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ON THE GROUP SSF(G), G A CYCLIC GROUP OF PRIME ORDER[1]

BY

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ABSTRACT. We extend the definition of the obstruction group SSF(G) in the case where G is a cyclic group of prime order. We show that an endomorphism of a free ZG-module is a direct summand of a virtual permutation if its characteristic polynomial has the appropriate form. Among these endomorphisms the virtual permutations are detected by K_0. The main application is in detecting Morse-Smale isotopy classes.

1. Introduction. A diffeomorphism of a compact manifold $f\colon M \to M$ is called *Morse-Smale* if it satisfies Axiom A and strong transversality, and the nonwandering set $\Omega(f)$ is finite. From the point of view of dynamics these are the simplest structurally stable diffeomorphisms (see **[9]** for more details). In this paper we study the algebraic obstruction to the existence of a Morse-Smale diffeomorphism in a given isotopy class, in the case where the fundamental group $G = \pi_1(M)$ is a cyclic group of prime order.

DEFINITION. A square integral matrix is called a virtual permutation (v.p.) if it has the block diagonal form

$$\begin{pmatrix} p_1 & & & * \\ & p_2 & & \\ & & \ddots & \\ & 0 & & \\ & & & p_r \end{pmatrix},$$

where each block p_i is either a signed permutation matrix or zero.

THEOREM (SHUB-SULLIVAN **[9]**). *Suppose* $\dim M \geqslant 6$ *and* $\pi_1(M) = 0$. $f\colon M \to M$ *is isotopic to a Morse-Smale diffeomorphism if and only if f can be represented on the integral chain level by virtual permutation matrices.* □

Franks and Shub **[3]** constructed an obstruction group, called SSF, which detects when this chain level condition can be satisfied. A linear map is called *quasi-unipotent* if all eigenvalues are roots of unity, and *quasi-idempotent* if the zero

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eigenvalue is also allowed. By a change of basis the permutation blocks p_i above can be put in the form

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & \dots & 1 \\ \pm 1 & \dots & \dots & \dots & 0 \end{pmatrix},$$

i.e. the companion matrix $C(x^n - 1)$ of the polynomial $(x^n - 1)$, so a Morse-Smale diffeomorphism must be quasi-unipotent on homology [**8**]. Let **QI** be the category whose objects are pairs (M, e), where M is a finitely generated Z-module and e: $M \to M$ is quasi-idempotent on $M/\mathrm{Torsion}(M)$. Let $\mathbf{P} \subset \mathbf{QI}$ be the full subcategory of objects which have finite resolutions by free v.p. endomorphisms. Then

$$\mathrm{SSF} = K_0(\mathbf{QI})/K_0(\mathbf{P}).$$

If E: $C_* \to C_*$ is a quasi-idempotent chain map, let $\chi(E) = \Sigma_{i=0}^{\dim C_*}(-1)^i[E_i]$ in SSF.

THEOREM (FRANKS-SHUB [**3**]). *An integral chain map E: $C_* \circlearrowleft$ is chain homotopy equivalent to a v.p. chain map if and only if E_*: $H_*(C_*) \supset$ is quasi-idempotent and* $\chi(E_*) = 0$ *in* SSF.

In [**4**] Lenstra proved that SSF is nontrivial.

For nonsimply connected manifolds the appropriate algebraic models are the chain maps in the universal cover $C_*(M)$ over the group ring ZG, $G = \pi_1(M)$ [**5**]. V.p. matrices have the same block diagonal form but the blocks p_i may have nonzero entries $\pm g$, $g \in G$; by a change of basis we can assume the p_i are companion matrices of polynomials $x^n \pm g$.

Let $\mathbf{Mod}(G)$ be the category with objects (M, e), where M is a finitely generated ZG-module and e: $M \to M$ is an operator homomorphism associated with an automorphism of G. A *morphism* h: $(M, e) \to (N, f)$ in $\mathbf{Mod}(G)$ is a ZG-homomorphism h: $M \to N$ such that $fh = he$. A *resolution* of (M, e) in $\mathbf{Mod}(G)$ is a long exact sequence

$$0 \to (N_k, f_k) \to \cdots \to (N_0, f_0) \to (M, e) \to 0.$$

Let $\mathbf{P} \subset \mathbf{Mod}(G)$ be the full subcategory of objects which have finite free v.p. resolutions. Let $\mathbf{Q} \supset \mathbf{P}$ be the full subcategory of $\mathbf{Mod}(G)$ of objects which are direct summands of objects of $\mathbf{P}$.

