

THE HOMOTOPY THEORY OF CYCLIC SETS¹

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ABSTRACT. The aim of this note is to show that the homotopy theory of the *cyclic* sets of Connes [3] is equivalent to that of *SO(2)-spaces* (i.e. spaces with a circle action) and hence to that of *spaces over* $K(\mathbb{Z}, 2)$.

1. Introduction.

1.1 *Summary.* The aim of this note is to study the homotopy theory of the cyclic sets of Connes [3]. These are simplicial sets with some additional structure, or more precisely, diagrams of sets indexed by a category Λ^{op} of cyclic operators which contains the category Δ^{op} of simplicial operators as a subcategory.

After a few preliminaries (§2), we construct a Quillen *closed model category structure* on the category of cyclic sets (§3) and show that the resulting homotopy theory is equivalent to (§4) that of *SO(2)-spaces* (i.e. spaces with a circle action) as well as to (§5) that of *simplicial sets over the nerve of* Λ^{op} , which has the homotopy type of $K(\mathbb{Z}, 2)$.

The first and the last of these results hold more generally, as their proofs depend only on certain properties (2.2 and 2.6) of the inclusion functor $j: \Delta^{\text{op}} \rightarrow \Lambda^{\text{op}}$. Some additional examples are (§6) a kind of covering \mathbf{K}^{op} of Λ^{op} and, for every simplicial group G , its flattening $\mathbf{b}G$. (The homotopy theory of $\mathbf{b}G$ -sets (i.e. functors $\mathbf{b}G \rightarrow \mathbf{sets}$) is equivalent to the homotopy theory of simplicial sets over the classifying complex of G .)

1.2 *Notation, terminology, etc.* (i) *The category* Δ^{op} *of simplicial operators.* This is the category with objects $\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots$ and generating maps

$$\begin{aligned} d_i: \mathbf{n} &\rightarrow \mathbf{n} - \mathbf{1}, & 0 \leq i \leq n, n > 0, \\ s_i: \mathbf{n} &\rightarrow \mathbf{n} + \mathbf{1}, & 0 \leq i \leq n, \end{aligned}$$

subject to the relations

$$(*) \quad \begin{aligned} d_i d_j &= d_{j-1} d_i, & 0 < j - i, \\ s_j s_i &= s_i s_{j-1}, & 0 < j - i, \\ d_i s_j &= s_{j-1} d_i: \mathbf{n} \rightarrow \mathbf{n}, & 0 < j - i \leq n, \\ &= \text{id}, & -1 \leq j - i \leq 0, \\ &= s_j d_{i-1}, & j - i < -1. \end{aligned}$$

Received by the editors November 9, 1984.

1980 *Mathematics Subject Classification.* Primary 55P15; Secondary 55U35.

¹This research was in part supported by the National Science Foundation.

(ii) *The category Λ^{op} of cyclic operators.* This is the category with objects $\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots$ and generating maps

$$\begin{aligned} d_i: \mathbf{n} &\rightarrow \mathbf{n} - \mathbf{1}, & 0 \leq i \leq n, n > 0, \\ s_i: \mathbf{n} &\rightarrow \mathbf{n} + \mathbf{1}, & 0 \leq i \leq n + 1, \end{aligned}$$

subject to the above relations (*) as well as the *cyclic relations*

$$(d_0 s_{n+1})^{n+1} = \text{id}: \mathbf{n} \rightarrow \mathbf{n}, \quad n \geq 0.$$

Note that, except for these cyclic relations, there is *no* relation involving the first face operator and last degeneracy operator, i.e. $d_0 s_{n+1} \neq s_n d_0: \mathbf{n} \rightarrow \mathbf{n}$.

(iii) *The categories \mathbf{S} of simplicial sets (i.e. Δ^{op} -sets) and \mathbf{S}^c of cyclic sets (i.e. Λ^{op} -sets).* These are the categories which have as objects the functions $\Delta^{\text{op}} \rightarrow \mathbf{sets}$ and $\Lambda^{\text{op}} \rightarrow \mathbf{sets}$, respectively, and as maps the natural transformations between them. A cyclic set X is thus a simplicial set together with an *extra degeneracy map* $s_{n+1}: X_n \rightarrow X_{n+1}$ in each dimension $n \geq 0$, with the cyclic property that $(d_0 s_{n+1})^{n+1} = \text{id}: X_n \rightarrow X_n$. However, in general, $d_0 s_{n+1} \neq s_n d_0: X_n \rightarrow X_n$.

