THE DETERMINANT OF THE EISENSTEIN MATRIX
AND HILBERT CLASS FIELDS

BY

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ABSTRACT. We compute the determinant of the Eisenstein matrix associated
to the Hilbert-Blumenthal modular group \( \text{PSL}_2(\mathcal{O}_K) \), and express it in terms
of the zeta function of the Hilbert class field of \( K \).

0. Introduction. The Eisenstein series for the modular group \( \Gamma = \text{PSL}_2(\mathbb{Z}) \) is
given by
\[
E(z, s) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} y(\gamma z)^s,
\]
where \( \Gamma_\infty = \{ (1, 0) | n \in \mathbb{Z} \} \), \( z \) is in the upper half-plane and \( y(z) = \text{Im}(z) \). The
Fourier expansion of \( E(z, s) \) is well known (see Hejhal [4, p. 76]):
\[
E(z, s) = y^s + \phi(s)y^{1-s} + \text{nonzero coefficients},
\]
where
\[
\phi(s) = \frac{\pi^{-(s-1/2)}\Gamma(s-1/2)\zeta(2s-1)}{\pi^{-s}\Gamma(s)\zeta(2s)},
\]
\( \Gamma(s) \) is the gamma function and \( \zeta(s) \) is the Riemann zeta function.

In the general case of a group with \( h \) cusps one has an Eisenstein series at each
cusp. The Fourier expansions of these series in the various cusps lead to an \( h \times h \)
matrix \( \Phi(s) \) (see (1.11) below). This matrix is the constant term of the Eisenstein
series, and is sometimes referred to as the scattering matrix [7], or simply the
\( \Phi \)-matrix.

The determinant \( \phi(s) = \det \Phi(s) \) plays an important role in the theory. In the
first place it controls the Eisenstein series in the sense that if \( \phi(s) \) is analytic at
some point, then so are all the Eisenstein series. More importantly, \( \phi(s) \) (or rather
its logarithmic derivative) accounts for the contribution of the continuous spectrum
to the Selberg Trace Formula. A classical situation in which multiplicity of cusps
occurs is that of congruence subgroups of the modular group, and the computation
of \( \phi(s) \) for these is quite involved (see Hejhal [4, Chapter 12] and Huxley [5]).

Our aim in this paper is to calculate \( \phi(s) \) for the Hilbert-Blumenthal modular
groups \( \text{PSL}_2(\mathcal{O}_K) \), \( \mathcal{O}_K \) being the ring of integers of a number field \( K \). In this
case the number of cusps is the class number of \( K \). What we show is that \( \phi(s) \) is
elegantly expressed in terms of the zeta function of the Hilbert class field of \( K \). An
interesting feature of our approach is that we do not use any Fourier developments

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in cusps. Instead we obtain our result purely from considerations of functional
equations. To keep notations to a minimum we shall deal only with imaginary
quadratic fields. The general case may be dealt with in a similar way (see Efrat [2]
for the set up for totally real fields).

1. General set up and results. Let \( h^3 \) be the hyperbolic three space \( \{ w = (y, x_1, x_2) = (y, z) \mid y > 0 \} \) with its Lobachevskii metric
\[
\frac{ds^2}{y^2} = \frac{dy^2 + dx_1^2 + dx_2^2}{y^2}.
\]

If we think of \( h^3 \) as the set of all quaternions \( x_1 + ix_2 + jy + kt \) for which \( t = 0 \),
then \( G = \text{PSL}_2(\mathbb{C}) \) is the group of all orientations preserving isometries of \( h^3 \). It
acts via linear fractional transformations, i.e., for \( \tau = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G, \)
\[
\tau(w) = (\alpha w + \beta)(\gamma w + \delta)^{-1}.
\]

