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ON THE SOLUTION OF CERTAIN SYSTEMS OF PARTIAL DIFFERENCE EQUATIONS AND LINEAR DEPENDENCE OF TRANSLATES OF BOX SPLINES

BY

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ABSTRACT. This paper is concerned with solving systems of partial difference equations which arise when studying linear dependence of translates of box splines.

1. Introduction. Let $X = \{x^1, \ldots, x^n\}$ be a set of (not necessarily distinct) vectors in $\mathbf{R}^s$, i.e. $x^i = (x^i_1, \ldots, x^i_s)$. The box spline $B(\cdot|X)$ is defined by requiring that

$$(1.1)\qquad \int_{\mathbf{R}^s} f(x) B(x|X)\,dx = \int_{[0,1]^n} f\left(t_1 x^1 + \cdots + t_n x^n\right) dt_1 \cdots dt_n$$

holds for any continuous function f on $\mathbf{R}^s$ [**1**]. Many of its basic properties were established in [**1**]. It is a special instance of the wider class of polyhedral splines discussed in [**4**]. For a detailed account of the properties of $B(\cdot|X)$ the reader is referred to [**1, 3, 4**]. Here it is sufficient for the moment to remark that $B(\cdot|X)$ is in general a distribution in $\mathscr{D}'(\mathbf{R}^s)$. If $\langle X \rangle = \operatorname{span} X = \mathbf{R}^s$, however, $B(\cdot|X)$ is a piecewise polynomial of (total) degree $\leqslant n - s$ with support

$$(1.2)\qquad B(X) = \left\{ x = \sum_{i=1}^{n} t_i x^i \colon 0 \leqslant t_i \leqslant 1,\ i = 1, \ldots, n \right\}.$$

More precisely, it is a polynomial in any region which is not intersected by any of the $(s-1)$-dimensional hyperplanes of the form

$$\left\{ \sum_{j=1}^{s-1} t_j x^{i_j} + \sum_{j=1}^{n-s+1} \sigma_j x^{i'_j} \colon \sigma_j \in \{0,1\},\ t_j \in \mathbf{R} \right\}$$

where $\{i'_1, \ldots, i'_{n-s+1}\}$ are the complementary integers for $\{i_1, \ldots, i_s\}$ in the set $\{1, \ldots, n\}$. We will briefly refer to these hyperplanes as the cut regions of $B(\cdot|X)$. The union of all cut regions of all translates $B(\cdot - \alpha|X)$, $\alpha \in Z^s$, will be denoted by $c(X)$.

The notion of box spline has proven to be very useful by providing a better understanding of many finite element spaces on regular grids. Moreover, the surprisingly rich structure associated with the simple notion (1.1) has lead to various problems which we believe are interesting in their own right. Here we concentrate on two spaces, one of functions and the other of sequences which play a central role for the understanding of the structure of splines generated by translates of box-splines.

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To this end, we have to introduce the following notation. For $y \in \mathbf{R}^s$, $D_y f = \sum_{j=1}^s y_j \partial f/\partial x_j$ denotes the usual directional derivative of f in the direction $y = (y_1, \ldots, y_s)$. Its discrete analogs are the forward and backward difference operator $\Delta_y f(\cdot) = f(\cdot + y) - f(\cdot)$, $\nabla_y f(\cdot) = f(\cdot) - f(\cdot - y)$. Generally, we will write, for any set $Y \in \mathbf{R}^s \setminus \{0\}$, $|Y| < \infty$, $|Y|$ denoting the cardinality of Y,

$$D_Y = \prod_{y \in Y} D_y, \qquad \Delta_Y = \prod_{y \in Y} \Delta_y.$$

Now, for any given finite set $X \subset \mathbf{R}^s \setminus \{0\}$, with $\langle X \rangle = \mathbf{R}^s$, let

$$\mathscr{Y}(X) = \{Y \colon Y \subset X, \langle X \setminus Y \rangle \neq \mathbf{R}^s\}. \tag{1.3}$$

These notions are used to introduce a subspace of distributions

$$D(X) = \left\{ f \in \mathscr{D}'(\mathbf{R}^s) \colon D_Y f = 0,\ Y \in \mathscr{Y}(X) \right\}, \tag{1.4}$$

as well as the space of sequences

$$\Delta(X) = \left\{ f \colon (\Delta_Y f)(\alpha) = 0, \alpha \in Z^s,\ Y \in \mathscr{Y}(X) \right\}, \tag{1.5}$$

a discrete analog of $D(X)$. In the following, we will use the same notation for *sequences* on Z^s as for functions on $\mathbf{R}^s$. Specifically, the restriction of some function f defined on $\mathbf{R}^s$ to Z^s is also denoted by f.