Define

$$\mathrm{SSF}(G) = K_0(\mathbf{Q})/K_0(\mathbf{P}).$$

In [**6**] we showed that if a ZG-chain map E: $C_* \to C_*$ has each $(C_i, E_i) \in \mathbf{Q}$, then (C_*, E) is chain homotopy equivalent to a v.p. endomorphism if and only if

$$\chi(E) = \sum_{i=0}^{\dim C_*} (-1)^i[E_i] = 0$$

in $\mathrm{SSF}(G)$. We were unable to directly identify the objects of **Q**.

When $G = 0$ it follows from [**3**, Proposition 2.7] that $\mathbf{Q} = \mathbf{QI}$. The proof depends on the fact that if ξ is a primitive kth root of unity, the ring $Z[\xi]$ is a Dedekind domain. In [**7**] we identified the objects of $\mathbf{Q}$ in a special case. By a *v.p. polynomial* in $ZG[x]$ we mean x^n or a factor of $(x^n \pm g)$, $g \in G$. When $G = Z^n$, M is a free ZG-module and e: $M \to M$ is ZG-linear, we showed that $(M, e) \in \mathbf{Q}$ if and only if the characteristic polynomial of e is a product of v.p. polynomials (equivalently the eigenvalues of e are zero or roots of unity).

In this paper, using techniques of integral representation theory, we prove the following

THEOREM. *Suppose G is a cyclic group of prime order, M is a free ZG-module and E: $M \to M$ is ZG-linear. Then $(M, e) \in \mathbf{Q}$ if and only if the characteristic polynomial of e is a product of v.p. polynomials.* □

This condition is clearly necessary. The rest of the paper is devoted to the proof of sufficiency. We would like to thank Michael Shub and John Franks for introducing us to this problem and for many helpful conversations.

2. Filtrations. A *filtration* of (M, e) in $\mathbf{Mod}(G)$ is a sequence of submodules $0 \subset M_1 \subset \cdots \subset M_r = M$ such that $e(M_i) \subset M_i$. By abuse of notation we will also denote as e the quotient self-map of M_i/M_{i-1}.

LEMMA 1. *If (M, e) has a filtration with each $(M_i/M_{i-1}, e) \in \mathbf{Q}$, then $(M, e) \in \mathbf{Q}$.*

PROOF. Consider the exact sequence in $\mathbf{Mod}(G)$

$$0 \to M_1 \to M_2 \to M_2/M_1 \to 0.$$

By assumption, M_1 and M_2/M_1 have inverses N and N' modulo $\mathbf{P}$. The sequence

$$0 \to M_1 \oplus N \to M_2 \oplus N \oplus N' \to M_2/M_1 \oplus N' \to 0$$

is exact and the end terms belong to $\mathbf{P}$. But $\mathbf{P}$ is closed under short exact sequences [**6**, Lemma 2]. (One verifies that under our hypothesis the M_i are projective.) Therefore $(M_2, e) \in \mathbf{Q}$ and the lemma follows by induction. □

Henceforth we assume that G is a cyclic group of prime order p, $G = \langle g | g^p = 1\rangle$. We begin by constructing projective filtrations for endomorphisms of free ZG-modules.

The integral group ring $ZG \cong Z[x]/x^p - 1$. Let $\phi(x) = 1 + x + \cdots + x^{p-1}$ so $x^p - 1 = (x-1)\phi(x)$, and let I be the augmentation ideal of ZG, the kernel of ω: $ZG \to Z$, $\omega(\Sigma a_i g^i) = \Sigma a_i$. Then $I = \{\alpha \in ZG | \phi(g)\alpha = 0\} = (g-1)ZG$ and $I \cong Z[x]/\phi(x) \cong Z[\xi]$ as ZG-modules (where g acts via x or ξ), ξ a primitive pth root of unity.

The short exact sequence

$$0 \to I \to ZG \xrightarrow{\omega} Z \to 0$$

is a nonsplit extension of $Z[\xi]$ by Z, but over Q the extension is split.