Consider next a discrete subgroup \( \Gamma \subset G \) for which \( \mathcal{F} = h/\Gamma \) is not compact
but is of finite volume. Let \( \kappa_1 = \infty, \kappa_2, \ldots, \kappa_n \) be a complete set of inequivalent
cusps, and let \( \Gamma_i \) be the subgroup of \( \Gamma \) that fixes \( \kappa_i \). We choose \( \rho_i \in G \) such that
\[
\rho_i(\kappa_i) = \infty \text{ and such that } \rho_i \Gamma_i \rho_i^{-1} = \left\{ \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \mid l \in L_i \right\},
\]
where \( L_i \) is a lattice in \( \mathbb{C} \) whose fundamental domain has volume 1. We let
\[
w^{(i)} = \rho_i w = (y(\rho_i w), z(\rho_i w)).
\]

**DEFINITION 1.1.** For each cusp \( \kappa_i \), define its Eisenstein series to be
\[
E_i(w, s) = \sum_{\gamma \in \Gamma_i \setminus \Gamma} y^{(i)}(\gamma w)^s, \quad \Re(s) > 2.
\]

Then Selberg’s theory shows that \( E_i(w, s) \) can be meromorphically continued in
\( s \) to all of \( \mathbb{C} \), and gives a \( \Gamma \)-automorphic eigenfunction of the Laplacian \( \Delta \) of \( h^3 \),
with
\[
\Delta E_i(w, s) + s(2 - s)E_i(w, s) = 0.
\]

Since \( E_i(w, s) \) is \( \Gamma \)-automorphic, it is invariant under the lattice at \( \kappa_j \). It thus
admits a Fourier expansion there, which is of the form
\[
E_i(w, s) = \delta_{ij} y^{(j)*} + \phi_{ij}(s)y^{(j)2-s}
\]
(1.1.1) + nonzero coefficients , rapidly decaying as \( y^{(j)} \to \infty \).

If we let \( \Phi(s) = (\phi_{ij}(s))_{i,j=1,\ldots,n} \) and
\[
\tilde{E}(w, s) = \begin{bmatrix} E_1(w, s) \\ \vdots \\ E_n(w, s) \end{bmatrix},
\]
then we have the functional equation
\[
\tilde{E}(w, s) = \Phi(s)\tilde{E}(w, 2 - s).
\]
(1.1.2)

(For these facts see Cohen and Sarnak [1], Hejhal [4] or Sarnak [8].) The following
lemma shows that \( \Phi(s) \) is characterized by this equation.
Lemma 1.2. If \( \vec{E}(w, s) = \Psi(s)\vec{E}(w, 2-s) \) for every \( w \in \mathfrak{h}^3 \), then \( \Psi(s) = \Phi(s) \).

Proof. We have

\[
E_i(w, s) = \sum_{k=1}^{h} \psi_{ik}(s) E_k(w, 2-s)
\]

(1.2.1)

\[
= \sum_{k=1}^{h} \psi_{ik}(s)(\delta_k y^{(j)^{2-s}} + \phi_{kj}(2-s)y^{(j)^{s}} + \ldots).
\]

Now choose \( s \) with \( \text{Re}(s) < 0 \). Then as \( y^{(j)} \to \infty \) only the terms involving \( y^{(j)^{2-s}} \) will increase. Comparing (1.2.1) with (1.1.1) gives

\[
\phi_{ij}(s) = \sum_{k=1}^{h} \psi_{ik} \delta_{kj} = \psi_{ij}(s). \quad \square
\]

Let \( \Gamma = \Gamma_D \) be the Hilbert modular group associated to the imaginary quadratic number field \( K = Q(\sqrt{-D}) \), i.e.,

\[
\Gamma_D = \text{PSL}_2(\mathcal{O}_K) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \mid a, b, c, d \in \mathcal{O}_K, \ ad - bc = 1 \right\} / \{ \pm 1 \},
\]

where \( \mathcal{O}_K \) is the ring of integers of \( K \). We also assume \( D \neq 1, 3 \).