The objective of this paper is to clarify the relation between $D(X)$, $\Delta(X)$ and $B(\cdot|X)$. We will briefly summarize in §2 a few known facts about $D(X)$ which will be needed in subsequent discussions. In §3 we will give a complete characterization of the elements of $\Delta(X)$. We show that $\Delta(X) = E(X) \oplus D(X)$ where $E(X)$ is spanned by certain sequences of the form $z^\alpha p(\alpha)$, $z \in \mathbf{C}^s$, $\alpha \in Z^s$, where p is a polynomial and $z \neq e = (1, \ldots, 1)$. Moreover, defining for $X \subset Z^s$ and any $x \notin c(X)$ the set

$$b(x|X) = \{\alpha \in Z^s \colon x \in \operatorname{supp} B(\cdot|X) + \alpha\},$$

we show that each finite sequence $\{d_\alpha \colon \alpha \in b(x|X)\}$ possesses a unique extension in $\Delta(X)$.

These results allow us to find linear relations among translates of box splines. Specifically, we show that any $f \in E(X)$ gives

$$\sum_{\alpha \in Z^s} f(\alpha) B(x - \alpha|X) = 0, \qquad x \in \mathbf{R}^s,$$

while the converse is true only under additional conditions on X. (If $\dim\langle X \rangle < s$ the above equation is interpreted in a distributional sense.)

2. Preliminaries and background material. The close relation between $D(X)$, $\Delta(X)$ and the box spline stems from the formula [1]

$$(\Delta_X f)(x) = \int_{\mathbf{R}^s} (D_X f(x+y)) B(y|X)\, dy. \tag{2.1}$$

In fact, this equation readily shows that

$$D(X) \subseteq \Delta(X). \tag{2.2}$$

On the other hand, when $f \in \Delta(X)$ and $X \subset Z^s \setminus \{0\}$, $\langle X \rangle = \mathbf{R}^s$, then the mapping

$$(Tf)(x) = \sum_{\alpha \in Z^s} f(\alpha) B(x - \alpha | X) \tag{2.3}$$

takes $f \in \Delta(X)$ into $Tf \in D(X)$. To see (2.3) we note that for $V \subset X$

$$D_V B(\cdot | X) = \nabla_V B(\cdot | X \setminus V) \tag{2.4}$$

so that using summation by parts

$$\begin{aligned}\int_{\mathbf{R}^s} D_V \phi(x)(Tf)(x)\, dx &= \sum_{\alpha \in Z^s} (\Delta_V f)(\alpha) \int_{\mathbf{R}^s} \phi(x) B(x - \alpha | X \setminus V)\, dx \\ &= 0.\end{aligned}$$

We will assume throughout the rest of the paper that $X \subset Z^s \setminus \{0\}$ and $\langle X \rangle = \mathbf{R}^s$.

Under additional assumptions on X, more can be said about the relationship of $D(X)$ and $\Delta(X)$. In particular, setting

$$\mathscr{B}(X) = \{ Y \subset X \colon |Y| = s, \langle Y \rangle = \mathbf{R}^s \},$$

the following was proven in [3]:

PROPOSITION 2.1. $D(X) = \Delta(X)$ *if and only if*

$$|\det Y| = 1 \quad \textit{for all } Y \in \mathscr{B}(X). \tag{2.5}$$

(For notational convenience we sometimes use Y also to denote the matrix whose column vectors are the elements of the indexed set Y.)

In [1] it was proved when $X \subset Z^s$ that, for any polynomial p in $D(X)$, $Tp = p + q$ where the degree of q is less than the degree of p. This fact was proved in [3] with the aid of the Poisson summation formula. Moreover, $D(X)$ is a finite-dimensional space of polynomials for any set $X \subset \mathbf{R}^s$ such that $\langle X \rangle = \mathbf{R}^s$ [3]. Combining these facts gives

PROPOSITION 2.2. *The mapping T defined in* (2.3) *is one-to-one and onto $D(X)$.*

$$D(X) \subset \Pi_{n-s} \tag{2.6}$$

where using standard multi-index notation,

$$\Pi_k = \left\{ \sum_{|\alpha| \leq k} c_\alpha x^\alpha \colon c_\alpha \in \mathbf{R} \right\}.$$

We also require the following facts proved in [5].

THEOREM 2.1. (i) $\dim D(X) = |\mathscr{B}(X)|$.