The categories \mathbf{S} and \mathbf{S}^c are connected by the rather useful *forgetful functor* $j^*: \mathbf{S}^c \rightarrow \mathbf{S}$, induced by the *inclusion functor* $j: \Delta^{\text{op}} \rightarrow \Lambda^{\text{op}}$.

(iv) *The category \mathbf{T}^c of SO(2)-spaces and the forgetful functor $u: \mathbf{T}^c \rightarrow \mathbf{T}$.* We denote by \mathbf{T} the category of topological spaces, by \mathbf{T}^c the category of SO(2)-spaces, i.e. topological spaces with a continuous SO(2)-action, and by $u: \mathbf{T}^c \rightarrow \mathbf{T}$ the functor which sends each SO(2)-space to its underlying topological space.

2. The standard cyclic sets $\Lambda[n]$. This is a brief discussion of the standard cyclic sets $\Lambda[n]$ (the cyclic analogs of the standard simplicial sets $\Delta[n]$), much of which is explicit or implicit in [8].

2.1 *The standard cyclic sets in $\Lambda[n]$.* For every integer $n \geq 0$, the *standard cyclic set* $\Lambda[n]$ is given by

$$\Lambda[n] = \text{hom}^{\Lambda^{\text{op}}}(\mathbf{n}, -): \Lambda^{\text{op}} \rightarrow \mathbf{sets}.$$

It has the *universal property* that, for every cyclic set X and every n -simplex $x \in X$, there is a unique map (1.2(iii)) $c_x: \Lambda[n] \rightarrow X \in \mathbf{S}^c$ such that $c_x i_n = x$ (where $i_n \in \Lambda[n]$ denotes the generating n -simplex, i.e. the identity map of \mathbf{n}). This gives rise to a natural isomorphism

$$\text{hom}^{\mathbf{S}^c}(\Lambda[n], X) \approx \text{hom}^{\mathbf{S}}(\Delta[n], j^*X) \approx X_n.$$

The $\Lambda[n]$ and the maps between them form a Λ -diagram of cyclic sets, i.e. a functor $\Lambda[-]: \Lambda \rightarrow \mathbf{S}^c$, with the following property:

2.2 PROPOSITION. *In the induced (1.2(iii)) diagram of simplicial sets $j^* \Lambda[-]: \Lambda \rightarrow \mathbf{S}$, all maps are weak (homotopy) equivalences.*

2.3 *The cyclic sets $\Lambda[n, k]$.* As in the simplicial theory [2, Chapter VIII, 3.3] we need the cyclic subsets $\Lambda[n, k] \subset \Lambda[n]$ ($0 \leq k \leq n$) which are spanned by the $(n - 1)$ -simplices $d_i i_n$ ($0 \leq i \leq n, i \neq k$), which clearly give rise to *natural isomorphisms*

$$\text{hom}^{\mathbf{S}^c}(\Lambda[n, k], X) \approx \text{hom}^{\mathbf{S}}(\Delta[n, k], j^*X), \quad 0 \leq k \leq n.$$

Moreover, it is not difficult to prove (using 2.2)

2.4 PROPOSITION. *The inclusions $\Lambda[n, k] \rightarrow \Lambda[n] \in \mathbf{S}^c$, ($0 \leq k \leq n$) induce weak equivalences $j^*\Lambda[n, k] \rightarrow j^*\Lambda[n] \in \mathbf{S}$.*

2.5 The cyclic sets $\dot{\Lambda}[n]$. Also useful are the cyclic subsets $\dot{\Lambda}[n] \subset \Lambda[n]$ ($n > 0$) spanned by the $(n - 1)$ -simplices $d_i i_n$ ($0 \leq i \leq n$). They give rise to *natural isomorphisms*

$$\text{hom}^{\mathbf{S}^c}(\dot{\Lambda}[n], X) \approx \text{hom}^{\mathbf{S}}(\dot{\Delta}[n], j^*X), \quad n > 0.$$

Moreover, each $\dot{\Lambda}[n]$ is closely related to the direct limit $\partial\Lambda[n]$ of the diagram in \mathbf{S}^c which consists of

