REGULAR LINEAR ALGEBRAIC MONOIDS

BY

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ABSTRACT. In this paper we study connected regular linear algebraic monoids. If \( \phi: G_0 \to GL(n, K) \) is a representation of a reductive group \( G_0 \), then the Zariski closure of \( K\phi(G_0) \) in \( \mathcal{M}_n(K) \) is a connected regular linear algebraic monoid with zero. In \( \S 2 \) we study abstract semigroup theoretic properties of a connected regular linear algebraic monoid with zero. We show that the principal right ideals form a relatively complemented lattice, that the idempotents satisfy a certain connectedness condition, and that these monoids are \( V \)-regular. In \( \S 3 \) we show that when the ideals are linearly ordered, the group of units is nearly simple of type \( A_1, B_1, C_1, F_4 \) or \( G_2 \). In \( \S 4 \), conjugacy classes are studied by first reducing the problem to nilpotent elements. It is shown that the number of conjugacy classes of minimal nilpotent elements is always finite.

1. Preliminaries. Throughout this paper \( Z^+ \) will denote the set of all positive integers. If \( X \) is a set, then \( |X| \) denotes the cardinality of \( X \). Let \( S \) be a semigroup. We let \( E(S) \) denote the set of all idempotents of \( S \). If \( a, b \in S \), then \( a \mid b \) (\( a \) divides \( b \)) if \( xay = b \) for some \( x, y \in S^1 \), \( J \subseteq b \) if \( a \mid b \mid a \), \( R \subseteq b \) if \( aS^1 = bS^1 \), \( L \subseteq b \) if \( S^1 a = S^1 b \), \( \mathcal{E} = R \cap L \), \( D = R \cdot L = L \cdot R \) (see [1, 7]). The semigroups we consider turn out to have the property that a power of each element lies in a subgroup and hence by [10], \( \mathcal{J} = D \). A \( \mathcal{J} \)-class with an idempotent is called regular. We let \( \mathcal{U}(S) \) denote the partially ordered set of all regular \( \mathcal{J} \)-classes of \( S \). If \( a \in S \), then \( J_a, H_a \) will denote the \( \mathcal{J} \)-class and \( \mathcal{H} \)-class of \( a \), respectively. If \( X \subseteq S \), then \( E(X) = E(S) \cap X \) and \( \langle X \rangle \) is the subsemigroup of \( S \) generated by \( X \). \( S \) is regular if \( a \in aSa \) for all \( a \in S \). A monoid \( S \) with group of units \( G \) is unit regular if for all \( a \in S \) there exist \( e \in E(S) \) and \( x \in G \) such that \( a = ex \). Clearly any submonoid of a unit regular monoid, containing the group of units, is also unit regular.

Let \( K \) be an algebraically closed field, \( K^* = K \setminus \{0\} \). We let \( \mathcal{M}_n(K) \) denote the monoid of all \( n \times n \) matrices over \( K \). We consider \( \mathcal{M}_n(K) \) with the Zariski topology on it (see [8]). By an algebraic monoid \( S \) we mean a closed submonoid of \( \mathcal{M}_n(K) \). The irreducible component of \( 1 \) in the Zariski topology of \( S \) will be denoted by \( S^c \). If \( S = S^c \), then we say that \( S \) is a connected monoid. Let \( S \) be a connected monoid with zero and group of units \( G \). Let \( T \) be a maximal torus of \( G \). Thus \( E(T) \) consists of the 'diagonal' idempotents in \( S \). Then \( E(T) \) is a finite relatively complemented lattice.
and \( \mathcal{P}(S) \) is a finite lattice. Moreover \( \dim T = \text{the length of any maximal chain in } E(T) \) is the length of any maximal chain in \( \mathcal{P}(S) \). We call this number the rank of \( S \). Let \( X, Y \subseteq S \). Then the centralizer in \( Y \) of \( X \) is \( C_Y(X) = \{a \in Y | ax = xa \text{ for all } x \in X \} \) and the normalizer in \( G \) of \( X \) is \( N_G(X) = \{a \in G|a^{-1}Xa = X \} \). Let \( \Gamma \subseteq E(T) \). Then the right centralizer of \( \Gamma \) in \( G \) is \( C_G^R(\Gamma) = \{a \in G|ae = eae \text{ for all } e \in \Gamma \} \). The left centralizer of \( \Gamma \) in \( G \) is \( C_G^L(\Gamma) = \{a \in G|ea = eae \text{ for all } e \in \Gamma \} \). The groups \( C_G^L(\Gamma), C_G^R(\Gamma), C_G(\Gamma) \) are all connected by [18, Corollary 4.2]. Let \( e \in E(S) \). Then \( S_e = \{a \in S|ea = ae = e \}^c \), \( G_e = G \cap S_e \) (see [17, 18] for further details). We refer to [8, 28] for the basic notions in the theory of algebraic groups. In particular, \( W = N_G(T)/C_G(T) \) is the Weyl group of \( G \).

Let \( S \) be a connected algebraic monoid with zero and group of units \( G \). If \( \text{char } K = 0 \), then the author [21] showed that \( G \) is a reductive group if and only if \( S \) is a regular semigroup. This result was extended to arbitrary characteristic by Renner [24]. Renner uses some results in algebraic geometry such as Zariski's main theorem to prove this result. Because of the central importance of this result in the theory of linear algebraic monoids, we present here an alternate proof.

**Theorem 1.1** [21, 24]. Let \( S \) be a connected algebraic monoid with zero and group of units \( G \). Then \( G \) is a reductive group if and only if \( S \) is a (unit) regular monoid.