(ii) *Let $\Omega \subset \mathbf{R}^s$ be any domain such that $\Omega \cap c(X) = \varnothing$. Then*

$$\operatorname{span}\{ B(x - \alpha | X) \colon \alpha \in Z^s, x \in \Omega \} = D(X).$$

(iii) *For any $x \notin c(X)$,*

$$|b(x|X)| = \sum_{Y \in \mathscr{B}(X)} |\det Y| = \operatorname{vol}_s B(X).$$

3. A characterization of $\Delta(X)$. The main result of this section is a complete characterization of the elements of $\Delta(X)$. For this purpose, let us introduce

$$A(X)=\{z: z\in \mathbf{C}^s \ni \exists\, Y\in \mathscr{B}(X)\ni z^y=1, \forall y\in Y\},$$

as well as

$$X_z=\{y\in X: z^y=1\}.$$

Therefore, when $z\in A(X)$ we have $|X_z|\geqslant s$.

THEOREM 3.1. $f\in \Delta(X)$ *if and only if*

$$f(\alpha)=\sum_{z\in A(X)} z^\alpha p(\alpha|z) \tag{3.1}$$

where $p(\cdot|z)$ *is some polynomial in* $D(X_z)$.

PROOF. Let us first point out that every sequence of the form (3.1) belongs to $\Delta(X)$. For every $V\in \mathscr{Y}(X)$ and any summand $f_z(\alpha)=z^\alpha p(\alpha|z)$ in (3.1) we may write

$$(\Delta_V f_z)(\alpha)=\sum_{\substack{V'\cup V''=V\\ V'\cap V''=\varnothing}} c_{V'V''}\Delta_{V'}z^\alpha \Delta_{V''}p(\alpha|z)$$

where $c_{V'V''}$ are certain constants. Note that since $V\in\mathscr{Y}(X)$ we have $V\cap W\neq\varnothing$ for each $W\in\mathscr{B}(X)$ and so too for each $W\in\mathscr{B}(X_z)$. Hence if $V'\cap X_z\neq\varnothing$, we otain

$$\Delta_{V'}z^\alpha=\Delta_{V'\setminus X_z}\left(\Delta_{V'\cap X_z}z^\alpha\right)=\Delta_{V'\setminus X_z}\left(z^\alpha\prod_{y\in V'\cap X_z}(z^y-1)\right)=0$$

since, by definition, $z^y=1$ for all $y\in X_z$. If, on the other hand, $V'\cap X_z=\varnothing$, we must have $V''\cap X_z\in\mathscr{Y}(X_z)$. In fact, since $V\in\mathscr{Y}(X)$ we conclude

$$\langle X_z\setminus V''\rangle\subset\langle X\setminus(V'\cup V'')\rangle\neq \mathbf{R}^s.$$

Recalling that by (2.2), $D(X_z)\subseteq\Delta(X_z)$, we get $\Delta_{V''}p(\alpha|z)=0$, confirming that even in this case $\Delta_V f_z=0$ and therefore $f\in\Delta(X)$.

The converse of Theorem 3.1 is proved through a series of ancillary results. First we determine the number of linearly independent sequences of the form (3.1). To this end, we observe that for any distinct $z^i\in\mathbf{C}^s$, $i=1,\dots,l$, and any choice of nontrivial polynomials p_i, $i=1,\dots,l$, the sequences $\{(z^i)^\alpha p_i(\alpha): \alpha\in Z^s\}$ are certainly linearly independent. Also, we require

LEMMA 3.1. *Let* $|X|=s$. *Then there exist exactly* $|\det X|$ *different solutions* z *of the system*

$$z^y=1,\qquad y\in X. \tag{3.2}$$

PROOF. As above let us denote the $s\times s$ matrix whose columns are the elements of X also by X. Then (3.2) implies $z^{X\alpha}=1$, $\alpha\in Z^s$. Hence by Cramer's rule,

$$z^{|\det X|\alpha}=1,\qquad \alpha\in Z^s,$$

which in particular implies

$$z_l^{|\det X|} = 1, \qquad l = 1,\ldots,s;\ z = (z_1,\ldots,z_s).$$

Thus each z_l must have the form $z_l = \exp\{2\pi ik/|\det X|\}$ for some k, $0 \leqslant k < |\det X|$. Since for $X = \{x^1,\ldots,x^s\}$ we must also have $z^{x^i} = 1$, $i = 1,\ldots,s$, any solution

$$z = \left(e^{2\pi i\alpha_1/|\det X|},\ldots,e^{2\pi i\alpha_s/|\det X|}\right) \tag{3.3}$$

where $0 \leqslant \alpha_i < |\det X|$, $\alpha \in Z^s$, must satisfy $z^{x^m} = e^{2\pi i\alpha\cdot x^m/|\det X|} = 1$, i.e.