Let QG be the group algebra, $QG \cong Q \otimes_Z ZG \cong Q[x]/x^p - 1$. Let J be the augmentation ideal of QG; then $J \cong Q[\xi] = Q(\xi)$. Let $K = \{\alpha \in QG | (g-1)\alpha = 0\}$; then $K \cong Q$, the trivial QG-module.

The sequence

$$0 \to J \to QG \xrightarrow{\omega} Q \to 0$$

splits and so $QG \cong Q(\xi) \oplus Q = J \oplus K$.

Let M be a free ZG-module of rank n, e: $M \to M$ a ZG-linear homomorphism and suppose the characteristic polynomial of e is $f_1(x)f_2(x) \cdots f_r(x)$, $f_i(x) \in ZG[x]$ and degree$(f_i(x)) = s_i$. We construct a filtration of M by first going up to QG.

Let $S = \{m \in M | \phi(g)m = 0\}$. If $\{m_1, \ldots, m_n\}$ is a basis for M as a free ZG-module, an element $m = \Sigma\alpha_i m_i \in M$ belongs to S if and only if each $\alpha_i \in I = (g-1)ZG$. Thus $S = (g-1)M \cong Z[\xi]^n$.

Let $QM = (Q \otimes_Z M) \circlearrowleft 1 \otimes e$ and let

$$QS = Q \otimes_Z S = \{v \in QM | \phi(g)v = 0\} \cong Q(\xi)^n.$$

Let $L = \{v \in QM | (g-1)v = 0\} \cong Q^n$. Then

$$QM \cong (QG)^n \cong QS \oplus L \cong Q(\xi)^n \oplus Q^n$$

and both summands are invariant under e. Suppose e is represented in the basis $\{m_1, \ldots, m_n\}$ by a ZG-matrix (a_{ij}). Then e: $L \to L$ is represented by the Q-matrix $(\omega(a_{ij}))$, where ω: $QG \to Q$, $g \to 1$, and e: $QS \to QS$ is represented by the $Q(\xi)$-matrix $(\theta(a_{ij}))$, where θ: $QG \to Q(\xi)$, $g \to \xi$.

Since $Q(\xi)$ and Q are fields there exist e-invariant filtrations of QS and L by subspaces

$$0 \subset QS_1 \subset \cdots \subset QS_r = QS, \qquad 0 \subset L_1 \subset \cdots \subset L_r = L$$

such that e: $[QS_i/QS_{i-1}] \circlearrowleft$ has characteristic polynomial $\theta(f_i(x)) \in Q(\xi)[x]$ and e: $(L_i/L_{i-1}) \circlearrowleft$ has characteristic polynomial $\omega(f_i(x)) \in Q[x]$.

Let $(QM)_i = QS_i \oplus L_i$. Then $0 \subset QM_1 \subset \cdots \subset QM_r = QM$ is a filtration, QM_i/QM_{i-1} is a free QG-module of rank $s_i =$ degree $f_i(x)$ and e: $(QM_i/QM_{i-1}) \circlearrowleft$ is annihilated by $f_i(x)$. Let $M_i = M \cap (QM)_i$, so $(QM)_i = Q \otimes M_i$, $0 \subset M_1 \subset \cdots \subset M_r = M$ is a filtration of M and e: $(M_i/M_{i-1}) \circlearrowleft$ is annihilated by $f_i(x)$. If q is any prime let $F_q = Z/qZ$, the finite field of q elements. If M is a ZG-module let $\overline{M} = M/qM$ be the reduced module over F_qG. Let Z_q denote Z localized at the prime ideal qZ and let $M_q = Z_q \otimes_Z M$. Recall that a *ZG-lattice* is a ZG-module which is free and finitely generated as a Z-module [**1**, p. 245]. We will need the following fact:

LEMMA 2. *Let A be a finite dimensional algebra over F_p containing F_pG, and let M be a ZG-lattice such that $\overline{M}$ is an A-module and $\overline{M}/(g-1)\overline{M} \cong (A)^r/(g-1)(A)^r$ as A-modules for some $r \geqslant 1$. If* $\operatorname{rank}_{F_p} \overline{M} = \operatorname{rank}_{F_p}(A)^r$, *then $\overline{M} \cong (A)^r$ as A-modules.*

PROOF. $\operatorname{rad}(F_pG) = (g-1)F_pG$ by [**1**, 5.24] and $(g-1)A \subset \operatorname{rad} A$. Therefore by [**1**, 5.1 and 5.29]

$$\frac{\operatorname{rad}(A)^r}{(g-1)(A)^r} = \operatorname{rad}\left[\frac{(A)^r}{(g-1)(A)^r}\right] \quad \text{and} \quad \frac{\operatorname{rad}\overline{M}}{(g-1)\overline{M}} = \operatorname{rad}\left[\frac{\overline{M}}{(g-1)\overline{M}}\right].$$

Thus $\overline{M}/\operatorname{rad}\overline{M} \cong (A)^r/\operatorname{rad}(A)^r$ as $(A/\operatorname{rad} A)$-modules.