Then the cusps of \( \Gamma \) are the numbers \( \alpha/\beta \in \mathbb{C} \) with \( \alpha, \beta \in \mathcal{O}_K \) and \( \infty \). Furthermore, \( \alpha/\beta \) and \( \alpha'/\beta' \) are equivalent under \( \Gamma \) as cusps if and only if \( (\alpha, \beta) \) and \( (\alpha', \beta') \) are equivalent ideals. So if \( \mathcal{A}_1, \ldots, \mathcal{A}_h, \mathcal{A}_i = (\gamma_i, \delta_i) \), are a complete set of representatives of the ideal classes of \( K \), then \( \kappa_i = -\delta_i/\gamma_i, i = 1, \ldots, h \), form a complete set of inequivalent cusps.

To define the \( \rho_i \)'s we first note that we can choose \( \alpha_i, \beta_i \in \mathcal{A}_i^{-1} \) such that

\[
\hat{\rho}_i = \left( \begin{array}{cc} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{array} \right) \in G.
\]

Note that \( \hat{\rho}_i(\kappa_i) = \infty \).

To guarantee the volume 1 condition, we scale \( \hat{\rho}_i \) and define

\[
\rho_i = \left( \begin{array}{cc} \lambda_i^{1/2} & 0 \\ 0 & \lambda_i^{-1/2} \end{array} \right) \hat{\rho}_i,
\]

where \( \lambda_i = \omega_K N \mathcal{A}_i, \omega_K = \sqrt{2/d_K^{1/4}} \) and \( d_K \) is the absolute value of the discriminant of \( K \).

By the discussion above, we now have our Eisenstein series defined for \( \Gamma_D \). Our main theorem is

Theorem 1. For \( \Gamma_D \),

\[
\phi(s) = (-1)^{(h-2^t-1)/2} \omega_K^{2g-2} \xi_H(s-1) / \xi_H(s),
\]

where \( \xi_H(s) = (d_H^{1/2}/(2\pi)^h)^2 \Gamma(s)^{h_5H(s)}, H \) is the Hilbert class field of \( K \) and \( t \) is the number of prime divisors of \( d_K \).

Remarks. (i) Since \( H \) is unramified everywhere it follows that \( \xi_H(s) \) is the usual \( \xi \) function for the Hilbert class field \( H \), so that \( \xi_H(1-s) = \xi_H(s) \).
(ii) The exponent \((h - 2t^{-1})/2\) of \(-1\) in the formula is always an integer by genus theory (see, for example, Hecke [3, p. 160]).

Besides the contribution \(\phi(s)\) to the trace formula there is also one other term that appears and which comes from the Eisenstein series. It is the term \(\text{tr}(\Phi(1))\) (see [1 or 4]).

Since \(\Phi(s)\) is real symmetric for real \(s\) and is unitary for \(s\) of the form \(1 + it\), it follows that at this special point (the middle of the critical strip) \(\Phi(1)\) is both unitary and real symmetric. Its eigenvalues are therefore \(\pm 1\), so that the trace is an integer which counts the excess of the eigenvalue +1 over -1 (or to put it another way we want the signature of \(\Phi(1)\)). We have

**Theorem 2.** \(\text{tr}(\Phi(1)) = 2^{t-1} - 2\), where, as above, \(t\) is the number of prime divisors of \(d_K\).

One final comment before turning to the proofs of these theorems: In the general number field case the formulas of Theorems 1 and 2 are similar, however the number \(2t^{-1}\) is replaced by the number of elements of order two in the class group. In the quadratic case this number may be determined in terms of the divisors of \(d\) as we have done.

### 2. Functional equations and the matrix \(\Phi(s)\).

We begin this section by deriving the functional equation for the Eisenstein series using theta functions. A nice discussion of Eisenstein series and Epstein zeta functions appears in Terras [9, especially Chapter 5]. Since these functional equations are central to the approach in this paper we derive them directly using the notation we have introduced.