- (i) for every integer i with $0 \leq i \leq n$, a copy $\Lambda[n]_i$ of $\Lambda[n - 1]$, and
- (ii) for every pair of integers (i, j) with $0 \leq i < j \leq n$, a copy $\Lambda[n]_{i,j}$ of $\Lambda[n - 2]$ together with a pair of maps

$$\Lambda[n]_i \xleftarrow{c_{d_{j-1}i_{n-1}}} \Lambda[n]_{i,j} \xrightarrow{c_{d_i i_{n-1}}} \Lambda[n]_j$$

and which has the following nontrivial property:

2.6 PROPOSITION. *For each integer $n > 0$, the obvious map $\partial\Lambda[n] \rightarrow \dot{\Lambda}[n] \in \mathbf{S}^c$ is an isomorphism.*

Propositions 2.2 and 2.6 follow readily from

2.7 PROPOSITION. *Let $\text{SO}(2) \times |\Delta[-1]|: \Delta \rightarrow \mathbf{T}^c$ (1.2(iv)) denote the obvious functor which sends \mathbf{n} ($n \geq 0$) to the product of $\text{SO}(2)$ and the geometric realization of $\Delta[n]$, with $\text{SO}(2)$ acting on the left on itself and trivially on $|\Delta[n]|$. Then there exists a functor $M: \Lambda \rightarrow \mathbf{T}^c$ such that the following diagram commutes up to natural equivalences:*

$$\begin{array}{ccc} \Delta & \xrightarrow{\text{SO}(2) \times |\Delta[-1]|} & \mathbf{T}^c \\ j^{\text{op}} \downarrow & \nearrow M & \downarrow u \\ \Lambda & \xrightarrow{|j^*\Lambda[-1]|} & \mathbf{T} \end{array}$$

In particular, $|j^\Lambda[n]|$ is homeomorphic to $\text{SO}(2) \times |\Delta[n]|$ ($n \geq 0$).*

As, for $X \in \mathbf{S}^c$, the geometric realization $|j^*X|$ can be expressed as a quotient of the disjoint union $\coprod_n X_n \times |j^*\Lambda[n]|$ by the obvious identifications, Proposition 2.7 readily implies

2.8 PROPOSITION. *There is a functor $L^c: \mathbf{S}^c \rightarrow \mathbf{T}^c$ such that the following diagram commutes up to a natural equivalence:*

$$\begin{array}{ccc} & & \mathbf{T}^c \\ & \nearrow L^c & \downarrow u \\ \mathbf{S}^c & & \mathbf{T} \\ & \searrow |j^* - | & \end{array}$$

It thus remains to give a

2.9 *Proof of Proposition 2.7.* Consider the “twisted” product $\Delta[1] \times, \Delta[n] \in \mathbf{S}$ which has as k -simplices ($k \geq 0$) the $(k + 1)$ -tuples of pairs of integers

$$((0, i_1), \dots, (0, i_a), (1, j_1), \dots, (1, j_b))$$

such that $0 \leq j_1 \leq \dots \leq j_b \leq i_1 \leq \dots \leq i_a \leq n$ and $a, b \geq 0$, with the obvious faces and degeneracies. Then there is a homeomorphism between the geometric realizations

$$|\Delta[1] \times, \Delta[n]| \approx |\Delta[1] \times \Delta[n]|$$

which is natural in n . As $j^* \Lambda[n]$ is exactly the simplicial set obtained from $\Delta[1] \times, \Delta[n]$ by identifying the simplices

$$((0, i_1), \dots, (0, i_a)) \quad \text{and} \quad ((1, i_1), \dots, (1, i_a))$$

for every sequence of integers (i_1, \dots, i_a) with $0 \leq i_1 \leq \dots \leq i_a \leq n$, the above homeomorphism induces a homeomorphism $|j^* \Lambda[n]| \approx \text{SO}(2) \times |\Delta[n]|$ and the desired result now follows readily.

