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THE DETERMINANT OF THE EISENSTEIN MATRIX AND HILBERT CLASS FIELDS

BY

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ABSTRACT. We compute the determinant of the Eisenstein matrix associated to the Hilbert-Blumenthal modular group $\mathrm{PSL}_2(\mathcal{O}_k)$, and express it in terms of the zeta function of the Hilbert class field of K.

0. Introduction. The Eisenstein series for the modular group $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$ is given by

$$E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y(\gamma z)^s,$$

where $\Gamma_\infty = \{ \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) | n \in \mathbf{Z}\}$, z is in the upper half-plane and $y(z) = \mathrm{Im}(z)$. The Fourier expansion of $E(z,s)$ is well known (see Hejhal [**4**, p. 76]):

$$E(z,s) = y^s + \phi(s) y^{1-s} + \text{nonzero coefficients},$$

where

$$\phi(s) = \frac{\pi^{-(s-1/2)}\Gamma(s-1/2)\varsigma(2s-1)}{\pi^{-s}\Gamma(s)\varsigma(2s)},$$

$\Gamma(s)$ is the gamma function and $\varsigma(s)$ is the Riemann zeta function.

In the general case of a group with h cusps one has an Eisenstein series at each cusp. The Fourier expansions of these series in the various cusps lead to an $h \times h$ matrix $\Phi(s)$ (see (1.11) below). This matrix is the constant term of the Eisenstein series, and is sometimes referred to as the scattering matrix [**7**], or simply the Φ-matrix.

The determinant $\phi(s) = \det \Phi(s)$ plays an important role in the theory. In the first place it controls the Eisenstein series in the sense that if $\phi(s)$ is analytic at some point, then so are all the Eisenstein series. More importantly, $\phi(s)$ (or rather its logarithmic derivative) accounts for the contribution of the continuous spectrum to the Selberg Trace Formula. A classical situation in which multiplicity of cusps occurs is that of congruence subgroups of the modular group, and the computation of $\phi(s)$ for these is quite involved (see Hejhal [**4**, Chapter 12] and Huxley [**5**]).

Our aim in this paper is to calculate $\phi(s)$ for the Hilbert-Blumenthal modular groups $\mathrm{PSL}_2(\mathcal{O}_K)$, $\mathcal{O}_K$ being the ring of integers of a number field K. In this case the number of cusps is the class number of K. What we show is that $\phi(s)$ is elegantly expressed in terms of the zeta function of the Hilbert class field of K. An interesting feature of our approach is that we do not use any Fourier developments

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in cusps. Instead we obtain our result purely from considerations of functional equations. To keep notations to a minimum we shall deal only with imaginary quadratic fields. The general case may be dealt with in a similar way (see Efrat [**2**] for the set up for totally real fields).

1. General set up and results. Let $\mathfrak{h}^3$ be the hyperbolic three space $\{w = (y, x_1, x_2) = (y, z) | y > 0\}$ with its Lobachevskii metric

$$ds^2 = \frac{dy^2 + dx_1^2 + dx_2^2}{y^2}.$$

If we think of $\mathfrak{h}^3$ as the set of all quaternions $x_1 + ix_2 + jy + kt$ for which $t = 0$, then $G = \mathrm{PSL}_2(\mathbf{C})$ is the group of all orientations preserving isometries of $\mathfrak{h}^3$. It acts via linear fractional transformations, i.e., for $\tau = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in G$,

$$\tau(w) = (\alpha w + \beta)(\gamma w + \delta)^{-1}.$$

Consider next a discrete subgroup $\Gamma \subset G$ for which $\mathcal{F} = \mathfrak{h}/\Gamma$ is not compact but is of finite volume. Let $\kappa_1 = \infty, \kappa_2, \ldots, \kappa_n$ be a complete set of inequivalent cusps, and let Γ_i be the subgroup of Γ that fixes κ_i. We choose $\rho_i \in G$ such that $\rho_i(\kappa_i) = \infty$ and such that

$$\rho_i \Gamma_i \rho_i^{-1} = \left\{ \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \middle| \, l \in L_i \right\},$$

where L_i is a lattice in $\mathbf{C}$ whose fundamental domain has volume 1. We let

$$w^{(i)} = \rho_i w = (y(\rho_i w), z(\rho_i w)).$$

DEFINITION 1.1. For each cusp κ_i, define its Eisenstein series to be

$$E_i(w,s) = \sum_{\gamma \in \Gamma_i \backslash \Gamma} y^{(i)}(\gamma w)^s, \qquad \mathrm{Re}(s) > 2.$$

