not assume that M^2 is compact or that the metric is complete. We let $x: \mathcal{F} \rightarrow M$ be the bundle of oriented orthonormal frames. Thus $f \in \mathcal{F}$ is a triple $f = (x; e_1, e_2)$, where $x \in M$ and $e_1, e_2 \in T_x M$ form an oriented orthonormal basis. The canonical 1-forms, ω^1, ω^2 , on \mathcal{F} are the unique 1-forms satisfying

$$dx = e_1 \omega^1 + e_2 \omega^2.$$

By the Fundamental Lemma of Riemannian Geometry, there exists a unique 1-form, ρ , satisfying

$$d\omega^1 = -\rho \wedge \omega^2, \quad d\omega^2 = \rho \wedge \omega^1.$$

The forms $\{\omega^1, \omega^2, \rho\}$ are a coframing of \mathcal{F} and we have the formula $d\rho = K\omega^1 \wedge \omega^2$, where K is a well-defined function on M called the Gaussian curvature of the metric ds^2 . From now on, we shall assume that K is a constant.

It will be convenient to use a complex coframing of \mathcal{F} instead of the given one. Thus, we set $\omega = \omega^1 + i\omega^2$ and rewrite the structure equations as

$$d\omega = i\rho \wedge \omega, \quad d\rho = (i/2)K\omega \wedge \bar{\omega},$$

where $ds^2 = \omega \circ \bar{\omega}$. Another useful convention will be that any function or mapping with domain M may be regarded by pull-back as a function or mapping with domain \mathcal{F} . We will usually omit the pull-back notation x^* and rely on context to make the statements clear. For example, if η is any complex-valued 1-form on M , the equation $\eta = A\omega + B\bar{\omega}$ will actually mean $x^*(\eta) = A\omega + B\bar{\omega}$ for some complex-valued functions A and B on \mathcal{F} (they will *not* be well defined on M).

Let $\tau \rightarrow M$ be the complex line bundle of 1-forms which are multiples of ω and let $\tau^{-1} \rightarrow M$ be the complex line bundle of 1-forms which are multiples of $\bar{\omega}$. For $m \geq 0$, we let $\tau^m \rightarrow M$ (resp. $\tau^{-m} \rightarrow M$) be the m th power of $\tau \rightarrow M$ (resp. $\tau^{-1} \rightarrow M$) as a complex line bundle. Using the identification $\omega^m = (\bar{\omega})^{-m}$ for all m , we have a canonical pairing $\tau^m \times \tau^k \rightarrow \tau^{m+k}$ for all m and k . If σ is any section of τ^m , then, on \mathcal{F} , we may write $\sigma = s(\omega)^m$ for a unique function s on \mathcal{F} . One easily computes that $ds = -m\rho s + s'\omega + s''\bar{\omega}$ for some unique functions s' and s'' on \mathcal{F} . Moreover, by differentiating this equation, we deduce that the forms $s'(\omega)^{m+1} = \sigma'$ and $s''(\omega)^{m-1} = \sigma''$ are well defined sections of τ^{m+1} and τ^{m-1} respectively. This allows us to define operators $\partial_m: C^\infty(\tau^m) \rightarrow C^\infty(\tau^{m+1})$ and $\bar{\partial}_m: C^\infty(\tau^m) \rightarrow C^\infty(\tau^{m-1})$ by $\partial_m\sigma = \sigma'$, $\bar{\partial}_m\sigma = \sigma''$. The reader is warned that this is the usual $\bar{\partial}$ only when $m = 0$. In particular, $\partial_m \circ \partial_{m-1} \neq 0$. Let $I_m: C^\infty(\tau^m) \rightarrow C^\infty(\tau^m)$ be the identity map. Set $\mathcal{T} = \bigoplus_m C^\infty(\tau^m)$ as a \mathbb{Z} -graded vector space and define the operators

$$X = \bigoplus_m \partial_m, \quad Y = \bigoplus_m \bar{\partial}_m, \quad H = \bigoplus_m mI_m.$$

PROPOSITION 1.1. *The operators X, Y , and H satisfy*

$$[H, X] = X, \quad [H, Y] = -Y, \quad [X, Y] = (-K/2)H.$$

Moreover, $\Delta = 2(XY + YX)$, where $\Delta: \mathcal{T} \rightarrow \mathcal{T}$ is the Laplace-Beltrami operator on each graded piece. Finally, $\Phi = \Delta - KH^2$ commutes with X, Y , and H .

PROOF. It suffices to verify each of these on graded pieces of \mathcal{T} . Let $\sigma \in C^\infty(\tau^m)$ and write $\sigma = s(\omega)^m$. The first formula is obvious since $X\sigma \in C^\infty(\tau^{m+1})$ so

$$[H, X]\sigma = H(X\sigma) - X(H\sigma) = (m + 1)(X\sigma) - X(m\sigma) = X\sigma.$$

The second formula is similar. For the third formula, let $X\sigma = s'(\omega)^{m+1}$ and $Y\sigma = s''(\omega)^{m+1}$, and write

$$\begin{aligned} ds' + i(m + 1)\rho s' &= (s')'\omega + (s')''\bar{\omega}, \\ ds'' + i(m - 1)\rho s'' &= (s'')'\omega + (s'')''\bar{\omega}. \end{aligned}$$

Differentiating the equation $ds + im\rho s = s'\omega + s''\bar{\omega}$ gives

$$-(m/2)Ks\omega \wedge \bar{\omega} = ((s'')' - (s')'')\omega \wedge \bar{\omega}$$

so $(s'')' - (s')'' = -m/2Ks$ which is clearly equivalent to $[X, Y]\sigma = (-K/2)H\sigma$.

The formula $\Delta = 2(XY + YX)$ should be regarded as the definition of Δ , though the reader might also compare [Simons]. The operator $\Phi = \Delta - KH^2$ is actually the Casimir operator of the Lie algebra defined by the above relations, so it commutes with X, Y and H . This may also be verified by direct computation. Q.E.D.

