

## GROUP-GRADED RINGS AND DUALITY

BY

DECLAN QUINN

**ABSTRACT.** We give an alternative construction of the duality between finite group actions and group gradings on rings which was shown by Cohen and Montgomery in [1]. This duality is then used to extend known results on skew group rings to corresponding results for large classes of group-graded rings. Finally we modify the construction slightly to handle infinite groups.

**Introduction.** In the first section we give an alternate construction of the duality between finite group actions and group gradings on rings which was shown by Cohen and Montgomery in [1]. This duality is then used to extend results on skew group rings and their modules to corresponding results for large classes of group-graded rings.

In the second section we modify the construction slightly to handle infinite groups and apply it to a theorem on prime and semiprime infinite crossed products.

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**1. Finite group-graded rings and the smash product.** Let  $G$  be a multiplicative group. An associative ring with identity is said to be  $G$ -graded if

$$(1) \quad R = \bigoplus_{x \in G} R(x)$$

is a direct sum of additive subgroups  $R(x)$ , with  $R(x)R(y) \subseteq R(xy)$ . It follows that necessarily  $1_R \in R(1)$  so that each  $R(x)$  is a unitary  $R(1)$ -bimodule. If  $r \in R$ , we write  $r(x)$  for the component of  $r$  in  $R(x)$  so that  $r = \sum_{x \in G} r(x)$ .  $R$  is said to be strongly  $G$ -graded if  $R(x)R(y) = R(xy)$  for all  $x, y \in G$ .

Throughout this section  $R$  is assumed to be  $G$ -graded, where  $G$  is a finite group with  $|G| = n$ .

Let  $M_G(R)$  denote the set of  $n \times n$  matrices over  $R$  with the rows and columns indexed by the elements of  $G$ . If  $\alpha \in M_G(R)$ , we write  $\alpha_{x,y}$  for the entry in the  $[x, y]$ -position of  $\alpha$ . Then if  $\alpha, \beta \in M_G(R)$ , the matrix product  $\alpha\beta$  is given by

$$(2) \quad (\alpha\beta)_{x,y} = \sum_{z \in G} \alpha_{x,z} \beta_{z,y}.$$

If  $U \subseteq G$  is any subset of  $G$ , let  $R(U) = \sum_{x \in U} R(x)$ . In particular,  $R = R(G)$ . Now suppose  $H \subseteq G$  is a subgroup of  $G$ . We define  $R\{H\} \subseteq M_G(R)$  by

$$R\{H\} = \left\{ \alpha \in M_G(R) \mid \alpha_{x,y} \in R(xHy^{-1}) \right\},$$

that is,  $R\{H\} = \sum_{x,y \in G} R(xHy^{-1})e(x,y)$ , where  $e(x,y)$  is the matrix unit with  $1_R$  in the  $[x,y]$ -position and zeroes elsewhere. Note that  $R\{G\} = M_G(R)$ .

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Since  $R(xHy^{-1})R(yHz^{-1}) \subseteq R(xHz^{-1})$ ,  $R\{H\}$  is a subring of  $M_G(R)$ . Furthermore,  $1_R \in R(1) \subseteq R(xHx^{-1})$  for each  $x \in G$ , so that  $I \in R\{H\}$ , where  $I$  is the identity matrix.

$R\{H\}$  can be viewed as a generalized conjugate of the matrix ring  $M_G(R(H))$ , where  $M_G(R(H)) \subseteq M_G(R)$  denotes the  $n \times n$  matrices over the ring  $R(H)$ . Let  $D$  and  $D^{-1}$  be the diagonal subsets of  $M_G(R)$  given by

$$(3) \quad D = \left\{ \alpha \in M_G(R) \mid \alpha_{x,x} \in R(xH) \text{ and } \alpha_{x,y} = 0 \text{ if } x \neq y \right\},$$

$$D^{-1} = \left\{ \beta \in M_G(R) \mid \beta_{x,x} \in R(Hx^{-1}) \text{ and } \beta_{x,y} = 0 \text{ if } x \neq y \right\}.$$

Then  $DM_G(R(H))D^{-1} \subseteq R\{H\}$  and  $D^{-1}R\{H\}D \subseteq M_G(R(H))$ . If  $R$  is strongly  $G$ -graded these inclusions are both equalities.

The ideals of  $R(H)$  correspond to the ideals of  $M_G(R(H))$  and we can extend this, using conjugation, to a correspondence between the ideals of  $R(H)$  and those of  $R\{H\}$  (see Lemma 1.4.).

Another description of  $R\{H\}$ , as a skew group ring over  $R\{1\}$ , is given in Lemma 1.2. We begin by studying the ring  $R\{1\}$ .

First we observe that  $R$  can be embedded in  $R\{1\}$  by a ring monomorphism  $\eta$ , with  $\eta(1_R) = I$ . We define  $\eta$  by  $\eta(r) = \sum_{x,y \in G} r(xy^{-1})e(x,y)$  and denote  $\eta(r)$  by  $\tilde{r}$ . If  $x, y \in G$  and  $r, s \in R$  then using equation (2) we find

$$(\tilde{r}\tilde{s})_{x,y} = \sum_{z \in G} \tilde{r}_{x,z}\tilde{s}_{z,y} = \sum_{z \in G} r(xz^{-1})s(zy^{-1}) = (rs)(xy^{-1}) = (\widetilde{rs})_{x,y}.$$

Thus  $\tilde{r}\tilde{s} = \widetilde{rs}$ . Similarly,  $\eta(r+s) = \eta(r) + \eta(s)$  and so  $\eta$  is a ring homomorphism. Clearly  $\eta(1_R) = I$ ,  $\eta$  is injective and  $\tilde{R} = \eta(R) \subseteq R\{1\}$ .

We call  $R\{1\}$  the smash product of  $\tilde{R}$  with  $G$  and denote it by  $\tilde{R}\#G$ . The following lemma shows that this definition coincides with the definition of the smash product given in [1]. Let  $p(x) = e(x,x)$ .

LEMMA 1.1.  $\tilde{R}\#G = \bigoplus \sum_{x \in G} \tilde{R}p(x)$  is a free  $\tilde{R}$ -module with basis  $\{p(x) \mid x \in G\}$ . Also,  $\tilde{r}p(x)\tilde{s}p(y) = \tilde{r}s(xy^{-1})p(y)$ , where  $r, s \in R$  and  $x, y \in G$ .

PROOF. Notice that  $\widetilde{r(x)p(y)} = r(x)e(xy,y)$ , so  $\tilde{r}p(y) = \sum_{x \in G} r(x)e(xy,y)$  and  $\tilde{R}p(y) = \sum_{x \in G} R(x)e(xy,y)$  is a free  $\tilde{R}$ -module with generator  $p(y)$ . Now,  $\sum_{y \in G} \tilde{R}p(y) = \sum_{y \in G} \sum_{x \in G} R(x)e(xy,y) = \tilde{R}\#G$  and the sum  $\sum_{y \in G} \tilde{R}p(y)$  is direct since  $I = \sum_{y \in G} p(y)$  is a decomposition of  $I$  into orthogonal idempotents. The final formula follows easily.  $\square$

The group  $G$  can be embedded in the matrix ring  $M_G(R)$ . Indeed, if  $g \in G$ , we let  $\bar{g} = \sum_{x \in G} e(x, xg)$ . Now  $\bar{g}$  is a unit of  $M_G(R)$  and  $G$  is isomorphic to  $\bar{G}$ , where  $\bar{G} = \{\bar{g} \mid g \in G\}$ , via the map taking  $g$  to  $\bar{g}$ . We now describe  $R\{H\}$  as a skew group ring over  $\tilde{R}\#G$ .

LEMMA 1.2. Let  $H$  be a subgroup of  $G$ . Then

(i)  $R\{H\} = \bigoplus \sum_{h \in H} (\tilde{R}\#G)\bar{h}$ , as a direct sum of abelian groups.