Then Selberg's theory shows that $E_i(w,s)$ can be meromorphically continued in s to all of $\mathbf{C}$, and gives a Γ-automorphic eigenfunction of the Laplacian Δ of $\mathfrak{h}^3$, with

$$\Delta E_i(w,s) + s(2-s)E_i(w,s) = 0.$$

Since $E_i(w,s)$ is Γ-automorphic, it is invariant under the lattice at κ_j. It thus admits a Fourier expansion there, which is of the form

$$(1.1.1) \qquad \begin{aligned} E_i(w,s) = \delta_{ij} {y^{(j)}}^s + \phi_{ij}(s) {y^{(j)}}^{2-s} \\ + \text{ nonzero coefficients}, \quad \text{rapidly decaying as } y^{(j)} \to \infty. \end{aligned}$$

If we let $\Phi(s) = (\phi_{ij}(s))_{i,j=1,\ldots,h}$ and

$$\vec{E}(w,s) = \begin{bmatrix} E_1(w,s) \\ \vdots \\ E_h(w,s) \end{bmatrix},$$

then we have the functional equation

$$(1.1.2) \qquad \vec{E}(w,s) = \Phi(s)\vec{E}(w, 2-s).$$

(For these facts see Cohen and Sarnak [**1**], Hejhal [**4**] or Sarnak [**8**].) The following lemma shows that $\Phi(s)$ is characterized by this equation.

LEMMA 1.2. *If $\vec{E}(w,s) = \Psi(s)\vec{E}(w, 2-s)$ for every $w \in \mathfrak{h}^3$, then $\Psi(s) = \Phi(s)$.*

PROOF. We have

$$\begin{aligned} E_i(w,s) &= \sum_{k=1}^{h} \psi_{ik}(s) E_k(w, 2-s) \\ &= \sum_{k=1}^{h} \psi_{ik}(s)(\delta_{kj} {y^{(j)}}^{2-s} + \phi_{kj}(2-s){y^{(j)}}^{s} + \cdots). \end{aligned} \tag{1.2.1}$$

Now choose s with $\mathrm{Re}(s) < 0$. Then as $y^{(j)} \to \infty$ only the terms involving ${y^{(j)}}^{2-s}$ will increase. Comparing (1.2.1) with (1.1.1) gives

$$\phi_{ij}(s) = \sum_{k=1}^{h} \psi_{ik}\delta_{kj} = \psi_{ij}(s). \qquad \square$$

Let $\Gamma = \Gamma_D$ be the Hilbert modular group associated to the imaginary quadratic number field $K = Q(\sqrt{-D})$, i.e.,

$$\Gamma_D = \mathrm{PSL}_2(\mathcal{O}_K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a,b,c,d \in \mathcal{O}_K,\ ad - bc = 1 \right\} \Big/ \{\pm 1\},$$

where $\mathcal{O}_K$ is the ring of integers of K. We also assume $D \neq 1, 3$.

Then the cusps of Γ are the numbers $\alpha/\beta \in \mathbf{C}$ with $\alpha, \beta \in \mathcal{O}_K$ and ∞. Furthermore, α/β and α'/β' are equivalent under Γ as cusps if and only if (α, β) and (α', β') are equivalent ideals. So if $\mathcal{A}_1, \dots, \mathcal{A}_h$, $\mathcal{A}_i = (\gamma_i, \delta_i)$, are a compelte set of representatives of the ideal classes of K, then $\kappa_i = -\delta_i/\gamma_i$, $i = 1, \dots, h$, form a complete set of inequivalent cusps.

To define the ρ_i's we first note that we can choose $\alpha_i, \beta_i \in \mathcal{A}_i^{-1}$ such that

$$\hat{\rho}_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \in G.$$

Note that $\hat{\rho}_i(\kappa_i) = \infty$.

