P POINTS WITH COUNTABLY MANY CONSTELLATIONS

BY

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Abstract. If the continuum hypothesis (CH) is true, then for any P point ultrafilter D (on the set of natural numbers) there exist initial segments of the Rudin-Keisler ordering, restricted to (isomorphism classes of) P points which lie above D, of order type $\aleph_1$. In particular, if D is an RK-minimal ultrafilter, then we have (CH) that there exist P-points with countably many constellations.

0. Introduction. Our main result is that in the presence of the continuum hypothesis (henceforth denoted CH), there exist P point ultrafilters on $\omega$ with exactly $\aleph_0$ many constellations. Actually, we prove a somewhat stronger theorem about initial segments of the Rudin-Keisler (RK) ordering on the class of P points; in order to state this result, we begin with a few definitions. All ultrafilters here are nonprincipal ultrafilters on $\omega = \{0, 1, 2, \ldots\}$. An ultrafilter D is a P point iff any function $f: \omega \to \omega$ is either constant or finite-to-one on a set in D. P points have been studied extensively, and we shall assume basic results about them and their RK ordering; good references are [Bl and Pu]. If D is a P point, let $<_{P,D}$ denote the RK ordering on (equivalence classes of) P points which lie above D in RK. An initial segment of $<_{P,D}$ means a downward closed subset, and the initial segment determined by E is $\{F: D < F < E\}$ (we use $<$ to denote the RK ordering).

In his thesis [Ec], Eck showed (CH) that if D is any P point, then there exist P points E immediately above D in RK in the strong sense that any strict RK predecessor of E is a predecessor of D; we call such an E a strong immediate successor (s.i.s) of D. Iterating Eck's theorem $\omega$ times yields the existence (CH), for any P point D, of initial segments of $<_{P,D}$ of order type $\omega$. Our main theorem is the existence (CH) of initial segments of $<_{P,D}$ of order type $\aleph_1$; the bulk of the article is devoted to its proof.

In [B3], Blass proved the result just stated without the restriction to P points; that is, he showed (CH) that for any ultrafilter D, there exist initial segments of "RK above D" of order type $\aleph_1$. The proof involved reformulating the problem in model theoretic terms, and we shall take the same approach. Let N be the complete first order structure on $\omega$ (i.e. the language for N contains names for every finitary function and relation on $\omega$). We use the term model to mean "nonstandard model of Th(N)", and we use *f to indicate the interpretation of the function $f: \omega \to \omega$ in whichever model is under consideration. If D is an ultrafilter, then $D-prod$ denotes

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the ultrapower of \( N \) by \( D \), and if \( f \) is a function from \( \omega \) to \( \omega \), then the corresponding element of the universe of \( D \)-prod (called the germ of \( f \)) is denoted \([f]_D\). In general, a model is isomorphic to an ultrapower iff it is finitely generated, which, due to the existence of pairing functions, is the same as saying that the model is generated by a single element in its universe.

There is an intimate relationship between the structure of an ultrapower \( D \)-prod and the RK ordering below \( D \); roughly, finitely generated submodels of \( D \)-prod correspond to RK predecessors of \( D \): the submodel generated by \([f]_D\) corresponds to the RK predecessor \( f(D) \) of \( D \) (the standard model is isomorphic to \( f(D) \)-prod by the map \( f^*: f(D) \)-prod \( \to \) \( D \)-prod defined by \( f^*(g[l(D)]) = [g \circ f]_D \)). The details for this construction are well known and can be found in \([B2]\). An ultrafilter \( D \) is a \( P \) point iff every (nonstandard) submodel of \( D \)-prod is cofinal in \( D \)-prod, and in general, we refer to models in which all nonstandard submodels are cofinal as “single-skied” (see \([Pu]\) for a discussion of skies). Then a model \( \mathcal{A} \) of \( Th(N) \) which is single-skied and finitely generated is isomorphic to \( D \)-prod for some \( P \) point \( D \). Note that if \( E = D \), then \( E \) is a s.i.s. of \( D \) iff the submodel \( \mathcal{E} \) of \( D \)-prod generated by \([f]_D \) (\( \mathcal{E} = f^*E \)-prod) is strictly maximal in \( D \)-prod (that is, every proper submodel of \( D \)-prod is a submodel of \( \mathcal{E} \)). Also, if \( \mathcal{D} \) is a strictly minimal extension of the model \( \mathcal{E} \) (that is, \( \mathcal{E} \) is strictly maximal in \( \mathcal{D} \)), then \( \mathcal{D} \) must be isomorphic to an ultrapower, since any element of \( \mathcal{D} - \mathcal{E} \) must generate \( \mathcal{D} \). The notation \( f^nX \) means the image of the set \( X \) under the function \( f \).

1. Extensions of countably generated models. The main result mentioned above will involve the construction of a sequence of \( P \) points \( \{D_\alpha: \alpha < \omega_1 \} \) for any \( P \) point \( D \), with \( D_0 = D \), which form the desired initial segment of \( \prec_{p,D} \). At successor stages, we construct a \( P \) point \( D_{\alpha + 1} \) which is a s.i.s. of \( D_\alpha \) with a modified version of Eck’s technique; the modifications are included to make the limit stages go through. Model theoretically, the ultrapower \( D_{\alpha + 1} \)-prod is a strictly minimal, cofinal extension of (the embedded image of) \( D_\alpha \)-prod. The strategy at limit stages requires a few more definitions. Suppose \( \lambda \) is a limit ordinal and \( \{D_\alpha: \alpha < \lambda \} \) is an RK-increasing sequence of ultrafilters. Call an ultrafilter \( E \) a strongly minimal upper bound (s.m.u.b.) for the given sequence if \( D_\alpha < E \) for all \( \alpha < \lambda \) and any strict RK predecessor of \( E \) is a predecessor of \( D_\alpha \) for some \( \alpha < \lambda \). Our construction will insure that \( D_\lambda \) is a \( P \) point and a s.m.u.b. for \( \{D_\alpha: \alpha < \lambda \} \), and it is easy to see then that we will obtain our desired sequence. The actual construction of \( D_\lambda \) for countable limit ordinals \( \lambda \) involves an excursion into model theory, which we now describe. Suppose we have \( P \) points \( D_\alpha \) for \( \alpha < \lambda \) satisfying the description above. Let \( \alpha_1, \alpha_2, \ldots \) be a cofinal \( \omega \)-sequence in \( \lambda \), let \( E_i = D_{\alpha_i} \) and let \( p_i: \omega \to \omega \) such that \( p_i(E_{i+1}) = E_i \). Then \( p_i^* \) embeds \( E_i \)-prod into \( E_{i+1} \)-prod, and so we can form the direct limit \( \mathcal{A} \) of the system \( \{p_i^*: i = 1, 2, \ldots \} \); let \( \mathcal{E} \) be the canonical image of \( E_i \)-prod in \( \mathcal{A} \), so that \( \mathcal{A} \) is the union of the \( \mathcal{E} \). Now \( \mathcal{A} \) is a model of \( Th(N) \) and \( \mathcal{A} \) is single-skied since each of the \( E_i \) are \( P \) points and hence the models \( \mathcal{E} \) are mutually cofinal. Note that \( \mathcal{A} \) is not finitely generated and hence not isomorphic to an ultrapower.