(ii)  $\bar{g}^{-1}(\tilde{R}\#G)\bar{g} = \tilde{R}\#G$  for all  $g \in G$ .

(iii)  $R\{G\} = (\tilde{R}\#G)\bar{G}$  is a skew group ring of the group  $\bar{G}$  over the ring  $\tilde{R}\#G$ .

Furthermore,  $R\{H\}$  is the naturally embedded sub skew group ring.

PROOF.

$$\begin{aligned}
 \text{(i)} \quad (\tilde{R}\#G)\bar{h} &= \sum_{x,y \in G} R(xy^{-1})e(x,y) \sum_{z \in G} e(z,zh) \\
 &= \sum_{x,y \in G} R(xy^{-1})e(x,yh) = \sum_{x,w \in G} R(xhw^{-1})e(x,w).
 \end{aligned}$$

Since for fixed  $x, w$  the elements  $xhw^{-1}$  are distinct for distinct  $h$ , it follows from equation (1) that  $\sum_{h \in H} (\tilde{R}\#G)\bar{h}$  is direct. Also,

$$\begin{aligned}
 \sum_{h \in H} (\tilde{R}\#G)\bar{h} &= \sum_{h \in H} \sum_{x,w \in G} R(xhw^{-1})e(x,w) \\
 &= \sum_{x,w \in G} \sum_{h \in H} R(xhw^{-1})e(x,w) \\
 &= \sum_{x,w \in G} R(xHw^{-1})e(x,w) = R\{H\}.
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \bar{g}^{-1}(\tilde{R}\#G)\bar{g} &= \sum_{u \in G} e(u,ug^{-1}) \sum_{x,y \in G} R(xy^{-1})e(x,y) \sum_{v \in G} e(v,vg) \\
 &= \sum_{x,y \in G} R(xy^{-1})e(xg,yg) \\
 &= \tilde{R}\#G \quad \text{since } (xg)(yg)^{-1} = xy^{-1}.
 \end{aligned}$$

(iii) Follows directly from (i) and (ii).  $\square$

The duality result of Cohen and Montgomery [1, Theorem 3.5] now follows. Observe that  $G \simeq \bar{G}$  acts by conjugation on  $\tilde{R}\#G$  and we let  $(\tilde{R}\#G)^G$  denote the set of fixed points.

**THEOREM 1.3.**  $M_G(R) = (\tilde{R}\#G)\bar{G}$  is a skew group ring of  $\bar{G} \simeq G$  over the ring  $\tilde{R}\#G$ . Furthermore,  $(\tilde{R}\#G)^G = \tilde{R}$ .

**PROOF.** Since  $M_G(R) = R\{G\}$ , the first statement is a special case of Lemma 1.1(iii).

If  $\alpha \in \tilde{R}\#G$  note that  $(\bar{g}^{-1}\alpha\bar{g})_{x,y} = \alpha_{xg^{-1},yg^{-1}}$ . So  $\alpha \in (\tilde{R}\#G)^G$  if and only if  $\alpha_{x,y} = \alpha_{xg^{-1},yg^{-1}}$  for all  $x, y, g \in G$  and hence if and only if  $\alpha = \eta(\sum_{x \in G} \alpha_{x,1}) \in \tilde{R}$ .  $\square$

A group-graded ring  $R$  is said to be left component regular if  $l_R(R(x)) = 0$  for all  $x \in G$ , where  $l_R(R(x))$  is the left annihilator of  $R(x)$  in  $R$ . Right component regular is defined similarly, and the term component regular is used to imply that both these conditions hold. If the left annihilator of  $R(x)$  in  $R(x^{-1})$ , written  $l_{R(x^{-1})}(R(x))$ , is zero for all  $x \in G$ , then we say  $R$  is left nondegenerate. It was shown in [2] that if  $R(1)$  is semiprime then left nondegenerate is equivalent to right nondegenerate. It was also shown that if  $G$  is finite, then semiprime implies nondegenerate.

We now use the generalized conjugation to give the ideal correspondence between the rings  $R(H)$  and  $R\{H\}$ . If  $T$  is any ring,  $\mathcal{I}(T)$  denotes the set of two-sided ideals of  $T$ .

LEMMA 1.4. *Let  $R$  be a  $G$ -graded ring with  $G$  finite and suppose  $H$  is subgroup of  $G$ . Then there exist maps*

$$\phi: \mathcal{F}(R(H)) \rightarrow \mathcal{F}(R\{H\}), \quad \psi: \mathcal{F}(R\{H\}) \rightarrow \mathcal{F}(R(H))$$

such that

- (i)  $\phi$  is injective and if  $A, B \triangleleft R(H)$  then  $\phi(A)\phi(B) \subseteq \phi(AB)$ ,
- (ii) if  $J, K \triangleleft R\{H\}$  then  $\psi(J)\psi(K) \subseteq \psi(JK)$ ,
- (iii)  $\psi \circ \phi$  is the identity on  $\mathcal{F}(R(H))$ .

Furthermore, let  $R$  be component regular and let  $I \triangleleft R\{H\}$ . Then  $\psi(I) = 0$  if and only if  $I = 0$ .

PROOF. Let  $A \triangleleft R(H)$  and recall from equation (3) that

$$D = \left\{ \alpha \in M_G(R) \mid \alpha_{x,x} \in R(xH), \alpha_{x,y} = 0 \text{ if } x \neq y \right\}$$

and

$$D^{-1} = \left\{ \beta \in M_G(R) \mid \beta_{x,x} \in R(Hx^{-1}), \beta_{x,y} = 0 \text{ if } x \neq y \right\}.$$

Take  $A^0 = DM_G(A)D^{-1}$ , where  $M_G(A)$  is the set of  $n \times n$  matrices over  $A$ .  $A^0 \subseteq DM_G(R(H))D^{-1} \subseteq R\{H\}$ . We claim that  $A^0$  is an ideal of  $R\{H\}$ . First we show that  $R\{H\}D \subseteq DM_G(R(H))$ . To see this consider the  $[x, y]$ -position of each side. For the left-hand side we get  $R(xHy^{-1})R(yH) \subseteq R(xH)$ , while on the right we have  $R(xH)R(H) = R(xH)$  since  $1 \in R(H)$ . Thus,

$$\begin{aligned} R\{H\}A^0 &= R\{H\}DM_G(A)D^{-1} \\ &\subseteq DM_G(R(H))M_G(A)D^{-1} = DM_G(A)D^{-1} = A^0. \end{aligned}$$

Hence  $A^0$  is a left ideal of  $R\{H\}$ . Similarly, it is also a right ideal.

Now suppose  $I \triangleleft R\{H\}$ . Take  $I_{1,1} = \{\alpha_{1,1} \mid \alpha \in I\} \subseteq R(H)$ . Clearly,  $I_{1,1}$  is an ideal of  $R(H)$  since  $R\{H\} \supseteq R(H)e(1, 1)$ .

We define  $\phi$  and  $\psi$  as follows: If  $A \triangleleft R(H)$ , let  $\phi(A) = A^0$  and, for  $I \triangleleft R\{H\}$ , take  $\psi(I) = I_{1,1}$ .

Note that  $A_{1,1}^0 = R(H)AR(H) = A$ . That is,  $\psi \circ \phi(A) = A$ , so (iii) is proved and also  $\phi$  is injective. To complete the proof of (i) note that  $D^{-1}D \subseteq \sum_{x \in G} R(H)e(x, x) \subseteq M_G(R(H))$ . Thus if  $A, B \triangleleft R(H)$  then

$$\begin{aligned} \phi(A)\phi(B) &= DM_G(A)D^{-1}DM_G(B)D^{-1} \\ &\subseteq DM_G(A)M_G(R(H))M_G(B)D^{-1} = DM_G(AB)D^{-1} = \phi(AB). \end{aligned}$$

To prove (ii) note that  $e(1, 1) \in R\{H\}$  and so if  $J, K \triangleleft R\{H\}$  then  $e(1, 1)Je(1, 1)Ke(1, 1) \subseteq JK$ . Evaluating both sides at the  $[1, 1]$ -position gives  $J_{1,1}K_{1,1} \subseteq (JK)_{1,1}$  or equivalently  $\psi(J)\psi(K) \subseteq \psi(JK)$ .