To guarantee the volume 1 condition, we scale $\hat{\rho}_i$ and define

$$\rho_i = \begin{pmatrix} \lambda_i^{1/2} & 0 \\ 0 & \lambda_i^{-1/2} \end{pmatrix} \hat{\rho}_i,$$

where $\lambda_i = \omega_K N\mathcal{A}_i$, $\omega_K = \sqrt{2}/d_K^{1/4}$ and d_K is the absolute value of the discriminant of K.

By the discussion above, we now have our Eisenstein series defined for Γ_D. Our main theorem is

THEOREM 1. *For Γ_D,*

$$\phi(s) = (-1)^{(h-2^{t-1})/2} \omega_K^{2s-2} \frac{\xi_H(s-1)}{\xi_H(s)},$$

where $\xi_H(s) = (d_H^{1/2}/(2\pi)^h)^s \Gamma(s)^h \varsigma_H(s)$, H is the Hilbert class field of K and t is the number of prime divisors of d_K.

REMARKS. (i) Since H is unramified everywhere it follows that $\xi_H(s)$ is the usual ξ function for the Hilbert class field H, so that $\xi_H(1-s) = \xi_H(s)$.

(ii) The exponent $(h-2^{t-1})/2$ of -1 in the formula is always an integer by genus theory (see, for example, Hecke [**3**, p. 160]).

Besides the contribution $\phi(s)$ to the trace formula there is also one other term that appears and which comes from the Eisenstein series. It is the term $\mathrm{tr}(\Phi(1))$ (see [**1** or **4**]).

Since $\Phi(s)$ is real symmetric for real s and is unitary for s of the form $1+it$, it follows that at this special point (the middle of the critical strip) $\Phi(1)$ is both unitary and real symmetric. Its eigenvalues are therefore ± 1, so that the trace is an integer which counts the excess of the eigenvalue $+1$ over -1 (or to put it another way we want the signature of $\Phi(1)$). We have

THEOREM 2. $\mathrm{tr}(\Phi(1)) = 2^{t-1} - 2$, *where, as above, t is the number of prime divisors of d_K.*

One final comment before turning to the proofs of these theorems: In the general number field case the formulas of Theorems 1 and 2 are similar, however the number 2^{t-1} is replaced by the number of elements of order two in the class group. In the quadratic case this number may be determined in terms of the divisors of d as we have done.

2. Functional equations and the matrix $\Phi(s)$. We begin this section by deriving the functional equation for the Eisenstein series using theta functions. A nice discussion of Eisenstein series and Epstein zeta functions appears in Terras [**9**, especially Chapter 5]. Since these functional equations are central to the approach in this paper we derive them directly using the notation we have introduced.

For a quaternion $w = x_1 + ix_2 + jy + kt$ we denote by $N(w)$ its norm: $N(w) = x_1^2 + x_2^2 + y^2 + t^2$. Its fundamental property is $N(w_1 w_2) = N(w_1)N(w_2)$. If $w \in \mathfrak{h}^3$ and $y(w)$ is its y-coordinate, then for $\tau \in \mathrm{PSL}_2(\mathbf{C})$, $\tau = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$, we have $y(\tau w) = y/N(\gamma w + \delta)$.

DEFINITION 2.1. Define $\tilde{E}_i(w,s)$ by

$$\tilde{E}_i(w,s) = \lambda_i^s \sum_{(c,d)=\mathcal{A}_i} \frac{y^s}{N(cw+d)^s}.$$

We will need the following, by now standard, lemma due to Selberg.

LEMMA 2.2. *Let $f(w)$ be a γ-automorphic function on $\mathfrak{h}^3$ satisfying*

$$\Delta f + s(2-s)f = 0$$

for some s with $\mathrm{Re}(s) > 2$ *and such that*

$$f(w) = \delta_{ij}(y^{(j)})^s + O(1) \quad \text{as } y^{(j)} \to \infty.$$

Then $f(w) = E_i(w,s)$.