For a quaternion \(w = x_1 + ix_2 + jy + kt\) we denote by \(N(w)\) its norm: \(N(w) = x_1^2 + x_2^2 + y^2 + t^2\). Its fundamental property is \(N(wiw_2) = N(w_1)N(w_2)\). If \(w \in \mathfrak{h}^3\) and \(y(w)\) is its \(y\)-coordinate, then for \(\tau \in \text{PSL}_2(\mathbb{C})\), \(\tau = \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right)\), we have \(y(\tau w) = y/N(\gamma w + \delta)\).

**Definition 2.1.** Define \(\tilde{E}_i(w, s)\) by

\[
\tilde{E}_i(w, s) = \lambda_i^s \sum_{(c,d) = A_i} \frac{y^s}{N(cw + d)^s}.
\]

We will need the following, by now standard, lemma due to Selberg.

**Lemma 2.2.** Let \(f(w)\) be a \(\gamma\)-automorphic function on \(\mathfrak{h}^3\) satisfying

\[\Delta f + s(2-s)f = 0\]

for some \(s\) with \(\text{Re}(s) > 2\) and such that

\[f(w) = \delta_{ij}(y^{(j)})^s + O(1) \text{ as } y^{(j)} \to \infty.\]

Then \(f(w) = E_i(w, s)\).

**Proof.** \(H_i(w) = E_i(w, s) - f(w)\) is a \(\Gamma\)-automorphic eigenfunction of \(\Delta\) with eigenvalue \(s(2-s)\). In view of the behaviour of \(f(w)\) in the cusps, \(H_i\) is in \(L^2(\Gamma \backslash \mathfrak{h}^3)\). However, from the selfadjointness and nonnegativity of \(\Delta\) it follows that its eigenvalue must be real and nonnegative. For \(s(2-s)\) with \(\text{Re}(s) > 2\) this is not the case, so \(H_i \equiv 0\).
Proposition 2.3. \( \tilde{E}_i(w, s) = E_i(w, s) \).

Proof. We verify the conditions of Lemma 2.2 for \( \tilde{E}_i \). First, \( \tilde{E}_i(w, s) \) is \( \Gamma \)-automorphic, because if \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \), then

\[
\tilde{E}_i(\gamma w, s) = \lambda_i^s \sum_{(c,d) = A_i} \frac{y^s / N(uw + v)^s}{N(c(xw + y)(uw + v) - 1 + d)^s}
= \lambda_i^s \sum_{(c,d) = A_i} \frac{y^s / N(uw + v)^s}{N(c(xw + y) + d(uw + v))^sN(uw + v)^{s}}
= \lambda_i^s \sum_{(c,d) = A_i} \frac{y^s}{N((cx + du)w + (cy + dv))^s}
\]

and \( cx + du \) and \( cy + dv \) run over the basis of \( A_i \) as \( c \) and \( d \) do.

To check the behaviour of \( \tilde{E}_i \) at the \( j \)th cusp, recall that

\[
\rho_j^{-1} = \begin{pmatrix} \delta_j & -\beta_j \\ -\gamma_j & \alpha_j \end{pmatrix}
= \begin{pmatrix} \lambda_j^{-1/2} & 0 \\ 0 & \lambda_j^{-1/2} \end{pmatrix}
\]

so that

\[
\tilde{E}_i(w, s) = \tilde{E}_i(\rho_j^{-1}w(j), s) = \lambda_i^s \sum_{(c,d) = A_i} \frac{y(\rho_j^{-1}w(j))^s}{N(c(\rho_j^{-1}w(j) + d)^s}
= \lambda_i^s \sum_{(c,d) = A_i} \frac{y(j)^s}{N(c(-\gamma_j\lambda_j^{-1/2}w(j) + \alpha_j\lambda_j^{1/2})^s\lambda_j^{-1/2}w(j) + (\alpha_j\lambda_j^{1/2} + d\alpha_j\lambda_j^{1/2}))^s
= \frac{\lambda_i^s}{\lambda_j^s} \sum_{(c,d) = A_i} \frac{y(j)^s}{N((c\delta_j\lambda_j^{1/2} - d\gamma_j\lambda_j^{1/2})w(j) + (-c\beta_j\lambda_j^{1/2} + d\alpha_j\lambda_j^{1/2}))^s}
\]