Finally, to prove the last statement, suppose  $R$  is component regular and let  $J \triangleleft R\{H\}$  with  $J \neq 0$ . Since  $J \neq 0$ , choose  $\alpha \in J \setminus 0$  and say  $\alpha_{x,y} \neq 0$ . Note that  $R(Hx^{-1})e(1, x) \subseteq R\{H\}$  and  $e(y, 1)R(yH) \subseteq R\{H\}$ , so  $R(Hx^{-1})e(1, x) \cdot \alpha \cdot e(y, 1)R(yH) \subseteq J$ . Thus,  $\psi(J) = J_{1,1} \supseteq R(Hx^{-1})\alpha_{x,y}R(yH) \neq 0$  since  $R$  is component regular.  $\square$

Now suppose  $H$  is a fixed subgroup of  $G$  and let  $x \in G$ . Then the conjugate  $xHx^{-1}$  is also a subgroup of  $G$  and we let  $\phi_x$  and  $\psi_x$  be the maps between  $\mathcal{I}(R(xHx^{-1}))$  and  $\mathcal{I}(R\{xHx^{-1}\})$ . Note that  $\bar{x}R\{H\}\bar{x}^{-1} = R\{xHx^{-1}\}$  since

$$\bar{x}R\{H\}\bar{x}^{-1} = \bar{x}(\tilde{R}\#G)\bar{H}\bar{x}^{-1} = (\tilde{R}\#G)\bar{x}\bar{H}\bar{x}^{-1} = R\{xHx^{-1}\}.$$

Thus if  $A \triangleleft R\{H\}$  then  $\bar{x}A\bar{x}^{-1} \triangleleft R\{xHx^{-1}\}$ .

LEMMA 1.5. *Let  $R$  be a left nondegenerate group-graded ring and let  $A \triangleleft R\{H\}$  be a nonzero ideal. Then there exists  $x \in G$  such that  $\psi_x(\bar{x}A\bar{x}^{-1}) \triangleleft R(xHx^{-1})$  is nonzero.*

PROOF. Since  $A \neq 0$ , there exists  $x, y \in G$  such that  $A_{x,y} \neq 0$ . Now  $A_{x,y}e(x, y) \subseteq A$  and  $R(yHx^{-1})e(y, x) \subseteq R\{H\}$  so that

$$A_{x,y}R(yHx^{-1})e(x, x) = A_{x,y}e(x, y)R(yHx^{-1})e(y, x) \subseteq A.$$

Thus  $A_{x,y}R(yHx^{-1}) \subseteq A_{x,x}$ . Now since  $0 \neq A_{x,y} \subseteq R(xHy^{-1})$ ,  $R$  is left nondegenerate, and  $yHx^{-1} = (xHy^{-1})^{-1}$ , we can conclude that  $A_{x,x} \neq 0$ . But  $A_{x,x} = (\bar{x}A\bar{x}^{-1})_{1,1} = \psi_x(\bar{x}A\bar{x}^{-1})$ .  $\square$

A crossed product  $R * G$ , where  $R$  is an arbitrary ring, is said to be weakly semiprime if whenever  $P$  is an elementary abelian  $p$ -subgroup of  $G$  where  $R$  has  $p$ -torsion, or  $P = 1$ , then  $R * P$  is semiprime. In [4, Theorems 1.8 and 1.9] Passman showed that if  $R * G$  is a crossed product with  $G$  finite then

(A) the semiprime condition on  $R * G$  is inherited by each subring  $R * H$  where  $H$  is a subgroup of  $G$ ;

(B) if  $R * G$  is weakly semiprime then it is semiprime.

We can now extend (A) and (B) to corresponding results for more general group-graded rings.

THEOREM 1.6. *Let  $R$  be a  $G$ -graded ring with  $G$  finite and suppose  $H$  is a subgroup of  $G$ . If  $R$  is semiprime then so is  $R(H)$ .*

PROOF. Since  $R$  is semiprime, so is the matrix ring  $M_G(R)$ . By Lemma 1.2(iii),  $M_G(R) = (\tilde{R}\#G)\bar{G}$  is a skew group ring, so (A) applies to give that  $R\{H\} = (\tilde{R}\#G)\bar{H}$  is also semiprime.

Now suppose  $I \triangleleft R(H)$  with  $I^2 = 0$ . Then by 1.4(i),  $\phi(I)^2 \subseteq \phi(I^2) = 0$ . But  $\phi(I) \triangleleft R\{H\}$  and  $R\{H\}$  is semiprime. Thus  $\phi(I) = 0$ , and since  $\phi$  is injective,  $I = 0$ .  $\square$

By analogy with the definition for crossed products, we say a  $G$ -graded ring  $R$  is weakly semiprime if  $R(P)$  is semiprime for  $P = 1$  or  $P$  an elementary abelian  $p$ -subgroup of  $G$ , where the ring  $R(1)$  has  $p$ -torsion.

THEOREM 1.7. *Let  $R$  be a  $G$ -graded ring with  $G$  finite. Then  $R$  is semiprime if and only if  $R$  is weakly semiprime and left nondegenerate.*

PROOF. Suppose  $R$  is semiprime. Then Theorem 1.6 implies  $R$  is weakly semiprime and [2, Proposition 1.2] implies  $R$  is nondegenerate.

Conversely, suppose  $R$  is weakly semiprime and left nondegenerate. To show  $R$  is semiprime it suffices to show that  $M_G(R)$  is semiprime. In turn, using (B), we need only show  $M_G(R) = (\tilde{R}\#G)\bar{G}$  is weakly semiprime.

Suppose  $\tilde{R}\#G$  has  $p$ -torsion. Then  $R$  has  $p$ -torsion, so there exists  $r \in R$ , with  $r \neq 0$ , such that  $pr = 0$ . Choose  $x \in G$  with  $r(x) \neq 0$  and note that  $pr(x) = (pr)(x) = 0$ . Hence  $p(r(x)R(x^{-1})) = 0$ . But  $0 \neq r(x)R(x^{-1}) \subseteq R(1)$  since  $R$  is left nondegenerate. Thus if  $\tilde{R}\#G$  has  $p$ -torsion then so also has  $R(1)$ .

Now consider  $P \subseteq G$ , an elementary abelian  $p$ -subgroup of  $G$ . We need to show that  $R\{P\} = (\tilde{R}\#G)\bar{P}$  is semiprime. Suppose  $I \triangleleft R\{P\}$  is a nonzero ideal with  $I^2 = 0$ . Then by Lemma 1.5 there exists  $x \in G$  such that  $\psi_x(\bar{x}I\bar{x}^{-1}) \neq 0$ . Now by 1.4(ii),  $\psi_x(\bar{x}I\bar{x}^{-1})^2 \subseteq \psi_x(\bar{x}I^2\bar{x}^{-1}) = \psi_x(0) = 0$ . But  $\psi_x(\bar{x}I\bar{x}^{-1}) \triangleleft R(xHx^{-1})$  and  $R(x^{-1}Hx)$  is semiprime since  $x^{-1}Hx$  is an elementary abelian  $p$ -subgroup of  $G$ . This is a contradiction and completes the proof.  $\square$

We now turn our attention to modules over finite group-graded rings. Let  $V$  be a left  $R$ -module and let  $\text{Col}_G V$  denote the set of  $n \times 1$  column vectors over  $V$  with the positions indexed by the elements of  $G$ . If  $v \in \text{Col}_G V$  we write  $v_x$  for the element of  $V$  in the  $[x]$ -position of  $v$ .  $\text{Col}_G V$  is naturally a left  $M_G(R)$ -module under matrix multiplication. The action of  $\alpha \in M_G(R)$  on  $v \in \text{Col}_G V$  is given by

$$(4) \quad (\alpha v)_x = \sum_{y \in G} \alpha_{x,y} v_y.$$

It can be shown that up to isomorphism all left  $M_G(R)$ -modules are of the form  $\text{Col}_G V$  for some left  $R$ -module  $V$ .