Suppose that \( \mathcal{A} \) admits a strictly minimal, cofinal extension \( \mathcal{B} \). Then \( \mathcal{B} \) must be (isomorphic to) an ultrapower, and \( \mathcal{B} \) must be single-skied since any proper
submodel of $\mathcal{B}$ is a submodel of, and hence cofinal in, $\mathcal{A}$ and $\mathcal{A}$ is cofinal in $\mathcal{B}$. Thus $\mathcal{B}$ is isomorphic to $F$-prod for some $P$ point $F$, and we set $D_{\lambda} = F$. It is easy to check that $D_{\lambda}$ is a s.m.u.b. for $\{D_{\alpha} : \alpha < \lambda\}$.

The discussion above shows that we can succeed at limit stages of our construction if we can find a strictly minimal, cofinal extension $\mathcal{B}$ of the countably generated model $\mathcal{A}$ which arises as a direct limit of previously constructed models. In [B3], Blass proved a characterization of those countably generated models of Th($\mathbb{N}$) which admit strictly minimal extensions. We shall require a number of modifications to that theorem, and what follows, through the proof of Theorem 2, is adapted from [B3].

Let $\mathcal{A}$ be the binary relation on $\omega$ defined by $m \in \langle n \rangle$ iff $2^m$ occurs in the binary expansion of $n$, so $n$ codes the finite set $\{m : m \in \langle n \rangle\}$. If $\mathcal{A}$ is a model and $a \in \mathcal{A}$, then let $a(\mathcal{A}) = \{b \in \mathcal{A} : \mathcal{A} = (b \in \langle a \rangle)\}$. If $a \in \mathcal{A} < \mathcal{B}$, then $a(\mathcal{A}) = a(\mathcal{B}) \cap \mathcal{A}$. Assume that the set of finite sequences from $\omega$ has been coded in some standard way, and let $\langle \ldots \rangle$ denote the coding function. Let $\text{Seq}$ be the set of codes, and for each $x$ in $\text{Seq}$, $lh(x)$ is the length of $x$ and $(x)_k$ is the $k$th component of $x$ if $k < lh(x)$ and $(x)_k = 0$ otherwise. Blass's result is

**Theorem 1 (Blass [B3]) (CH).** Let $\mathcal{A}$ be a countably generated model of Th($\mathbb{N}$). $\mathcal{A}$ admits a strictly minimal extension if and only if for any sequence $\{a_i : i \in \omega\}$ with $a_i(\mathcal{A})$ nonempty and $a_0(\mathcal{A}) \supseteq a_1(\mathcal{A}) \supseteq a_2(\mathcal{A}) \supseteq \cdots$, either

(i) $\bigcap_{i \in \omega} a_i(\mathcal{A}) \neq \emptyset$, or
(ii) for any $b$ in $\mathcal{A}$, there is a $c$ in $a_0(\mathcal{A})$ and an $f : \omega \to \omega$ such that $f(c) = b$.

**Remarks.** The "only if" direction does not use CH. If $\mathcal{A}$ is finitely generated, and hence isomorphic to an ultrapower, then (i) always holds since ultrapowers are $\aleph_1$-saturated [CK, p. 305], and so (CH) ultrapowers always admit strictly minimal extensions.

Our first modification takes care of insuring that the new model is a cofinal extension.

**Theorem 2 (CH).** Let $\mathcal{A}$ be a single-skied, countably generated model of Th($\mathbb{N}$), and suppose $\mathcal{A}$ satisfies the conditions of Theorem 1. Then $\mathcal{A}$ admits a single-skied, strictly minimal extension.

**Proof.** Most of this proof is identical to the proof of Theorem 1 given in [B3]. First, if $\mathcal{A}$ is finitely generated, and hence isomorphic to a $P$ point ultrapower, then this theorem is simply Eck's result that any $P$ point has strong immediate successors which are $P$ points. Assume therefore that $\mathcal{A}$ is not finitely generated, and let $\{a_n : n = 1, 2, \ldots\}$ be a set which generates $\mathcal{A}$; without loss of generality, the sequence $\langle a_n : n = 1, 2, \ldots\rangle$ is not redundant, that is, $a_{n+1}$ is not in the submodel of $\mathcal{A}$ generated by $\langle a_1, a_2, \ldots, a_n \rangle$, and let $g_n = \langle a_1, \ldots, a_n \rangle$. Let $\text{TR}_n : \text{Seq} \to \text{Seq}$ be the map which truncates sequences by removing all but the first $n$ components (and leaves shorter sequences fixed). Then $\text{TR}_n(g_m) = g_n$ for all $m > n$; note also that
$g_m$ is not in the submodel of $\mathcal{A}$ generated by $g_n$ if $n < m$ (by the nonredundancy of the $a_i$'s). Let $\mathcal{G}_n$ be the submodel generated by $g_n$.

Let $G_n$ be the type of $g_n$ in $\mathcal{A}$, that is $G_n = \{ X \subseteq \text{Seq}: \mathcal{A} \models *X(g_n) \}$. Then $G_n$ is an ultrafilter on $\text{Seq}$, in fact a $P$ point (since $\mathcal{A}$, and hence each of its submodels, is single-skied), and $\text{TR}_n(G_m) = G_n$ for $m > n$. $G_n$ concentrates on sequences of length $n$, that is $\{ x \subseteq \text{Seq}: \text{l}(x) = n \} \in G_n$, and the nonredundancy implies that $\text{TR}_n$ is not one-to-one on any set in $G_m$ for $m > n$.

To obtain a strictly minimal, single-skied extension of $\mathcal{A}$, it suffices to construct an ultrafilter $E$ on $\text{Seq}$ such that

1. for all $n \geq 1$, $\text{TR}_n(E) = G_n$,
2. for all $f: \text{Seq} \to \omega$, there is a set $A$ in $E$ such that either $f$ is one-to-one on $A$ or, for some $n, f$ is $\text{TR}_n$-fiberwise constant on $A$ (that is, $f$ is constant on sets of the form $A \cap \text{TR}_n(i)$), and
3. for some set $B$ in $E$, $\text{TR}_1$ is finite-to-one on $B$.