The only term in this sum which grows with \( y(j)^s \) is the one for which \( c\delta_j - d\gamma_j = 0, -c\beta_j + d\alpha_j = 1 \), which occurs if and only if \( i = j \), so that the above is \( \delta_{ij}[y(j)^s + O(1)] \). Since \( \tilde{E}_i \) is clearly an eigenfunction with eigenvalue \( s(2 - s) \), Lemma 2.2 implies our claim. □

Definition 2.4.

\[
F_i(w, s) = \lambda_i^s \sum_{c,d \in A_i (mod \pm 1)} \frac{y^s}{N(cw + d)^s}
\]

(so that we run over half the lattice \( A_i \times A_i \)).

Proposition 2.5.

\[
\tilde{F}(w, s) = \begin{bmatrix} F_1(w, s) \\ \vdots \\ F_h(w, s) \end{bmatrix} = [\zeta_{A^{-1}}(s)] \begin{bmatrix} E_1(w, s) \\ \vdots \\ E_h(w, s) \end{bmatrix}
\]

where \( \zeta_A(s) \) is the Dedekind zeta function of the ideal class of \( A \).
Proof (see Terras [9]).

\[ F_i(w, s) = \lambda_i^s \sum_{\substack{A_i | B \ (c,d) = A_j \ B \sim A_j}} \frac{y^s}{N(cw + d)^s} = \lambda_i^s \sum_{j=1}^{h} \sum_{B \sim A_j} \frac{1}{N^{\theta_B}} \sum_{(c,d) = A_j} \frac{y^s}{N(cw/\theta_B + d/\theta_B)^s} \]

Write \( B = (\theta_B)_A j \) for some \( \theta_B \in K \). Then

\[ \sum_{j=1}^{h} \left( \sum_{B \sim A_j} \frac{1}{NB^s} \right) N^{\theta_B} \sum_{(c,d) = A_j} \frac{y^s}{N(cw + d)^s} \]

Now let \( C = A_i^{-1} B \) so that \( C \sim A_i^{-1} A_j, NB = NC \cdot NA_i \); then

\[ \sum_{j=1}^{h} \left( \sum_{C \sim A_i^{-1} A_j} \frac{1}{NC^s} \right) \lambda_j^s \sum_{(c,d) = A_j} \frac{y^s}{N(cw + d)^s}. \]

Proposition 2.6. The function \( \omega_K^{-s} (d^{1/2}/2\pi)^s \Gamma(s) F_i(w, s) \) is invariant under \( s \rightarrow 2 - s \) and \( A_i \rightarrow (\partial_KA_i)^{-1} \), where \( \partial_K \) is the different of \( K \).

Proof. Using the formula

\[ \frac{\Gamma(s)}{a^s} = \int_0^{\infty} e^{-at} t^s \frac{dt}{t} \]

with \( a = 2\pi/d_K^{1/2}N A_i \cdot N(cw + d)/y \), we get

\[ \omega_K^{-s} \cdot \left( \frac{d^{1/2}/2\pi}{N A_i} \right)^s \Gamma(s) (\omega_K N A_i)^s \sum_{c,d \in A_i} \frac{y^s}{N(cw + d)^s} \]