If  $v \in V$  then we let  $vf(x)$  denote the element of  $\text{Col}_G V$  with  $v$  in the  $[x]$ -position and zeroes elsewhere. Thus if  $u \in \text{Col}_G V$  then  $u = \sum_{x \in G} u_x f(x)$ . If  ${}_R W \subseteq {}_R V$  are  $R$ -modules we write  $W \text{ess}_R V$  to indicate the  $W$  is an essential  $R$ -submodule of  $V$ . Also  $\text{rank}_R V$  is used to denote the Goldie rank of  $V$  as an  $R$ -module.

The following two results are known for modules over crossed products. (See [3].) Let  $R$  be an arbitrary ring and let  $R * G$  be a crossed product with  $G$  finite. Then

(C) if  $V$  is  $|G|$ -torsion free then  $W \text{ess}_R V$  if and only if  $W \text{ess}_{R * G} V$ .

(D)  $\text{rank}_{R * G} V \leq \text{rank}_R V \leq |G| \cdot \text{rank}_{R * G} V$ .

As before we can find analogs of these results for the group-graded rings. Suppose  $V$  is a module over a  $G$ -graded ring  $R$ . We say  $V$  is component regular if whenever  $v \in V$ , with  $v \neq 0$ , then  $R(x)v \neq 0$  for all  $x \in G$ .

**THEOREM 1.8.** *Let  $R$  be a  $G$ -graded ring with  $G$ -finite and let  ${}_R W \subseteq {}_R V$  be  $R$ -modules where  $V$  is component regular with no  $|G|$ -torsion. Then  $W \text{ess}_R V$  if and only if  $W \text{ess}_{R(1)} V$ .*

*Note.* Two proofs of this result are given. The first proof uses the smash product  $\tilde{R}\#G$  to obtain this result directly from (C). In Theorem 1.9 an equivalent result is proved by adapting the proof of (C) given in [3].

**PROOF.** It is clear that  $W \text{ess}_{R(1)} V$  implies  $W \text{ess}_R V$ .

Conversely suppose  $W \text{ess}_R V$ . We can form the  $M_G(R)$ -modules  $\text{Col}_G W$  and  $\text{Col}_G V$ . Note that  $\text{Col}_G W \subseteq \text{Col}_G V$ , and by considering the diagonal of  $M_G(R)$  acting on  $\text{Col}_G V$  we have that  $\text{Col}_G W$  is an essential  $M_G(R)$ -submodule of  $\text{Col}_G V$ . Since  $V$  has no  $|G|$ -torsion, neither has  $\text{Col}_G V$ . Thus we can apply (C) to  $\text{Col}_G W \subseteq \text{Col}_G V$  as modules over the skew group ring  $M_G(R) = (\tilde{R}\#G)\bar{G}$  to give that  $\text{Col}_G W$  is an essential  $\tilde{R}\#G$ -submodule of  $\text{Col}_G V$ .

To prove  $W\text{ess}_{R(1)}V$  we need to show that, if  $v \in V$  with  $v \neq 0$ , then  $R(1)v \cap W \neq 0$ . Consider  $vf(1) \in \text{Col}_G V$ . Since  $\text{Col}_G W$  is an essential  $\tilde{R}\#G$ -submodule of  $\text{Col}_G V$ , there exists  $\alpha \in \tilde{R}\#G$  such that  $\alpha vf(1) \neq 0$  and  $\alpha vf(1) \in \text{Col}_G W$ . Using equation (4) we get  $\alpha vf(1) = \sum_{x \in G} \alpha_{x,1}vf(x)$ . So  $\alpha_{x,1}v \in W$  for each  $x \in G$  and since  $\alpha vf(1) \neq 0$  we can find  $x \in G$  such that  $\alpha_{x,1}v \neq 0$ . But  $\alpha_{x,1} \in R(x)$  since  $\alpha \in \tilde{R}\#G$ , and  $W$  is an  $R$ -module. Thus  $R(x^{-1})\alpha_{x,1}v \subseteq R(1)v \cap W$  and  $R(x^{-1})\alpha_{x,1}v \neq 0$  because  $V$  is component regular.  $\square$

**THEOREM 1.9.** *Let  $W \subseteq V$  be left  $R$ -modules and suppose  $V$  is component regular with no  $|G|$ -torsion. Then there exists  $K \subseteq V$ , an  $R$ -submodule, such that  $W \cap K = 0$  and  $W \oplus K$  is an essential  $R(1)$ -submodule of  $V$ .*

**PROOF.** Using Zorn's Lemma there exists an  $R(1)$ -submodule  $U \subseteq V$  such that  $W \oplus U$  is an essential  $R(1)$ -submodule of  $V$ . If  $x \in G$  it is easily seen, since  $V$  is component regular, that  $W \oplus R(x)U$  is an essential  $R(1)$ -submodule of  $V$ . Thus, since  $G$  is finite, we have that  $E = \bigcap_{x \in G} (W + R(x)U)$  is again an essential  $R(1)$ -submodule of  $V$ . Note that

$$R(y)E \subseteq \bigcap_{x \in G} [R(y)W + R(y)R(x)U] \subseteq \bigcap_{x \in G} (W + R(yx)U) = E.$$

Hence  $E$  is an  $R$ -submodule of  $V$ .

Since  $W \subseteq E \subseteq W \oplus R(x)U$ ,  $E$  can be written  $E = W \oplus U_x$ , where  $U_x = R(x)U \cap E$ . Let  $\pi_x$  be the projection from  $E$  to  $W$  relative to this decomposition. Now let  $\theta: E \rightarrow W$  be given by  $\theta(t) = \sum_{x \in G} \pi_x(t)$ , where  $t \in E$ .

Let  $t \in E$  and write  $t = w + s$  where  $w \in W$  and  $s \in R(x)U$ , so that  $\pi_x(t) = w$ . Now, if  $r(y) \in R(y)$ , we get  $r(y)t = r(y)w + r(y)s$ . Note that  $r(y)w \in W$  and  $r(y)s \in R(y)R(x)U \subseteq R(yx)U$ . Thus,  $\pi_{yx}(r(y)t) = r(y)w = r(y)\pi_x(t)$ . Summing over all  $x \in G$ , we obtain  $\sum_{x \in G} \pi_{yx}(r(y)t) = r(y) \cdot \sum_{x \in G} \pi_x(t)$ , that is  $\theta(r(y)t) = r(y)\theta(t)$ . Since  $\theta$  is additive, we conclude that  $\theta$  is  $R$ -linear.

Let  $K = \ker \theta$ , so  $K$  is an  $R$ -submodule of  $E$ . If  $w \in W$  then  $\pi_x(w) = w$ , so  $\theta(w) = nw$ . But  $V$  is  $|G|$ -torsion free, so  $K \cap W = 0$ . Also if  $t \in E$  and  $\theta(t) = w$ , we note that  $nt - w \in \ker \theta = K$ . Thus  $n \cdot E \subseteq W \oplus K$  and so  $W \oplus K$  is an essential  $R(1)$ -submodule of  $E$  by the torsion free assumption. Finally, since  $E$  is an essential  $R(1)$ -submodule of  $V$  we conclude that  $W \oplus K$  is essential as an  $R(1)$ -submodule of  $V$ .  $\square$

In Theorem 1.11 we prove a generalization of (D) but first we require a lemma. A collection  $W_1, \dots, W_k$  of submodules of a module  $V$  is said to be independent if  $W_1 + \dots + W_k$  is a direct sum.