Of course, if V is a real vector space with a Euclidean inner product $\langle \cdot, \cdot \rangle: V \times V \rightarrow \mathbb{R}$, we may set $\mathcal{V} = V \otimes_{\mathbb{R}} \mathcal{T}$ and extend the operators X, Y , and H to \mathcal{V} in the natural way. We also have a pairing $\langle \cdot, \cdot \rangle: \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{T}$ gotten by extending the given $\langle \cdot, \cdot \rangle$ in the obvious fashion. An element in the m th graded piece, say σ , may be written in the form $\sigma = s(\omega)^m$, where s is a well-defined function on \mathcal{F} with values in $V_{\mathbb{C}}$; we define conjugation in \mathcal{V} by setting $\bar{\sigma} = \bar{s}(\omega)^{-m}$. Then we have $X\bar{\sigma} = \bar{Y}\sigma$, $Y\bar{\sigma} = \bar{X}\sigma$, $H\bar{\sigma} = -\bar{H}\sigma$.

PROPOSITION 1.2. *Let V be an Euclidean vector space of dimension $n + 1$ and let $S^n \subset V$ be the unit sphere in V . Let $f: M \rightarrow V$ be a smooth map. In order that f be an isometric immersion, it is necessary and sufficient that $\langle Xf, Xf \rangle \equiv 0$, $\langle Xf, \bar{X}f \rangle \equiv \frac{1}{2}$. In addition, $f(M) \subset S^n$ if and only if $\langle f, f \rangle \equiv 1$. Finally, $f(M) \subset S^n$ is minimal iff $\Delta f \equiv -2f$.*

PROOF. Everything is clear except the last part about minimality. This calculation may be found in [Simons] or done directly. Q.E.D.

PROPOSITION 1.3. *Suppose that $f: M \rightarrow V$ satisfies $\Delta f = -2f$. Then, for $m > 0$,*

$$YX^mf = \frac{1}{2} \left[\binom{m}{2} K - 1 \right] X^{m-1}f \quad \text{and} \quad XY^mf = \frac{1}{2} \left[\binom{m}{2} K - 1 \right] Y^{m-1}f.$$

PROOF. Since $f \in \mathcal{V}$ has degree 0, $Hf = 0$. Thus

$$\Delta f = 2(XY + YX)f = -2f, \quad Hf = (XY - YX)f = 0.$$

It follows that

$$XYf = YXf = -\frac{1}{2}f.$$

Since $\binom{m}{2} = m(m-1)/2$, and $\binom{1}{2} = 0$, this verifies our claim when $m = 1$. Now suppose that

$$YX^mf = \frac{1}{2} \left[\binom{m}{2} K - 1 \right] X^{m-1}f.$$

We compute

$$\begin{aligned} YX^{m+1}f &= YX(X^mf) = \left(XY + \frac{1}{2}KH \right) (X^mf) \\ &= \frac{1}{2} \left[\binom{m}{2} K - 1 \right] X^mf + \frac{1}{2} K m X^mf \\ &= \frac{1}{2} \left[\binom{m+1}{2} K - 1 \right] X^mf \end{aligned}$$

so the induction is complete. Since $\bar{f} = f$, we may conjugate the first equation to get the second. Q.E.D.

PROPOSITION 1.4. *Suppose $f: M \rightarrow V$ satisfies $\Delta f = -2f$ and $\langle f, f \rangle \equiv 1$. Then, for all $m \geq 0$,*

$$\langle X^{m+1}f, Y^mf \rangle = \langle X^mf, Y^{m+1}f \rangle = 0 \quad \text{and} \quad \langle X^mf, Y^mf \rangle = A_m,$$

where A_m is a constant depending only on m and K satisfying

$$A_0 = 1, \quad A_{m+1} = \frac{1}{2} \left[1 - \binom{m+1}{2} K \right] A_m.$$

PROOF. We are assuming that $\langle f, f \rangle = \langle X^0f, Y^0f \rangle \equiv 1$. Applying X and Y to this equation gives

$$\langle Xf, f \rangle = \langle f, Yf \rangle = 0.$$

Thus, the above equations are verified for $m = 0$. Suppose we know them to be true for $m = p$. Applying Y to $\langle X^{p+1}f, Y^pf \rangle = 0$, we get

$$\langle YX^{p+1}f, Y^pf \rangle + \langle X^{p+1}f, Y^{p+1}f \rangle = 0$$

so by Proposition 1.3, we have

$$\langle X^{p+1}f, Y^{p+1}f \rangle = \frac{1}{2} \left[1 - \binom{p+1}{2} K \right] \langle X^p f, Y^p f \rangle = A_{p+1}$$

(by induction and the definition of A_{p+1}). Applying X to this equation, we get

$$\langle X^{p+2}f, Y^{p+1}f \rangle + \langle X^{p+1}f, XY^{p+1}f \rangle = XA_{p+1} = 0$$

or

$$\langle X^{p+2}f, Y^{p+1}f \rangle = \left\langle X^{p+1}f, \frac{1}{2} \left[\binom{p+1}{2} K - 1 \right] Y^p f \right\rangle = 0$$

(by induction and conjugation). Since

$$\langle Y^{p+2}f, X^{p+1}f \rangle = \langle \overline{X^{p+2}f}, \overline{Y^{p+1}f} \rangle$$

we are done. Q.E.D.

THEOREM 1.5. *If $K > 0$ and $K^{-1} \neq \binom{m}{2}$ for some $m \geq 2$, then there is no solution of the overdetermined system $\langle f, f \rangle = 1, \Delta f = -2f$.*

PROOF. If $K > 0$ and $K^{-1} \neq \binom{m}{2}$ for some $m \geq 2$, then clearly $A_p < 0$ for some $p \gg 0$. However, if there were a solution $f: M \rightarrow V$ satisfying $\langle f, f \rangle = 1$ and $\Delta f = -2f$, we would have

$$\langle X^p f, \overline{X^p f} \rangle = \langle X^p f, Y^p f \rangle = A_p < 0$$

which is a contradiction. Q.E.D.