**Proof.** If \( S \) is a regular semigroup, then by [18, Theorem 2.11], \( G \) is a reductive group. We prove by induction on \( \dim S \) that \( S \) is regular. If \( e \in E(S) \), then \( G_e \) and \( H_e \) are reductive groups by [21, Theorem 2.3]. Thus \( e \neq 0 \) implies that \( S_e \) is regular, and \( e \neq 1 \) implies that \( eSe \) is regular. Let \( S \subseteq \mathcal{M}_n(K) \), \( \dim S = p \). Consider det: \( S \to K \) where 'det' denotes the matrix determinant. Let \( Y = S \setminus G = \det^{-1}(0) \). Then \( Y = SYS \). So by [8, Theorem 4.1], every irreducible component of \( Y \) is an ideal of \( S \) by dimension \( p - 1 \). Let \( Y_0 \) be an irreducible component of \( Y \). Then \( \dim Y_0 = p - 1 \). We first show that \( Y_0 \) is not nil. For suppose otherwise. Let \( B, B^+ \) be opposite Borel subgroups of \( G \) with respect to some maximal torus \( T \). By [8, Proposition 27.2] we can choose an appropriate change of basis such that every element of \( B \) is upper triangular and every element of \( B^+ \) is lower triangular. Thus the same is true for \( \overline{B} \) and \( \overline{B}^+ \). Thus every element of \( \overline{B} \cap \overline{B}^+ \) is a diagonal matrix. Let \( W \) denote the Weyl group of \( G \). So \( B^+ = \sigma^{-1}B_\sigma \) for some \( \sigma \in W \). Now \( B \cap B^+ \) and hence \( B\sigma B \) is an open subset of \( G \) by [8, Proposition 28.5]. So \( X = G \setminus B\sigma B \) is a closed subset of \( G \). Thus \( \dim X \leq p - 1 \). We claim that \( Y_0 \subseteq X \). So let \( a \in Y_0 \), \( a \notin X \). Since \( 0 \notin \overline{B} \subseteq \overline{X} \), \( a \neq 0 \). By the Bruhat decomposition [8, Theorem 28.3], \( G \) is the disjoint union of \( B\theta B \) (\( \theta \in W \)). Thus \( a \notin B\theta B \) for any \( \theta \in W, \theta \neq \sigma \). So \( a \in \overline{B}\overline{\theta}B \cup \overline{\theta}BB \) for \( \theta \in W, \theta \neq \sigma \). Also by [29, p. 68 or 23, Proposition 3.4.1] \( S = GB = BG \). Hence \( a \in \overline{B}B\sigma B \cap \overline{B}\sigma B \subseteq \overline{B}B \cap BOB \). Let \( \sigma = gT, g \in N_G(T) \). Then there exist \( b_1, b_2 \in B, u_1, u_2 \in \overline{B} \) such that \( a = u_1gb_1 = b_2gu_2 \). So \( u_2b_1^{-1} = g^{-1}b_1^{-1}u_2g \in \overline{B} \cap \overline{B}^+ \). Thus \( u_2b_1^{-1} \) is a diagonal matrix. But \( u_2b_1^{-1} = g^{-1}b_2^{-1}u_1g^{-1} \) in \( Y_0 \setminus \{0\} \). This contradicts the assumption that \( Y_0 \) is nil. Thus \( Y_0 \subseteq \overline{X} \). Let \( X_1, \ldots, X_i \) be the irreducible components of \( X \). Then \( Y_0 \subseteq \overline{X}_i \) for some \( i \). Since \( \dim Y_0 = p - 1 \) and \( \dim X_i \leq p - 1 \) we see that \( Y_0 = \overline{X}_i \). This is a contradiction since \( Y_0 \subseteq S \setminus G, X_i \subseteq G \). Therefore \( Y_0 \) is not nil. So there exists \( e \in E(Y_0), \)
e ≠ 0. Choose e such that \( J_e \) is maximal in \( \mathcal{U}(Y_0) \). Let \( Y_1 = \{ y \in Y_0 | y + e \} \). Then \( Y_1 \) is closed by [13, Lemma 2.1]. Let \( u \in Y \setminus Y_1 \). Then \( u|e \). By [21, Theorem 1.4], \( u \in GS_G \). Hence \( u \) is regular. By the maximality of \( J_e \), \( u \not\in J_e \). Thus \( u \in SeS \). Hence \( Y_0 = SeS \cup Y_1 \). Since \( e \not\in Y_1 \), and since \( Y_0 \) is irreducible, we see that \( Y_0 = SeS \).

Now \( e \in \Gamma \) for some maximal chain in \( E(S) \). Let \( B_1 = C_G^e(\Gamma) \), \( B_2 = C_G^f(\Gamma) \). Then \( B_1 \), \( B_2 \) are Borel subgroups of \( G \) by [18, Theorem 4.5]. Clearly, \( B_1e = eB_1e \), \( eB_2 = eB_2e \). Thus \( B_1eS = eB_1Se \). So by [29, p. 68], \( GeSeG \) is closed in \( S \). Since \( GeG \subseteq GeSeG \), we see that \( Y_0 = SeS = GeG \subseteq GeSeG \). Since \( eSe \) is regular, so is \( Y_0 \).

Thus \( Y = S \setminus G \) is regular. This proves the theorem.

Let \( S \) be a connected regular monoid with zero and group of units \( G \). Let \( T \) be a maximal torus of \( G \). If \( \Gamma \) is a maximal chain in \( E(\bar{T}) \), then by [18] \( C_G^e(\Gamma) \) is a Borel subgroup of \( G \). However, the map: \( \Gamma \to C_G^e(\Gamma) \) is not one-to-one. To obtain a better correspondence, the author [18–20] considers cross-section lattices. A subset \( A \) of \( \mathcal{E}(f) \) is a cross-section lattice if: (1) \( |A \cap \mathcal{J}| = 1 \) for all \( \mathcal{J} \in \mathcal{U}(S) \), and (2) for all \( e, f \in \Lambda, J_e \succeq J_f \) implies \( e \succeq f \). Thus a cross-section lattice is a diagonal idempotent cross-section of \( J \)-classes preserving the \( J \)-class ordering. The author [18–20] has shown that cross-section lattices correspond one-to-one with Borel subgroups of \( G \) containing \( T \). The correspondence is as follows:

\[
\Lambda \rightarrow B = C_G^e(\Lambda),
\]

\[
B \rightarrow \Lambda = \{ e \in E(\bar{T}) | ae = eae \text{ for all } a \in B \}
\]

\[
= \{ e \in E(\bar{T}) | \text{for all } f \in E(S), e \not\in J \text{ implies } f \not\in B \}.
\]

Let \( C \) denote the center of the reductive group \( G \). Then by [8] \( G = CG_0 \), where \( G_0 \) is a semisimple group (the commutator subgroup of \( G \)). We will say that \( G \) is nearly semisimple if \( \dim C = 1 \). In such a case Renner [25, 26] calls the monoid \( S \) semisimple. We avoid the terminology in this paper because the term 'semisimple' has a completely different meaning in abstract semigroup theory. With the additional assumption that \( S \) is normal, Renner classifies these monoids geometrically. Let \( X(T), X(\bar{T}) \) denote the character group and character monoid of \( T \) and \( \bar{T} \), respectively. Let \( \Phi \) denote the root system of \( G \). Then Renner [26] shows that the map \( S \rightarrow (X(T), \Phi, X(\bar{T})) \) classifies normal 'semisimple' monoids up to isomorphism. Since \( X(T), \Phi, X(\bar{T}) \) are all geometrical objects, this classification may be thought of as being geometrical. This theorem of Renner is quite deep, having many important consequences. For example, Renner uses his classification to show that any such monoid \( S \) has an involution fixing \( \bar{T} \). A key step in the proof of Renner's classification theorem is the generalization of Chevalley's big cell from groups to monoids. Renner uses cross-section lattices in the construction of his big cell.