**LEMMA 1.10.** *Let  $V$  be an  $R$ -module. Then*

(i)  $\text{rank}_R V = \text{rank}_{M_G(R)} \text{Col}_G V$ .

*Furthermore if  $V$  is component regular then*

(ii)  $\text{rank}_{R(1)} V = \text{rank}_{\tilde{R}\#G} \text{Col}_G V$ .

**PROOF.** (i) It is easy to see that  $M_G(R)$ -submodules of  $\text{Col}_G V$  are of the form  $\text{Col}_G W$  where  $W \subseteq V$  is an  $R$ -submodule. Note that  $W_1, \dots, W_k$  is an independent set of  $R$ -submodules of  $V$  if and only if  $\text{Col}_G W_1, \dots, \text{Col}_G W_k$  forms an independent set of  $M_G(R)$ -submodules of  $\text{Col}_G V$ . This proves (i).

(ii) Suppose  $V_1, \dots, V_s$  form an independent set of nonzero  $R(1)$ -submodules of  $V$ . Then we claim  $V_1^0, \dots, V_s^0$  is an independent collection of nonzero  $(\tilde{R}\#G)$ -submodules of  $\text{Col}_G V$  where

$$V_i^0 = \sum_{x \in G} R(x)V_i f(x) = (\tilde{R}\#G)V_i f(1).$$

Clearly,  $V_i^0$  is nonzero for each  $i$ . Also, it is easily checked, since  $V$  is component regular, that  $R(x)V_1, \dots, R(x)V_s$  is an independent set of subgroups of  $V$ . Hence  $V_1^0 + \dots + V_s^0$  is direct, and so  $\text{rank}_{\tilde{R}\#G} \text{Col}_G V \geq \text{rank}_{R(1)} V$ .

To obtain the reverse inequality, suppose that  $U_1, \dots, U_t$  is an independent collection of nonzero  $\tilde{R}\#G$ -submodules of  $\text{Col}_G V$ . Let  $U_i' = \{u \in V \mid uf(1) \in U_i\}$  and note that  $U_1' + \dots + U_t'$  is a direct sum. Thus, we need only show that each  $U_i'$  is nonzero.

Since  $U_i \neq 0$  and  $U_i = \sum_{x \in G} e(x, x)U_i$ , we have that  $e(x, x)U_i \neq 0$  for some  $x \in G$ . Thus  $R(x^{-1})e(1, x)U_i = R(x^{-1})e(1, x) \cdot e(x, x)U_i \neq 0$  since  $V$  is component regular. Also, since  $R(x^{-1})e(1, x) \subseteq \tilde{R}\#G$ , we have that  $R(x^{-1})e(1, x)U_i \subseteq U_i$ . But  $R(x^{-1})e(1, x)U_i \subseteq Vf(1)$ , which implies  $U_i' \neq 0$ . Thus  $\text{rank}_{\tilde{R}\#G} \text{Col}_G V = \text{rank}_{R(1)} V$ .  $\square$

**THEOREM 1.11.** *Let  $V$  be a component regular  $R$ -module. Then*

$$\text{rank}_R V \leq \text{rank}_{R(1)} V \leq |G| \cdot \text{rank}_R V.$$

**PROOF.** Using that  $M_G(G) = (\tilde{R}\#G)\bar{G}$  is a skew group ring, we can apply (D) to conclude

$$\text{rank}_{M_G(R)} \text{Col}_G V \leq \text{rank}_{\tilde{R}\#G} \text{Col}_G V \leq |G| \cdot \text{rank}_{M_G(R)} \text{Col}_G V.$$

Now applying Lemma 1.10 gives the result.  $\square$

We conclude this section with a note on graded modules. A left  $R$ -module  $V$  is said to be  $G$ -graded if  $V = \bigoplus_{x \in G} V(x)$  is a direct sum of additive groups such that  $R(x)V(y) \subseteq V(xy)$ . If  $v \in V$  we write  $v(x)$  for the component of  $v$  in  $V(x)$ . Clearly,  $R$  is graded as a left module over itself.

If  $V$  is a  $G$ -graded  $R$ -module then we can embed  $V$  in  $\text{Col}_G V$  by a map  $\rho$ , where  $\rho(v) = \sum_{x \in G} v(x)f(x)$  for all  $v \in V$ . We write  $\bar{v}$  for  $\rho(v)$  and denote the image of  $\rho$  by  $\tilde{V}$ . It is easily checked that  $\tilde{V}$  is an  $\tilde{R}\#G$ -submodule of  $\text{Col}_G V$ . If we let  $r \cdot \bar{v} = \bar{r}\bar{v}$  then  $\tilde{V}$  becomes an  $R$ -module and  $\rho$  is an  $R$ -module isomorphism.

In [1] it was noted that the  $\tilde{R}\#G$ -modules correspond to the graded  $R$ -modules. The map taking  $V$  to  $\tilde{V}$  gives one direction. Conversely, if  $W$  is an  $\tilde{R}\#G$ -module one can check that taking  $W(x) = e(x, x)W$  and  $r \cdot w = \bar{r}w$  for all  $w \in W$  gives  $W$  a graded  $R$ -module structure.

The modules  $\tilde{V}$  and  $\text{Col}_G V$  are related as follows:

**LEMMA 1.12.** *Let  $R$  be a  $G$ -graded ring with  $G$  finite and let  $V$  be a graded left  $R$ -module. Then  $\text{Col}_G V = \bigoplus_{g \in G} \bar{g}\tilde{V}$  and each  $\bar{g}\tilde{V}$  is an  $\tilde{R}\#G$ -submodule of  $\text{Col}_G V$ . Thus  $\text{Col}_G V$  is the  $M_G(R)$ -module induced from the  $\tilde{R}\#G$ -module  $\tilde{V}$ .*

PROOF. Note that  $e(x, y)V(z)f(z) = V(y)f(x)$  if  $y = z$  and is zero otherwise. Hence

$$\bar{g}\tilde{V} = \sum_{x \in G} e(x, xg) \sum_{y \in G} V(y)f(y) = \sum_{x \in G} V(xg)f(x),$$

and since  $\sum_{x \in G} V(x)$  is a direct sum, it follows that  $\sum_{g \in G} \bar{g}\tilde{V}$  is also direct. Also,

$$\sum_{g \in G} \bar{g}\tilde{V} = \sum_{g \in G} \sum_{x \in G} V(xg)f(x) = \sum_{x \in G} Vf(x) = \text{Col}_G V.$$

Finally, from Lemma 1.2(ii), we have that  $(\tilde{R}\#G)\bar{g} = \bar{g}(\tilde{R}\#G)$ , and so  $(\tilde{R}\#G)\bar{g}\tilde{V} = \bar{g}(\tilde{R}\#G)\tilde{V} = \bar{g}\tilde{V}$ . Thus  $\bar{g}V$  is an  $\tilde{R}\#G$ -submodule of  $\text{Col}_G V$ .  $\square$

**2. Infinite group graded rings.** The constructions of §1 can be repeated, with some minor changes, when  $R$  is  $G$ -graded with  $G$  infinite. Here we work inside  $M_G(R)$ —the set of row and column finite matrices over  $R$  with the rows and columns again indexed by the elements of  $G$ . We use  $M_G^*(R)$  to denote the matrices with finitely many nonzero entries.

As before,  $R$  can be embedded in  $M_G(R)$  by a ring monomorphism  $\eta$ , taking  $r = \sum_{x \in G} r(x)$  to  $\tilde{r} \in M_G(R)$ , where  $\tilde{r}_{x,y} = r(xy^{-1})$ . Now  $\tilde{R}\#G$  is defined to be the subring of  $M_G(R)$  generated by  $\tilde{R}$  and  $\{p(x)\}_{x \in G}$ , where  $p(x) = e(x, x) \in M_G(R)$ .