It follows that $(A)^r$ is the projective cover of $\overline{M}$ as an A-module [**1**, 6.23]. Therefore there exists an epimorphism ε: $(A)^r \twoheadrightarrow \overline{M}$. Since $\operatorname{rank}_{F_p}(A)^r = \operatorname{rank}_{F_p} \overline{M}$, $\ker(\varepsilon) = 0$ and ε is an isomorphism. □

LEMMA 3. *M_i/M_{i-1} is a projective ZG-module.*

PROOF. Observe first that M_i/M_{i-1} is torsion-free over Z. Therefore M_i/M_{i-1} is a ZG-lattice and

$$\operatorname{rank}_Z\left(\frac{M_i}{M_{i-1}}\right) = \operatorname{rank}_Q\left(Q \otimes \left(\frac{M_i}{M_{i-1}}\right)\right) = \operatorname{rank}_Q\left(\frac{QM_i}{QM_{i-1}}\right) = s_i p,$$

where $s_i = \operatorname{degree}(f_i(x))$. Let $S_i = M \cap QS_i = M_i \cap S = \{m \in M_i | \phi(g)m = 0\}$. Then S_i is a free Z-module of rank $s_i(p-1)$.

By [**1**, 31.3 and 30.11] it is enough to show that $\overline{[M_i/M_{i-1}]} = \overline{M}_i/\overline{M}_{i-1}$ is projective over F_pG. We show that $\overline{M}/\overline{M}_{r-1}$ is projective and the lemma follows by a finite induction.

Now F_pG is a local ring and so by [**1**, 5.24 and 5.19]

$$\begin{aligned} \operatorname{rad}\frac{\overline{M}}{\overline{M}_{r-1}} &= \operatorname{rad}(F_pG)\frac{\overline{M}}{\overline{M}_{r-1}} = (g-1)\frac{\overline{M}}{\overline{M}_{r-1}} \\ &= \frac{(g-1)\overline{M} + \overline{M}_{r-1}}{\overline{M}_{r-1}} = \frac{\overline{S} + \overline{M}_{r-1}}{\overline{M}_{r-1}}. \end{aligned}$$

Since

$$\frac{S + M_{r-1}}{M_{r-1}} \cong \frac{S}{M_{r-1} \cap S} = \frac{S}{S_{r-1}}$$

is a free Z-module of rank $s_r(p-1)$ we obtain $(\overline{S} + \overline{M}_{r-1})/\overline{M}_{r-1}$ is an F_p-vector space of dimension $s_r(p-1)$. Similarly $\overline{M}/\overline{M}_{r-1}$ is an F_p-vector space of dimension $s_r p$. Therefore $(\overline{M}/\overline{M}_{r-1})/\operatorname{rad}(\overline{M}/\overline{M}_{r-1})$ is an F_p-vector space of dimension s_r. Let B be a free F_pG-module of rank s_r. Then

$$\frac{B}{\operatorname{rad} B} \cong \frac{\overline{M}/\overline{M}_{r-1}}{\operatorname{rad}(\overline{M}/\overline{M}_{r-1})}$$

as $(F_pG/\operatorname{rad}(F_pG))$-modules and by Lemma 2, $\overline{M}/\overline{M}_{r-1} \cong B$ as F_pG-modules. Hence M/M_{r-1} is projective as a ZG-module. Therefore M_{r-1} is projective, and $S_{r-1} = M_{r-1} \cap S = (g-1)M_{r-1}$ and the induction continues. □

Next we show that as in the case $G = 0$, going up to QG the obstruction vanishes.