$$X^T\alpha = |\det X|\mu \tag{3.4}$$

must hold for some $\mu \in Z^s$. Since $\mu = X^Tt$ for some $t \in \mathbf{R}^s$ we see that (3.4) is valid for $0 \leqslant \alpha_i < |\det X|$, $i = 1,\ldots,s$, if and only if $0 \leqslant t_i < 1$, i.e. $\mu \in P = \{X^Tt\colon t \in [0,1)^s\}$, the "half open" parallelepiped generated by X^T. However, we show in [5] that $|P \cap Z^s| = |\det X|$, and so if $P \cap Z^s = \{\mu^1,\ldots,\mu^l\}$, $l = |\det X|$, then all the solutions to (3.2) are given by

$$\left(e^{2\pi i\alpha_1^k/l},\ldots,e^{2\pi i\alpha_s^k/l}\right), \qquad k = 1,\ldots,l, \tag{3.5}$$

where $X^T\alpha^k = |\det X|\mu^k$, $k = 1,\ldots,l$.

LEMMA 3.2. *There exist at least* $\Sigma_{Y\in\mathscr{B}(X)}|\det Y|$ *linearly independent sequences of the form* (3.1) *in* $\Delta(X)$.

PROOF. To prove Lemma 3.2 we recall from Theorem 2.1 (see also [5]) that there exist $|\mathscr{B}(X_z)|$ linearly independent polynomials $P_j(x|z)$, $j = 1,\ldots,|\mathscr{B}(X_z)|$, in $D(X_z)$. Thus, by the remark preceding Lemma 3.1, we have at least $\Sigma_{z\in A(X)}|\mathscr{B}(X_z)|$ linearly independent sequences of the form (3.1) in $\Delta(X)$. Let us enumerate the elements of $\mathscr{B}(X)$ and $A(X)$ as $\mathscr{B}(X) = \{Y_1,\ldots,Y_l\}$ and $A(X) = \{z^1,\ldots,z^m\}$, respectively. Furthermore, we introduce the $l \times m$ matrix $A = (A_{ij})$ given by

$$A_{ij} = \begin{cases} 1, & Y_i \in \mathscr{B}(X_{z^j}), \\ 0, & \text{otherwise.} \end{cases}$$

Since by Lemma 3.1 for each Y_i there are exactly $|\det Y_i|$ solutions $z^y = 1$, $y \in Y_i$, we see that $\Sigma_j A_{ij} = |\det Y_i|$. Similarly, by definition, $\Sigma_i A_{ij} = |\mathscr{B}(X_{z^j})|$. Hence we conclude

$$\sum_{z\in A(X)} |\mathscr{B}(X_z)| = \sum_{Y\in\mathscr{B}(X)} |\det Y| \tag{3.6}$$

proving Lemma 3.2.

Thus the proof of Theorem 3.1 is complete once we have shown

THEOREM 3.2. $\dim \Delta(X) = \Sigma_{Y\in\mathscr{B}(X)}|\det Y| = \operatorname{vol}_s(\operatorname{supp} B(\cdot|X))$.

PROOF. The second equation is just a restatement of Theorem 2.1(iii). Thus in view of Lemma 3.2 it is sufficient to show that

$$\dim \Delta(X) \leqslant \sum_{Y\in\mathscr{B}(X)} |\det Y|. \tag{3.7}$$

We will use induction on $|X|$ to prove this inequality. We begin by studying the case $|X| = s$ for later use. For this purpose we define $P(X) = \{Xt\colon t \in [0,1)^s\}$.

LEMMA 3.3. *Let* $|X| = s$. *Then* $\dim \Delta(X) = |\det X|$. *Moreover*, $\Delta(X)$ *is spanned by the sequences*

$$f_\mu(\alpha) = \begin{cases} 1, & \alpha \in Z_\mu, \\ 0, & \alpha \notin Z_\mu, \end{cases}$$

where, for $\mu \in P(X) \cap Z^s$, $Z_\mu = \{\mu + X\beta \colon \beta \in Z^s\}$.

PROOF. From [5] we recall that $|P(X) \cap Z^s| = |\det X|$. Thus f_μ, $\mu \in P(X) \cap Z^s$, are clearly $|\det X|$ linearly independent sequences which are in $\Delta(X)$ because $\mathscr{Y}(X) = \{\{y\} \colon y \in X\}$. On the other hand, it is clear that any $f \in \Delta(X)$ has to be constant on each sublattice Z_μ, $\mu \in P(X) \cap Z^s$. Hence the assertion follows.