Given such an ultrafilter $E$, we can embed $\mathcal{A}$ into $E$ by mapping $g_n$ to $[\text{TR}_n]E$ (by (1)), and for simplicity we identify $A$ with its embedded image in $E$ by (2). Since every element of $E$ either generates $E$ or is in the submodel $\mathcal{G}_n$ for some $n$, and so $E$ is a strictly minimal extension of $\mathcal{A}$. By (3), the submodel $\mathcal{G}_1$, and hence $\mathcal{A}$, is cofinal in $E$-prod. It follows that $E$ is single-skied since any proper submodel of $E$ is a submodel of (and hence cofinal in) $\mathcal{A}$, and $\mathcal{A}$ is cofinal in $E$-prod.

The existence proof for $E$ is a typical sort of inductive construction for ultrafilters on $\omega$. Call a subset $L$ of $\text{Seq}$ large if $\text{TR}_n''L \in G_n$ for all $n$; otherwise $L$ is small. Any ultrafilter consisting entirely of large sets satisfies (1). Thus it suffices to construct a filter $F$ consisting of large sets, such that $F$ contains a set $B$ satisfying (3) and for each $f: \text{Seq} \to \omega$, $F$ contains a set $A$ satisfying (2). Then let $E$ be any ultrafilter extending $F$ and containing the complements of all small sets. To construct $F$, first order $\text{Seq}_\omega$ in a $\aleph_1$-sequence (by CH) and then inductively define large sets $L_\alpha$ for $\alpha < \aleph_1$ such that $L_0$ satisfies (3), $L_{\alpha+1}$ works as $A$ in (2) for the $\alpha$th function $f$ and $L_\alpha - L_\beta$ is small for $\alpha \geq \beta$. It is easy to check that the finite union of small sets is small, and it follows that $\{ L_\alpha: \alpha < \aleph_1 \}$ generates a filter; let $F$ be this filter.

To construct $L_0$, first find a set $B_n$ in $G_n$ for $n \geq 1$ such that $B_n$ consists of sequences of length (exactly) $n$, $\text{TR}_1$ is finite-to-one on $B_n$ and $\text{TR}_1''B_n \subseteq \{ n, n + 1, n + 2, \ldots \}$. Such $B_n$ exist since the $G_n$'s are $P$ points (and $G_1$ is nonprincipal). Set $L_0 = \bigcup_{n \geq 1} B_n$, and then $\text{TR}_n''L_0$ includes $B_n$ for all $n$, so $L_0$ is large. For any $k \geq 1$, $\text{TR}_1''(k)$ is the union of $k$ finite sets, so $\text{TR}_1$ is finite-to-one on $L_0$.

The construction of $L_\lambda$ for limit $\lambda$ uses only that finite unions of small sets are small. The successor stages use the hypotheses of the theorem (that is, the conditions on $\mathcal{A}$ given in Theorem 1) to construct a large subset $A$ of $L$ which satisfies (2) for any large set $L$ and any $f: \text{Seq} \to \omega$. The details for both the limit and successor cases can be found in [B3, pp. 154–155].

By Theorem 2, we will succeed at limit stages in the construction of our sequence $\{ D_\alpha: \alpha < \aleph_1 \}$ if the associated (countably generated) model satisfies the conditions of Theorem 1. The model $\mathcal{A}$ arising in our construction has a special structure in that
those submodels of $\mathcal{A}$ which include (the embedded copy of) $D_0\text{-prod}$ are linearly (in fact, well) ordered by inclusion. This makes it somewhat easier to satisfy the conditions of Theorem 1, as the next two lemmas show.

**Lemma 3 (CH).** Suppose $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \mathcal{E}_3 \subseteq \cdots$ is an ascending chain of countably generated models of $\text{Th}(\mathbb{N})$ such that

(a) each $\mathcal{E}_n$ admits a strictly minimal extension,
(b) for any $b \in (\mathcal{E}_{n+1} - \mathcal{E}_n)$, the submodel generated by $b$ includes $\mathcal{E}_n$, and
(c) $\mathcal{A} = \bigcup_{n \geq 1} \mathcal{E}_n$ does not admit a strictly minimal extension.

Then there is an element $a \in \mathcal{A}$ and some $j \in \omega$ such that $a(\mathcal{A}) \subseteq \mathcal{E}_j$ but $a \notin \mathcal{E}_j$.

**Proof.** First note that (b) says that for any $b$ in $\mathcal{E}_{n+1} - \mathcal{E}_n$ and any $c$ in $\mathcal{E}_n$, there is an $f: \omega \to \omega$ with $f(b) = c$. By Theorem 1, there is a sequence $\{a_i\}$ such that $a_i(\mathcal{A}) \supseteq a_{i+1}(\mathcal{A})$, $a_i(\mathcal{A}) \neq \emptyset$, $\bigcap_{i \geq 0} a_i(\mathcal{A}) = \emptyset$ and $a_0(\mathcal{A})$ does not generate $\mathcal{A}$ by standard unary functions. Then, for some $j$, $a_0(\mathcal{A}) \subseteq \mathcal{E}_j$, since otherwise, for arbitrarily large $k$, $a_0(\mathcal{A}) \cap (\mathcal{E}_{k+1} - \mathcal{E}_k)$ is nonempty, and then it follows from (b) that every element of $\mathcal{A}$ is obtainable from an element of $a_0(\mathcal{A})$ by a standard unary function, thus contradicting the choice of $\{a_i\}$. We have then that, for all $i$, $a_i(\mathcal{A}) \subseteq \mathcal{E}_j$, and if $a_i$ is an element of $\mathcal{E}_j$ then $a_i(\mathcal{A}) = a_i(\mathcal{E}_j) = a_i(\mathcal{A})$.

We now claim that for some $i$, $a_i$ is not an element of $\mathcal{E}_j$ (and then the proof is complete by setting $a = a_i$). Suppose not; then for all $i$, $a_i$ is in $\mathcal{E}_j$ and so $a_i$ is also in $\mathcal{E}_{j+1}$. Since $\mathcal{E}_{j+1}$ admits a strictly minimal extension, and since $a_i(\mathcal{E}_{j+1}) = a_i(\mathcal{A}) \supseteq a_{i+1}(\mathcal{E}_{j+1}) = a_{i+1}(\mathcal{A})$, we have by Theorem 1 that either $a_0(\mathcal{E}_{j+1})$ generates $\mathcal{E}_{j+1}$ or $\bigcap_{i \geq 0} a_i(\mathcal{E}_{j+1}) = \emptyset$. The latter conclusion is impossible by the choice of $\{a_i\}$ and the fact that $a_i(\mathcal{A}) \subseteq \mathcal{E}_{j+1}$, and the former conclusion says that a subset of $\mathcal{E}_j$ generates $\mathcal{E}_{j+1}$, which is impossible since $\mathcal{E}_j$ is a proper submodel of $\mathcal{E}_{j+1}$.