Let \( S \) be a connected regular monoid with zero and let \( \hat{S} \) be its normalization (see [23]). Then there exists a finite homomorphism \( \phi: \hat{S} \rightarrow S \). The term 'finite' is being used in the sense of algebraic geometry [8, §4.2]. Finite homomorphisms are of abstract semigroup theoretic interest also, because of the following result of Renner [23, Propositions 3.4.13, 4.1.6].
Theorem 1.2 (Renner). Let $\phi: S \to S'$ be a homomorphism between connected regular monoids with zero such that $\phi(S) = S'$. Then $\phi$ is finite if and only if it is idempotent separating.

Following Renner [26] we say that a connected regular monoid $S$ with zero is irreducible if $\mathcal{B}(S) \setminus \{0\}$ has a minimum element. The following result is due to Renner [26, Corollary 8.3.3].

Theorem 1.3 (Renner). Let $S$ be a connected regular monoid with zero. Then $S$ is irreducible if and only if there exists an idempotent separating matrix representation. In particular the group of units of such a monoid is nearly semisimple.

We will say that $G = CG_0$ is nearly simple if $\dim C = 1$ and $G_0$ is almost simple, i.e. $G_0$ has no nontrivial normal closed connected subgroups. We denote by Ren($S$) the finite inverse monoid $\widetilde{NC(T)}/T$. Renner [27] has used this monoid to obtain a Bruhat decomposition for $S$. Ren($S$) is a fundamental (i.e. has no nontrivial idempotent separating congruences) unit regular monoid with group of units $W = NC(T)/T$ and idempotent set $E(T)$.

2. Abstract properties. Let $S$ be a regular semigroup. We let $S/\mathcal{R}$ denote the partially ordered set of all principal right ideals of $S$ under inclusion. Then $S/\mathcal{R}$ can be identified with the partially ordered set of all $\mathcal{R}$-classes of $S$.

Theorem 2.1. Let $S$ be a connected regular monoid with zero and group of units $G$. Then $S/\mathcal{R}$ is a relatively complemented, complete lattice with all maximal chains having the same finite length equal to rank $S$.

Proof. By [14] all maximal chains in $E(S)$ have the same finite length. Thus the same is true for $S/\mathcal{R}$. Let $eS, fS \in S/\mathcal{R}$ where $e, f \in E(S)$. By [18, Theorem 4.6] $C_\mathcal{R}(e)$ and $C_\mathcal{R}(f)$ are parabolic subgroups of $G$. So by [8, Corollary 28.3] $C_\mathcal{R}(e) \cap C_\mathcal{R}(f)$ contains a maximal torus $T$ of $G$. By [14] there exist $a \in C_\mathcal{R}(e), b \in C_\mathcal{R}(f)$ such that $e' = a^{-1}ea, f' = b^{-1}fb \in T$. Then $eS = e'S, fS = f'S, e'f' = f'e'$. So $eS \cap fS = e'f'S$. Since $S/\mathcal{R}$ has a maximum element, and since all maximal chains in $S/\mathcal{R}$ have the same finite length, it follows that $S/\mathcal{R}$ is a complete lattice.

Let $e, f, h \in E(S)$ such that $eS \supseteq fS \supseteq hS$. Without loss of generality we can assume that $e \geq f \geq h$. By [14] there exists a maximal torus $T$ of $G$ such that $e, f, h \in E(T)$. Since $E(T)$ is relatively complemented [13], there exists $f' \in E(T)$ such that $e \geq f' \geq h$ and $ff' = h$. Then $eS \supseteq f'S \supseteq hS$ and $fS \cap f'S = hS$. It follows that $S/\mathcal{R}$ is relatively complemented.

Let $S$ be a regular semigroup, $a \in S$. Then $V(a) = \{x \in S | axa = a, xax = x\}$ denote the set of inverses of $a$ in $S$. Let $e, f \in E(S)$. Of crucial importance in the theory of regular semigroups is the sandwich set $\mathcal{S}(e, f) = V(ef) \cap fSe$ which is a rectangular band (see [11]).

Theorem 2.2. Let $S$ be a connected regular monoid with zero and group of units $G$. Let $e, f \in E(S)$. Then

(i) $C_\mathcal{R}(f) \cap C_\mathcal{R}(e)$ is a connected group and $fSe \subseteq C_\mathcal{R}(f) \cap C_\mathcal{R}(e)$.

(ii) If $h \in \mathcal{S}(e, f)$, then $\mathcal{S}(e, f) = \{x^{-1}hx | x \in C_\mathcal{R}(f) \cap C_\mathcal{R}(e)\}$. 

(iii) $\mathcal{P}(e, f) = \{ e'f' | e', f' \in E(\bar{T}) \text{ for some maximal torus } T \text{ of } G \text{ such that } e \mathcal{L} e', f \mathcal{R} f' \} = \{ e'f' | e', f' \in E(S), e \mathcal{L} e', f \mathcal{R} f', e'f' = f'e' \}.$

**Proof.** By [18, Theorem 4.6], $C'_G(f), C'_G(e)$ are parabolic subgroups of $G.$ By [8, Corollary 28.3], there exists a maximal torus $T$ of $G$ contained in $C'_G(f) \cap C'_G(e).$ By [14], there exist $a \in C'_G(e), b \in C'_G(f)$ such that $e' = a^{-1}e, f' = b^{-1}f \in E(\bar{T}).$ Then $f \mathcal{R} f', e \mathcal{L} e'.$ Hence by [15, Theorems 1, 9], $C'_G(e) = C'_G(e'), C'_G(f) = C'_G(f').$ Since $f' \in \bar{T} \subseteq C'_G(e'),$ it follows from [18, Theorem 4.1] that $G' = C'_G(e) \cap C'_G(f)$ is a connected group. Let $S' = G'$, $S_1 = C'_G(e').$ By [15, Theorem 1] $S' \subseteq S_1.$ By the same theorem, $f'S_1 \subseteq S'.$ Clearly $f'S \subseteq f'S_1$ and $fSe = f'Se'.$ Thus $fSe \subseteq S'.$ Since $e'f' = f'e',$ we see that $e'f' \in \mathcal{P}(e, f).$ Let $x \in G'.$ Then $x^{-1}e'f'x = e''f''$ where $e'' = x^{-1}e'x, f'' = x^{-1}f'x \in x^{-1}T_x, e \mathcal{L} e'', f \mathcal{R} f''.$ Hence $e''f'' \in \mathcal{P}(e, f).$ Finally let $h \in \mathcal{P}(e, f).$ Since $\mathcal{P}(e, f)$ is a rectangular band, and since $\mathcal{P}(e, f) \subseteq fSe \subseteq S',$ we see that $h \mathcal{J} e'f'$ in $S'.$ By [15, Theorem 9], there exists $x \in G'$ such that $h = x^{-1}e'f'x.$ This proves the theorem.