**LEMMA 2.1.** *Let  $R$  be a  $G$ -graded ring with  $G$  infinite. If  $r, s \in R$ , then  $\tilde{r}p(x)\tilde{s}p(y) = \tilde{r}s(xy^{-1})p(y)$  and  $\tilde{R}\#G = \tilde{R} \oplus [\oplus_{x \in G} \tilde{R}p(x)]$  is a free  $\tilde{R}$ -module with basis  $\{I\} \cup \{p(x)\}_{x \in G}$ . Furthermore,  $\sum_{x \in G} \tilde{R}p(x) = \sum_{x,y \in G} R(xy^{-1})e(x, y)$  is an ideal of  $\tilde{R}\#G$  which is essential both as a left ideal and as a right ideal.*

**PROOF.** The argument of Lemma 1.1 applies again to give that  $\tilde{r}p(x)\tilde{s}p(y) = \tilde{r}s(xy^{-1})p(y)$  and that  $\tilde{R}p(x)$  is a free  $\tilde{R}$ -module with generator  $p(x)$ . The sum  $\sum_{x \in G} \tilde{R}p(x)$  is direct since  $\{p(x)\}_{x \in G}$  is a set of orthogonal idempotents. It is easily checked that  $\sum_{x \in G} \tilde{R}p(x) = \sum_{x,y \in G} R(xy^{-1})e(x, y)$  and so  $\sum_{x \in G} \tilde{R}p(x) \cap \tilde{R} = 0$ , since nonzero elements of  $\tilde{R}$  have infinitely many nonzero entries.

It is now clear that  $\tilde{R}\#G = \tilde{R} + \sum_{x \in G} \tilde{R}p(x)$ . Note that

$$\tilde{R}\#G \subseteq \left\{ \alpha \in M_G(R) \mid \alpha_{x,y} \in R(xy^{-1}) \right\}$$

and

$$\tilde{R}\#G \supseteq \left\{ \alpha \in M_G^*(R) \mid \alpha_{x,y} \in R(xy^{-1}) \right\} = \sum_{x,y \in G} R(xy^{-1})e(x, y).$$

Thus  $\sum_{x \in G} \tilde{R}p(x) = (\tilde{R}\#G) \cap M_G^*(R)$ , and since  $M_G^*(R) \triangleleft M_G(R)$ , this implies  $\sum_{x \in G} \tilde{R}p(x) \triangleleft \tilde{R}\#G$ . To show that it is essential as a left ideal, let  $\alpha \in \tilde{R}\#G$  with  $\alpha \neq 0$ . We can find  $x \in G$  such that  $p(x)\alpha \neq 0$ . Note that  $p(x)\alpha$  has finitely many nonzero entries, and so  $p(x)\alpha \in \sum_{x \in G} \tilde{R}p(x)$ . Thus,  $\sum_{x \in G} \tilde{R}p(x)$  is essential as a left ideal and similarly it is essential as a right ideal.  $\square$

Now  $G$  is embedded in  $M_G(R)$  by the map taking  $g$  to  $\bar{g}$ , where  $\bar{g} = \sum_{x \in G} e(x, xg)$ . As in §1 this map is a group isomorphism from  $G$  to  $\bar{G} = \{\bar{g} \mid g \in G\}$  and  $\bar{G}$  is a subgroup of the units of  $M_G(R)$ .

LEMMA 2.2. *Let  $R$  be a  $G$ -graded ring with  $G$  infinite. Then*

- (i)  $\bar{g}^{-1}(\tilde{R}\#G)\bar{g} = \tilde{R}\#G$  and  $(\tilde{R}\#G)^G = \tilde{R}$ ,
- (ii)  $\sum_{g \in G} (\tilde{R}\#G)\bar{g}$  is a direct sum of additive groups,
- (iii)  $(\tilde{R}\#G)\bar{G}$  is a skew group ring of the group  $\bar{G}$  over the ring  $\tilde{R}\#G$ ,
- (iv)  $(\tilde{R}\#G)\bar{G} = (\oplus \sum_{g \in G} \tilde{R}\bar{g}) \oplus M_G^*(R)$  as additive groups and  $M_G^*(R)$  is an ideal of  $(\tilde{R}\#G)\bar{G}$  which is essential both as a left ideal and as a right ideal.

PROOF. Parts (i), (ii) and (iii) are proved as in the case of  $G$  finite.

$$\begin{aligned}
 \text{(iv)} \quad (\tilde{R}\#G)\bar{G} &= \sum_{g \in G} (\tilde{R}\#G)\bar{g} \\
 &= \sum_{g \in G} \left[ \tilde{R} + \sum_{x, y \in G} R(xy^{-1})e(x, y) \right] \bar{g} \quad \text{by Lemma 2.1} \\
 &= \sum_{g \in G} \tilde{R}\bar{g} + \sum_{x, y, g \in G} R(xy^{-1})e(x, y)\bar{g}.
 \end{aligned}$$

Note that

$$\begin{aligned}
 \sum_{x, y, g \in G} R(xy^{-1})e(x, y)\bar{g} &= \sum_{x, y, g \in G} R(xy^{-1})e(x, yg) \\
 &= \sum_{x, w, g \in G} R(xgw^{-1})e(x, w) = \sum_{x, w \in G} Re(x, w) = M_G^*(R).
 \end{aligned}$$

Thus  $(\tilde{R}\#G)\bar{G} = \sum_{g \in G} \tilde{R}\bar{g} + M_G^*(R)$ .

$\sum_{g \in G} \tilde{R}\bar{g}$  is a direct sum from (ii), and since nonzero elements of  $\sum_{g \in G} \tilde{R}\bar{g}$  have infinitely many nonzero entries,  $\sum_{g \in G} \tilde{R}\bar{g} \cap M_G^*(R) = 0$ .

Clearly,  $M_G^*(R)$  is an ideal of  $(\tilde{R}\#G)\bar{G}$ . Now if  $\alpha \in (\tilde{R}\#G)\bar{G}$  with  $\alpha \neq 0$ , we can find  $x \in G$  such that  $p(x)\alpha \neq 0$ . Note that  $p(x)\alpha \in M_G^*(R)$ , and so  $M_G^*(R)$  is essential as a left ideal. Similarly, it is essential as a right ideal.  $\square$

We now define  $R\{H\}$  to be  $(\tilde{R}\#G)\bar{H} = \sum_{h \in H} (\tilde{R}\#G)\bar{h}$ . Note that  $R\{H\} \subseteq \{\alpha \in M_G(R) \mid \alpha_{x,y} \in R(xHy^{-1})\}$ . Now let

$$R^*\{H\} = \{\alpha \in M_G^*(R) \mid \alpha_{x,y} \in R(xHy^{-1})\}.$$

The following lemma is proved in the same way as Lemma 2.2(iv):

LEMMA 2.3. *With the above notation,  $R\{H\} = \sum_{h \in H} \tilde{R}\bar{h} \oplus R^*\{H\}$ . Furthermore,  $R^*\{H\}$  is a two-sided ideal which is essential on either side.  $\square$*

LEMMA 2.4. *Let  $R$  be a  $G$ -graded ring with  $G$  infinite and suppose  $H$  is a subgroup of  $G$ . Then there exist maps*

$$\phi: \mathcal{I}(R(H)) \rightarrow \mathcal{I}(R\{H\}), \quad \psi: \mathcal{I}(R\{H\}) \rightarrow \mathcal{I}(R(H)),$$

such that

- (i)  $\phi$  is injective and if  $A, B \triangleleft R(H)$  then  $\phi(A)\phi(B) \subseteq \phi(AB)$ ,
- (ii) if  $J, K \triangleleft R\{H\}$  then  $\psi(J)\psi(K) \subseteq \psi(JK)$ ,
- (iii)  $\psi \circ \phi$  is the identity on  $\mathcal{I}(R(H))$ .

Furthermore, let  $R$  be component regular, and let  $I \triangleleft R\{H\}$ . Then  $\psi(I) = 0$  if and only if  $I = 0$ .