REMARK. According to Lemma 3.1, there are $|\det X|$ linearly independent sequences assuming constant values on each sublattice Z_{μ^k}, $P(X) \cap Z^s = \{\mu^1, \ldots, \mu^l\}$, $l = |\det X|$. Alternatively, these can be obtained by restricting the functions

$$g_{\mu^k}(x) = e^{2\pi i (x - \mu^k)\cdot \alpha^k / l}, \qquad k = 1, \ldots, l,$$

to Z^s where the α^k, $k = 1, \ldots, l$, are given by (3.5). In fact, $g_{\mu^k}(\alpha) = e^{2\pi i(\mu^j - \mu^k)\cdot\alpha^k/l}$, $\alpha \in Z_{\mu^j}$.

To advance the induction hypothesis for the proof of Theorem 3.2 let us assume that $|X| \geqslant s$, $\langle X \rangle = \mathbf{R}^s$ and (3.7) holds for any subset of cardinality $\leqslant |X|$. Now, we will show that (3.7) holds for the set $\{y\} \cup X$ where y is any vector in $Z^s \setminus \{0\}$. Setting

$$\mathscr{A}(X|y) = \left\{ V \subset X \colon |V| = s - 1, \dim\langle V \rangle = s - 1, \langle \{y\} \cup V \rangle = \mathbf{R}^s \right\}$$

we observe that for any $Y \in \mathscr{A}(X|y)$ the set $\{y\} \cup (X \setminus \langle Y \rangle)$ belongs to

$$\mathscr{Y}(\{y\} \cup X).$$

Thus for any $Y \in \mathscr{A}(X|y)$, $f \in \Delta(\{y\} \cup X)$ must satisfy by definition

$$\Delta_y(\Delta_{X\setminus\langle Y\rangle} f) = 0. \tag{3.8}$$

Defining now as in [5], for $l \leqslant \dim\langle X \rangle$,

$$\mathscr{Y}_l(X) = \{ Y \subset X \colon \dim\langle X \setminus Y \rangle < l, \dim\langle X \setminus V \rangle \geqslant l, \text{ for all } V \subsetneqq Y \},$$

we recall from [5] that for any $W \in \mathscr{Y}_{s-1}(X \cap \langle Y \rangle)$ one has $W \cup (X \setminus \langle Y \rangle) \in \mathscr{Y}(\{y\} \cup X)$ (see (2.18) in the proof of Theorem 2.4 in [5]). Thus we infer from this observation that $R_Y = \Delta_{X\setminus\langle Y\rangle} f$ satisfies

$$\Delta_W R_Y = 0 \quad \text{for all } W \in \mathscr{Y}_{s-1}(X \cap \langle Y \rangle). \tag{3.9}$$

We appeal now to Proposition 2.1 in [5] which states that any $V \in \mathscr{Y}_s(\{y\} \cup X)$ either has the form $V = \{y\} \cup (X \setminus \langle U \rangle)$ for some $U \in \mathscr{A}(X|y)$, or, if $y \notin V$, V contains an element of $\mathscr{Y}_{s-1}(X \cap \langle U \rangle)$ for all $U \in \mathscr{A}(X|y)$. Thus combining (3.8) and (3.9) yields $R_Y \in \Delta(\{y\} \cup (X \cap \langle Y \rangle))$. Since $\dim\langle X \cap \langle Y \rangle\rangle = \dim\langle Y \rangle = s - 1$ we have $|\{y\} \cup (X \cap \langle Y \rangle)| = 1 + |X \cap \langle Y \rangle| \leqslant |X|$. Also, since $Y \in \mathscr{A}(X|y)$, $\langle \{y\} \cup (X \cap \langle Y \rangle)\rangle = \mathbf{R}^s$, our induction hypothesis assures that there are at most

$$l_Y = \sum_{V \in \mathscr{B}(\{y\} \cup (X \cap \langle Y \rangle))} |\det V| \tag{3.10}$$

linearly independent solutions $R_{Y,j}$, $j = 1,\dots,l_Y$, of (3.9). Now let $\mathscr{R}(X|y) \subseteq \mathscr{A}(X|y)$ be any fixed set of representers for the equivalence classes induced by the relation $Y \sim Y'$ iff $\langle Y \rangle = \langle Y' \rangle$. Notice that

$$\mathscr{Y}_s(X) = \{X \setminus \langle Y \rangle : Y \in \mathscr{R}(X|y)\} \cup \{V \subset X : V \in \mathscr{Y}_s(\{y\} \cup X)\}.$$

Thus the number of independent solutions satisfying $\Delta_{X \setminus \langle Y \rangle} f_j = R_{Y,j}$, $Y \in \mathscr{R}(X|y)$, $\Delta_V f_j = 0$, $V \subset X$, $V \in \mathscr{Y}_s(\{y\} \cup X)$, is by induction hypothesis less than or equal to $\sum_{Y \in \mathscr{B}(X)} |\det Y|$. Hence

$$\begin{aligned}\dim \Delta(\{y\} \cup X) &\leqslant \sum_{Y \in \mathscr{B}(X)} |\det Y| + \sum_{Y \in \mathscr{R}(X|y)} \left(\sum_{V \in \mathscr{B}(\{y\} \cup (X \cap \langle Y \rangle))} |\det V| \right) \\ &= \sum_{Y \in \mathscr{B}(\{y\} \cup X)} |\det Y|\end{aligned}$$

comfirming (3.7). This finishes the proof of Theorem 3.2 and 3.1.