PROOF. $H_i(w) = E_i(w,s) - f(w)$ is a Γ-automorphic eigenfunction of Δ with eigenvalue $s(2-s)$. In view of the behaviour of $f(w)$ in the cusps, H_i is in $L^2(\Gamma\backslash\mathfrak{h}^3)$. However, from the selfadjointness and nonnegativity of Δ it follows that its eigenvalue must be real and nonnegative. For $s(2-s)$ with $\mathrm{Re}(s) > 2$ this is not the case, so $H_i \equiv 0$.

PROPOSITION 2.3. $\tilde{E}_i(w,s) = E_i(w,s)$.

PROOF. We verify the conditions of Lemma 2.2 for $\tilde{E}_i$. First, $\tilde{E}_i(w,s)$ is Γ-automorphic, because if $\gamma = \left(\begin{smallmatrix} x & y \\ u & v \end{smallmatrix}\right) \in \Gamma$, then

$$\begin{aligned} \tilde{E}_i(\gamma w, s) &= \lambda_i^s \sum_{(c,d)=\mathcal{A}_i} \frac{y^s/N(uw+v)^s}{N(c(xw+y)(uw+v)^{-1}+d)^s} \\ &= \lambda_i^s \sum_{(c,d)=\mathcal{A}_i} \frac{y^s/N(uw+v)^s}{N(c(xw+y)+d(uw+v))^s N(uw+v)^{-s}} \\ &= \lambda_i^s \sum_{(c,d)=\mathcal{A}_i} \frac{y^s}{N((cx+du)w+(cy+dv))^s} \end{aligned}$$

and $cx+du$ and $cy+dv$ run over the basis of $\mathcal{A}_i$ as c and d do.

To check the behaviour of $\tilde{E}_i$ at the jth cusp, recall that

$$\rho_j^{-1} = \begin{pmatrix} \delta_j & -\beta_j \\ -\gamma_j & \alpha_j \end{pmatrix} \begin{pmatrix} \lambda_j^{-1/2} & 0 \\ 0 & \lambda_j^{1/2} \end{pmatrix}$$

so that

$$\begin{aligned} \tilde{E}_i(w,s) &= \tilde{E}_i(\rho_j^{-1}w^{(j)},s) = \lambda_i^s \sum_{(c,d)=\mathcal{A}_i} \frac{y(\rho_j^{-1}w^{(j)})^s}{N(c\rho_j^{-1}w^{(j)}+d)^s} \\ &= \lambda_i^s \sum_{(c,d)=\mathcal{A}_i} \frac{y^{(j)^s}/N(-\gamma_j\lambda_j^{-1/2}w^{(j)}+\alpha_j\lambda_j^{1/2})^s}{N(c(\delta_j\lambda_j^{-1/2}w^{(j)}-\beta_j\lambda_j^{1/2})(-\gamma_j\lambda_j^{-1/2}w^{(j)}+\alpha_j\lambda_j^{1/2})^{-1}+d)^s} \\ &= \lambda_i^s \sum_{(c,d)=\mathcal{A}_i} \frac{y^{(j)^s}}{N((c\delta_j\lambda_j^{-1/2}-d\gamma_j\lambda_j^{-1/2})w^{(j)}+(-c\beta_j\lambda_j^{1/2}+d\alpha_j\lambda_j^{1/2}))^s} \\ &= \frac{\lambda_i^s}{\lambda_j^s} \sum_{(c,d)=\mathcal{A}_i} \frac{y^{(j)^s}}{N(\lambda_j^{-1}(c\delta_j-d\gamma_j)w^{(j)}+(-c\beta_j+d\alpha_j))^s}. \end{aligned}$$

The only term in this sum which grows with $y^{(j)}$ is the one for which $c\delta_j - d\gamma_j = 0$, $-c\beta_j + d\alpha_j = 1$, which occurs if and only if $i = j$, so that the above is $\delta_{ij}[y^{(j)^s} + O(1)]$. Since $\tilde{E}_i$ is clearly an eigenfunction with eigenvalue $s(2-s)$, Lemma 2.2 implies our claim. $\square$

DEFINITION 2.4.

$$F_i(w,s) = \lambda_i^s \sideset{}{'}\sum_{\substack{c,d\in\mathcal{A}_i \\ (\mathrm{mod}\ \pm 1)}} \frac{y^s}{N(cw+d)^s}$$

(so that we run over half the lattice $\mathcal{A}_i \times \mathcal{A}_i$).