**Corollary 2.3.** Let $S$ be a connected regular monoid with zero and group of units $G.$ Let $a \in S.$ Then there exists a maximal torus $T$ of $G,$ $e, f \in E(\bar{T})$ such that $e \mathcal{R} a \mathcal{L} f.$ In particular the number of conjugacy classes of the $\mathcal{K}$-classes of $S$ is finite.

Let $S$ be a regular semigroup. Then $S$ is $V$-regular if $V(ab) \subseteq V(b)V(a)$ for all $a, b \in S.$ $S$ is strongly $V$-regular if for all $e, f \in E(S), h \in \mathcal{P}(e, f),$ there exist $e', f' \in E(S)$ such that $e \mathcal{L} e', f \mathcal{R} f', h \in \mathcal{P}(f', e').$ Nambooripad and Pastijn [12] show that every strongly $V$-regular semigroup is $V$-regular. They also show that many classes of naturally arising regular semigroups are strongly $V$-regular. We add to this list. The following theorem is immediate from Theorem 2.2.

**Theorem 2.4.** Every connected regular monoid with zero is strongly $V$-regular.

Let $S$ be a regular semigroup. Then Fitz-Gerald [5] showed that $\langle E(S) \rangle$ is a regular semigroup. The determination of $\langle E(S) \rangle$ represents an important step in the study of $S$ (see [6, 11]).

**Problem 2.5.** Let $S$ be a connected regular monoid with zero. Determine $\langle E(S) \rangle.$

We solve the above problem in the special case when $S \setminus G$ has a maximum $\mathcal{J}$-class. Erdos [4] (see also Dawlings [2, 3]) showed that the singular matrices in $\mathcal{M}_n(K)$ can always be written as products of idempotent matrices.

**Lemma 2.6.** Let $S$ be a connected regular monoid with zero, $S_1$ a closed connected submonoid of $S.$ If $E(S) = E(S_1),$ then $S = S_1.$

**Proof.** We prove the lemma by induction on $\dim S.$ Let $S \subseteq \mathcal{M}_n(K).$ Consider $\det: S \rightarrow K$ where ‘det’ denotes the determinant. Let $V$ be an irreducible component of $\det^{-1}(0).$ Then $V$ is clearly an ideal of $S$ and by [8, Theorem 4.1], $\dim V = \dim S - 1.$ Since $S$ is regular, $V = SeS$ for some $e \in E(S), e \neq 1.$ Now $eSe \subseteq eSe, E(eSe) = E(eSe),$ and $eSe$ is regular. By the induction hypothesis, $eSe = eSe.$ Let $X = \{ f \in E(S) | e \mathcal{R} f \}, Y = \{ f \in E(S) | e \mathcal{L} f \}.$ By [15, Theorem 14], $\dim S_1 eSe = \dim X + \dim Y + \dim eSe, \dim SeS = \dim X + \dim Y + \dim eSe.$ Thus $\dim SeS = \dim S_1 eS_1 < \dim S_1.$ So $\dim S \leq \dim S_1$ and $S = S_1.$
Theorem 2.7. Let $S$ be a connected regular monoid with zero and group of units $G$. Suppose $S \setminus G$ has a maximum $\mathcal{J}$-class. Then $S \setminus G \subseteq E(S)^m = \langle E(S) \rangle$ for some $m \in \mathbb{Z}^+$. 

Proof. Let $J$ denote the maximum $\mathcal{J}$-class of $S \setminus G$, $E_0 = E(S) \cap J$. Then $E_0$ is closed and irreducible by [15, Theorem 8]. We have the product maps $E_0 \times \cdots \times E_0 \to E_0^k$. Thus each $E_0^k$ is also irreducible. Clearly $E_0 \subseteq E_0^2 \subseteq E_0^3 \subseteq \cdots$. 

So there exists $p \in \mathbb{Z}^+$, $p \leq \dim S$ such that $E_0^q = E_0^p$ for all $q \in \mathbb{Z}^+$, $q \geq p$. Let $S_0 = E_0^p$. Since $E(T)$ is relatively complemented for any maximal torus $T$ of $G$, it follows that $E(S) \setminus \{1\} \subseteq \langle E_0 \rangle$. Thus $E(S_0) = E(S) \setminus \{1\}$. Let $e \in E_0$. Then $E(es_0e) = E(es_0e)$. By Lemma 2.6, $es_0e = es_0e$. Let $X = E_0 \times \cdots \times E_0$, the $p$-fold direct product. Define $\phi : X \to eSe$ as $\phi(e_1, \ldots, e_p) = ee_1 \cdots e_p e$. Thus $\phi(X) = eE_0^p e$. So $\phi(X) = eE_0^p e = eS_0 e = eSe$.

By [8, Theorem 4.4], $\phi(X)$ is constructible and hence contains a nonempty open subset $U$ of $eSe$. Let $H$ denote the group of units of $eSe$. Then $H$ is an open subset of $eSe$. Thus $U_1 = U \cap H$ is a nonempty open subset of $H$. By [8, Lemma 7.4], $U_1^{p+3} = H$. Thus $H \subseteq E_0^{2p+3}$.

Now let $a \in J$. Then there exist $e, f \in E_0$ such that $e \mathcal{R} a \mathcal{P} f$. By [16, Lemma 1.12] there exist $e_1, f_1 \in E_0$ such that $e \mathcal{R} e_1 \mathcal{P} f_1 \mathcal{R} f$. Then $e_1 f \mathcal{J} a$. So $a \in H_1 = H \cdot e_1 f \subseteq E_0^{2p+5}$. Finally, let $a \in S \setminus G$. Then $a \mathcal{R} f$ for some $f \in E(S)$, $f \neq 1$. Now $f' \geq f$ for some $f' \in E_0$ (see [15]). By [15], $a = fg$ for some $g \in G$. So $a = f(f'g)$, $f'g \in J \subseteq E_0^{2p+5}$. Thus $a \in E_0^{2p+6}$. This proves the theorem.

Example 2.8. Let $S = \{ A \mathcal{B} B \mid A, B \in \mathcal{M}_2(K) \}$. Then for $B \in \text{GL}(2, K)$, $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \mathcal{B} B$ cannot be written as a product of idempotents in $S$ unless $B$ is a scalar.

Example 2.9. Let $S = KG_0$ where $G_0 = \{ A \mathcal{B} (A^{-1})^t \mid A \in \text{SL}(3, K) \}$. Then $\mathcal{U}(S)$ is the same as in Example 2.8. However $S \setminus G$ is generated by idempotents where $G = K \ast G_0$. Note that $S \setminus G$ has two maximal $\mathcal{J}$-classes.

Problem 2.10. Let $S$ be a connected regular monoid with zero and group of units $G$. Find necessary and sufficient conditions for $S \setminus G$ to be idempotent generated.