PROOF. Let  $D = \sum_{x \in G} R(xH)e(x, x)$  and let  $D^{-1} = \sum_{x \in G} R(Hx^{-1})e(x, x)$ . If  $A \triangleleft R(H)$ , we take  $\phi(A)$  to be  $DM_G^*(A)D^{-1}$  where  $M_G^*(A)$  is the set of matrices over  $A$  with finitely many nonzero entries. Next, if  $I \triangleleft R\{H\}$  we define  $\psi(I)$  to be  $I_{1,1} = \{\alpha_{1,1} | \alpha \in I\}$ . The remainder of the proof is identical to that of Lemma 1.4.  $\square$

As in the case where  $G$  was finite, results on infinite skew group rings can be translated into results on infinite group-graded rings. In Theorems 2.7 and 2.8 we apply the above construction to a result of Passman [5, Theorem 1.3]. While the theorem was proved for strongly group-graded rings, we state it for skew group rings, since that is all we require here.

(E) THEOREM. *Let  $R * G$  be a skew group ring with  $G$  infinite. Then  $R * G$  contains nonzero ideals  $A$  and  $B$  with  $AB = 0$  if and only if there exist*

- (i) *subgroups  $N \triangleleft H \subseteq G$  with  $N$  finite,*
  - (ii) *an  $H$ -invariant ideal  $I$  of  $R$  with  $I^x I = 0$  for  $x \in G \setminus H$ ,*
  - (iii) *nonzero  $H$ -invariant ideals  $\tilde{A}, \tilde{B} \triangleleft R * N$  with  $\tilde{A}, \tilde{B} \subseteq I * N$  and  $\tilde{A}\tilde{B} = 0$ .*
- Furthermore,  $A = B$  if and only if  $\tilde{A} = \tilde{B}$ .  $\square$*

Let  $R$  be a  $G$ -graded ring and suppose  $N \triangleleft H \subseteq G$  are subgroups. Then if  $I \triangleleft R(N)$  and  $x \in H$ , we let  $I^x = R(x^{-1}N)IR(Nx)$ .  $I^x$  is again an ideal of  $R(N)$ . This may not be a group action on the ideals of  $R(N)$  since it is not true in general that  $(I^x)^y = I^{xy}$ , but we do have  $I^1 = I$  and  $(I^x)^y \subseteq I^{xy}$ . We say  $I \triangleleft R(N)$  is  $H$ -stable if  $I^h \subseteq I$  for all  $h \in H$ . If  $I \triangleleft R(N)$  is  $H$ -stable then  $\bar{I} = R(H)IR(H) \triangleleft R(H)$  and  $I = \bar{I} \cap R(N)$ . In particular, an ideal  $I \triangleleft R(N)$  is  $H$ -stable if and only if  $I = J \cap R(N)$  for some ideal  $J \triangleleft R(H)$ . An ideal  $K \triangleleft R(N)$  is said to be a graded ideal if  $K = \bigoplus_{x \in N} K(x)$ , where  $K(x) = K \cap R(x)$  for each  $x \in N$ .

LEMMA 2.5. *Let  $R$  be a  $G$ -graded ring and let  $N \triangleleft H \subseteq G$  be subgroups of  $G$ . Also suppose  $A$  is an  $\bar{H}$ -stable ideal of  $R\{N\}$ . Then  $\psi(A) \triangleleft R(N)$  is an  $H$ -stable ideal of  $R(N)$ . Furthermore, let  $I \triangleleft \bar{R}\#G$  be an  $\bar{H}$ -stable ideal of  $\bar{R}\#G$  such that  $I^t I = 0$  for all  $t \in G \setminus H$ . Then  $I\bar{N} \triangleleft R\{N\}$  and  $\psi(I\bar{N})$  is a graded  $H$ -stable ideal of  $R(N)$  with the property that  $\psi(I\bar{N}) \cdot R(t) \cdot \psi(I\bar{N}) = 0$  for all  $t \in G \setminus H$ .*

PROOF. Let  $A \triangleleft R\{N\}$  and let  $x \in H$ . Notice that

$$R(x^{-1}N)e(1, 1)\bar{x}^{-1} = R(x^{-1}N)e(1, x^{-1}) \subseteq R\{N\}$$

since  $N \triangleleft H$  implies  $x^{-1}N = Nx^{-1}$ . Similarly,  $\bar{x}e(1, 1)R(Nx) \subseteq R\{N\}$ . Since  $A \triangleleft R\{N\}$ , this yields that

$$R(x^{-1}N)e(1, 1)\bar{x}^{-1} \cdot A \cdot \bar{x}e(1, 1)R(Nx) \subseteq A.$$

But  $A$  is  $\bar{H}$ -stable, so that  $\bar{x}^{-1}A\bar{x} = A$  and  $e(1, 1)Ae(1, 1) = \psi(A)e(1, 1)$ . Thus  $R(x^{-1}N)\psi(A)R(Nx)e(1, 1) \subseteq A$  or, equivalently,  $R(x^{-1}N)\psi(A)R(Nx) \subseteq \psi(A)$ , and so  $\psi(A)$  is  $H$ -stable.

Next let  $I \triangleleft \bar{R}\#G$  be  $\bar{H}$ -stable so that  $J = I\bar{N}$  is an  $\bar{H}$ -stable ideal of  $R\{N\}$ . From  $I^t I = 0$  for all  $t \in G \setminus H$  we get that  $J^t J = 0$ . This follows since  $H \supseteq N$  implies  $Nt \subseteq G \setminus H$ . Notice that  $e(1, 1)R(t)\bar{t}^{-1} \subseteq \bar{R}\#G \subseteq R\{N\}$ , and so since  $J \triangleleft R\{N\}$ ,

we have that  $Je(1, 1)R(t)J = Je(1, 1)R(t)\bar{t}^{-1} \cdot \bar{t}J \subseteq \bar{t}J\bar{t} = 0$ . But  $e(1, 1)Je(1, 1) = J_{1,1}e(1, 1) = \psi(J)e(1, 1)$ , so that  $J_{1,1}R(t)J_{1,1} = 0$  or  $\psi(J)R(t)\psi(J) = 0$ . Finally,

$$\psi(J) = (I\bar{N})_{1,1} = \left( \sum_{h \in N} \bar{I}h \right)_{1,1} = \sum_{h \in N} (I\bar{h})_{1,1} = \sum_{h \in N} I_{1,h^{-1}}$$

and, since  $I \subseteq \bar{R}\#G$ ,  $I_{1,h^{-1}} \subseteq R(h)$ , so that  $\psi(J)$  is a graded ideal.  $\square$

LEMMA 2.6. *Let  $R$  be a left component regular  $G$ -graded ring with  $G$  finite and suppose  $A$  is a nonzero ideal of  $R$ . Then  $A^0 = \bigcap_{x \in G} AR(x)$  is a nonzero ideal of  $R$ . Furthermore, if  $A \subseteq I$ , where  $I$  is a graded ideal of  $R$ , then  $A^0 \subseteq I(1)R$ .*

PROOF. It is clear that  $A^0$  is an ideal of  $R$ . To show  $A^0 \neq 0$ , let  $x_1, \dots, x_n$  be a listing of the elements of  $G$  where  $n = |G|$ . Now let

$$B = AR(x_n x_{n-1}^{-1})R(x_{n-1} x_{n-2}^{-1}) \cdots R(x_2 x_1^{-1})R(x_1).$$

Note that  $B \neq 0$  since  $R$  is left component regular. Also,

$$\begin{aligned} B &= [AR(x_n x_{n-1}^{-1}) \cdots R(x_{i+1} x_i^{-1})][R(x_i x_{i-1}^{-1}) \cdots R(x_1)] \\ &\subseteq AR(x_i x_{i-1}^{-1} x_{i-1} x_{i-2}^{-1} \cdots x_2 x_1^{-1} x_1) = AR(x_i) \quad \text{for } i = 1, 2, \dots, n. \end{aligned}$$

Thus  $0 \neq B \subseteq \bigcap_{x \in G} AR(x) = A^0$ .