(2.6.1)

\[ = \int_0^{\infty} \sum_{c,d \in A_i} \exp \left( -\frac{2\pi}{d^{1/2}/N A_i} \frac{N(cw + d)}{y} \right) t^s \frac{dt}{t} \]

\[ = \int_0^{\infty} \sum_{l \in A_i \times A_i} \exp \left( -\frac{2\pi}{d^{1/2}/N A_i} (l, A_l) t \right) t^s \frac{dt}{t} \]

where \( A \) is the Hermitian matrix

\[ A = \begin{pmatrix} Nw/y & z/y \\ \bar{z}/y & 1/y \end{pmatrix}. \]
We wish to apply the Poisson summation formula to this last expression. To this end, let
\[f(z_1, z_2) = e^{-r(Z,AZ)}, \quad Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},\]
be thought of as a function of four real variables. Then for \((\beta, \gamma) \in \mathbb{C}^2,\)
\[\hat{f}(\beta, \gamma) = \int_{\mathbb{C}^2} f(z_1, z_2) e^{-2\pi i \text{Re}(\beta z_1 + \gamma z_2)} \, dZ,\]
where \(dZ = -\frac{1}{4} \cdot dz_1 \wedge \overline{dz_1} \cdot dz_2 \wedge \overline{dz_2},\)
\[= \int_{\mathbb{C}^2} e^{-r(Z,AZ)} e^{-2\pi i \text{Re}(\beta, \gamma), Z} \, dZ.\]

Since \(A\) is positive definite we can change variables \(W = A^{1/2}Z.\) Since \(|A| = 1\) this leads to
\[\int_{\mathbb{C}^2} e^{-r(W,W)} e^{-2\pi i \text{Re}(A^{-1/2}(\beta),W)} \, dW.\]

Writing
\[W = \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad A^{-1/2} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \nu_1 \\ \nu_2 \end{pmatrix},\]
we have
\[\int \ldots \int_{\mathbb{R}^8} e^{-r(u_1^2 + u_2^2 + v_1^2 + v_2^2)} e^{-2\pi i (\mu_1 u_1 + \mu_2 u_2 + \nu_1 v_1 + \nu_2 v_2)} \, du_1 \, du_2 \, dv_1 \, dv_2.\]

Recalling that
\[\int_{-\infty}^{\infty} e^{-rx^2} e^{-2\pi i xy} \, dx = \sqrt{\frac{\pi}{r}} e^{-\pi^2 y^2 / r},\]
we obtain
\[\frac{\pi^2}{r^2} e^{-\pi^2 (\mu_1^2 + \mu_2^2 + \nu_1^2 + \nu_2^2) / r} = \frac{\pi^2}{r^2} e^{-\pi^2 (A^{-1/2}(\beta), A^{-1/2}(\gamma)) / r} = \frac{\pi^2}{r^2} e^{-\pi^2 (\beta, A^{-1}(\gamma)) / r}.\]

Going back to (2.6.1), we separate the integral into two parts to obtain
\[\int_1^{\infty} \sum'_{l \in \mathcal{A}_i \times \mathcal{A}_i} \exp \left( -\frac{2\pi}{d_K^{1/2} N A_{l_i}} \langle l, A l \rangle t \right) \frac{t^s \, dt}{t} \]
\[+ \int_0^1 \sum_{l \in \mathcal{A}_i \times \mathcal{A}_i} \exp \left( -\frac{2\pi}{d_K^{1/2} N A_{l_i}} \langle l, A l \rangle t \right) \frac{t^s \, dt}{t} - \frac{1}{s} \]
\[= I + II.\]
Applying the Poisson summation formula to II we get from our computation above

\[
II = \frac{1}{v} \int_0^1 \pi^2 \sum_{l \in \mathcal{A}'_i \times \mathcal{A}_i} \exp \left( -\frac{\pi^2}{r} (l, A^{-1}l) \right) t^{s-2} dt - \frac{1}{s},
\]

where \( r = 2\pi t / d_K^{1/2} \mathcal{N} \mathcal{A}_i \) and \( \mathcal{A}'_i \) is the dual lattice of \( \mathcal{A}_i \), so that \( \mathcal{A}'_i = 2(\partial_K \mathcal{A}_i)^{-1} \).