LEMMA 4. *Suppose $f_i(x) = x^s \pm g$, $g \in G$. Then $(QM_i/QM_{i-1}) \circlearrowleft e$ is represented by a v.p. matrix over QG.*

PROOF. We have

$$\frac{QM_i}{QM_{i-1}} = \left(\frac{QS_i}{QS_{i-1}}\right) \oplus \frac{L_i}{L_{i-1}} \cong Q(\xi)^s \oplus Q^s,$$

with each term invariant under e, and e: $Q(\xi)^s \circlearrowleft$ has characteristic polynomial $\theta(f_i(x)) = (x^s \pm \xi)$, e: $Q^s \circlearrowleft$ has characteristic polynomial $x^s \pm 1$. Let $t \in Z$ be respectively ps, $2ps$, $s, 2s$ if $f_i(x)$ is $(x^s - g)$, $(x^s + g)$, where $g \neq 1$, $(x^s - 1)$, $(x^s + 1)$. Let $H = \langle h | h^t = 1 \rangle$. Then we may regard QS_i/QS_{i-1} as a module over the ring

$$\frac{Q(\xi)[x]}{(x^t - 1)} \cong Q(\xi)H$$

and L_i/L_{i-1} as a module over the ring

$$\frac{Q[x]}{(x^t - 1)} \cong QH,$$

where multiplication by x is given in each case by e.

Let N be a free ZG-module of rank s and e': $N \to N$ an endomorphism represented by the companion matrix of $f_i(x)$; i.e. v.p. matrix. Then QN has a corresponding decomposition $QN \cong QS' \oplus L'$, where QS' and L' are modules over $Q(\xi)H$ and QH.

For each element $a \in Q(\xi)H$ consider the actions $\hat{a}$: $[QS_i/QS_{i-1}] \circlearrowleft$ and $\hat{a}'$: $(QS')\circlearrowleft$. We claim the two actions have the same characteristic polynomials as $Q(\xi)$-homomorphisms. If $a \in Q(\xi)$ this is obvious, while if h occurs in a it follows since $\operatorname{char}(e) = \operatorname{char}(e') = x^s \pm \xi$. Similarly, for $a \in QH$ the actions on L_i/L_{i-1} and L' have the same characteristic polynomials as Q-homomorphisms.

Now by Maschke's theorem [**1**, 3.14] $Q(\xi)H$ and QH are semisimple rings. Therefore by a theorem of Brauer (see [**10**, p. 9, proof of Theorem 1.8]) modules over these rings are determined up to isomorphism by the characteristic polynomials of the actions of ring elements. Thus $(QS_i/QS_{i-1}) \cong QS'$ as $Q(\xi)H$-modules and $(L_i/L_{i-1}) \cong L'$ as QH-modules. Therefore

$$\frac{QM_i}{QM_{i-1}} = \frac{QS_i}{QS_{i-1}} \oplus \frac{L_i}{L_{i-1}} \cong QS' \oplus L' \cong N$$

as $(QG[x]/x^t - 1)$-modules. Hence e: $[QM_i/QM_{i-1}] \circlearrowleft$ is represented over QG by the v.p. matrix of e'. □

3. Proof of theorem. We restate our main result.

THEOREM. *Let G be a cyclic group of prime order p, M a free ZG-module and e: $M \to M$ ZG-linear. $(M, e) \in \mathbf{Q}$ if and only if the characteristic polynomial of e is a product of v.p. polynomials.*

PROOF. The condition is clearly necessary, we prove sufficiency. We have the characteristic polynomial of e equals $f_1(x)f_2(x) \cdots f_r(x)$, where each $f_i(x) | x^{n_i} \pm h_i$, $h_i \in G$. By Lemma 1 it suffices to prove that each factor module $(M_i/M_{i-1}, e) \in \mathbf{Q}$. Observe that $x^n \pm h | x^m - 1$, where $m = 2np$. Let N be a free ZG-module and e': $N \to N$ an endomorphism such that $f_i(x)\operatorname{ch}(e') = x^m - 1$.

Let $T = M_i/M_{i-1} \oplus N$ and $k = e \oplus e'$: $T \to T$. By Lemma 3 T is a projective ZG-module and by Lemma 4 QT is a free QG-module with $\operatorname{ch}(k) = x^m - 1$ on QT. We prove that $(T, k) \in \mathbf{Q}$ through a series of lemmas, which will complete the proof of the theorem. □

DEFINITION. Let R be a Dedekind domain with quotient field K, and A a finite-dimensional K-algebra. A *commutative R-order* in A is a commutative subring $\Lambda \subset A$ such that Λ is finitely generated as an R-module and $K\Lambda = A$ [1, 23.2].