PROPOSITION 2.5.

$$\vec{F}(w,s) = \begin{bmatrix} F_1(w,s) \\ \vdots \\ F_h(w,s) \end{bmatrix} = [\zeta_{\mathcal{A}_i^{-1}\mathcal{A}_j}(s)] \begin{bmatrix} E_1(w,s) \\ \vdots \\ E_h(w,s) \end{bmatrix},$$

where $\zeta_{\mathcal{A}}(s)$ is the Dedekind zeta function of the ideal class of $\mathcal{A}$.

PROOF (SEE TERRAS [**9**]).

$$F_i(w,s) = \lambda_i^s \sum_{\mathcal{A}_i|\mathcal{B}} \sum_{(c,d)=\mathcal{B}} \frac{y^s}{N(cw+d)^s}$$

$$= \lambda_i^s \sum_{j=1}^{h} \sum_{\substack{\mathcal{A}_i|\mathcal{B}\\ \mathcal{B}\sim\mathcal{A}_j}} \sum_{(c,d)=\mathcal{B}} \frac{y^s}{N(cw+d)^s}.$$

Write $\mathcal{B} = (\theta_{\mathcal{B}})\mathcal{A}_j$ for some $\theta_{\mathcal{B}} \in K$. Then

$$\lambda_i^s \sum_{j=1}^{h} \sum_{\substack{\mathcal{A}_i|\mathcal{B}\\ \mathcal{B}\sim\mathcal{A}_j}} \frac{1}{N\theta_{\mathcal{B}}^s} \sum_{(c/\theta_{\mathcal{B}},d/\theta_{\mathcal{B}})=\mathcal{A}_j} \frac{y^s}{N(cw/\theta_{\mathcal{B}}+d/\theta_{\mathcal{B}})^s}$$

$$= \lambda_i^s \sum_{j=1}^{h} \left(\sum_{\substack{\mathcal{A}_i|\mathcal{B}\\ \mathcal{B}\sim\mathcal{A}_j}} \frac{1}{N\mathcal{B}^s} \right) N\mathcal{A}_j^s \sum_{(c,d)=\mathcal{A}_j} \frac{y^s}{N(cw+d)^s}.$$

Now let $\mathcal{C} = \mathcal{A}_i^{-1}\mathcal{B}$ so that $\mathcal{C} \sim \mathcal{A}_i^{-1}\mathcal{A}_j$, $N\mathcal{B} = N\mathcal{C}\cdot N\mathcal{A}_i$; then

$$= \sum_{j=1}^{h} \left(\sum_{\mathcal{C}\sim\mathcal{A}_i^{-1}\mathcal{A}_j} \frac{1}{N\mathcal{C}^s} \right) \lambda_j^s \sum_{(c,d)=\mathcal{A}_j} \frac{y^s}{N(cw+d)^s}. \qquad \square$$

PROPOSITION 2.6. *The function* $\omega_K^{-s}(d_K^{1/2}/2\pi)^s\Gamma(s)F_i(w,s)$ *is invariant under* $s \to 2-s$ *and* $\mathcal{A}_i \to (\partial_K \mathcal{A}_i)^{-1}$, *where* ∂_K *is the different of* K.

PROOF. Using the formula

$$\frac{\Gamma(s)}{a^s} = \int_0^\infty e^{-at} t^s \frac{dt}{t}$$

with $a = 2\pi/d_K^{1/2} N\mathcal{A}_i \cdot N(cw+d)/y$, we get

$$\omega_K^{-s}\cdot\left(\frac{d_K^{1/2}}{2\pi}\right)^s \Gamma(s)(\omega_K N\mathcal{A}_i)^s \sum_{c,d\in\mathcal{A}_i}{}' \frac{y^s}{N(cw+d)^s}$$

$$= \int_0^\infty \sum_{c,d\in\mathcal{A}_i}{}' \exp\left(-\frac{2\pi}{d_K^{1/2}N\mathcal{A}_i}\frac{N(cw+d)}{y}t\right) t^s \frac{dt}{t} \tag{2.6.1}$$