We also have the following result

THEOREM 1.6. *Suppose that $K = \binom{m}{2}$ for some $m \geq 2$ and that $f: M \rightarrow V$ satisfies both $\langle f, f \rangle = 1$ and $\Delta f = -2f$. Then $f(M)$ lies linearly fully in a $(2m - 1)$ -dimensional vector space $W \subseteq V$ and $f: M \rightarrow S^{2m-2}$ is a minimal isometric immersion. Moreover, $f(M)$ is an open subset of the Boruwka sphere $S^2 \rightarrow S^{2m-2}$.*

PROOF. By Proposition 1.4, we see that, for $r \geq m$, we have $\langle X^r f, \overline{X^r f} \rangle = \langle X^r f, Y^r f \rangle = A_r = 0$, so $X^r f \equiv 0$ when $r \geq m$ (of course, we also have $Y^r f \equiv 0$ for $r \geq m$). Since $\langle X^{m-1} f, Y^{m-2} f \rangle = 0$ (again by Proposition 1.4) we may apply X to conclude that $\langle X^{m-1} f, Y^{m-3} f \rangle = 0$ (if $m \geq 3$). By induction, it follows that $\langle X^{m-1} f, Y^q f \rangle = 0$ for all $0 \leq q < m - 1$. Similarly, starting with $\langle X^{m-2} f, Y^{m-3} f \rangle = 0$ we get $\langle X^{m-2} f, Y^q f \rangle = 0$ for all $0 \leq q < m - 2$. In this way, we conclude that $\langle X^p f, Y^q f \rangle \equiv 0$ when $p \neq q$ and $p, q \geq 0$. In particular $\langle X^p f, f \rangle \equiv 0$ for $p > 0$. Applying X , we conclude that $\langle X^p f, Xf \rangle \equiv 0$ for $p > 0$. Again, by induction we see that $\langle X^p f, X^q f \rangle \equiv 0$ when $p, q \geq 0$ and $p + q > 0$. In particular, note that $\langle Xf, Xf \rangle = 0$ and $\langle Xf, \overline{Xf} \rangle = \frac{1}{2}$ so by Proposition 1.2 $f: M \rightarrow V$ is an isometric immersion into the unit sphere.

Now, for $0 < k \leq m - 1$, set $X^k f = \sqrt{2A_k} f_k(\omega)^k$. This defines the $V_{\mathbb{C}}$ -valued functions f_k for $k > 0$. The formulae for the inner products $\langle X^p f, X^q f \rangle$ and $\langle X^p f, \overline{X^q f} \rangle$ now translate into the formulae

$$\langle f, f_k \rangle = \langle f_k, f_j \rangle = 0, \quad \langle f_k, \overline{f_j} \rangle = \frac{1}{2} \delta_{kj}.$$

It follows that if we set $e_0 = f$ and $\frac{1}{2}(e_{2k-1} - ie_{2k}) = f_k$ for $0 \leq k \leq m - 1$, where the e_0, \dots, e_{2m-2} are V -valued functions on \mathcal{F} , then these $(2m - 1)$ vectors are orthonormal at every point of \mathcal{F} . Moreover, the equations

$$df = f_1\omega + \bar{f}_1\bar{\omega}, \quad df_1 = -i\rho f_1 + \sqrt{2A_2}f_2\omega - \frac{1}{2}f_0\bar{\omega}$$

and

$$df_k + kipf_k = \sqrt{\frac{A_{k+1}}{A_k}} f_{k+1}\omega + \frac{1}{2} \left[\binom{k}{2} \binom{m}{2}^{-1} - 1 \right] \sqrt{\frac{A_{k-1}}{A_k}} f_{k-1}\bar{\omega}$$

for $k > 1$, which are consequence of the definitions of f and Proposition 1.3, together with $f_m \equiv 0$ can now be written in the form

$$d(e_0, e_1, \dots, e_{2m-2}) = (e_0, e_1, \dots, e_{2m-2})\psi,$$

where the matrix of 1-forms ψ is given by

$$\psi = \begin{bmatrix} 0 & -i\Omega & 0 & 0 & \cdots & 0 \\ \Omega & J & -c_2'\Psi & 0 & \cdots & \vdots \\ 0 & c_2\Psi & 2J & -c_3'\Psi & & \vdots \\ 0 & 0 & c_3\Psi & 3J & 0 & \vdots \\ \vdots & & 0 & & \ddots & -c_{m-1}'\Psi \\ 0 & 0 & \cdots & \cdots & 0 & c_{m-1}\Psi & (m-1)J \end{bmatrix},$$

where

$$\Omega = \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & \rho \\ -\rho & 0 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \omega^1 & -\omega^2 \\ \omega^2 & \omega^1 \end{pmatrix}$$

and $c_k = \sqrt{A_k/A_{k-1}}$ for $k = 2, \dots, m - 1$. Since the c_k are constants, it is easy to verify that $d\psi + \psi \wedge \psi \equiv 0$. Clearly $d(e_0 \wedge \cdots \wedge e_{2m-2}) \equiv 0$, so the space spanned by $\{e_0, \dots, e_{2m-2}\}$, say $W \subseteq V$, is constant on \mathcal{F} . Clearly $f(M) = e_0(M) \subseteq W$ and W is the smallest subspace with this property. Finally, the fact that the entries of ψ are constant linear combinations of ω^1, ω^2, ρ implies that, when we write $\psi = F_0\rho + F_1\omega^1 + F_2\omega^2$, then the matrices $\{F_1, F_2, F_0\}$ span a Lie algebra isomorphic to $\mathfrak{so}(3)$. It is easy to see that this representation of $\mathfrak{so}(3)$ into $\mathfrak{so}(2m - 1)$ is irreducible. It follows that $f(M)$ is an open subset of a minimal orbit of dimension 2 of the irreducible representation of $SO(3)$ into $SO(2m - 1)$. It is well known that this minimal orbit is the Boruvka sphere in S^{2m-2} . See [Calabi]. Q.E.D.