3. A homotopy theory for cyclic sets. In Theorem 3.1 below we turn the category \mathbf{S}^c of cyclic sets (see subsection 1.2) into a *closed model category* in the sense of Quillen, i.e. [2, p. 241] we define notions of *weak equivalence*, *fibration* and *cofibration* such that the following five axioms are satisfied:

CM1. *The category is closed under finite and direct and inverse limits.*

CM2. *If f and g are maps such that gf is defined and if two of f, g and gf are weak equivalences, then so is the third.*

CM3. *If f is a retract of g (i.e. if there are, in the category of maps, maps $a: f \rightarrow g$ and $b: g \rightarrow f$ such that $ba = \text{id}_f$) and g is a weak equivalence, a fibration or a cofibration, then so is f .*

CM4. *Given a commutative solid arrow diagram*

$$\begin{array}{ccc} U & \rightarrow & X \\ i \downarrow & \nearrow & \downarrow p \\ V & \rightarrow & Y \end{array}$$

in which i is a cofibration, p is a fibration, and either p or i is a weak equivalence, then the dotted arrow exists (and one says that i has the left lifting property with respect to p , or equivalently, that p has the right lifting property with respect to i).

CM5. *Any map f can be factored in two ways:*

(i) $f = pi$, where i is a cofibration and p is a trivial fibration (i.e. a fibration as well as a weak equivalence), and

(ii) $f = pi$, where p is a fibration and i is a trivial cofibration (i.e. a cofibration as well as a weak equivalence).

3.1 THEOREM. *The category \mathbf{S}^c of cyclic sets admits a closed model category structure in which a map $X \rightarrow Y \in \mathbf{S}^c$ is a weak equivalence or a fibration whenever the induced (1.2) map $j^* X \rightarrow j^* Y \in \mathbf{S}$ is so, and in which the cofibrations are (see also 3.4) the retracts of the (possibly transfinite) compositions of cobase extensions of the inclusions $\Lambda[n] \rightarrow \hat{\Lambda}[n]$ ($n \geq 0$).*

To prove this we first note that 2.3 and 2.5 imply

3.2 PROPOSITION. *A map in S^c is a fibration iff it has the right lifting property with respect to the inclusions $\Lambda[n, k] \rightarrow \Lambda[n]$ ($0 \leq k \leq n$).*

3.3 PROPOSITION. *A map in S^c is a trivial fibration iff it has the right lifting property with respect to the inclusions $\dot{\Lambda}[n] \rightarrow \Lambda[n]$ ($n \geq 0$).*

PROOF OF THEOREM 3.1. Verification of CM1, CM2, CM3 and the first part of CM4 is easy. To prove CM5 one combines the small object argument of [9, Chapter II, §3] with 3.3 or 2.4 and 3.2. This proof of CM5 implies that a trivial cofibration $U \rightarrow V$ admits a factorization $U \rightarrow V' \rightarrow V$ in which (by construction) the map $U \rightarrow V'$ is a trivial cofibration which has the left lifting property with respect to the fibrations and $V' \rightarrow V$ is a (necessarily trivial) fibration. The remaining part of CM4 now follows from the fact that the map $U \rightarrow V$ is (easily seen to be) a retract of the map $U \rightarrow V'$.

We end with some alternate descriptions of the cofibrations and the weak equivalences.

3.4 Cofibrations in S^c . Call a map $X \rightarrow Y \in S^c$ free if it is 1-1 and if, for every integer $n \geq 0$, the group of automorphisms of \mathbf{n} (which is cyclic of order $n + 1$) acts freely on the n -simplices of Y which are not in the image of X . Clearly the inclusions $\Lambda[n, k] \rightarrow \Lambda[n]$ and $\dot{\Lambda}[n] \rightarrow \Lambda[n]$ are free and so are their cobase extensions. In fact one readily verifies

3.5 PROPOSITION. *The free maps in S^c are exactly the (possibly transfinite) compositions of cobase extensions of the inclusions $\dot{\Lambda}[n] \rightarrow \Lambda[n]$ ($n \geq 0$). Hence a map in S^c is a cofibration iff it is a free map.*

3.6 Weak equivalences in S^c . Given a cyclic set X , i.e. a functor $X: \Lambda^{op} \rightarrow \mathbf{sets}$, one can compose it with the inclusion functor $i: \mathbf{sets} \rightarrow \mathbf{S}$ and take the homotopy direct limit [2, Chapter XIII] $\text{holim}^{\Lambda^{op}} iX \in \mathbf{S}$. Then one has

3.7 PROPOSITION *A map $X \rightarrow Y \in S^c$ is a weak equivalence iff the induced map $\text{holim}^{\Lambda^{op}} iX \rightarrow \text{holim}^{\Lambda^{op}} iY \in \mathbf{S}$ is so.*