Here \( v = \text{vol}(\mathbb{C}^2/\mathcal{A}_i \times \mathcal{A}_i) = d_K/4 \cdot \mathcal{N} \mathcal{A}_i^2 \). Thus

\[
II = \int_0^1 \sum_{l \in (\partial_K \mathcal{A}_i)^{-1} \times (\partial_K \mathcal{A}_i)^{-1}} \exp \left( -\frac{\pi^2}{r} (2l, 2A^{-1}l) t^{s-2} dt - \frac{1}{s} \right).
\]

Now change variables \( t \rightarrow t^{-1} \)

\[
= \int_1^\infty \sum_{l \in (\partial_K \mathcal{A}_i)^{-1} \times (\partial_K \mathcal{A}_i)^{-1}} \exp(-2\pi d_K^{1/2} \mathcal{N} \mathcal{A}_i (l, A^{-1}l) t) t^{2-s} dt - \frac{1}{s} = \frac{1}{2-s}.
\]

Finally we note that

\[
d_K^{1/2} \mathcal{N} \mathcal{A}_i = \frac{1}{d_K^{1/2} \mathcal{N}(\partial_K \mathcal{A}_i)^{-1}}
\]

and comparing our last expression with (2.6.1) completes the proof. □

**Corollary 2.7.** Let \( P \) be the permutation matrix of the permutation of the ideal class group given by \( \mathcal{A}_i \rightarrow (\partial_K \mathcal{A}_i)^{-1} \). Then

\[
\tilde{F}(w, s) = \omega_K^{2s-2} \left( d_K^{1/2} / 2\pi \right)^{2-s} \Gamma(2-s) \cdot P \cdot \bar{F}(w, 2-s).
\]

Putting Proposition 2.5 and Corollary 2.7 together we obtain

\[
[\zeta^{-1}_1 \zeta^{-1}_j(s)] \tilde{E}(w, s) = \omega_K^{2s-2} \left( d_K^{1/2} / 2\pi \right)^{2-s} \Gamma(2-s) \cdot P \cdot [\zeta^{-1}_1 \zeta^{-1}_j(2-s)] \tilde{E}(w, 2-s).
\]

But then Lemma 1.2 implies

**Corollary 2.8.**

\[
\Phi(s) = \omega_K^{2s-2} \left( d_K^{1/2} / 2\pi \right)^{2-s} \Gamma(2-s) \cdot P \cdot [\zeta^{-1}_1 \zeta^{-1}_j(2-s)]^{-1} \cdot \bar{E}(w, 2-s).
\]

**3. Computations of \( \phi(s) \) and \( \text{tr} \Phi(1) \).** We can finally turn to the computation of \( \phi(s) \). We first find \( \text{det} [\zeta^{-1}_1 \zeta^{-1}_j(s)] \).

**Proposition 3.1.** Let \( \chi \) be a character of the ideal class group and let \( L(s, \chi) \) be its \( L \)-function. Then

\[
[\zeta^{-1}_1 \zeta^{-1}_j(s)] \begin{bmatrix} \chi(A_1) \\ \vdots \\ \chi(A_h) \end{bmatrix} = L(s, \chi) \begin{bmatrix} \chi(A_1) \\ \vdots \\ \chi(A_h) \end{bmatrix}.
\]
Proof.

\[
\sum_{j=1}^{h} \zeta_{A_i^{-1}A_j}(s) \cdot \chi(A_j) = \sum_{j=1}^{h} \sum_{B \sim A_i^{-1}A_j} \frac{\chi(A_j)}{NB^s} = \chi(A_i) \sum_{j=1}^{h} \sum_{B \sim A_i^{-1}A_j} \frac{\chi(B)}{NB^s} = \chi(A_i) \cdot L(s, \chi). \]

Since the determinant is the product of the eigenvalues, we infer using class field theory (see Lang [6, XII, §1])

**Corollary 3.2.** \(\det[\zeta_{A_i^{-1}A_j}(s)] = \prod_{\chi} L(s, \chi) = \zeta_H(s)\), where \(H\) is the Hilbert class field of \(K\).