REMARK. The reader should compare [Wallach] where Theorem 1.6 is proved under the additional hypothesis that f be an isometric immersion.

2. The case $K \leq 0$. Throughout this section, we assume $K \leq 0$. It follows that $A_p > 0$ for all p . We also assume that $f: M \rightarrow V$ satisfies $\langle f, f \rangle = 1$ and $\Delta f = -2f$. For $p > 0$, we define $X^p f$ and $Y^p f$ as before, but we set $X^{-p} f = (-1)^p A_p^{-1} Y^p f$ (for $p \geq 0$).

PROPOSITION 2.1. For every $m \geq 0$ and for all p , $X^m(X^p f) = X^{m+p} f$.

PROOF. If $p \geq 0$, this is obvious. Induction reduces us to the case $p < 0$ and $m = 1$. We compute

$$\begin{aligned} X(X^{-q}f) &= (-1)^q A_q^{-1} X Y^q f = (-1)^q A_q^{-1} \frac{1}{2} \left[\binom{q}{2} K - 1 \right] Y^{q-1} f \\ &= (-1)^{q-1} A_{q-1}^{-1} Y^{q-1} f = X^{1-q} f \end{aligned}$$

if $q > 0$. Q.E.D.

PROPOSITION 2.2. For every $m \geq 2$, there exists a polynomial in s , $R_m(s)$, of degree at most $m - 2$ in s , with coefficients in $C^\infty(\tau^m)$, and satisfying

$$\langle X^{p+m} f, X^{-p} f \rangle = (-1)^p R_m(p)$$

for all p . Moreover

$$\langle X^p f, X^{-p} f \rangle = (-1)^p, \quad \langle X^{p+1} f, X^{-p} f \rangle = 0$$

for all p .

PROOF. The last two statements are just Proposition 1.4 restated. To get the first statement, we apply X^{m-1} to the equation $\langle X^{p+1} f, X^{-p} f \rangle = 0$ and use the Leibnitz rule to get

$$\sum_{r=1}^m \binom{m-1}{r-1} \langle X^{p+r} f, X^{m-(p+r)} f \rangle \equiv 0.$$

This equation says that the $(m - 1)$ th difference sequence of the sequence $\{(-1)^p \langle X^{p+m} f, X^{-p} f \rangle \mid p \in \mathbf{Z}\}$ vanishes identically. This is well known to be equivalent to the statement that this sequence is polynomial in p of degree at most $m - 2$. Q.E.D.

THEOREM 2.3. For $K < 0$ there is no solution to the over-determined system $\Delta f = -2f$, $\langle f, f \rangle = 1$ for maps $f: M \rightarrow V$. It follows that S^n has no (local) minimal surfaces of constant negative Gaussian curvature.

PROOF. For $p \geq 0$, define $Z_p = X^p f / \sqrt{A_p}$. Then we compute from Proposition 2.2 that for $p \geq 0$

$$\langle Z_p, \bar{Z}_p \rangle \equiv 1, \quad \langle Z_{p+1}, \bar{Z}_p \rangle \equiv 0$$

and, when $m \geq 2$

$$\langle Z_{p+m}, \bar{Z}_p \rangle = \sqrt{A_p/A_{p+m}} R_m(p).$$

The last equation is gotten as follows:

$$\begin{aligned} \langle Z_{p+m}, \bar{Z}_p \rangle &= \frac{1}{\sqrt{A_{p+m} A_p}} \langle X^{p+m} f, \bar{X}^p f \rangle = \frac{1}{\sqrt{A_{p+m} A_p}} \langle X^{p+m} f, Y^p f \rangle \\ &= \sqrt{\frac{A_p}{A_{p+m}}} (-1)^p \langle X^{p+m} f, X^{-p} f \rangle = \sqrt{\frac{A_p}{A_{p+m}}} R_m(p) \quad (m \geq 2). \end{aligned}$$

Since $K < 0$, we easily compute that

$$\sqrt{A_p/A_{p+m}} < C_m/p^m$$

for some constant C_m (which depends on K). Since $R_m(p)$ has p -degree at most $m - 2$ it follows that for any $m > 0$

$$\mathcal{L}_{p \rightarrow \infty} \langle Z_{p+m}, \bar{Z}_p \rangle = 0$$

pointwise on \mathcal{F} . Let $y_0 \in \mathcal{F}$ be fixed and define the vectors W_k in V_C by $Z_k(y_0) = W_k(\omega|_{y_0})^k$.

Let $r > n$ be any integer and let $\epsilon > 0$ be small. By the above argument, there exists a p so large that

$$|\langle W_{p+k}, \overline{W_{p+l}} \rangle| < \epsilon$$

for all $k \neq l, 0 \leq k, l \leq r$, while $\langle W_{p+k}, \overline{W_{p+k}} \rangle = 1$ for all k . Taking ϵ sufficiently small, this implies that the $r + 1$ vectors $\{W_p, \dots, W_{p+r}\}$ are linearly independent in V_C (which has dimension $n + 1$). Since $r > n$, this is impossible. Q.E.D.