Let $S$ be a regular semigroup. The fundamental congruence $\mu$ on $S$ is the largest congruence on $S$ contained in $\mathcal{J}$. If $\mu$ equals the equality, then $S$ is said to be fundamental (see [6, 7, 11]). We now show that when $S$ is a connected regular monoid with zero, then the extension $S$ of $S/\mu$ is a special 'central' extension and hence susceptible to a cohomological approach [9].

Theorem 2.11. Let $S$ be a connected regular monoid with zero and group of units $G$. Consider the natural (semigroup) homomorphism $\phi : S \to S/\mu$. Then for all $e \in E(S)$, $\phi^{-1}\phi(e)$ is the center of $H_e$.

Proof. Let $e \in E(S)$. Then by Hall [6, Corollary 6], $\mu$ restricted to $eSe$ is the fundamental congruence on $eSe$. Thus without loss of generality we can assume that
$e = 1$. Let $C$ denote the center of $G$. Let $x \in G$ such that $x \mu 1$. Then for all $f \in C(S)$, $fx \mathcal{H} f_x$. So $fx = xf$ for all $f \in C(S)$. By [18, Theorem 4.5], $x \in B$ for all Borel subgroups $B$ of $G$. It follows [8, p. 162, Exercise 2] that $x \in C$. Now the congruence $\theta$ defined by $a \theta b$ if $a \in Cb$, is idempotent separating and hence $\theta \subseteq \mu$. Thus $c x_1$ for all $c \in C$.

Since all maximal chains in $C(S)$ have the same finite length, we can define an obvious height function $ht$ on $C(S)$ such that $ht(1) = \text{rank } S$ and $ht(0) = 0$.

**Theorem 2.12.** Let $S$ be a connected regular monoid with zero and group of units $G$. Let $e, e' \in C(S)$ with $ht(e) = ht(e') = p > 0$. Then there exist $e = e_0, e_1, \ldots, e_k = e'$, $f_1, \ldots, f_k \in C(S)$ such that $ht(e_i) = p$, $ht(f_i) = p - 1$, $e_i > f_i, e_{i-1} > f_i, i = 1, \ldots, k$.

**Proof.** By [16, Lemma 1.15], the theorem is true when $G$ is a torus. Since any idempotent in $S$ is in the closure of a maximal torus and since any two maximal tori are conjugate, we are reduced to the case when $e \mathcal{H} e'$. By [16, Lemma 1.12] we are then further reduced to the case when $e \mathcal{R} e'$. There exists a cross-section lattice $\Lambda$ of $S$ such that $e \in \Lambda$. Let $B = C_G(\Lambda), T = C_T(\Lambda)$. Then $B$ is a Borel subgroup of $G$, $T$ is a maximal torus of $G$, $T \subseteq B, e \in E(T), e' \in B$. By [20, Theorem 1.2], there exists a cross-section lattice $\Lambda' \subseteq E(T)$ such that $B = C_G(\Lambda')$. There exists $u \in \Lambda'$ such that $e R u$ in $S$. Then $ht(e) = hu(u)$. So there exist $e = u_0, u_1, \ldots, u_k = u, v_1, \ldots, v_k \in E(T)$ such that $ht(u_i) = p, ht(v_i) = p - 1, u_i > v_i, u_{i-1} > v_i, i = 1, \ldots, k$. By [18, Proposition 4.3], $e, e', v_1 \in C_B(e, v_1)$. There exists $x \in C_B(e, v_1)$ such that $e' = x^{-1}ex$. Let $v_1' = x^{-1}v_1x \in E(B)$. Then $v_1 \mathcal{R} v_1', e' \mathcal{R} e'$. Now $v_1, v_1' \in C_B(u_2, v_1)$. There exists $y \in C_B(u_2, v_1)$ such that $v_1' = y^{-1}v_1y$. Let $u_2' = y^{-1}u_2y \in E(B)$. Then $u_2' \mathcal{R} u_1', u_2 \mathcal{R} u_2'$. Continuing, we find $e' = u_0', u_1', \ldots, u_k' = u', v_1', \ldots, v_k' \in E(B)$ such that $ht(u_i') = p, ht(v_1') = p - 1, u_i' > v_1', u_{i-1} > v_1', u_i \mathcal{R} u_i', v_i \mathcal{R} v_i', i = 1, \ldots, k$. In particular, $u \mathcal{R} u'$ and $u' = zuz^{-1}$ for some $z \in C_B(u)$. Since $B \subseteq C_G(u)$, we see that $u = u'$. This proves the theorem.

3. $\mathcal{H}$-linear monoids. The monoid $\mathcal{H}_n(K)$ has the very nice property that the $\mathcal{H}$-classes are linearly ordered. We will call such a monoid $\mathcal{H}$-linear. We start by obtaining a useful test for $\mathcal{H}$-linearity.

**Theorem 3.1.** Let $S$ be a connected regular monoid with zero and group of units $G$. Let rank $S = m$. Then the following conditions are equivalent.

(i) $S$ is $\mathcal{H}$-linear.

(ii) There exists $a \in S$ such that $a^m = 0, a^{m-1} \neq 0$.

(iii) There exists $a \in \text{ren}(S)$ such that $a^m = 0, a^{m-1} \neq 0$.

**Proof.** We first show that (ii) $\Rightarrow$ (i). We proceed by induction on dim $S$. If dim $S = 1$, then $|\mathcal{H}(S)| = 2$ and hence $S$ is $\mathcal{H}$-linear. So let dim $S > 1$. Let $a \in S$ such that $a^m = 0, a^{m-1} \neq 0$. Let $J_i$ denote the $\mathcal{H}$-class of $a^i$. Then $J_i \neq J_k$ for $1 \leq i < k \leq m$ (recall that $S$ is a matrix semigroup). Since $S$ is regular, each $J_i$ is regular. Thus $\{G > J_1 > J_2 > \cdots > J_m = 0\}$ is a maximal chain in $\mathcal{H}(S)$. By Corollary 2.3, there exists a maximal torus $T$ of $G, e, f \in E(T)$ such that $e \mathcal{R} a \mathcal{L} f$. So there exist $x, y \in G$ such that $a = ex = yf$. Now rank $eSe = m - 1, (exe)^{m-1} = 0, (exe)^{m-2} \neq 0$. By the induction hypothesis, $eSe$ is $\mathcal{H}$-linear. Hence by [17, §4], the
interval \([0, J_1] = \{0, J_{m-1}, \ldots, J_1\}\). Now \(a^2 = yfex \mathcal{J} ef\). So \(ef = fe \in J_2, e \neq f\). By [16, Proposition 1.14(6)], the interval \([J_2, G] = \{J_2, J_1, G\}\). It then follows from [16, Lemma 1.15] that \(\mathcal{F}(S) = \{0, J_{m-1}, \ldots, J_1, G\}\). Hence \(S\) is \(\mathcal{J}\)-linear.