$$= \int_0^\infty \sum_{l\in\mathcal{A}_i\times\mathcal{A}_i}{}' \exp\left(-\frac{2\pi}{d_K^{1/2}N\mathcal{A}_i}\langle l, Al\rangle t\right) t^s \frac{dt}{t},$$

where A is the Hermitian matrix

$$A = \begin{pmatrix} Nw/y & z/y \\ \bar{z}/y & 1/y \end{pmatrix}.$$

We wish to apply the Poisson summation formula to this last expression. To this end, let

$$f(z_1, z_2) = e^{-r\langle Z, AZ\rangle}, \qquad Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

be thought of as a function of four real variables. Then for $(\beta, \gamma) \in \mathbf{C}^2$,

$$\hat{f}(\beta, \gamma) = \iint_{\mathbf{C}^2} f(z_1, z_2) e^{-2\pi i \mathrm{Re}(\overline{\beta} z_1 + \overline{\gamma} z_2)}\, dZ,$$

where $dZ = -\frac{1}{4} \cdot dz_1 \wedge \overline{dz_1} \cdot dz_2 \wedge \overline{dz_2}$,

$$= \iint_{\mathbf{C}^2} e^{-r\langle Z, AZ\rangle} e^{-2\pi i \mathrm{Re}\langle (\beta,\gamma), Z\rangle}\, dZ.$$

Since A is positive definite we can change variables $W = A^{1/2}Z$. Since $|A| = 1$ this leads to

$$\iint_{\mathbf{C}^2} e^{-r\langle W, W\rangle} e^{-2\pi i \mathrm{Re}\langle A^{-1/2}\binom{\beta}{\gamma}, W\rangle}\, dW.$$

Writing

$$W = \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad A^{-1/2}\begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \nu_1 \\ \nu_2 \end{bmatrix}$$

we have

$$\int \underset{\mathbf{R}^4}{\cdots} \int e^{-r(u_1^2+u_2^2+v_1^2+v_2^2)} e^{-2\pi i(\mu_1 v_1 + \mu_2 u_2 + \nu_1 v_1 + \nu_2 v_2)} du_1\, du_2\, dv_1\, dv_2.$$

Recalling that

$$\int_{-\infty}^{\infty} e^{-rx^2} e^{-2\pi i x y}\, dx = \sqrt{\frac{\pi}{r}} e^{-\pi^2 y^2/r},$$

we obtain

$$\begin{aligned} &\frac{\pi^2}{r^2} e^{-\pi^2(\mu_1^2+\mu_2^2+\nu_1^2+\nu_2^2)/r} \\ &\quad = \frac{\pi^2}{r^2} e^{-\pi^2\langle A^{-1/2}\binom{\beta}{\gamma}, A^{-1/2}\binom{\beta}{\gamma}\rangle/r} \\ &\quad = \frac{\pi^2}{r^2} e^{-\pi^2\langle \binom{\beta}{\gamma}, A^{-1}\binom{\beta}{\gamma}\rangle/r}. \end{aligned}$$

Going back to (2.6.1), we separate the integral into two parts to obtain

$$\begin{aligned} &\int_1^{\infty} \sum_{l \in \mathcal{A}_i \times \mathcal{A}_i}{}' \exp\left(-\frac{2\pi}{d_K^{1/2} N\mathcal{A}_i}\langle l, Al\rangle t\right) t^s \frac{dt}{t} \\ &\quad + \int_0^1 \sum_{l \in \mathcal{A}_i \times \mathcal{A}_i} \exp\left(-\frac{2\pi}{d_K^{1/2} N\mathcal{A}_i}\langle l, Al\rangle t\right) t^s \frac{dt}{t} - \frac{1}{s} \\ &= \mathrm{I} + \mathrm{II}. \end{aligned}$$