3. The case $K = 0$. In the case $K = 0$, we may introduce considerable simplifications. In the first place, we may choose a framing of M along which $\rho = 0$. In other words, we may introduce (at least locally) a complex function $z: M \rightarrow \mathbb{C}$ so that $ds^2 = dz \circ d\bar{z}$. Thus, we may take $\omega = dz$ and $\rho = 0$. The operator X is just $\partial/\partial z$ and Y is just $\partial/\partial \bar{z}$. The sequence $\{A_m\}$ becomes $A_m = 2^{-m}$. The formula for Z_m becomes

$$Z_m = 2^{m/2}(\partial f/\partial z^m)(dz)^m \quad (m \geq 0).$$

Moreover, the formulas

$$\langle Z_m, \bar{Z}_m \rangle \equiv 1, \quad \langle Z_{m+1}, \bar{Z}_m \rangle \equiv 0, \quad \langle Z_{p+m}, \bar{Z}_p \rangle = 2^{m/2} \tilde{R}_m(p)(dz)^m$$

(for $m > 0$) still hold, where, now, $\tilde{R}(p)$ is a polynomial in p of degree at most $m - 2$. Because of the triangle inequality we must have $2^{m/2}|\tilde{R}_m(p)| \leq 1$ for all p and m . This clearly implies that $\tilde{R}_m(p)$ is of degree 0 in p . Thus, the formula becomes

$$\langle Z_{p+m}, \bar{Z}_p \rangle = 2^{m/2} \tilde{R}_m(dz)^m,$$

where \tilde{R} is a function of z (and \bar{z}) alone. Recalling Proposition 2.2, we see that this implies that

$$\langle X^{p+m}f, X^{-p}f \rangle = (-1)^p \tilde{R}_m(dz)^m$$

for all $p \in \mathbb{Z}$. If m is odd, say $m = 2q + 1$, then setting $p = -q$ and $p = -q - 1$, we get

$$\begin{aligned} \langle X^{q+1}f, X^qf \rangle &= (-1)^{-q} \tilde{R}_{2q+1}(dz)^{2q+1}, \\ \langle X^qf, X^{q+1}f \rangle &= (-1)^{q+1} \tilde{R}_{2q+1}(dz)^{2q+1} \end{aligned}$$

so it follows that $\tilde{R}_{2q+1} \equiv 0$ for all $q \geq 0$. Thus $\langle X^{p+m}f, X^{-p}f \rangle \equiv 0$ for m odd. For m even, differentiating

$$\langle X^{p+2q}f, X^{-p}f \rangle = (-1)^p \tilde{R}_{2q}(dz)^{2q}$$

with X and then with Y and using the result for m odd, we see that \tilde{R}_{2q} must be a constant independent of z (or \bar{z}). Let us define

$$H_{p\bar{q}} = 2^{(p+q)/2} \langle \partial^p f / \partial z^p, \partial^q f / \partial z^q \rangle = \overline{H_{q\bar{p}}}.$$

Then, for every $m \geq 0$, the $(m + 1) \times (m + 1)$ Hermitian matrix obtained by considering $\{H_{p\bar{q}} | 0 \leq p, q \leq m\}$ is a positive semidefinite Hermitian matrix of constants. Since the $\partial^k f / \partial z^k$ all lie in a finite-dimensional vector space, it follows that there must be an integer $r > 0$ so that $\{H_{p\bar{q}} | 0 \leq p, q \leq r\}$ is positive definite while $\{H_{p\bar{q}} | 0 \leq p, q \leq r + 1\}$ has zero determinant. It follows that the vectors $\{f, \partial f / \partial z, \dots, \partial^r f / \partial z^r\}$ are linearly independent at all points of M while there must exist a relation of the form

$$2^{(r+1)/2} \frac{\partial^{r+1} f}{\partial z^{r+1}} + \sum_{k=0}^r 2^{k/2} B^k \frac{\partial^k f}{\partial z^k} = 0,$$

where the B_k are smooth functions on M . Taking the inner product of this equation with $2^{k/2} \partial^k f / \partial \bar{z}^k$ gives

$$H_{r+1, \bar{k}} + \sum_{j=0}^r B^j H_{j\bar{k}} = 0.$$

Thus $(B^0, B^1, \dots, B^r, 1)$ is a zero eigenvector of the matrix $\{H_{p\bar{q}} | 0 \leq p, q \leq r + 1\}$. Since the matrix has rank r by definition (and has constant entries), it follows that the B^j are constants. Obviously, this is the only nontrivial linear relation on $\{f, \dots, \partial^{r+1} f / \partial z^{r+1}\}$. Now, since $\Delta f = -2f = 4\partial^2 f / \partial z \partial \bar{z}$, and since f is real, we may apply the operator $2^{(r+1)/2} \partial^{r+1} / \partial z^{r+1}$ to the above equation and conjugate to get

$$2^{(r+1)/2} \bar{B}^0 \frac{\partial^{r+1} f}{\partial z^{r+1}} + \sum_{k=1}^r 2^{k/2} (-1)^{r+1-k} \bar{B}^{r+1-k} \frac{\partial^k f}{\partial z^k} + (-1)^{r+1} f = 0.$$

It follows that $B^0 \neq 0$ and that $B^k = (-1)^{r+1-k} \bar{B}^{r+1-k} / \bar{B}^0$. Setting $k = 0$ gives $B^0 \bar{B}^0 = (-1)^{r+1}$. Thus r is odd. Our equation now simplifies to $B^k = (-1)^k B^0 \bar{B}^{r+1-k}$.