Noting that, for $e = (1,\dots,1)$, $e \in A(X)$ and $X_e = X$, we immediately infer from Theorem 3.1.

Corollary 3.1. *$f \in \Delta(X)$ if and only if f has a unique representation $f = f_1 + f_2$ where $f_1 \in D(X)$ and f_2 has the form*

$$f_2(\alpha) = \sum_{z \in A(X) \setminus \{e\}} z^\alpha p(\alpha|z)$$

where $p(x|z)$ is some polynomial in $D(X_z)$.

Note that Proposition 2.1 is an immediate consequence of Corollary 3.1, since (2.5) implies, in view of Lemma 3.1, that $A(X) = \{e\}$.

For our convenience later, let us introduce the set $E(X) = \{f_2 : f \in \Delta(X)\}$. Thus Corollary 3.1 says that $\Delta(X) = D(X) \oplus E(X)$.

We are now in a position to solve the initial value problem:

$$\begin{aligned} \Delta_V f &= 0, \quad \text{for all } V \in \mathscr{Y}(X), \\ f(\alpha) &= d_\alpha, \quad \alpha \in b(x|X), \end{aligned} \tag{3.11}$$

where x is any point in $\mathbf{R}^s \setminus c(X)$.

Theorem 3.3. *Let $x \in \mathbf{R}^s \setminus c(X)$. Then any finite sequence $\{d_\alpha : \alpha \in b(x|X)\}$ has a unique extension $\{f(\alpha) : \alpha \in Z^s\}$ in $\Delta(X)$.*

Proof. From Theorem 2.1(iii) and Theorem 3.2 we know already that $|b(x|X)| = \sum_{Y \in \mathscr{B}(X)} |\det Y| = \dim \Delta(X)$. Therefore, it suffices to show that whenever $f \in \Delta(X)$ satisfies $f(\alpha) = 0$, $\alpha \in b(x|X)$, for some $x \in \mathbf{R}^s \setminus c(X)$, then $f = 0$.

We proceed to prove this fact by induction on $|X|$. When $|X| = s$, we know by Lemma 3.3 that any $f \in \Delta(X)$ must be constant on each of the $|b(x|X)|$ sublattices Z_μ, $\mu \in P(X) \cap Z^s$. Since $b(x|X) \cap Z_\mu \neq \varnothing$, for $\mu \in P(X) \cap Z^s$ we conclude $f = 0$ if $f(\alpha) = 0$, $\alpha \in b(x|X)$.

So assume the assertion holds for all subsets of Z^s of cardinality less than $|X|$. Observe that, for any $y \in X$,

$$b(x|X\setminus\{y\}) \cup (b(x|X\setminus\{y\}) - y) \subseteq b(x|X).$$

Hence $f(\alpha) = 0$, $\alpha \in b(x|X)$, implies $(\nabla_y f)(\alpha) = 0$, $\alpha \in b(x|X\setminus\{y\})$. Clearly, when $f \in \Delta(X)$ we have $\nabla_y f \in \Delta(X\setminus\{y\})$. Thus by our induction assumption we conclude that $\nabla_y f = 0$, $y \in X$. In particular, $f \in \Delta(Y)$, where Y is any element of $\mathscr{B}(X)$. Choose some such Y and again apply Lemma 3.3 to conclude that f is constant on each sublattice Z_μ, $\mu \in P(X) \cap Z^s$. Since $b(x|X) \cap Z_\mu \neq \varnothing$, we conclude $f = 0$. This proves Theorem 3.3.

In the case that $D(X) = \Delta(X)$, which is, according to Proposition 2.1, equivalent to the requirement that $|\det Y| = 1$, for all $Y \in \mathscr{B}(X)$, this result was obtained in **[5]** by different means.

In the general case, where (2.5) does not necessarily hold, we have

PROPOSITION 3.1. *For any $x \notin c(X)$ the set $b(x|X)$ contains a unisolvent subset for interpolation by elements of $D(X)$.*

PROOF. Suppose there exists some $p \in D(X)$ such that $p(\alpha) = 0$, $\alpha \in b(x|X)$. Since $p \in \Delta(X)$ Theorem 3.3 gives $p(\alpha) = 0$, $\alpha \in Z^s$, and hence $p = 0$.