PROOF. Let \mathbf{X} be the category which has as objects the simplices of X and as maps the cyclic operators between them. Then the nerve $N\mathbf{X}$ is clearly naturally isomorphic to $\text{holim}^{\Lambda^{op}} iX$. Similarly, if $j^*\mathbf{X} \subset \mathbf{X}$ is the subcategory consisting of the simplicial operators (the objects remain the same), then the nerve $Nj^*\mathbf{X}$ is naturally isomorphic to $\text{holim}^{\Lambda^{op}} ij^*\mathbf{X}$. Moreover, by [2, Chapter XII, 4.3] the latter simplicial set is naturally weakly equivalent to j^*X . The under categories of the inclusion functor $j^*\mathbf{X} \rightarrow \mathbf{X}$ are readily verified to be isomorphic to the under categories $\mathbf{n} \downarrow j$ of the inclusion functor $j: \Delta^{op} \rightarrow \Lambda^{op}$ and, by the above arguments, the nerves of the latter are naturally isomorphic to $\text{holim}^{\Lambda^{op}} ij^*\Lambda[n]$ and hence naturally weakly equivalent to $j^*\Lambda[n]$. The desired result now follows immediately from 2.2 and Quillen's theorem B [10, p. 97].

4. Comparison with SO(2)-spaces. In order to show that the homotopy theory of cyclic sets of §3 is equivalent, in the strong sense of [5, §5], to the homotopy theory of SO(2)-spaces (resulting from its usual model category structure (see Theorem 4.1)), we verify in Theorem 4.2 the existence of a pair of adjoint functors

$$L^c: (\text{cyclic sets}) \leftrightarrow (\text{SO}(2)\text{-spaces}): R^c$$

satisfying the *equivalence conditions* of Quillen [9, Chapter I, Theorem 3]:

EQ1. *The left adjoint L^c sends cofibrations into cofibrations and weak equivalences between cofibrant objects into weak equivalences.*

EQ2. *The right adjoint R^c sends fibrations into fibrations and weak equivalences between the fibrant objects into weak equivalences.*

EQ3. *For every cofibrant cyclic set X and every fibrant SO(2)-space Y , a map $X \rightarrow R^c Y$ is a weak equivalence iff its adjoint $L^c X \rightarrow Y$ is so.*

First we recall from [6, 1.2 and 2.2]

4.1 THEOREM. *The category \mathbf{T}^c of SO(2)-spaces (see subsection 1.2) admits a closed model category structure in which a map $X \rightarrow Y \in \mathbf{T}^c$ is a weak equivalence or a fibration whenever the underlying map of topological spaces $uX \rightarrow uY \in \mathbf{T}$ is a weak homotopy equivalence or a Serre fibration, and in which the cofibrations are the retracts of the (possibly transfinite) compositions of cobase extensions of the inclusions $\text{SO}(2) \times |\dot{\Delta}[n]| \rightarrow \text{SO}(2) \times |\Delta[n]|$ ($n \geq 0$).*

In view of Proposition 2.8 one can then formulate

4.2 THEOREM. *The functor $L^c: \mathbf{S}^c \rightarrow \mathbf{T}^c$ has as right adjoint the functor $R^c = \text{hom}(L^c \wedge [-], -): \mathbf{T}^c \rightarrow \mathbf{S}^c$. Moreover, this pair of adjoint functors satisfies the (above) equivalence conditions of Quillen.*

4.3 COROLLARY. *The categories \mathbf{S}^c and \mathbf{T}^c have equivalent homotopy theories in the strong sense that [5] their simplicial localizations with respect to the weak equivalences are weakly equivalent in the sense of [5, §2].*

PROOF. One verifies successively and without much difficulty the following properties:

- (i) the adjointness;
- (ii) L^c preserves weak equivalences;
- (iii) for every object $Y \in \mathbf{T}^c$, the simplicial set $j^* R^c Y$ is just the singular complex of the underlying topological space uY ;
- (iv) R^c preserves weak equivalences and fibrations;
- (v) L^c preserves cofibrations (use (iv) and adjointness);
- (vi) for every object $Y \in \mathbf{T}^c$, the underlying map $uL^c R^c Y \rightarrow uY$ of the adjunction map is the usual adjunction map from the geometric realization of the singular complex of uY back to uY ;
- (vii) EQ3 is satisfied.