Let $\mathcal{M}$ be a finitely generated model. We say that $\mathcal{M}$ is element generated iff for every generator $a$ of $\mathcal{M}$, there is a generator $b$ of $\mathcal{M}$ with $b \in a(\mathcal{M})$.

**Lemma 4 (CH).** Suppose that $\{\mathcal{E}_i: i = 1, 2, \ldots\}$ form an ascending chain of countably generated models, $\mathcal{A} = \bigcup_{i \geq 1} \mathcal{E}_i$, and that all the hypotheses of Lemma 3 are satisfied. Then there is a finitely generated submodel $\mathcal{M}$ of $\mathcal{A}$ which includes $\mathcal{E}_1$ and is not element generated.

**Proof.** Let $a$ and $\mathcal{E}_j$ satisfy the conclusion of Lemma 3, and let $\mathcal{M}$ be the submodel of $\mathcal{A}$ generated by $a$. Then $\mathcal{E}_1 \subseteq \mathcal{E}_j \subseteq \mathcal{M}$ (the second inclusion follows from hypothesis (b) of Lemma 3), and $\mathcal{M}$ is not element generated since $a(\mathcal{M}) \subseteq \mathcal{E}_j$ and so $a(\mathcal{M})$ does not contain a generator of $\mathcal{M}$.

**2. Simple combinatorics.** Let $\langle A_1, A_2, \ldots, A_n \rangle$ be a sequence of nonempty sets. A complete set of distinct representatives (CDR) is the image of a one-to-one choice function on the set $\{A_1, \ldots, A_n\}$, that is, a sequence $\langle a_1, \ldots, a_n \rangle$ with $a_i \in A_i$ and $a_i \neq a_j$ if $i \neq j$. We will need the following simple combinatorial lemma in order to construct ultrapowers which are element generated.

**Lemma 5.** Let $\langle A_0, \ldots, A_{n-1} \rangle$ be a sequence of distinct, nonempty sets. Then there is a subsequence of length at least $\log_2(n)$ which admits a CDR.
Proof. The lemma is obvious by inspection for \( n < 5 \), so assume \( n \geq 5 \). Let \( K = |\bigcup_{0 \leq i < n} A_i| \). We prove the lemma by induction on \( K \). Since \( n > 4 \), we have \( K > 1 \), so assume the lemma for smaller \( K \), and fix an arbitrary \( x \in \bigcup A_i \). Reorder the \( A_i \) so that \( x \in A_i \) for \( i < p \) and \( x \notin A_i \) for \( i \geq p \), where \( p \) is the number of sets among the \( A_i \) which contain \( x \). Then \( 0 < p \leq n \).

Case 1: \( 0 < p < (n/2) \). In this case, \( \langle A_p, A_{p+1}, \ldots, A_{n-1} \rangle \) satisfies the induction hypothesis, so there is a subsequence \( \langle B_0, \ldots, B_{m-1} \rangle \) of \( \langle A_p, \ldots, A_{n-1} \rangle \) which admits a CDR \( \langle b_0, \ldots, b_{m-1} \rangle \), and \( m \geq \log_2(n-p) \geq \log_2(n/2) = \log_2(n) - 1 \). Then \( \langle x, b_0, \ldots, b_{m-1} \rangle \) is a CDR for \( \langle A_0, B_0, \ldots, B_{m-1} \rangle \), which has length \( m + 1 \geq \log_2(n) \).

Case 2: \( (n/2) \leq p \leq n \). For \( i < p \), let \( B_i = (A_i - \{x\}) \). By the induction hypothesis, there is a CDR \( \langle d_0, \ldots, d_{m-1} \rangle \) for some subsequence of the \( B_i \)'s with \( m \geq \log_2(p) \geq \log_2(n) - 1 \); without loss of generality, assume the \( A_i \) were ordered so that the subsequence of the \( B_i \)'s admitting this CDR is \( \langle B_0, \ldots, B_{m-1} \rangle \). If \( m \geq \log_2(n) \), we are done; otherwise, since \( n > 4 \) we have \( p \geq n/2 \geq \log_2(n) > m \). Thus \( \langle d_0, \ldots, d_{m-1}, x \rangle \) is a CDR for \( \langle A_0, \ldots, A_{m-1}, A_p \rangle \).

3. The successor case. The point of the theorem in this section is to insure the existence of a \( P \) point \( D_{a+1} \) which is a s.i.s. of \( D_a \) and such that \( D_{a+1} \)-prod is element generated. If \( E \) is an ultrafilter, then the generators of \( E \)-prod are the germs \([f]_E\) of one-to-one (mod \( E \)) functions \( f \), and so it follows that \( E \)-prod is element generated iff for any such one-to-one germ \( a = [f]_E \), there is a one-to-one (mod \( E \)) function \( g \) with \([g]_E \in a(E\text{-prod})\), which means there is a set \( A \) in \( E \) such that for all \( x \) in \( A \), \( g(x) \equiv f(x) \mod E \).

The construction of \( D_{a+1} \) will be done within the framework of Theorem 2.2 of [Ro]. For the convenience of the reader we state this theorem below after supplying the requisite definitions. If \( X \) is a set and \( p \) is a function which is finite-to-one on \( X \), then the \emph{cardinality function} of \( X \) with respect to \( p \), denoted \( C_{X,p} \), is defined by \( C_{X,p}(n) = |X \cap p^{-1}(n)| \). We will omit reference to \( p \) when there is no ambiguity. A \emph{(Dedekind) cut} in an ultrapower \( D\text{-prod} \) is a partition of \( D\text{-prod} \) into convex sets \( S \) and \( L \) such that every element of \( S \) precedes every element of \( L \). A cut is \emph{fair} if \( S \) and \( L \) are nonempty and \( L \) has no countable coinitial subset. If \( E \) is a \( P \) point and \( \omega \to \omega \) with \( p(E) = D \), then the cut in \( D\text{-prod} \) \emph{associated to} \( p \) and \( E \) is defined by putting into \( L \) those \( D \)-germs \( [C_{X,p}]_D \) for \( X \in E \) (and all larger \( D \)-germs), and setting \( S = D\text{-prod} - L \) (see [B2] for a thorough discussion of Dedekind cuts in ultrapowers). Finally, a \emph{condition} on \( X \) is simply a statement about \( X \).