For convenience, we set $w = z / \sqrt{2}$. Then we get $H_{p\bar{q}} = \langle \partial^p f / \partial w^p, \partial^q f / \partial \bar{w}^q \rangle$ and

$$\frac{\partial^{r+1} f}{\partial w^{r+1}} + \sum_{k=0}^r B^k \frac{\partial^k f}{\partial w^k} = 0.$$

Now $H_{p\bar{q}} = 0$ if $p - q$ is odd. It follows that the even derivatives $\{f, \partial^2 f / \partial w^2, \dots, \partial^{r+1} f / \partial w^{r+1}\}$ are orthogonal to all of the odd derivatives $\{\partial f / \partial w, \dots, \partial^r f / \partial w^r\}$. From this, it immediately follows that $B^k = 0$ when k is odd. If we replace w by λw , where $\lambda^{r+1} = B_0$, then we see that B_0 becomes 1. Thus, we assume $B_0 = 1$ from now on. We have

$$\frac{\partial^{r+1} f}{\partial w^{r+1}} + B^{r-1} \frac{\partial^{r-1} f}{\partial w^{r-1}} + \dots + B^2 \frac{\partial^2 f}{\partial w^2} + f = 0.$$

Let $\Lambda \subseteq \mathbb{C}$ be the set of roots of $x^{r+1} + B^{r-1} x^{r-1} + \dots + B^2 x^2 + 1 = 0$. It follows from the facts that $B^{2k+1} = 0$ and $B^{2k} = \bar{B}^{r+1-2k}$, that the set Λ does not

contain $0 \in \mathbf{C}$, and that, if $\lambda \in \Lambda$, then $-\lambda$ and $\bar{\lambda}^{-1}$ must also belong to Λ . Let $\{\lambda_\alpha | 1 \leq \alpha \leq s\} = \Lambda$. The above equation plus the reality condition $f = \bar{f}$ implies that

$$f(w) = \sum_{\alpha, \beta} p_{\alpha\bar{\beta}} e^{\lambda_\alpha w + \bar{\lambda}_\beta \bar{w}},$$

where $p_{\alpha\bar{\beta}} = \overline{p_{\beta\bar{\alpha}}}$ is a $V_{\mathbf{C}}$ -valued polynomial in w and \bar{w} of degree less than or equal to the multiplicity of λ_α plus the multiplicity of λ_β . Obviously, we may now regard f as being defined on all of \mathbf{C} by this formula. Applying the relation $\Delta f = -2f = 2\partial^2 f / \partial w \partial \bar{w}$, we see that $p_{\alpha\bar{\beta}} = 0$ unless $\lambda_\alpha \bar{\lambda}_\beta = -1$. Moreover, since $\langle f, f \rangle \equiv 1$, in particular, it follows that $\langle f, f \rangle$ is bounded. It is not hard to show that this implies that $p_{\alpha\bar{\beta}} \equiv 0$ unless $\text{Re}(\lambda_\alpha w + \bar{\lambda}_\beta \bar{w}) \equiv 0$ for all $w \in \mathbf{C}$. This can only happen if $\lambda_\beta = -\lambda_\alpha$. Combined with the above result, we see that $p_{\alpha\bar{\beta}} \equiv 0$ unless $\lambda_\alpha \bar{\lambda}_\alpha = 1$. Let $\{\lambda_1, \dots, \lambda_k\} \subseteq \Lambda$ be such that we may write

$$f(w) = \sum_{j=1}^k p_j e^{\lambda_j w - \bar{\lambda}_j \bar{w}} + \bar{p}_j e^{-\lambda_j w + \bar{\lambda}_j \bar{w}},$$

where $\lambda_i \neq \pm \lambda_j$ when $i \neq j$ and the p_j are nonzero polynomials in w and \bar{w} . The independence of the exponentials and the boundedness of $f(w)$ then again imply that the p_j are bounded polynomials and hence are constant vectors in $V_{\mathbf{C}}$. In fact, the $2k^2$ numbers $\{\lambda_i + \lambda_j, -(\lambda_i + \lambda_j) | 1 \leq i, j \leq k\} \cup \{\lambda_i - \lambda_j | i \neq j\}$ are all nonzero and distinct (as is easily seen since the $\pm \lambda_j$ are distinct points on the unit circle). If we expand $\langle f, f \rangle \equiv 1$ and use the independence of the exponentials we see immediately that $\langle p_i, p_j \rangle = 0$ for all i, j , that $\langle p_i, \bar{p}_j \rangle \equiv 0$ for all $i \neq j$, and that

$$\sum_{j=1}^k 2\langle p_i, \bar{p}_i \rangle \equiv 1.$$

Conversely, if $\{\lambda_i | 1 \leq i \leq k\}$ are complex numbers of unit norm so that $\{\pm \lambda_i | 1 \leq i \leq k\}$ is a set of $2k$ distinct numbers, and $p_1, \dots, p_k \in V_{\mathbf{C}}$ satisfy the above three conditions, then $f(w)$ as defined above gives a map $f: \mathbf{C} \rightarrow S^n$ satisfying $\langle f, f \rangle = 1$ and $\Delta f = -2f$ (where $ds^2 = 2dw \circ d\bar{w}$). Obviously f is linearly full in a real $2k$ -dimensional subspace of V . Thus $f(\mathbf{C})$ lies linearly fully in an odd-dimensional sphere. We record this as

THEOREM 3.1. *Let D be a small disk about $0 \in \mathbf{C}$ and let $f: D \rightarrow S^n \subseteq V$ be a smooth mapping satisfying $\Delta f = -2f$, where we compute the Laplacian with respect to a metric $ds^2 = 2dw \circ d\bar{w}$ where w is a linear complex coordinate on \mathbf{C} . Assume that $f(D) \subseteq V$ does not lie in any proper subspace of V . Then*

- (1) n is odd,
- (2) f extends uniquely to all of \mathbf{C} as a map satisfying $\Delta f = -2f$ and $\langle f, f \rangle = 1$,
- (3) after rotating w if necessary, f can be written in the form

$$f(w) = \sum_{k=1}^{(n+1)/2} p_k e^{\lambda_k w - \bar{\lambda}_k \bar{w}} + \bar{p}_k e^{-\lambda_k w + \bar{\lambda}_k \bar{w}},$$

where the $\{\pm\lambda_i | 1 \leq i \leq (n+1)/2\}$ are $n+1$ distinct complex numbers of norm 1 and the $p_k \in V_C$ are nonzero vectors satisfying

$$\langle p_k, p_j \rangle = 0, \quad \langle p_k, \bar{p}_j \rangle = 0 \text{ if } j \neq k \quad \text{and} \quad \sum_{k=1}^{(n+1)/2} \langle p_k, \bar{p}_k \rangle = \frac{1}{2}.$$

Moreover, the pairs (p_k, λ_k) are unique up to permutations and by replacing (p_k, λ_k) by $(\bar{p}_k, -\lambda_k)$.