LEMMA 5. *Let Λ be a commutative Z_q-order in a finite dimensional Q-algebra A. For any pair of projective Λ-lattices C and D, $QC \cong QD \Leftrightarrow C \cong D$ as Λ-modules.*

PROOF. By [1, 32.5] it is sufficient to show that $\overline{\Lambda}/\text{rad}\,\overline{\Lambda}$ is a separable F_q-algebra (where reduction is mod q).

By [1, 30.3] $\Lambda/\text{rad}\,\Lambda \cong \overline{\Lambda}/\text{rad}\,\overline{\Lambda}$ is a semisimple artinian ring. Since F_q is perfect, every finite extension field E of F_q is separable over F_q. Let $E \otimes \overline{\Lambda} = \overline{\Lambda}^E$. Then $\text{rad}(\overline{\Lambda}^E) = (\text{rad}\,\overline{\Lambda})^E$ by [1, 7.9(i)] and

$$\left(\frac{\overline{\Lambda}}{\text{rad}\,\overline{\Lambda}}\right)^E = \frac{\overline{\Lambda}^E}{(\text{rad}\,\overline{\Lambda})^E} = \frac{\overline{\Lambda}^E}{\text{rad}\,\overline{\Lambda}^E}$$

is a semisimple artinian ring. Hence $(\overline{\Lambda}/\text{rad}\,\overline{\Lambda})^E$ is semisimple for each finite extension field E over F_q. Therefore $\overline{\Lambda}/\text{rad}\,\overline{\Lambda}$ is a separable F_q-algebra by [1, 7.3]. $\square$

COROLLARY. *For all primes q, T_q is a free Z_qG-module.*

PROOF. T is a projective ZG-module and QT is a free QG-module. The result follows since Z_qG is a Z_q-order in QG. $\square$

DEFINITION. Let R be a Dedekind domain with quotient field K and let Λ be an R-order in a separable finite-dimensional K-algebra A. Two Λ-lattices C and D are in the same genus ($C \vee D$) if $C_p \cong D_p$ as Λ_p-modules for all maximal ideals P of R [1, p. 642].

LEMMA 6. *Let $\Phi(x) \in Z[x]$ be a cyclotomic polynomial and let C be a projective ZG-module, e: $C \to C$ with C annihilated by $\Phi(e)$. Let D be a free ZG-module, f: $D \to D$ an endomorphism represented by the companion matrix $C(\Phi(x))$. If $QC \cong QD$ as $(QG[x]/\phi(x))$-modules, then $C \vee D$ as $(ZG[x]/\Phi(x))$-modules.*

PROOF. We must show that $C_q \cong D_q$ for all primes q.

Case 1. $q = p$. $C_p/(g-1)C_p$ is a torsion-free $(Z_p[x]/\Phi(x))$-module. Since $Z_p[x]/\Phi(x) \cong Z_p[\eta]$, η a primitive rth root of unity, it follows that $Z_p[x]/\Phi(x)$ is a Dedekind domain and hence $C_p/(g-1)C_p$ is projective as a $(Z_p[x]/\Phi(x))$-module. By hypothesis $QC \cong QD$. As before

$$QC \cong \frac{Q[x]}{\Phi(x)} \oplus \frac{Q(\xi)[x]}{\Phi(x)} \quad \text{and} \quad \frac{QC}{(g-1)QC} \cong \frac{Q[x]}{\Phi(x)}.$$

It follows that $Q(C_p/(g-1)C_p) \cong Q[Z_p[x]/\Phi(x)]$ and by Lemma 5 $C_p/(g-1)C_p \cong Z_p[x]/\Phi(x)$. Hence

$$\frac{\overline{C}}{(g-1)\overline{C}} \cong \frac{F_p[x]}{\Phi(x)} \cong \frac{A}{(g-1)A}$$

a A-modules where $A = F_pG[x]/\Phi(x)$. By Lemma 2 $A \cong \bar{C}$ as A-modules. Hence C_p is a projective $(Z_pG[x]/\Phi(x))$-module [**1**, 30.11] and by Lemma 5 $C_p \cong D_p$.