Theorem 6 [Ro] (CH). Let \( D \) be a \( P \) point, \( \langle S, L \rangle \) a fair cut in \( D\text{-prod} \) such that \( S \) is closed under addition in \( D\text{-prod} \), \( p \) the first projection from \( \omega^2 \) to \( \omega \) and \( \{C_i; i \in I\} \) a set of at most \( 2^{\aleph_0} \) conditions on subsets of \( \omega^2 \). Call a set \( X \subseteq \omega^2 \) \emph{large} if it contains a subset \( Y \) on which \( p \) is finite-to-one and \( [C_Y]_D \) is in \( L \), and suppose that for any large \( X \) and condition \( C_i \), there is a large subset \( Y \) of \( X \) which satisfies \( C_i \).

Then there exist \( 2^{\aleph_0} \) many (pairwise nonisomorphic) \( P \) points \( E \) on \( \omega^2 \) with \( p(E) = D \) and associated cut \( \langle S, L \rangle \), such that for all \( i \) in \( I \), \( E \) contains a set satisfying condition \( C_i \).
Proof. This is a special case of Theorem 2.2 in [Ro] obtained by taking the fiber measure of that theorem to be the cardinality function (see also [B2]).

Theorem 7 (CH). Let D be a P point, p the first projection from $\omega^2$ to $\omega$ and $[h]_D$ any element of D-prod. There exist $2^{\aleph_1}$ P points $E$ with $p(E) = D$ such that

1) $E$ is a s.i.s. of $D$,
2) $E$-prod is element generated, and
3) $[h]_D$ is in the small part $S$ of the cut in $D$-prod associated to $p$ and $E$.

Proof. Begin by defining functions $s_j: \omega \to \omega$ as follows;

$$
\begin{align*}
    &s_0(0) = h(0), \\
    &s_{i+1}(k) = 2^{s_i(k)}, \\
    &s_k(k + 1) = \max \left\{ h(k + 1), \sum_{j < k} s_j(j) \right\}.
\end{align*}
$$

Define a cut $\langle S, L \rangle$ in $D$-prod by $a \in S$ iff $a \leq \lfloor s_j \rfloor_D$ for some $j$. Then $[h]_D \in S$ and $\langle S, L \rangle$ is a fair cut since the existence of a countable cofinal set in $S$ implies that there is no countable cofinal set in $L$ (by the $\mathfrak{S}_1$-saturation of $D$-prod). It is easy to check that $S$ is closed under addition, multiplication and exponentiation. For each $f: \omega^2 \to \omega$, let $C_f$ be the following condition on $Y$: ($f$ is $p$-fiberwise constant on $Y$) or ($f$ is one-to-one on $Y$ and there is a $p$-fiberwise one-to-one function $g$ on $Y$ such that for all $y \in Y$, $g(y) \in f(y)$).

If we show that, for any $f$, any large set $X$ includes a large subset $Y$ satisfying $C_f$, then by Theorem 6 we will have shown the existence of $2^{\aleph_1}$ many $P$ points $E$ with $p(E) = D$, associated cut $\langle S, L \rangle$, such that $E$ contains sets satisfying $C_f$ for all $f$. Thus every $f$ is either $p$-fiberwise constant or (globally) one-to-one on a set in $E$; so every $[f]_E$ in $E$-prod either is in the embedded image (by $p^*$) of $D$-prod or is a generator of $E$-prod. It follows that $E$ is a s.i.s. of $D$. To see that $E$-prod is element generated, let $a$ be a generator of $E$-prod; so $a = [f]_E$ for some function $f$ which is one-to-one on a set in $E$. Let $Y \subseteq E$ satisfy $C_f$, then $f$ cannot be $p$-fiberwise constant on $Y$ (since $[f]_E$ is a generator of $E$-prod), so $Y$ satisfies the second part of $C_f$. Then there is a $p$-fiberwise one-to-one function $g$ on $X$ such that for all $y \in Y$, $g(y) \in f(y)$, and so $[g]_E$ is in $a(E$-prod). Now $[g]_E$ is not in $p^{**}D$-prod since if it were, then $g$ would be $p$-fiberwise constant as well as $p$-fiberwise one-to-one on a set in $E$, which would imply that $p$ is one-to-one (mod $E$), contradicting the fairness of the associated cut. By the strict maximality of $p^{**}D$-prod, it follows then that $[g]_E$ is a generator of $E$-prod. Thus $E$-prod is element generated.

It remains to show that we can find a large subset satisfying any given condition $C_f$ for any large $X$. We can assume (by cutting down $X$ if necessary) that $p$ is finite-to-one on $X$. For each nonempty fiber $X_n = (X \cap p^{-1}\{n\})$, we can find a set $Z_n \subseteq X_n$ such that $f$ is constant on $Z_n$ or $f$ is one-to-one on $Z_n$ and $|Z_n|^2 \geq |X_n|$. Partition the fibers into sets $W_c$ and $W_1$, where $W_c$ consists of those $n$ such that $f$ is constant on $Z_n$ and $W_1$ consists of those $n$ such that $f$ is one-to-one on $Z_n$. One of these sets is in $D$ (else $X$ was not large). If $W_c$ is in $D$, then let $Y = \bigcup_{n \in W_c} Z_n$ and
clearly is fiberwise constant on $Y$; for all $n$ in $W$, $(C_Y(n))^2 \geq C_X(n)$, and so $[C_Y]_D$ is in $L$ since $[C_X]_D$ is in $L$ and $S$ is closed under multiplication.

If, on the other hand, $W$ is in $D$, then we must cut down $Z = \bigcup_{n \in W} Z_n$ to a large $Y$ on which $f$ is one-to-one and for which we can find the desired function $g$. First define sets $B_i$ in $D$ ($i = 1, 2, 3, \ldots$) by

$$B_i = \{ m \in p''Z : |Z_m| \geq s_{i+1}(m) \} - \{0, 1, \ldots, m\}.$$  

Then $B_i$ is in $D$ since $Z$ is large (by the argument given above for $Y$) and hence, for all $j$, $C_Y(m)$ cannot be less than $s_j(m)$ for $D$-many $m$. Note also that $B_i \subseteq B_i$, and we may assume that $p''Z \subseteq B_0$ (throw away some fibers of $Z$ if necessary). Define $t(m) = \max i$ such that $m \in B_i$ for $m$ in $B_0$. Then, for all $m \in B_0$, $t(m) < m$, $m \in B_t(m)$, and $|Z_m| \geq s_t(m)(m)$. Let $M$ be the minimal element of $B_0$, and let $Q_M$ be any subset of $Z_M$ with $|Q_M| = s_t(M)(M)$. Note that $f$ is one-to-one on $Q_M$.