Conversely, any f as described in (3) satisfies $\langle f, f \rangle = 1$ and $\Delta f = -2f$.

REMARK. The reader will note the similarities with Calabi's result for minimal surfaces of constant positive Gauss curvature. There, of course, n turns out to be even.

COROLLARY 3.2. Keeping the same notations as in Theorem 3.1, the map $f: \mathbb{C} \rightarrow S^n$ is a minimal immersion if and only if $\sum_k \lambda_k^2 \langle p_k, \bar{p}_k \rangle = 0$.

PROOF. One immediately computes that

$$\langle df, df \rangle = 2dw \circ d\bar{w} + 2 \sum_{k=1}^{(n+1)/2} \lambda_k^2 \langle p_k, \bar{p}_k \rangle (dw)^2 + \bar{\lambda}_k^2 \langle p_k, \bar{p}_k \rangle (d\bar{w})^2.$$

Now apply Proposition 1.2. Q.E.D.

REMARKS. Combining Theorem 3.1 and Corollary 3.2, it is possible to show that every minimal immersion $f: \mathbb{R}^2 \rightarrow S^{2m-1}$ satisfying $\langle df, df \rangle = \frac{1}{2}(dx^2 + dy^2)$ can be rotated in S^{2m-1} to the normal form $f = (f^1, \dots, f^{2m})$ where, for $1 \leq k \leq m$,

$$f^{2k-1}(x, y) = r_k \cos(x \cos \theta_k + y \sin \theta_k),$$

$$f^{2k}(x, y) = r_k \sin(x \cos \theta_k + y \sin \theta_k),$$

where (r_k, θ_k) are m real numbers satisfying $r_k > 0$, $\theta_j \not\equiv \theta_k \pmod{\pi}$ for $j \neq k$, $r_1^2 + \dots + r_m^2 = 1$, and $e^{2i\theta_1} r_1^2 + \dots + e^{2i\theta_m} r_m^2 = 0$. Moreover, these constants are almost uniquely determined: One can permute the pairs (r_k, θ_k) and displace the θ_k by the same angle θ_0 , i.e. $\{(r_k, \theta_k)\}$ and $\{(r_k, \theta_k + \theta_0)\}$ give the same minimal surface in S^{2m-1} . It follows that the linearly full minimal isometric immersions $f: \mathbb{R}^2 \rightarrow S^{2m-1}$ form a $(2m-4)$ -dimensional family after reducing mod the isometries in the domain and range.

Our second remark concerns the image $f(\mathbb{R}^2) \subseteq S^{2m-1}$ for such an f . From our formula above, we see that $f(x, y) = f(x', y')$ iff

$$(x' - x) \cos \theta_k + (y' - y) \sin \theta_k \equiv 0 \pmod{2\pi}$$

for $k = 1, \dots, m$. We define

$$\Lambda_f = \{(x, y) \in \mathbb{R}^2 | x \cos \theta_k + y \sin \theta_k \equiv 0 \pmod{2\pi}, 1 \leq k \leq m\}.$$

Clearly $\Lambda_f \subseteq \mathbb{R}^2$ is a discrete lattice and $f(\mathbb{R}^2)$ is a 1-1 immersion of \mathbb{R}^2/Λ_f into S^{2m-1} . If Λ_f has rank 2, then \mathbb{R}^2/Λ_f is imbedded as a minimal flat torus in S^{2m-1} .

It is of some interest to determine which tori \mathbb{R}^2/Λ (where Λ has rank 2 and is discrete) admit minimal flat imbeddings into some S^{2m-1} . We can answer this question as follows: Let $\Lambda^* = \{(a, b) \in \mathbb{R}^2 | ax + by \equiv 0 \pmod{2\pi} \text{ for all } (x, y) \in \Lambda\}$.

Then $\Lambda^* \subseteq \mathbf{R}^2$ is a rank 2, discrete lattice in \mathbf{R}^2 . The following proposition now follows easily.

PROPOSITION 3.3. *Let $\Lambda \subseteq \mathbf{R}^2$ be a rank 2, discrete lattice. For each $r > 0$, let $H(\Lambda, r)$ be the convex hull of the (possibly empty) set*

$$c(\Lambda, r) = \{(a + ib)^2 \mid a^2 + b^2 = r^2 \text{ and } (a, b) \in \Lambda^*\} \subseteq \mathbf{C}.$$

Then \mathbf{R}^2/Λ admits a minimal immersion $f: \mathbf{R}^2/\Lambda \rightarrow S^{2m-1}$ for some m with $\langle df, df \rangle = r^2/2(dx^2 + dy^2)$ iff $0 \in H(\Lambda, r)$. If 0 is on the boundary of $H(\Lambda, r)$, then $m = 2$. If 0 is in the interior of $H(\Lambda, r)$ then m is at most the cardinality of $c(\Lambda, r)$.

REMARK. It is possible to show that $0 \in H(\Lambda, r)$ for some r if and only if Λ^* is generated as a lattice by two periods $\alpha, \beta \in \mathbf{C}$ with the property that $\operatorname{Re}(\alpha/\beta)$ and $\alpha\bar{\alpha}/\beta\bar{\beta}$ are both rational. In this case $0 \in H(\Lambda, r_i)$ for an infinite sequence of $r_i > 0$ and the cardinality of $c(\Lambda, r_i)$ can be made arbitrarily large. On the other hand, the generic Λ^* will have $H(\Lambda, r)$ either empty or a single point for all $r > 0$. Thus, the generic torus \mathbf{R}^2/Λ cannot be minimally, flatly, and conformally immersed in any S^n .