Case 2. $q \neq p$. Since $\Phi(x)$ is a factor of $x^s - 1$ for some s, we may view C_q as a $Z_q\tilde{G}$-module, where

$$\tilde{G} = \langle x, y | x^s = 1, y^p = 1, [x, y] = 1 \rangle$$

and identifying G with $\langle y \rangle$.

Let $H = \langle x \rangle$. Then $[\tilde{G} : H] = p$ which is invertible in Z_q. Hence C_q is a direct summand of $Z_q\tilde{G} \otimes_{Z_qH} C_{qH}$, where C_{qH} denotes C_q viewed as an H-module [**1**, 33.5]. As above C_{qH} is a torsion-free $(Z_q[x]/\Phi(x))$-module and hence is projective. Since

$$Z_q\tilde{G} \otimes_{Z_qH} \frac{Z_q[x]}{\Phi(x)} \cong \frac{Z_qG[x]}{\Phi(x)}$$

as $(Z_qG[x]/\Phi(x))$-modules it follows that C_q is projective as a $(Z_qG[x]/\Phi(x))$-module. Lemma 5 then implies that $C_q \cong D_q$. □

LEMMA 7. *Let (C, e) and (D, f) be as in Lemma* 6. *If $C \vee D$ as $(ZG[x]/\Phi(x))$-modules, then $(C, e) \in \mathbf{Q}$.*

PROOF. We first show that $(D, f) \in \mathbf{P}$. There exist polynomials $f_1(x) f_2(x) \cdots f_t(x)$ and $h(x)$ such that

$$f_1(x) \cdots f_t(x)\Phi(x) = h(x), \quad f_i(x) = x^{s_i} - 1,$$

and $h(x) = x^n - 1$. By [**2**] $C(h(x))$ is similar over Z to

$$Y = \begin{pmatrix} C(f_1(x)) & & & * \\ & \ddots & & \\ & & C(f_t(x)) & \\ 0 & & & C(\Phi(x)) \end{pmatrix}.$$

If N_1 and N_2 are free ZG-modules with endomorphisms k_1 and k_2 represented by Y and $C(h(x))$ respectively, then $N_1 \cong N_2$ as $(ZG[x]/h(x))$-modules, i.e. $(N_1, k_1) \in \mathbf{P}$. Let N_3 be the free ZG-module with endomorphism k_3 represented by

$$\begin{pmatrix} C(f_1(x)) & & * \\ & \ddots & \\ 0 & & C(f_t(x)) \end{pmatrix}.$$

There exists a short exact sequence of $(ZG[x]/h(x))$-modules

$$0 \to N_3 \to N_1 \to D \to 0.$$

Since $N_3, N_1 \in \mathbf{P}$ we obtain $D \in \mathbf{P}$ since $\mathbf{P}$ is closed under short exact sequences [**6**]. But $C \vee D$, hence there exists a $(ZG[x]/\Phi(x))$-module L such that $L \oplus C \cong D \oplus D$ [**1**, 31.7], i.e. $(C, e) \in \mathbf{Q}$ as required. □

LEMMA 8. *Let T be a projective ZG-lattice with k: $T \to T$ a ZG-homomorphism. Suppose QT is a free QG-module with* $\mathrm{ch}(k \otimes 1) = x^m - 1$ *on QT. Then* $(T, k) \in \mathbf{Q}$.

PROOF. QT is a free QG-module by [**1**, 32.11]. Factor $x^m - 1$ into a product of irreducible factors in $Q[x]$. Then there exists a filtration

$$0 \subseteq U_1 \subseteq U_2 \subseteq \cdots \subseteq U_{r-1} \subseteq U_r = QT$$

such that U_i/U_{i-1} is a free QG-module with $k \otimes 1$ having characteristic polynomial $\Phi_i(x)$ on U_i/U_{i-1}, $\Phi_i(x)$ cyclotomic. Then $k \otimes 1$ may be represented by $C(\Phi_i(x))$ on U_i/U_{i-1}.

Let $T_i = T \cap U_i$. By Lemma 3 T_i/T_{i-1} is a projective ZG-lattice. By Lemmas 6 and 7 $(T_i/T_{i-1}, k) \in \mathbf{Q}$ and so by Lemma 1 $(T, k) \in \mathbf{Q}$. $\square$

This completes the proof of the theorem.

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