Applying the Poisson summation formula to II we get from our computation above

$$\mathrm{II} = \frac{1}{v} \int_0^1 \frac{\pi^2}{r^2} \sum_{l \in \mathcal{A}_i' \times \mathcal{A}_i'} \exp\left(-\frac{\pi^2}{r} \langle l, A^{-1} l \rangle \right) t^s \frac{dt}{t} - \frac{1}{s},$$

where $r = 2\pi t / d_K^{1/2} N\mathcal{A}_i$ and $\mathcal{A}_i'$ is the dual lattice of $\mathcal{A}_i$, so that $\mathcal{A}_i' = 2(\partial_K \mathcal{A}_i)^{-1}$. Here $v = \operatorname{vol}(\mathbf{C}^2 / \mathcal{A}_i \times \mathcal{A}_i) = d_K / 4 \cdot N\mathcal{A}_i^2$. Thus

$$\mathrm{II} = \int_0^1 \sum_{l \in (\partial_K \mathcal{A}_i)^{-1} \times (\partial_K \mathcal{A}_i)^{-1}} \exp\left(-\frac{\pi d_K^{1/2} N\mathcal{A}_i}{2} \langle 2l, 2A^{-1} l \rangle t^{-1} \right) t^{s-2} \frac{dt}{t} - \frac{1}{s}.$$

Now change variables $t \to t^{-1}$

$$= \int_1^\infty \sum_{l \in (\partial_K \mathcal{A}_i)^{-1} \times (\partial_K \mathcal{A}_i)^{-1}} \exp(-2\pi d_K^{1/2} N\mathcal{A}_i \langle l, A^{-1} l \rangle t) t^{2-s} \frac{dt}{t} - \frac{1}{s}$$

$$= \int_1^\infty \sum_{c,d \in (\partial_K \mathcal{A}_i)^{-1}}{}' \exp\left(-2\pi d_K^{1/2} N\mathcal{A}_i \frac{N(cw+d)}{y} t \right) t^{2-s} \frac{dt}{t} - \frac{1}{s} - \frac{1}{2-s}.$$

Finally we note that

$$d_K^{1/2} N\mathcal{A}_i = \frac{1}{d_K^{1/2} N(\mathcal{A}_i \partial_K)^{-1}}$$

and comparing our last expression with (2.6.1) completes the proof. $\square$

COROLLARY 2.7. *Let P be the permutation matrix of the permutation of the ideal class group given by $\mathcal{A}_i \to (\partial_K \mathcal{A}_i)^{-1}$. Then*

$$\vec{F}(w,s) = \omega_K^{2s-2} \frac{(d_K^{1/2}/2\pi)^{2-s} \Gamma(2-s)}{(d_K^{1/2}/2\pi)^s \Gamma(s)} \cdot P \cdot \vec{F}(w, 2-s).$$

Putting Proposition 2.5 and Corollary 2.7 together we obtain

$$[\zeta_{\mathcal{A}_i^{-1}\mathcal{A}_j}(s)] \vec{E}(w,s) = \omega_K^{2s-2} \frac{(d_K^{1/2}/2\pi)^{2-s} \Gamma(2-s)}{(d_K^{1/2}/2\pi)^s \Gamma(s)} \cdot P \cdot [\zeta_{\mathcal{A}_i^{-1}\mathcal{A}_j}(2-s)] \vec{E}(w, 2-s).$$

But then Lemma 1.2 implies

COROLLARY 2.8.

$$\Phi(s) = \omega_K^{2s-2} \frac{(d_K^{1/2}/2\pi)^{2-s} \Gamma(2-s)}{(d_K^{1/2}/2\pi)^s \Gamma(s)} [\zeta_{\mathcal{A}_i^{-1}\mathcal{A}_j}(s)]^{-1} \cdot P \cdot [\zeta_{\mathcal{A}_i^{-1}\mathcal{A}_j}(2-s)].$$

3. Computations of $\phi(s)$ and $\operatorname{tr}\Phi(1)$. We can finally turn to the computation of $\phi(s)$. We first find $\det[\zeta_{\mathcal{A}_i^{-1}\mathcal{A}_j}(s)]$.