5. Comparison with simplicial sets over $K(Z, 2)$. As the homotopy theory of SO(2)-spaces is (well known to be [4]) equivalent to that of simplicial sets over $K(Z, 2)$, Theorem 4.2 implies that the same holds for the homotopy theory of cyclic

sets of §3. We will now give a direct proof of this result (Theorem 5.1) which only uses the fact that the functor $j: \Delta^{op} \rightarrow \Lambda^{op}$ has the properties described in Propositions 2.2 and 2.6.

To formulate Theorem 5.1, let $N\Lambda^{op}$ be the nerve of Λ^{op} (which [3] is weakly equivalent to $K(Z, 2)$), let the category $\mathbf{S}/N\Lambda^{op}$ of simplicial sets over $N\Lambda^{op}$ have the closed model category structure induced by the usual one on \mathbf{S} [9, Chapter II], let $*$: $\Lambda^{op} \rightarrow \mathbf{sets} \in \mathbf{S}^c$ be the terminal object and let $L^{\Lambda^{op}}: \mathbf{S}^c \rightarrow \mathbf{S}/N\Lambda^{op}$ be the functor which sends an object $X \in \mathbf{S}^c$ to the map (see 3.6)

$$\text{holim}_{\rightarrow}^{\Lambda^{op}} iX \rightarrow \text{holim}_{\rightarrow}^{\Lambda^{op}} i* = N\Lambda^{op}.$$

Then one has

5.1 THEOREM. *The functor $L^{\Lambda^{op}}: \mathbf{S}^c \rightarrow \mathbf{S}/N\Lambda^{op}$ has as right adjoint the functor*

$$R^{\Lambda^{op}} = \text{hom}(L^{\Lambda^{op}}\Lambda[-], -): \mathbf{S}/N\Lambda^{op} \rightarrow \mathbf{S}^c.$$

Moreover, this pair of adjoint functors satisfies the (see §4) equivalence conditions of Quillen.

5.2 COROLLARY. *The categories \mathbf{S} and $\mathbf{S}/N\Lambda^{op}$ have equivalent homotopy theories in the strong sense that [5] their simplicial localizations with respect to the weak equivalences are weakly equivalent in the sense of [5, §2].*

PROOF. The adjointness is obvious and Proposition 3.7 implies that a map $X \rightarrow Y \in \mathbf{S}^c$ is a weak equivalence iff the induced map $L^{\Lambda^{op}}X \rightarrow L^{\Lambda^{op}}Y \in \mathbf{S}/N\Lambda^{op}$ is so. Moreover, $L^{\Lambda^{op}}$ preserves cofibrations and hence it follows from the adjointness that $R^{\Lambda^{op}}$ preserves fibrations and trivial fibrations. By [1, 1.2 and 1.3], this implies EQ2. Finally, to prove EQ3, it suffices to show that, for every fibrant object $Y \in \mathbf{S}/N\Lambda^{op}$, the adjunction map $L^{\Lambda^{op}}R^{\Lambda^{op}}Y \rightarrow Y \in \mathbf{S}/N\Lambda^{op}$ is a weak equivalence.

To do this let $Y' = (Nj)*Y \in \mathbf{S}/N\Delta^{op}$ be the pull back over $Nj: N\Delta^{op} \rightarrow N\Lambda^{op}$ and let

$$L^{\Delta^{op}}: \mathbf{S} \leftrightarrow \mathbf{S}/N\Delta^{op}: R^{\Delta^{op}}$$