That (iii) \(\Rightarrow\) (ii) is obvious. We are therefore left with showing (i) \(\Rightarrow\) (iii). So assume that \(S\) is \(\mathcal{J}\)-linear. Let \(T\) be a maximal torus of \(G\). If \(e \in E(T), \sigma \in W\), then let \(e^n = \sigma^{-1} e \sigma\). Let \(\{1 = e_0 > e_1 > \cdots > e_m = 0\}\) be a maximal chain in \(E(T)\). Since \(E(T)\) is relatively complemented, there exist \(1 = f_0, f_1, \ldots, f_m = 0 \in E(T)\) such that \(e_{i+1} = f_i, e_{f_i} = e_{i+1}, i = 1, \ldots, m - 1\). Since \(S\) is \(\mathcal{J}\)-linear, \(e_i \mathcal{F} f_i, i = 1, \ldots, m - 1\). Thus by [17, §4], there exist \(\sigma_i \in W, i = 1, \ldots, m - 1,\) such that \(e^n_{\sigma_i} = f_i, f^n_{\sigma_i} = f_j \) for \(j < i, e^n_{\sigma_i} = e_k \) for \(k > i\). Let \(\sigma = \sigma_1 \cdots \sigma_{m-1}\). Then

\[
e_s = (e_1^{\sigma_1} \cdots e_{m-1}^{\sigma_{m-1}})^{\sigma} = e_1^{\sigma_1} \cdots e_{m-1}^{\sigma_{m-1}} = f_1^{\sigma_1} \cdots f_{m-1}^{\sigma_{m-1}} = f_i, i = 1, \ldots, m - 1.
\]

So \(e_i e^n_s = e_{i+1} = 1, \ldots, m - 1\). Thus

\[
e_i e^n_1 = e_2, e_i e^n_1 e^n_2 = (e_1^{e_1} e^n_1)(e^n_1 e^n_1) = e_2 e^n_2 = e_3, \ldots, e_i e^n_1 \cdots e^n_{i-1} = e_{k+1}, \ldots.
\]

Now \(e_i e^n_i \cdots e^n_k = (e_1^{e_1} e^n_1 \cdots e^n_{k-1}) = 0, (e_1^{e_1} e^n_1 \cdots e^n_{k-1}) = 0\). This proves the theorem.

**Remark 3.2.** Let \(G_0\) be an almost simple group of type \(A_1, B_1\) or \(C_1\) with the representation given in Humphreys [8, §7.2]. Let \(S = KG_0\). Then Theorem 3.1 can be used to show that \(S\) is \(\mathcal{J}\)-linear. The \(\mathcal{J}\)-linearity of \(S\) does indeed very much depend on the particular representation of \(G_0\). For example, let \(G_0 = (A \otimes (A^{-1}))/A \in SL(3, K)\). Then \(S = KG_0\) is certainly not \(\mathcal{J}\)-linear.

**Theorem 3.3.** Let \(S\) be a connected regular \(\mathcal{J}\)-linear monoid with zero and group of units \(G\). Then \(G\) is a nearly simple group of type \(A_1, B_1, C_1, D_4\) or \(G_2\).

**Proof.** Let \(R\) denote the radical of \(G\). Then \(R\) lies in the center of \(G\) and \(G = RG_0\) where \(G_0\) is a semisimple group. By Theorem 1.3, \(\dim R = 1\). Suppose that \(G_0\) is not almost simple. Then by [8, Theorem 27.5], \(G_0 = G_1G_2\) where \(G_1, G_2\) are nontrivial semisimple subgroups of \(G_0\) and \(G_1\) centralizes \(G_2\). Let \(T\) be a maximal torus of \(G\). Then \(T_1 = T \cap G_1, T_2 = T \cap G_2\) are maximal tori of \(G_1\) and \(G_2\), respectively. Let \(e_i\) be a nonzero minimal idempotent of \(RT_i, i = 1, 2\). Let \(J_i\) denote the \(\mathcal{J}\)-class of \(e_i\) in \(S, i = 1, 2\). By symmetry suppose that \(J_1 \geq J_2\). Then \(e_1 \geq e'_2\) for some \(e'_2 \in E(J_2)\). There exists \(x \in G\) such that \(x^{-1} e_2 x = e'_2\). Now \(x = x_1 x_2\) for some \(x_1 \in G_1, x_2 \in RG_2\). Since \(e_2 \in S_2 = RG_2\) and \(G_1\) centralizes \(S_2\), we see that \(e'_2 = x_2^{-1} e_2 x_2 \in S_2\). Now since \(RG_2\) is nearly semisimple, we see by [18, Theorems 2.3, 2.13] that there exists \(e'_i \in E(RT_i), e'_i \neq e'_i\) such that \(e'_i \mathcal{J} e'_i \in S_i = RG_i\). Thus \(e'_i = y^{-1} e'_i y\) for some \(y \in G_1\). Then \(e'_i \geq y^{-1} e'_i y = e'_2\). So \(0 = e'_1 e'_i \geq e'_2, a contradiction. This shows that \(G\) is nearly simple.

Let \(T\) be a maximal torus of \(G\) and let \(\Lambda = \{0 < e_1 < \cdots < e_k < 1\}\) be a maximal chain in \(E(T)\). Then \(\Lambda\) is a cross-section lattice. Let \(B = C_0^G(\Lambda)\). For \(i = 1, \ldots, k\), there exists \(\sigma_i \in W, \sigma_i^2 = 1\) such that \(e_i^{\sigma_i} \neq e_i, e_i^\sigma = e_j\) for \(j \neq i\). By [20, Corollary 2.8], \(\mathcal{F} = \{\sigma_i | i = 1, \ldots, k\}\) is the set of simple reflections with respect to \(B\). Let \(\mathcal{F}_i = \mathcal{F} \setminus \{\sigma_i\}, P_i = C_0^G(\sigma_i) \supseteq B\). Then \(W_i = \langle \mathcal{F}_i \rangle\) is the Weyl group of \(P_i\) and hence \(P_i = BW_i B\). Clearly \(C_0^G(\sigma_i)\) is a Levi factor of \(P_i\). Let \(H_i\) denote the \(\mathcal{K}\)-class of
e_i, G_i = G_{e_i}. Then H_i, G_i are the groups of units of e_iS_e_i and S_{e_i}, respectively. By [17, Theorem 4.6], e_iS_{e_i}, S_{e_i} are linear monoids. Thus H_i, G_i are nearly simple groups. By [15, Theorem 4], the homomorphism ψ: CG(e_i) → H_i given by ψ(a) = e_ia is surjective. Clearly G_i is the identity component of the kernel of ψ. It follows that for i ≠ 1, k, W_i is reducible. Hence G is not of type D_l, E_6, E_7 or E_8. This proves the theorem.

**Theorem 3.4.** Let S be a connected regular linear monoid with zero and group of units G. Let T be a maximal torus of G. Then:

(i) Cross-section lattices are just the maximal chains in E(T).