We could also achieve this conclusion by noting that, for any neighborhood Ω of x such that $\Omega \cap c(X) = \varnothing$ and $y \in \Omega$,

$$(Tp)(y) = \sum_{\alpha \in b(x|X)} p(\alpha)B(y-\alpha|X) = 0, \qquad y \in \Omega.$$

Therefore, Proposition 2.2 gives $Tp = 0$ and hence $p = 0$. This was the argument used in **[5]**.

As an application of Proposition 3.1 we have

COROLLARY 3.2. *Let $X \subset Z^s\setminus\{0\}$ be finite such that $\langle X\rangle = \mathbf{R}^s$. Then for any $y \in \mathbf{R}^s\setminus c(X)$ there is a sequence $\{d_\alpha\colon \alpha \in b(y|X)\}$ such that the mapping $(Sf)(x) = \sum_{\alpha\in Z^s} f(\alpha)\phi(x-\alpha)$, where $\phi(x) = \sum_{\beta \in b(y|X)} d_\beta B(x+\beta|X)$, reproduces $D(X)$.*

PROOF. Since by Proposition 2.2 the mapping $Tf = \sum_{\alpha\in Z^s} f(\alpha)B(\cdot-\alpha|X)$ is one-to-one and onto $D(X)$, we can introduce $G = T^{-1}|_{D(X)}$. Let $E_\beta f(\cdot) = f(\cdot+\beta)$ be the forward shift by $\beta \in Z^s$. Clearly, T commutes with E_β and therefore so too does G. Here we are using the fact that $D(X)$ is an invariant subspace of E_β. By Proposition 3.1 the common null space in $D(X)$ of the linear functionals $f(\alpha)$, $\alpha \in b(y|X)$, is contained in the null space of $(Gf)(0)$, $f \in D(X)$. Hence there are constants $\{d_\beta\colon \beta \in b(y|X)\}$ such that

$$(Gf)(0) = \sum_{\beta \in b(y|X)} d_\beta f(\beta), \qquad f \in D(X).$$

Consequently, for any $\alpha \in Z^s$, $f \in D(X)$, we have

$$\begin{aligned}(Gf)(\alpha) &= (E_\alpha Gf)(0) = (GE_\alpha f)(0)\\ &= \sum_{\beta\in b(y|X)} d_\beta f(\alpha+\beta).\end{aligned}$$

This gives for $f \in D(X)$

$$f(x) = \sum_{\alpha \in Z^s} (Gf)(\alpha) B(x - \alpha | X)$$

$$= \sum_{\alpha \in Z^s} \left(\sum_{\beta \in b(y|X)} d_\beta f(\alpha + \beta) \right) B(x - \alpha | X) = \sum_{\alpha \in Z^s} f(\alpha) \phi(x - \alpha),$$

as claimed.

4. Linear dependence relations for the translates of a box spline. Let us finally discuss the linear dependence relations for the translates $B(\cdot - \alpha | X)$, $\alpha \in Z^s$. It was shown in [**3**, **5**] that (2.5) is equivalent to the global as well as to the local linear independence of the $B(\cdot - \alpha | X)$, $\alpha \in Z^s$.

In general, we can expect linear relations among the translates $B(\cdot - \alpha | X)$. Specifically, we have

THEOREM 4.1. *If $f \in E(X)$ then*

$$(Tf)(x) = \sum_{\alpha \in Z^s} f(\alpha) B(x - \alpha | X) = 0, \qquad x \in \mathbf{R}^s.$$

PROOF. It is sufficient to prove the assertion for $f(\alpha) = z^\alpha p(\alpha | z)$, where $z \in A(X) \setminus \{e\}$ and $p(\cdot | z) \in D(X_z)$. Recall from Lemma 3.1 that $z \in A(X) \setminus \{e\}$ has the form

$$z_k = \exp\{2\pi i \gamma_k\}, \qquad z = (z_1, \ldots, z_s),$$

where for some $Y \in \mathscr{B}(X_z)$, $\mu \in P(Y^T) \cap Z^s$, γ is given by $\gamma = \beta / |\det Y|$ and

$$Y^T \beta = |\det Y| \mu. \tag{4.1}$$

By Poisson's summation formula we therefore get

$$\sum_{\alpha \in Z^s} z^\alpha p(\alpha | z) B(x - \alpha | z)$$

$$= \sum_{\alpha \in Z^s} \exp\{2\pi i x \cdot (\gamma - \alpha)\} (p(x - iD | z) \hat{B}(\cdot | X))(2\pi(\gamma - \alpha)).$$