be the pair of adjoint functors analogous to $L^{\Lambda^{op}}$ and $R^{\Lambda^{op}}$. Then it is not difficult to verify that the adjunction map $L^{\Delta^{op}}R^{\Delta^{op}}Y' \rightarrow Y' \in \mathbf{S}/N\Delta^{op}$ is a weak equivalence and that this map admits an obvious factorization $L^{\Delta^{op}}R^{\Delta^{op}}Y' \rightarrow L^{\Delta^{op}}j^*R^{\Lambda^{op}}Y \rightarrow Y'$. Using the argument of Proposition 3.7 and the fact that $Y \in \mathbf{S}/N\Lambda^{op}$ is fibrant, one now proves that the adjunction map $L^{\Delta^{op}}R^{\Delta^{op}}Y \rightarrow Y \in \mathbf{S}/N\Delta^{op}$ is a weak equivalence iff the map $L^{\Delta^{op}}j^*R^{\Lambda^{op}}Y \rightarrow Y' \in \mathbf{S}/N\Delta^{op}$ is so. It thus remains to prove that the map $L^{\Delta^{op}}R^{\Delta^{op}}Y' \rightarrow L^{\Delta^{op}}j^*R^{\Lambda^{op}}Y \in \mathbf{S}/N\Delta^{op}$ is a weak equivalence and to do this one observes [4, §5] that the obvious maps $\text{holim}_{\rightarrow}^{\Delta^{op}} i\Delta[n] \rightarrow \text{holim}_{\rightarrow}^{\Delta^{op}} i\Lambda[n] \in \mathbf{S}$ ($n \geq 0$) are weak equivalences and that the Δ -diagram $L^{\Delta^{op}}\Lambda[-] \circ j^{\text{op}}: \Delta \rightarrow \mathbf{S}/N\Delta^{op}$ is a cosimplicial resolution [5, 4.3]. To prove this last statement one notes that Proposition 2.6 implies that the Δ -diagram $\Lambda[-] \circ j^{\text{op}}: \Delta \rightarrow \mathbf{S}^c$ is a cosimplicial resolution and that the functor $L^{\Delta^{op}}$ satisfies EQ1 and commutes with the direct limits in question.

5.3 REMARK. The last step in the above proof is the only place where we used Proposition 2.6.

6. A generalization to other categories under Δ^{op} . As mentioned in 1.1, Theorems 3.1 and 5.1 hold more generally, as their proofs only use the fact that the functor $j: \Delta^{\text{op}} \rightarrow \Lambda^{\text{op}}$ has the properties described in Propositions 2.2 and 2.6. In order to formulate these generalizations, we start with a brief discussion of

6.1 *Categories under Δ^{op} .* Let Σ be a small category and let $j: \Delta^{\text{op}} \rightarrow \Sigma^{\text{op}}$ be a functor. For every integer $n \geq 0$, one can then define the *standard* Σ^{op} -set $\Sigma[n]$ by

$$\Sigma[n] = \text{hom}^{\Sigma^{\text{op}}}(jn, -): \Sigma^{\text{op}} \rightarrow \mathbf{sets}.$$

Clearly, these Σ^{op} -sets give rise to a Σ -diagram $\Sigma[-]: \Sigma \rightarrow \Sigma^{\text{op}}\text{-sets}$ of Σ^{op} -sets as well as an induced Σ -diagram $j^*\Sigma[-]: \Sigma \rightarrow \mathbf{S}$ of simplicial sets. One can also consider the Σ^{op} -subset $\check{\Sigma}[n] \subset \Sigma[n]$ spanned by the “faces” of the generating element, i.e. the identity map of jn , and the Σ^{op} -set $\partial\Sigma[n]$, defined by means of a direct limit as in 2.5.

We will assume that the functor $j: \Delta^{\text{op}} \rightarrow \Sigma^{\text{op}}$ satisfies one or both of the following conditions:

- (i) *In the diagram of simplicial sets $j^*\Sigma[-]: \Sigma \rightarrow \mathbf{S}$, all maps are weak equivalences.*
- (ii) *For every integer $n > 0$, the obvious map $\partial\Sigma[n] \rightarrow \check{\Sigma}[n] \in \Sigma^{\text{op}}\text{-sets}$ is an isomorphism.*

The arguments of the proof of Theorem 3.1 then yield

6.2 THEOREM. *Let Σ be a small category and let: $\Delta^{\text{op}} \rightarrow \Sigma^{\text{op}}$ be a functor satisfying 6.1(i). Then the category $\Sigma^{\text{op}}\text{-sets}$ (of functors $\Sigma^{\text{op}} \rightarrow \mathbf{sets}$ and natural transformations between them) admits a closed model category structure in which a map $X \rightarrow Y \in \Sigma^{\text{op}}\text{-sets}$ is a weak equivalence or a fibration whenever the induced map $j^*X \rightarrow j^*Y \in \mathbf{S}$ is so and in which the cofibrations are the retracts of the (possibly transfinite) compositions of cobase extensions of the inclusions $\check{\Sigma}[n] \rightarrow \Sigma[n]$ ($n \geq 0$).*