Let $m \in B_0$, let $H_m = \{ j \in B_0 : j < m \}$, and suppose, as an induction hypothesis, that for all $j$ in $H_m$, we have defined $Q_j \subseteq Z_j$ such that $|Q_j| = s_t(j)(j)$ and $f$ is one-to-one on $(\bigcup_{j \in H_m} Q_j)$. Let $R_m = \{ x \in Z_m : f(x) = f(y)$ for some $y$ in $(\bigcup_{j \in H_m} Q_j) \}$. Since $f$ is one-to-one on $Z_m$, we have

$$|R_m| \leq \sum_{j \in H_m} |Q_j| = \sum_{j \in H_m} s_t(j)(j) \leq \sum_{j < m} s_j(j) \leq s_0(m),$$

and thus

$$|Z_m - R_m| \geq s_{t(m)+1}(m) - s_0(m) = 2^{s_t(m)(m)} - s_0(m) \geq s_t(m)(m).$$

Let $Q_m$ be any subset of $Z_m - R_m$ of size $s_t(m)(m)$; this completes the induction construction of the $Q_m$ for $m$ in $B_0$, for it is clear by the definition of $R_m$ that $f$ is one-to-one on $(\bigcup_{j \in H_m} Q_j) \cup Q_m$. Let $Q = \bigcup_{j \in B_0} Q_j$, then $f$ is one-to-one on $Q$ and for any $k$ in $\omega$, we have that for all $m$ in $B_k$, $|Q_m| = s_{t(m)}(m) \geq s_k(m)$, and so $[C_0]_D \geq [s_k]_D$. Thus $Q$ is large. The argument just given to make $f$ one-to-one on a large set is due to Eck [Ec].

If $f$ assumes the value 0 on $Q$, then remove that point from $Q$. Temporarily fix $m \in B_0 = p''Q$, and write $Q_m (= p^{-1}(m) \cap Q)$ as $\{a_1, a_2, \ldots, a_n\}$. Let $A_i = \{ t : t \in 'f(a_i) \}$; since $f$ is one-to-one on $Q$, $\{ A_i : 1 \leq i \leq n \}$ is a collection of distinct nonempty sets, and so by Lemma 5, there is a subsequence $\langle B_1, \ldots, B_J \rangle$ of $\langle A_1, \ldots, A_n \rangle$ which admits a CDR $\langle u_1, \ldots, u_J \rangle$ with $J \geq \log_2(n)$. Reorder the $\{a_i\}$ so that $\langle u_1, \ldots, u_J \rangle$ is a CDR for $\langle A_1, \ldots, A_J \rangle$, and let $Y_m = \{ a_1, \ldots, a_J \}$. Define $g$ on $Y_m$ by $g(a_i) = u_i$, so $g$ is one-to-one on $Y_m$.

Let $Y = \bigcup_{m \in B_0} Y_m$; clearly $Y$ satisfies condition $C_f$ and so we only need show that $Y$ is large. For all $m$ in $B_0$, $|Y_m| \geq \log_2(|Q_m|)$, whereupon the largeness of $Y$ follows from the largeness of $Q$ and the closure of $S$ under exponentiation.

4. The limit case. In order to apply Lemma 4 at limit stage $\lambda$ in the construction of our sequence of $P$ points, we need to know that (beyond some point in the sequence) each finitely generated submodel of the associated countably generated model $\mathcal{A}$ is element generated. The previous theorem insures this for finitely generated models
arising at successor stages in the construction; the following theorem gives us this result for those arising at limit stages previous to λ.

**Theorem 8 (CH).** Let \( \{ E_i : i = 1, 2, \ldots \} \) be an RK-increasing sequence of \( P \) points with \( p_i(E_{i+1}) = E_i \), and let \( A \) be the direct limit of \( \langle E_i \text{-prod}, p_i^* \rangle \). For \( i \geq 2 \), let \( q_i = p_1 \circ p_2 \circ \cdots \circ p_{i-1} \) and let \( \langle S', L' \rangle \) be the cut in \( E_1 \text{-prod} \) associated to \( q_i \) and \( E_i \). Assume that:

(a) \( A \) admits a strictly minimal extension,
(b) \( S' \subseteq S'^{i+1} \) for all \( i \geq 2 \), and
(c) \( E_i \text{-prod} \) is element generated for all \( i \geq 1 \).

Then \( A \) admits a single-skied, element generated, strictly minimal extension.

**Proof.** The proof is a modification of the proof of Theorem 2. Let \( \mathcal{A}_i \) be the canonical image of \( E_i \text{-prod} \) in \( \mathcal{A} \), and let \( a_i \) be a generator of \( \mathcal{A}_i \) in \( \mathcal{A} \). Let \( g_i \) and \( G_i \) be as in the proof of Theorem 2 (i.e. \( g_i = \langle a_1, \ldots, a_i \rangle \) and the ultrafilter \( G_i \) is the type of \( g_i \) in \( \mathcal{A} \)). Since the ultrafilters \( E_i \) form an increasing RK-sequence, it follows that \( g_i \) generates \( \mathcal{A}_i \) and hence \( G_i = E_i \); thus \( G_i \text{-prod} \) is element generated and if \( \langle S_i, L_i \rangle \) is the cut in \( G_i \text{-prod} \) associated to \( \mathcal{T}_1 \) and \( G_i \), then \( S_i \subseteq S_{i+1} \). Since \( A \) admits a strictly minimal extension, the hypotheses of Theorem 1 are satisfied.

For each \( f: \text{Seq} \to \omega \), let \( \mathcal{C}_f \) be the following condition on a subset \( A \subseteq \text{Seq} \): \( f \) is \( \mathcal{T}_n \)-fiberwise constant on \( X \) for some \( n \) or \( f \) is one-to-one on \( X \) and there is a \( \mathcal{T}_i \)-fiberwise one-to-one function \( g \) on \( X \) such that \( g(x) \in f(x) \) for all \( x \in X \).

As in Theorem 2, we will construct a filter \( F \) on \( \text{Seq} \) consisting of large (in the sense of Theorem 2) sets such that \( F \) contains a set on which \( \mathcal{T}_1 \) is finite-to-one as well as sets satisfying \( \mathcal{C}_f \) for each \( f \); let \( E \) be an ultrafilter including \( F \) and the complements of all large sets. Let \( \mathcal{B} = E \text{-prod} \), and then \( \mathcal{B} \) is a single-skied, strictly minimal extension of the (embedded image of) \( \mathcal{A} \); in addition, \( \mathcal{B} \) is element generated, since the function \( g \) of condition \( \mathcal{C}_f \) is \( \mathcal{T}_1 \) (and hence \( \mathcal{T}_n \) for all \( n \)) fiberwise one-to-one (mod \( E \)), and thus is not \( \mathcal{T}_n \)-fiberwise constant (mod \( E \)) for any \( n \), insuring (by strict minimality) that \( \{g\}_E \) is a generator of \( E \text{-prod} \).