(ii) The Weyl group of G is isomorphic to the abstract automorphism group of E(T).

(iii) Ren(S) is isomorphic to the Munn semigroup of E(T).

**Proof.** (i) is clear and (ii) follows from [18, Theorem 3.9]. We now prove (iii). The fundamental congruence μ on the inverse monoid N_G(T) is given by aμb if and only if b ∈ Ta. Thus Ren(S) is isomorphic to the submonoid of the Munn semigroup of E(T) generated by W and E(T). It therefore suffices to show that any isomorphism between two principal ideals of E(T) extends to an automorphism of E(T). Let φ: eTE(T) → fE(T) where e, f ∈ E(T). Then e / f and so σ^{-1}eσ = f for some σ ∈ W. Thus we are reduced to the case when e = f. Then φ is an automorphism of eE(T) = E(eT) and eSe is a linear monoid. So by (ii) φ belongs to the Weyl group E וג W. Hence φ extends to an automorphism of E(T). This proves the theorem.

4. Conjugacy classes. We start with the following preliminary result.

**Theorem 4.1.** Let S be a connected regular monoid with zero and group of units G, rank S = m. Let e ∈ E(S). Then Ne denote the set of nilpotent elements of S, and let G^e = C_G(S_e)^e. Let a, b ∈ S such that a^m H e H b^m. Then there exist η ∈ N_e, g ∈ G^e such that a = ηg. Moreover a is conjugate to b if and only if b = η'g' for some η' ∈ N_e, g' ∈ G^e such that η is conjugate to η' in S, and g is conjugate to g' in G^e.

**Proof.** Now G_e < C_G(e) and C_G(e) is a reductive group. Thus by [8, Theorem 27.5], C_G(e) = G^eG^e. Now ea ∈ Sa^{m+1} ⊆ Se, ae ∈ a^{m+1}S ⊆ eS. So ea = ae H e. By [15, Theorem 4] there exists x ∈ G^e such that ex = xe. So eax^{-1} = ax^{-1}e = e. Thus ax^{-1} ∈ S_e = {z ∈ S|ez = ze = e}. Let G'_e = G ∩ S'_e. Then by [21, Theorem 1.3], S'_e = G'_e. By [21, Lemma 1.2], S'_e = S_eS'_e ⊆ S_eC_G(e) = S_eG^e = S_eG^e. So there exists η ∈ S_e, g_1 ∈ G^e such that ax^{-1} = ηg_1. Let g = g_1x ∈ G^e. Then a = ηg. Now η'' H f for some f ∈ E(S_e). So f > e. Now e H a^m = (ηg)^m = η''g'' H f. Thus e / f, f > e. Hence e = f. Since η ∈ S_e, η'' = e. Thus η ∈ N_e. Now suppose a is conjugate to b. Then y^{-1}ay = b for some y ∈ G. Then e H b^m = y^{-1}a^my H y^{-1}ey. So y^{-1}ay = e and y ∈ C_G(e). Hence y = y_1y_2 for some y_1 ∈ G_e and y_2 ∈ G^e. Then b = η'g' where η' = y^{-1}ηy = y_1^{-1}ηy_1, g' = y^{-1}gy = y_2^{-1}g_2y_2. Conversely let b = η'g', η'' ∈ N_e, g'' ∈ G^e, such that η' = y_1^{-1}ηy_1 for some y_1 ∈ G_e and g'' = y_2^{-1}g_2y_2 for some y_2 ∈ G^e. Then η'' = y^{-1}ηy, g'' = y^{-1}gy where y = y_1y_2. So y^{-1}ay = b. This proves the theorem.

Let S be a connected regular monoid with zero and group of units G, rank S = m. Let a, b ∈ S. Then for some e, f ∈ E(S), a^m H e, b^m H f. If a is conjugate to b
then e is conjugate to f. If e = f, then by Theorem 4.1, the conjugacy problem for a
and b reduces to that for nilpotent elements in S, and that for elements in the
reductive group G. Thus the conjugacy problem in S reduces to the following three
problems:

(A) Conjugacy problem for idempotents.
(B) Conjugacy problem within a reductive group.
(C) Conjugacy problem for nilpotent elements.

Much is known about (A) and (B). So we are left with problem (C). An obvious
question is whether the number of conjugacy classes of nilpotent elements in S is
always finite. That this is not always the case was pointed out to the author by L.
Renner. His example is: S = \{(aA, ßA)\mid a, ß \in K, A \in M_2(K)\}. We now prove
the following general result.

**Theorem 4.2.** Let S be a connected regular monoid with zero and group of units G.
Suppose that S is \(\mathcal{J}\)-irreducible. If the number of conjugacy classes of nilpotent elements
in S is finite, then G is nearly simple.

**Proof.** Let J denote the nonzero minimum \(\mathcal{J}\)-class of S. Let R denote the radical
of G. Then \(G = RG_G\) where \(G_G\) is a semisimple group. By Theorem 1.3, \(\dim R = 1\).
Also R is contained in the center of G. Suppose \(G_G\) is not almost simple. Then by [8,
Theorem 27.5], \(G_G = G_1G_2\) where \(G_1\) is a nontrivial semisimple subgroup of \(G_G\) and
\(G_2\) is a nontrivial almost simple subgroup of \(G_G\) and \(G_1\) centralizes \(G_2\). Let \(S_1 = RG_G\)
and let \(J_J\) be a nonzero minimal \(\mathcal{J}\)-class of \(S_1\). Since \(RG_G\) is nearly simple, we see
by [18, Theorem 2.3] that \(J_J^2 \notin J_J\). So there exists \(a \in J_J\) such that \(a^2 = 0\). Suppose
\(aG_2 \subseteq S_1aS_1\). Then \(aG_2 \subseteq S_1aS_1\) whereby \(SaS = S_1aS_1\). So \(J \subseteq SaS = J_J \cup \{0\}.
Thus \(J = J_J\). Let \(S_2 = RG_{G_2}\). Since \(RG_{G_2}\) is nearly simple we see by [18, Theorem 2.3]
that there exist nonzero minimal idempotents \(e_1, e_1', e_1 \neq e_1'\) in the closure of a
maximal torus of \(RG_{G_2}\) such that \(e_1 \mathcal{J} e_1'\) in \(S_2\). Then \(e_1e_1' = 0 = e_1'e_1\). Since J is the
minimum nonzero \(\mathcal{J}\)-class of S, there exists \(e_2 \in E(J)\) such that \(e_1 \geq e_2\). Now
\(e_1' = x^{-1}e_1x\) for some \(x \in RG_{G_2}\). Then since \(e_2 \in S_1, e_1' \geq x^{-1}e_2x = e_2\). So \(0 = e_1e_1'\)
\(\geq e_2\), a contradiction. Hence \(aG_2 \subseteq S_1aS_1\). Let

\[V = \{ g \in G_2 \mid ag \in S_1aS_1 \}.\]