Similarly, if the category $\mathbf{S}/N\Sigma^{\text{op}}$ of simplicial sets over $N\Sigma^{\text{op}}$ has the closed model category structure induced by the usual one on \mathbf{S} [9, Chapter II], if $*$ $\in \Sigma^{\text{op}}\text{-sets}$ is the terminal object and if $L^{\Sigma^{\text{op}}}\text{-sets} \rightarrow \mathbf{S}/N\Sigma^{\text{op}}$ denotes the functor which sends an object $X \in \Sigma^{\text{op}}\text{-sets}$ to the map (see 3.6)

$$\text{holim}_{\rightarrow}^{\Sigma^{\text{op}}} iX \rightarrow \text{holim}_{\rightarrow}^{\Sigma^{\text{op}}} i* = N\Sigma^{\text{op}},$$

then the arguments of the proof of Theorem 5.1 yield

6.3 THEOREM. *Let Σ be a small category with connected nerve and let $j: \Delta^{\text{op}} \rightarrow \Sigma^{\text{op}}$ be a functor satisfying 6.1(i) and (ii). Then the functor $L^{\Sigma^{\text{op}}}$ has as right adjoint the functor*

$$R^{\Sigma^{\text{op}}} = \text{hom}(L^{\Sigma^{\text{op}}}\Sigma[-], -): \mathbf{S}/N\Sigma^{\text{op}} \rightarrow \Sigma^{\text{op}}\text{-sets}$$

and this pair of adjoint functors satisfies the (see §4) equivalence conditions of Quillen.

6.4 COROLLARY. *The categories $\Sigma^{\text{op}}\text{-sets}$ and $\mathbf{S}/N\Sigma^{\text{op}}$ have equivalent homotopy theories in the strong sense that [5] their simplicial localizations with respect to the weak equivalences are weakly equivalent in the sense of [5, §2].*

6.5 EXAMPLE. If in the definition of Λ^{op} (see 1.2(ii)) one omits the cyclic relations $(d_0 s_{n+1})^{n+1} = \text{id}: \mathbf{n} \rightarrow \mathbf{n}$ ($n \geq 0$), one gets a category \mathbf{K}^{op} which is a kind of covering of Λ^{op} . There is an obvious functor $j: \Delta^{\text{op}} \rightarrow \mathbf{K}^{\text{op}}$ which satisfies 6.1(i) and (ii). Moreover, the nerve $N\mathbf{K}^{\text{op}}$ is contractible and *the homotopy theory of \mathbf{K}^{op} -sets thus is equivalent to the usual homotopy theory of simplicial sets.*

6.6 EXAMPLE. Given a simplicial group G , one can form its *flattening* $\mathbf{b}G$, i.e. [7, §7] the category which has as objects the \mathbf{n} ($n \geq 0$) and as maps $\mathbf{k} \rightarrow \mathbf{n}$ the pairs (e, g) , where e is a map $e: \mathbf{k} \rightarrow \mathbf{n} \in \Delta^{\text{op}}$ and g is an n -simplex of G . There is an obvious functor $j: \Delta^{\text{op}} \rightarrow \mathbf{b}G$ given by $je = (e, 1)$ for all $e \in \Delta^{\text{op}}$. Moreover, one readily verifies that this functor satisfies 6.1(i) and (ii) and that $N\mathbf{b}G$ is weakly equivalent to the classifying complex of G . Hence, *the resulting homotopy theory of $\mathbf{b}G$ -sets is equivalent to the homotopy theory of simplicial sets over the classifying complex of G .* This is, however, not surprising as $\mathbf{b}G$ -sets are essentially the same as *simplicial sets with a G -action.*

We end with observing that in all three cases (i.e. Δ^{op} , \mathbf{K}^{op} and $\mathbf{b}G$) *the functor j is onto on objects.* Actually this is not surprising in view of

6.7 PROPOSITION. *Let Σ be a small category with connected nerve and let $j: \Delta^{\text{op}} \rightarrow \Sigma^{\text{op}}$ be a functor satisfying 6.1(i) and (ii). If $\Sigma_0^{\text{op}} \subset \Sigma^{\text{op}}$ denotes the full subcategory spanned by the image of j , then the resulting functor $j_0: \Delta^{\text{op}} \rightarrow \Sigma_0^{\text{op}}$ also satisfies 6.1(i) and (ii) and the inclusion $\Sigma_0^{\text{op}} \rightarrow \Sigma^{\text{op}}$ induces an equivalence between the homotopy theory of Σ^{op} -sets and that of Σ_0^{op} -sets.*

PROOF. This follows immediately from Theorem 6.3 and Quillen's Theorem B [10, p. 97].

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