It remains to find \(\det(P)\). First, for an imaginary quadratic field the different is a principal ideal, so that the permutation is actually \(A_i \rightarrow A_i^{-1}\), and therefore \(\det(P) = (-1)^{(h-m)/2}\), where \(m\) is the number of elements in the ideal class group of order 2. This, however, can be made more explicit using genus theory (see Hecke [3, p. 176]), which asserts that \(m = 2t_1\), where \(t\) is the number of prime divisors of \(d_K\).

Combining Corollaries 2.8 and 3.2 and the above remark concludes the proof of Theorem 1.

We now prove Theorem 2. Let \(M = (\chi_i(A_j))_{i,j=1,\ldots,h}\) so that by Proposition 3.1

\[
[\zeta_{A_i^{-1}A_j}(s)] = M^{-1} \begin{bmatrix} L(s, \chi_1) & \cdots & \cdot \cdot \cdot & \cdot \cdot \cdot & L(s, \chi_h) \end{bmatrix} M.
\]

Then

\[
\text{Tr}(\Phi(s)) = \text{Tr} \left[ \omega_K^{2s - 2} \frac{2^{1/2}/2\pi}{(d_K^{1/2}/2\pi)^s} \Gamma(2-s) \cdot P \cdot [\zeta_{A_i^{-1}A_j}((2-s))][\zeta_{A_i^{-1}A_j}(s)]^{-1} \right]
\]

\[
= \text{Tr} \left[ \omega_K^{2s-2} \frac{(d_K^{1/2}/2\pi)^{2-s}\Gamma(2-s)}{(d_K^{1/2}/2\pi)^s} \cdot P \cdot M^{-1} \begin{bmatrix} L(2-s, \chi_1) & \cdots & \cdot \cdot \cdot & \cdots & L(2-s, \chi_h) \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ L(s, \chi_1)^{-1} & \cdots & \cdot \cdot \cdot & \cdot \cdot \cdot & L(s, \chi_h)^{-1} \end{bmatrix} M \right].
\]

If we assume that \(\chi_1 = 1\), then

\[
\lim_{s \to 1} \frac{L(2-s, \chi_1)}{L(s, \chi_1)} = -1
\]

so that

\[
\text{Tr}(\Phi(1)) = \lim_{s \to 1} \text{Tr}(\Phi(s))
\]

\[
= \text{Tr} \left( t^* M^{-1} \begin{bmatrix} -1 & 1 & \cdots & \cdot \cdot \cdot & 1 \\ 1 & 1 & \cdots & \cdot \cdot \cdot & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 1 & 1 & \cdots & \cdot \cdot \cdot & 1 \end{bmatrix} M \right) = \text{Tr} \left( P^{-2} \begin{bmatrix} 1 & 1 & \cdots & \cdot \cdot \cdot & 1 \\ 1 & 1 & \cdots & \cdot \cdot \cdot & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 1 & 1 & \cdots & \cdot \cdot \cdot & 1 \end{bmatrix} + P \begin{bmatrix} -1 & \cdots & \cdot \cdot \cdot & 1 \\ \cdot \cdot \cdot & \cdots & \ddots & \ddots & \ddots \\ \cdot \cdot \cdot & \ddots & \cdots & \ddots & \ddots \\ \cdot \cdot \cdot & \ddots & \cdots & \ddots & \ddots \\ 1 & \cdots & \cdot \cdot \cdot & 1 \end{bmatrix} \right).
\]
Recalling the definition of \( P \) we have \( \text{Tr}(\Phi(1)) = m - 2 = 2^{t-1} - 2 \), which is Theorem 2.

REFERENCES