Clearly \(V\) is a closed subset of \(G_2\). Let \(g, g' \in V\). Then \(ag, ag' \in S_1aS_1\). So
\(agg' \in S_1aS_1g' = S_1agS_1 \subseteq S_1aS_1\). Now let \(x \in G_2, g \in V\). Then \(ag \in S_1aS_1\) and
\(ax^{-1}gx = x^{-1}agx \in x^{-1}S_1aS_1x = S_1aS_1\). Thus \(V\) is a closed normal subgroup of \(G_2\).
By the above, \(V \neq G_2\). Since \(G_2\) is almost simple, \(V\) is a finite subgroup of \(G_2\) lying
in the center of \(G_2\). Let \(|V| = m\). We can find a sequence of positive integers
\(n_1 < n_2 < \cdots, a_1, a_2, \ldots \in K^*\) such that the order of \(a_i\) is \(n_i\) and \(n_{i+1} > mn_i, i = 1, 2, \ldots\) Thus we can find \(u_1, u_2, \ldots\) in a maximal torus of \(G_2\) such that the order
of \(u_i\) is \(n_i, i = 1, 2, \ldots\). Now \(au_i\) is a nilpotent element of S for any \(i \in \mathbb{Z}^+\). Thus it
suffices to show that \(au_i\) is not conjugate to \(au_j\) for \(i < j\). So suppose to the contrary.
Then there exists \(x \in G\) such that \(x^{-1}au_i x = au_j\). Now \(x = yz\) for some \(y \in RG_1, z \in G_2\). Then \(au_i yz = yzau_j\). Thus \(au_i z = yzau_j\). So \(a(uz_i z^{-1}u_i^{-1}) = y^{-1}ay \in S_1aS_1\)
and \(zu_i z^{-1}u_i^{-1} \in G_2\). Hence \(zu_i z^{-1}u_i^{-1} \in V\) and \(zu_j z^{-1} \in Vu_i\). Since \(u_i^n = 1, |V| = m,\)
we see that $(zu_jz^{-1})^{n,m} = 1$. Hence $u_j^{n,m} = 1$ and $n,m \leq n_j$, a contradiction. This proves the theorem.

**Example 4.3.** Theorem 4.2 shows that the irreducible monoid $S = \{ A \otimes B | A, B \in \mathcal{M}_2(K) \}$ has infinitely many conjugacy classes of nilpotent elements.

**Example 4.4.** Let $G_0 = \{ A \otimes (A^{-1})' | A \in \text{SL}(3, K) \}$, $S = \overline{KG_0}$, $G = K^*G_0$. Then $G$ is nearly simple. Also $S$ is irreducible. However the number of conjugacy classes of nilpotent elements in $S$ can be shown to be infinite.

**Conjecture 4.5.** Let $S$ be a irreducible connected regular monoid with zero. Then the number of conjugacy classes of nilpotent elements is finite if and only if $S$ is linear.

**Conjecture 4.6.** Let $S$ be a connected irreducible regular monoid with zero. Then the number of conjugacy classes of nilpotent elements in $S$ is equal to the number of conjugacy classes of nilpotent elements in the finite inverse monoid $\text{Ren}(S)$ (Munn semigroup of $E(T)$).

**Remark 4.7.** The Jordan canonical form shows the above conjecture to be true for $\mathcal{M}_n(K)$.

If $S$ is a regular semigroup with zero and if $a$ is a nilpotent element in $S$, then we say that $a$ is minimal if $a$ lies in a nonzero minimal class of $S$.

**Theorem 4.8.** Let $S$ be a connected regular monoid with zero. Then the number of conjugacy classes of minimal nilpotent elements in $S$ equals the number of conjugacy classes of minimal nilpotent elements in $\text{Ren}(S)$.

**Proof.** Let $T$ be a maximal torus of $G$. Let $X = \{(e, f) | e, f$ are nonzero minimal idempotents of $T, e \neq f, e \not\sim f \}$. Define an equivalence relation $\equiv$ on $X$ as follows: $(e, f) \equiv (e', f')$ if there exists $\theta \in W$ such that $e^\theta = e'$, $f^\theta = f'$. If $a$ is a minimal nilpotent element of $\text{Ren}(S)$, then there exists a unique $(e, f) \in X$ such that $e \equiv a \not\sim f$. It follows that the number of $\equiv$-classes of $X$ equals the number of conjugacy classes of minimal nilpotent elements in $\text{Ren}(S)$. Now by Corollary 2.3, any minimal nilpotent element in $S$ is conjugate to an element of $eSf$ for some $(e, f) \in X$. Now let $(e, f) \in X$, $U = U(e, f) = eSf \setminus \{0\}$. Then every element of $U$ is a minimal nilpotent element and $U$ is an $H$-class of $S$. Since $eSe$ is regular and $E(eSe) = \{(0, e)\}$ we see that $\dim eSe = 1$. There exists $y \in G$ such that $f = y^{-1}ey$. Then $eSf = eSey$. It follows that $\dim U = 1$ and that $U$ is affine and irreducible. Now $T$ acts on $U$ by conjugation. By [8, Proposition 8.3] some orbit $A$ is closed in $U$. Suppose $\dim A = 0$. Then $A = \{a\}, a \in \mathcal{C}_S(T) = \overline{T}$ by [23, Theorem 4.4.4]. This is a contradiction since $a^2 = 0, a \neq 0$. Hence $\dim A = 1$ and $A = U$. Thus any two elements in $U$ are conjugate. Now let $(e, f), (e', f') \in X, a \in U(e, f), b \in U(e', f')$ such that $a$ is conjugate to $b$. We must show that $(e, f) \equiv (e', f')$. Now $e \equiv a \not\sim b \equiv e'$. Hence by [16, Lemma 1.7], there exists $\sigma \in W$ such that $\sigma^\sigma = e'$. Thus without loss of generality we can assume that $e = e'$. Now $b = x^{-1}ax$ for some $x \in G$. Then $x^{-1}ex \not\sim x^{-1}ax = b \equiv e$. So $e \equiv x^{-1}ex$ and $x \in \mathcal{C}_S(e)$. By [15, Theorem 1], $a, b \in eS \subseteq S' = \mathcal{C}_S(e)$. Now in $S, f' \not\sim b = x^{-1}ax \not\sim x^{-1}fx$. Hence $f' \not\sim x^{-1}fx$ in $S$ and hence in $S'$. Thus $f' \not\sim f$ in $S'$. So by [16, Lemma 1.7] there exists
\[ u \in N_G(T) \cap C_G(e) = N_G(T) \cap C_G(e) \] such that \( f' = u^{-1}fu \). Thus there exists \( \theta \in W \) such that \( e^\theta = e, f^\theta = f' \). This proves the theorem.

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