Finally note that if  $A \subseteq I$  then  $A^0 \subseteq I^0$  and, since  $I$  is graded,

$$I^0 = \sum_{x \in G} \left[ \bigcap_{y \in G} I(xy^{-1})R(y) \right] \subseteq \sum_{x \in G} I(1)R(x) = I(1)R. \quad \square$$

If  $I \triangleleft R(1)$  is an  $H$ -stable ideal then  $R(H)IR(H) \triangleleft R(H)$ , and  $\hat{I} = R(H)IR(H) \cap R(N)$  is an  $H$ -stable graded ideal of  $R(N)$  with  $\hat{I} \cap R(1) = I$ .

THEOREM 2.7. *Let  $R$  be a left component regular  $G$ -graded ring with  $G$  infinite. Then there exists a nonzero ideal  $A \triangleleft R$  with  $A^2 = 0$  if and only if there exist*

- (i) subgroups  $N \triangleleft H \subseteq G$ , with  $N$  finite,
- (ii) an  $H$ -stable ideal  $I \triangleleft R(1)$  with  $IR(t)I = 0$  for all  $t \in G \setminus H$ ,
- (iii) a nonzero  $H$ -stable ideal  $\hat{A} \triangleleft R(N)$  with  $\hat{A} \subseteq \hat{I}$  and  $\hat{A}^2 = 0$ , where  $\hat{I} = R(H)IR(H) \cap R(N)$ .

PROOF. Assume  $R$  has a nonzero ideal  $A$  with  $A^2 = 0$ . Then  $M_G^*(A)$  is a nonzero ideal of  $(\bar{R}\#G)\bar{G}$  which has square zero. We can apply (E) to give

- (a) subgroups  $N \triangleleft H \subseteq G$  with  $N$  finite,
- (b)  $I'$  an  $\bar{H}$ -stable ideal of  $\bar{R}\#G$  such that  $I'\bar{t}I' = 0$  for all  $t \in G \setminus H$ ,
- (c) a nonzero  $\bar{H}$ -stable ideal  $A' \triangleleft (\bar{R}\#G)\bar{H}$  with  $A' \subseteq I'\bar{N}$  and  $(A')^2 = 0$ .

Assume  $\psi(A') \neq 0$  and let  $B = \psi(A')$  and  $J = \psi(I'\bar{N})$ . By Lemma 2.5,  $B$  and  $J$  are  $H$ -stable ideals of  $R(N)$  and  $J$  is graded. Also  $B \subseteq J$ , since  $A' \subseteq I'\bar{N}$ , and by Lemma 2.4(ii),  $B^2 = \psi(A')^2 \subseteq \psi(0) = 0$ . Now let  $I = J \cap R(1)$  so that  $I$  is an  $H$ -stable ideal of  $R(1)$ . Also  $\hat{I} = R(H)IR(H) \cap R(N)$  is an  $H$ -stable ideal of  $R(N)$ , and so if we take  $A = \hat{I} \cap B$  then  $A \triangleleft R(N)$  is again  $H$ -stable and  $A^2 \subseteq B^2 = 0$ . Note that  $A \neq 0$  since applying Lemma 2.6 to the group  $N$  gives that  $B \cap IR(N) \neq 0$  and  $A = B \cap \hat{I} \supseteq B \cap IR(N)$ .

To complete the proof in this direction it remains to show that we can assume  $\psi(A') \neq 0$ . Note that since  $A' \neq 0$ , there exist  $x, y \in G$  such that  $A'_{x,y} \neq 0$ .  $A'_{x,y}e(x, y) \subseteq A' \triangleleft R\{N\}$  and  $R(yNx^{-1})e(y, x) \subseteq R\{N\}$  so that  $A'_{x,y}R(yNx^{-1})e(x, x) = A'_{x,y}e(x, y)R(yNx^{-1})e(y, x) \subseteq A'$ .

Since  $A'_{x,y}$  is nonzero and  $R$  is left component regular, it follows that  $A'_{x,y}R(yNx^{-1}) \neq 0$  and hence  $A'_{x,x} \neq 0$ . Now

$$\bar{x}A'\bar{x}^{-1} \triangleleft \bar{x}R\{N\}\bar{x}^{-1} = R\{xNx^{-1}\} \quad \text{and} \quad \psi_x(\bar{x}A'\bar{x}^{-1}) = A'_{x,x} \neq 0.$$

Thus replacing  $I', A', N$  and  $H$  by  $\bar{x}I'\bar{x}^{-1}, \bar{x}A'\bar{x}^{-1}, xNx^{-1}$  and  $xHx^{-1}$ , respectively, gives the desired result.

Conversely, suppose conditions (i), (ii) and (iii) are satisfied. Let  $A = R\hat{A}R \triangleleft R$ . Since  $\hat{A} \neq 0$ , we have that  $A \neq 0$ . If  $t \in G \setminus H$  then  $HtH \subseteq G \setminus H$  and thus

$$\begin{aligned} \hat{I}R(t)\hat{I} &\subseteq R(H)IR(H) \cdot R(t) \cdot R(H)IR(H) \\ &\subseteq R(H)IR(HtH)IR(H) = 0 \quad \text{by (ii)}. \end{aligned}$$

Hence,  $\hat{A}R(t)\hat{A} \subseteq \hat{I}R(t)\hat{I} = 0$ . Next let  $x \in H$ . Since  $\hat{A}$  is  $H$ -stable,  $R(xN)\hat{A}R(Nx^{-1}) \subseteq \hat{A}$ . Thus  $\hat{A}R(xN)\hat{A}R(Nx^{-1}) \subseteq \hat{A}^2 = 0$ . But since  $R$  is left component regular, this implies that  $\hat{A}R(xN)\hat{A} = 0$ . In particular,  $\hat{A}R(x)\hat{A} = 0$ . Thus we have that  $\hat{A}R\hat{A} = 0$ , and so  $A^2 = R\hat{A}R\hat{A}R = 0$ .  $\square$

The proof of the final result is left to the reader.

**THEOREM 2.8.** *Let  $R$  be a component regular  $G$ -graded ring with  $G$  infinite. Then there exist nonzero ideals  $A, B \triangleleft R$  with  $AB = 0$  if and only if there exist*

- (i) *subgroups  $N \triangleleft H \subseteq G$  with  $N$  finite,*
- (ii) *an  $H$ -stable ideal  $I \triangleleft R(1)$  with  $IR(t)I = 0$  for all  $t \in G \setminus H$ ,*
- (iii) *nonzero  $H$ -stable ideals  $\hat{A}, \hat{B} \triangleleft R(N)$  with  $\hat{A}, \hat{B} \subseteq \hat{I}$  and  $\hat{A}\hat{B} = 0$ , where  $\hat{I} = R(H)IR(H) \cap R(N)$ .  $\square$*

If  $M$  is a graded left  $R$ -module, where  $R$  is  $G$ -graded, then  $M$  is said to be graded Noetherian if  $M$  satisfies the ascending condition on graded submodules. We write  $\text{K-dim}_R M$  for the Krull dimension of  $M$  and  $\text{gr K-dim}_R M$  for the graded Krull dimension of  $M$ . The methods of this paper can be used to obtain the following theorem which extends a result of C. Nastasescu.

**THEOREM.** *Let  $R$  be a  $G$ -graded ring where  $G$  is a polycyclic-by-finite group. If  $M$  is a graded Noetherian  $R$ -module, then  $M$  is Noetherian as an  $R$ -module. Furthermore*

$$\text{K-dim}_R M \leq \text{gr K-dim}_R M + h(G), \quad \text{where } h(G) \text{ is the Hirsch number of } G.$$

This result will be included in a subsequent paper.

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