List the conditions in an \( \aleph_1 \)-sequence. The proof proceeds exactly as for Theorem 2 (the definition of \( L_0 \) and the construction of \( L_\lambda \) for limit \( \lambda \) are identical), except that we must satisfy the stronger condition of the present theorem at successor stages. So assume that \( L \) is large and let \( C_f \) be the \( \lambda \)th condition. Since the hypotheses of Theorem 1 are satisfied by \( \mathcal{A} \), we can find a large \( A \subseteq L_\alpha \) such that \( f \) is one-to-one on \( A \), or, for some \( n \), \( f \) is \( \mathcal{T}_n \)-fiberwise constant on \( A \). If the latter, then let \( L_{\alpha+1} = A \). Assume therefore that \( f \) is one-to-one on \( A \); we must cut down \( A \) further so that we can define the desired function \( g \).

For \( n \geq 1 \), let \( R_n = \mathcal{T}_n^{-1} \cap \{ x \in \text{Seq}: \text{lh}(x) = n \} \). Since \( A \) is large, \( R_n \) is in \( G_n \). Let \( B_1 = R_1 \), and for each \( x \) in \( B_1 \), let \( s(x, 1) \in A \) such that \( \mathcal{T}_1(s(x, n)) = x \). Assume we have defined \( B_j \) for \( j < K \) such that \( B_j \subseteq R_j \), \( B_j \subseteq G_j \), with \( \mathcal{T}_j^{-1}(B_{j+1}) \subseteq B_j \), and we have defined a function \( s \) such that \( s(x, j) \in A \) and \( \mathcal{T}_j(s(x, j)) = x \) for each \( x \) in \( B_j \). Let

\[
B_K = \left( R_K \cap \mathcal{T}^{-1}_{K-1}(B_{K-1}) \right) - W,
\]
where $W = \{z \in R_K: z = \text{TR}_K(s(x, K - 1)) \text{ for some } x \in B_{K-1}\}$. It is easy to see that $\text{TR}_{K-1}$ is one-to-one on $W$ (two different $z$'s which map to $x$ under $\text{TR}_{K-1}$ can not both be truncations of $s(x, K - 1)$), and so $W$ is not in the ultrafilter $G_K$ (since $\text{TR}_{K-1}$ maps $G_K$ to $G_{K-1}$ and is not an isomorphism). It follows that $B_K$ is in $G_K$, since both $R_K$ and $\text{TR}_{K-1}^{-1}(B_{K-1})$ are in $G_K$. For each $x$ in $B_K$, let $s(x, K)$ be an element of $A$ such that $\text{TR}_K(s(x, K)) = x$. This completes the inductive construction of the $B_K$.

Let $P = \{s(x, n): n \geq 1 \text{ and } x \in B_n\}$. Then $P \subseteq A$, $P$ is large, and by the construction we have that for all $s$ in $P$, there is a unique $n$ and a unique $x$ in $B_n$ with $s = s(x, n)$. Define functions $f_n$ on $B_n$ by $f_n(x) = f(s(x, n))$. Then $f_n$ is one-to-one on $B_n$ since $f$ is one-to-one on $A$. Since $G_n$-prod is element generated, we can find, for each $n$, a set $Z_n$ in $G_n$, $Z_n \subseteq B_n$, and a one-to-one function $g_n$ on $Z_n$ such that for all $x$ in $Z_n$, $g_n(x) \in f_n(x)$.

Define $g: P \to \omega$ by $g(s(x, n)) = g_n(x)$. Then $g$ is well defined, and for all $s$ in $P$, $g(x) \in f(x)$. We need only cut down $P$ to a large $L_{\alpha+1}$ on which $g$ is $\text{TR}_1$-fiberwise one-to-one.

Since $S_i \subseteq S_{i+1}$, we can find functions $h_i: \text{Seq}_i \to \omega$ such that $[h_i] \in L_i - L_{i+1}$ (Seq$_i$ is the set of codes for sequences of length $i$). Let $P_1 = Z_1$, and assume that we have defined sets $P_i \subseteq Z_i$ for $1 \leq i < K$ such that

1. $P_i \subseteq G_i$, and $\text{TR}_{j-1}(P_i) \subseteq P_{j-1}$,
2. for all $i \geq 2$, $[\text{TR}_i^{-1}(p) \cap P_i] \subseteq h_i(p)$ for all $p$ in $P_i$, and
3. $(\forall (p) \in P_i)(\forall i, j < K)(\forall y, z \in (\text{TR}_i^{-1}(p) \cap (\bigcup_{n<K}P_n))) (g_i(y) = g_j(z) \iff i = j \text{ and } y = z).

To define $P_K$, first let $V = \{z \in Z_K \cap \text{TR}_K^{-1}(P_{K-1}): (\exists j < K)(\exists y \in P_j)(g_K(z) = g_j(y) \text{ and } \text{TR}_1(y) = \text{TR}_1(z))\}.$

If $p \in P_i$, then $|\text{TR}_i^{-1}(p) \cap V| \leq 1 + \sum_{j<K} |P_j \cap \text{TR}_i^{-1}(p)| \leq 1 + \sum_{j<K} h_j((p))$ (the first inequality follows since $g_K$ is one-to-one on $Z_K$; the second follows from (2)). Thus $[C_V]_G \leq 1 + \sum_{j<K} [h_j]_G$, and since $S_K$ is closed under addition (Theorem 2 of [B2]), it follows that $[C_V]_G \in S_K$ (as $[h_j]_G \in S_K$ for all $j < K$). Thus $V$ is not in $G_K$. Since $[h_K]_G \in L_K$, we can find $W$ in $G_K$ such that for all $\langle p \rangle$ in $P_1$, $C_W(\langle p \rangle) \leq h_K(\langle p \rangle)$. Let

$$P_K = (Z_K \cap W \cap \text{TR}_K^{-1}(P_{K-1})) - V.$$ Then $P_K$ obviously satisfies (1) and (2), and it is easy to check, from the definition of $V$, that $P_K$ satisfies (3).