REMARK. The results of this section, under the additional hypothesis that f be an isometric immersion are essentially due to [Kenmotsu, 1976].

4. Generalizations to other ambient sectional curvatures. The n -sphere is the standard model for n -manifolds of constant sectional curvature $c = 1$. In this section, we sketch how the classification can be extended to other constant sectional curvatures. By scaling, we are reduced to considering the three cases where the sectional curvature c is either 1, 0, or -1 . We have already treated the case $c = 1$, so now we consider $c = 0$ or -1 . It is well known that if M is a minimal surface in a space of constant sectional curvature c , then $K \leq c$ (where K is the Gauss curvature of M) with equality holding iff M is totally geodesic in the ambient space.

The following proposition is due to [Pinn]. We give a simple proof based on some results of E. Calabi which are reported on in [Lawson, Chapter IV].

PROPOSITION 4.1. *Let $M^2 \subseteq \mathbf{E}^n$ be a connected minimal surface of constant Gaussian curvature K . Then $K \equiv 0$ and M is an open subset of a 2-plane in \mathbf{E}^n .*

PROOF. We quote [Lawson, Theorem 11, p. 157] and maintain his notation. If the induced metric on M is written locally in isothermal form $ds^2 = 2Fdz \circ d\bar{z}$, then the assumption that ds^2 has Gauss curvature K is equivalent to the equation $\partial^2(\log F)/\partial z \partial \bar{z} = -KF$. Assume that $K < 0$. It follows that the sequence $\{F_j\}$ is given by $F_j = c_j F^{j(j+1)/2}$, where the c_j are constants satisfying $c_0 = 1$, $c_1 = 1$ and the recursion formula

$$c_{k+1}c_{k-1}/c_k^2 = (-K)k(k+1)/2 \quad \text{for } k \geq 1.$$

Clearly F_j never vanishes, so $K < 0$ is impossible. Since $c \equiv 0$ in this case, we must have $K \equiv 0$ and $M \subseteq \mathbf{E}^n$ is totally geodesic. Q.E.D.

We now turn to the more interesting case where $c \equiv -1$. Let V be an $(n + 1)$ -dimensional real vector space endowed with an inner product of type $(n, 1)$. Let H^n be one of the two components of the hyperboloid of 2 sheets given by $\langle x, x \rangle = -1$. Then H^n is the classical model for a space of constant sectional curvature -1 .

THEOREM 4.2. *Suppose that $M^2 \subseteq H^n$ is a connected minimal surface of constant Gaussian curvature K . Then $K \equiv -1$ and M^2 is totally geodesic in H^n .*

PROOF. It suffices to prove that $K \geq -1$ since $c = -1$. Note that the inclusion map $f: M^2 \rightarrow H^n \subseteq V$ satisfies $\langle f, f \rangle = -1$ and $\Delta f = 2f$ (where the Laplacian is computed with respect to the induced metric). Following the calculations in §§1 and 2, we easily establish the formulae

$$\begin{aligned} YX^mf &= \frac{1}{2} \left[1 + \binom{m}{2} K \right] X^{m-1}emf & (m > 0), \\ XY^mf &= \frac{1}{2} \left[1 + \binom{m}{2} K \right] Y^{m-1}f \\ \langle X^{m+1}f, Y^mf \rangle &= \langle X^mf, Y^{m+1}f \rangle = 0 & (m \geq 0), \\ \langle X^mf, Y^mf \rangle &= B_m & (m \geq 0), \end{aligned}$$

where $B_0 = -1$ and $B_m = -\frac{1}{2}[1 + \binom{m}{2}K]B_{m-1}$ for $m > 0$. Now assume that $K < -1$. Then $B_m > 0$ for all $m > 0$. Setting $X^{-p}f = (-1)^{p+1}B_p^{-1}Y^pf$ as before, we derive identities similar to those of §2. Again, we define $Z_p = X^pf/\sqrt{B_p}$ for $p > 0$ and we set $Z_0 = f$. The same estimate $\mathcal{L}_p \rightarrow \infty \langle Z_{p+m}, \bar{Z}_p \rangle = 0$ for $m > 0$ (pointwise on M) continues to hold while $\langle Z_p, \bar{Z}_p \rangle \equiv \pm 1$. Thus, we again derive a contradiction from the finite dimensionality of V . Hence $K < -1$ is impossible. Q.E.D.

REMARK. With more care, it is possible to show that if (M, ds^2) is any connected Riemannian surface of constant Gauss curvature K and V is as above, then the system of equations $\langle f, f \rangle \equiv -1, \Delta f = 2f$ for maps $f: M \rightarrow V$ has no solutions at all unless $K = 0$ or -1 . When $K = -1$, the only such f are isometric imbeddings and hence are totally geodesic. When $K \equiv 0$, the situation is more complicated with a solution space like that for the $c = 1$ case treated by Theorem 3.1.

Finally, we remark that [Kenmotsu, 1983] has given a proof of Theorem 4.2 in the case $n = 4$.

BIBLIOGRAPHY

E. Calabi, *Minimal immersions of surfaces in Euclidean spheres*, J. Differential Geometry **1** (1967), 111–125.
 K. Kenmotsu, *On minimal immersions of \mathbf{R}^2 into S^n* , J. Math. Soc. Japan **28** (1976), 182–191.
 ———, *Minimal surfaces with constant curvature in 4-dimensional space forms*, Proc. Amer. Math. Soc. **89** (1983), 133–138.
 B. Lawson, *Lectures on minimal submanifolds*, vol. 1, Publish or Perish, Berkeley, Calif., 1980.
 M. Pinl, *Minimalflächen fester Gaußscher Krümmung*, Math. Ann. **136** (1958), 34–40.
 J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. (2) **88** (1968), 62–105.
 N. Wallach, *Extension of locally defined minimal immersions of spheres into spheres*, Arch. Math. **21** (1970), 210–213.