PROPOSITION 3.1. *Let χ be a character of the ideal class group and let $L(s,\chi)$ be its L-function. Then*

$$[\zeta_{\mathcal{A}_i^{-1}\mathcal{A}_j}(s)] \begin{bmatrix} \chi(\mathcal{A}_1) \\ \vdots \\ \chi(\mathcal{A}_h) \end{bmatrix} = L(s,\chi) \begin{bmatrix} \chi(\mathcal{A}_1) \\ \vdots \\ \chi(\mathcal{A}_h) \end{bmatrix}.$$

PROOF.

$$\sum_{j=1}^{h} \varsigma_{\mathcal{A}_i^{-1}\mathcal{A}_j}(s)\cdot\chi(\mathcal{A}_j) = \sum_{j=1}^{h}\sum_{\mathcal{B}\sim\mathcal{A}_i^{-1}\mathcal{A}_j} \frac{\chi(\mathcal{A}_j)}{N\mathcal{B}^s}$$
$$= \chi(\mathcal{A}_i)\sum_{j=1}^{h}\sum_{\mathcal{B}\sim\mathcal{A}_i^{-1}\mathcal{A}_j}\frac{\chi(\mathcal{B})}{N\mathcal{B}^s} = \chi(\mathcal{A}_i)\cdot L(s,\chi). \qquad \square$$

Since the determinant is the product of the eigenvalues, we infer using class field theory (see Lang [**6**, XII, §1])

COROLLARY 3.2. $\det[\varsigma_{\mathcal{A}_i^{-1}\mathcal{A}_j}(s)] = \prod_\chi L(s,\chi) = \varsigma_H(s)$, *where H is the Hilbert class field of K.*

It remains to find $\det(P)$. First, for an imaginary quadratic field the different is a principal ideal, so that the permutation is actually $\mathcal{A}_i \to \mathcal{A}_i^{-1}$, and therefore $\det(P) = (-1)^{(h-m)/2}$, where m is the number of elements in the ideal class group of order 2. This, however, can be made more explicit using genus theory (see Hecke [**3**, p. 176]), which asserts that $m = 2^{t-1}$, where t is the number of prime divisors of d_K.

Combining Corollaries 2.8 and 3.2 and the above remark concludes the proof of Theorem 1.

We now prove Theorem 2. Let $M = (\overline{\chi}_i(\mathcal{A}_j))_{i,j=1,\dots,h}$ so that by Proposition 3.1

$$[\varsigma_{\mathcal{A}_i^{-1}\mathcal{A}_j}(s)] = M^{-1}\begin{bmatrix} L(s,\chi_1) & & \\ & \ddots & \\ & & L(s,\chi_h)\end{bmatrix} M.$$

Then

$$\operatorname{Tr}(\Phi(s)) = \operatorname{Tr}\left[\omega_K^{2s-2}\frac{(d_K^{1/2}/2\pi)^{2-s}\Gamma(2-s)}{(d_K^{1/2}/2\pi)^s\Gamma(s)}\cdot P\cdot[\varsigma_{\mathcal{A}_i^{-1}\mathcal{A}_j}(2-s)][\varsigma_{\mathcal{A}_i^{-1}\mathcal{A}_j}(s)]^{-1}\right]$$
$$= \operatorname{Tr}\left[\omega_K^{2s-2}\frac{(d_K^{1/2}/2\pi)^{2-s}\Gamma(2-s)}{(d_K^{1/2}/2\pi)^s\Gamma(s)}\cdot P\cdot M^{-1}\begin{bmatrix} L(2-s,\chi_1) & & \\ & \ddots & \\ & & L(2-s,\chi_h)\end{bmatrix}\right.$$
$$\left.\times\begin{bmatrix} L(s,\chi_1)^{-1} & & \\ & \ddots & \\ & & L(s,\chi_h)^{-1}\end{bmatrix} M\right].$$

If we assume that $\chi_1 = 1$, then

$$\lim_{s\to 1}\frac{L(2-s,\chi_1)}{L(s,\chi_1)} = -1$$

so that

$$\operatorname{Tr}(\Phi(1)) = \lim_{s\to 1}\operatorname{Tr}(\Phi(s))$$
$$= \operatorname{Tr}\left(PM^{-1}\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{bmatrix}M\right) = \operatorname{Tr}\left(P\frac{-2}{h}\begin{bmatrix} 1 & 1 & \cdots & 1\\ 1 & 1 & \cdots & 1\\ \vdots & \vdots & & \vdots\\ 1 & 1 & \cdots & 1\end{bmatrix} + P\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1\end{bmatrix}\right).$$

Recalling the definition of P we have $\operatorname{Tr}(\Phi(1)) = m-2 = 2^{t-1}-2$, which is Theorem 2.

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