We have inductively defined $\{P_j: j = 1, 2, 3, \ldots\}$, and we set $L = s''((\bigcup(P_n \times \{n\}))$. Then $\text{TR}_1''L$ includes $P_n$, and $P_n$ in $G_n$, so $L$ is large. If $s$ and $t$ are in $L$ with $g(s) = g(t)$ and $\text{TR}_1(s) = \text{TR}_1(t)$, then there are unique $x, y, m, n$ with $x \in P_m$, $y \in P_n$, $s = s(x, m)$ and $t = s(y, n)$ (by the construction of the set $P$). By the definition of $g$, $g_m(x) = g_n(y)$, and by (3) above, $m = n$ and $x = y$, and so $s = t$. Thus $g$ is $\text{TR}_1$-fiberwise one-to-one on $L$, and the proof is complete by setting $L_{\alpha+1} = L$. 
5. The finale. We now combine the previous results to produce our desired sequence of \( P \) points.

**Theorem 9 (CH).** Let \( D \) be a \( P \) point. There exist initial segments of \( \preceq_{P,D} \) of order type \( \mathbb{N}_1 \).

**Proof.** We construct a sequence of \( P \) points \( \{ D_\alpha : \alpha < \mathbb{N}_1 \} \) with \( D_0 = D \) such that:

1. \( D_{\alpha+1} \) is a strong immediate successor of \( D_\alpha \),
2. for limit \( \lambda < \mathbb{N}_1 \), \( D_\lambda \) is a strongly minimal upper bound for \( \{ D_\alpha : \alpha < \lambda \} \),
3. \( D_\alpha \)-prod is element generated for \( \alpha \geq 1 \), and
4. if \( q_\alpha \) is the map from \( D_\alpha \) to \( D_1 \), and \( \langle S_\alpha, L_\alpha \rangle \) is the cut in \( D_1 \)-prod associated to \( q_\alpha \) and \( D_\alpha \), then \( S_\beta \subseteq S_\alpha \) whenever \( \beta < \alpha \).

By (1) and (2), such a sequence forms an initial segment of \( \preceq_{P,D} \).

If \( \alpha = 1 \), let \( D_1 \) be a s.i.s. of \( D \) such that \( D_1 \)-prod is element generated; we can find such a \( D_1 \) by Theorem 7.

If \( \alpha = \beta + 1 \), fix \( g : \omega \to \omega \) such that \([g]_{D_\beta}\) is in the large part \( L_\beta \) of the cut in \( D_1 \)-prod, \( D_\beta \) is a s.i.s. of \( D_\beta \) and such that \( q_\beta ([g]_{D_\beta}) \) is in the small part \( S' \) of the cut in \( D_1 \)-prod associated to \( p_\beta \). We can find such a \( D_\beta \) by Theorem 7.

If \( \lambda \) is a countable limit ordinal, let \( \alpha, \alpha_2, \ldots \) be an increasing \( \omega \)-sequence cofinal in \( \lambda \), let \( E_i = D_\alpha \), and let \( p_i \) be the map from \( E_{i+1} \) to \( E_i \), let \( \mathcal{M} \) be the direct limit of \( \{ (E_r \text{-prod}, p_r^*) \} \), and let \( \mathcal{E}_i \) be the canonical image of \( E_r \text{-prod} \) in \( \mathcal{M} \). We claim first that \( \mathcal{M} \) admits a strictly minimal extension.

To prove this claim, first note that each \( \mathcal{E}_i \) admits a strictly minimal extension (since each \( E_i \) is one of the previously constructed \( D_\alpha \)'s) and that any element of \( \mathcal{E}_{i+1} - \mathcal{E}_i \) generates a submodel which includes \( \mathcal{E}_i \) (since the \( D_\alpha \)'s already constructed form an initial segment of \( \preceq_{P,D} \)). If \( \mathcal{M} \) did not admit a strictly minimal extension, then by Lemma 4 there would be a finitely generated submodel \( \mathcal{M} \) of \( \mathcal{M} \) which includes \( \mathcal{E}_1 \) but is not element generated. But \( \mathcal{M} \) is the embedded image of one of the \( D_\alpha \)-prod (again since they form an initial segment) and so \( \mathcal{M} \) must be element generated by the induction hypothesis. This contradiction proves the claim.

By the induction hypothesis, each of the \( \mathcal{E}_i \) is element generated and if \( i < j \), then \( S_{\alpha_i} \subseteq S_{\alpha_j} \). By Theorem 8, \( \mathcal{M} \) admits a strictly minimal extension \( \mathcal{M} \) which is single-skied and element generated. Let \( D_\lambda \) be an ultrafilter such that \( D_\lambda \)-prod is isomorphic to \( \beta \); then \( D_\lambda \) is a \( P \) point and is a s.m.u.b. for \( \{ D_\alpha : \alpha < \lambda \} \). Finally, if \( \alpha < \lambda \), it is easy to check that \( S_{\alpha+1} \subseteq S_\lambda \) (since the map \( q_\lambda \) factors through \( q_{\alpha+1} \)) and so \( S_\alpha \subseteq S_\lambda \), showing that \( D_\lambda \) satisfies (4).

This completes the inductive construction of \( \{ D_\alpha : \alpha < \mathbb{N}_1 \} \), and with it, the proof of the theorem.
Corollary 10 (CH). There exist initial segments of the RK ordering, restricted to (isomorphism classes of) P points, of order type $\aleph_1$.

Proof. This is simply Theorem 9 with $D$ an RK-minimal ultrafilter. The existence of such $D$ follows from CH (see [Pu], for example); ultrafilters which are minimal in RK are also called selective or Ramsey, and they are all P points.

Corollary 11 (CH). There exist P points with countably many constellations; in fact, for any countable ordinal $\alpha$, there exist P points $E$ such that the initial segment of RK determined by $E$ has order type $\alpha$.

Proof. Immediate by the previous corollary.

Corollary 12 (CH). For any P point $D$, there exist initial segments $T$ of $<_D$ such that $T$ is a tree with $\aleph_1$ levels, each node has $2^{\aleph_1}$ immediate successors, and each countable increasing sequence in $T$ has a unique upper bound in $T$.

Proof. The proof is the same as the proof of Theorem 9, except at successor stages, use Theorem 7 to generate $2^{\aleph_1}$ immediate successors.

6. Two questions. We conclude with two natural questions.

1. Given CH, is there an RK-increasing $\omega$-sequence of P points which does not admit a s.m.u.b. which is a P point? In other words, do we really need element generated models to prove the theorems here?

2. Can our result extend beyond $\aleph_1$? That is, do there exist (CH) initial segments of $<_D$ of order type $\aleph_1 + 1$ or perhaps $\aleph_2$? In general, the successor cardinal of the continuum would be the best possible.

References


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