Thus the assertion follows once we have shown

LEMMA 4.1. *Let $p \in D(X_z)$ and γ be defined by* (4.1). *Then*

$$p(D) \hat{B}(2\pi(\gamma - \alpha) | X) = 0, \qquad \alpha \in Z^s.$$

PROOF. Setting $\rho(t) = (1 - e^{-it})/it$, it is easy to verify from (1.1) that $\hat{B}(x|X) = \prod_{y \in X} \rho(x \cdot y)$. Using this formula we showed in [**3**] that

$$(p(D) \hat{B}(\cdot | X))(x) = \sum_{\beta \in Z_+^n} \frac{(D_{X^\beta} p)(0)}{\beta!} \omega_\beta(x) \tag{4.2}$$

where, for $\beta = (\beta_1, \ldots, \beta_n)$,

$$X^\beta = \{\underbrace{x^1, \ldots, x^1}_{\beta_1}, \ldots, \underbrace{x^n, \ldots, x^n}_{\beta_n}\}$$

and

$$\omega_\beta(x) = \prod_{i=1}^{n} \rho^{(\beta_i)}(x \cdot x^i).$$

Now suppose $x = 2\pi(\gamma - \alpha)$, $\alpha \in Z^s$. For any β such that

(4.3) $$V_\beta = \{x^i \colon \beta_i \neq 0, x^i \in X_z\} \notin \mathscr{Y}(X_z)$$

we have $(\gamma - \alpha) \cdot y \neq 0$ for some $y \in X_z \setminus V_\beta$. Otherwise, because $\langle X_z \setminus V_\beta \rangle = \mathbf{R}^s$ we would have $\alpha = \gamma$, contradicting $\gamma \notin Z^s$. However, $\gamma \cdot y$ is an integer for all $y \in X_z$ and so $\omega_\beta(2\pi(\gamma - \alpha)) = 0$, $\alpha \in Z^s$, for any β satisfying (4.3). On the other hand, if $V_\beta \in \mathscr{Y}(X_z)$ then $(D_{X^\beta} p)(0) = 0$ because $p \in D(X_z)$ and $V_\beta \subset X^\beta$. This proves Lemma 4.1 and Theorem 4.1.

The converse of Theorem 4.1 is generally not true. In fact, consider the following

EXAMPLE 4.1. Let $e^1 = (1,0)$, $e^2 = (0,1)$ and $X = \{e^1, e^1, 2e^2, 2e^2\}$, so that $\mathscr{Y}_2(X) = \{\{e^1, e^1\}, \{2e^2, 2e^2\}\}$, $\dim \Delta(X) = 4$ and $\dim D(X) = 2$ (see the proof of Theorem 3.1 for the definition of $\mathscr{Y}_l(X)$). In this case the box spline $B(\cdot|X)$ is the tensor product of the univariate box splines corresponding to the sets $X_1 = \{1,1\}$, $X_2 = \{2,2\}$, i.e.

(4.4) $$B(x, y|X) = B(x|X_1)B(y|X_2).$$

Since $B(x|X_1)$ is the univariate piecewise linear B-spline it has linearly independent integer translates. On the other hand, the box spline $B(y|X_2)$ has dependent translates. In fact, $A(X_2) = \{1, -1\}$ and the set $E(X_2)$ is spanned by the sequences $(-1)^j$, $j(-1)^j$, $j \in Z$, since $D((X_2)_1) = D((X_2)_{-1}) = D(X_2) = \{f \colon f'' = 0\}$. Consequently, by Theorem 4.1, we have for instance that

$$\sum_{j \in Z} (-1)^j B(y - j|X_2) = 0, \qquad y \in \mathbf{R}.$$

Therefore, for *any* sequence $h(j)$ on Z we observe that

$$\sum_{(k,j) \in Z^2} h(k)(-1)^j B(x - k, y - j|X) = 0, \qquad (x, y) \in \mathbf{R}^2.$$

But the space of sequences $h(k)(-1)^j$ is clearly infinite dimensional.

This example indicates that, in order to obtain a converse of Theorem 4.1, one should require that the translates of all the box splines $B(\cdot|X \setminus V)$, $V \in \mathscr{Y}_s(X)$, are linearly independent on $\langle X \setminus V \rangle$.

In fact, let us point out that when this condition fails for some $V \in \mathscr{Y}_s(X)$ then there are infinitely many independent linear relations among the translates of $B(\cdot|X)$. To see this, we suppose that for some $V \in \mathscr{Y}_s(X)$

$$\sum_{\alpha \in Z^s} a_\alpha B(x - \alpha|X \setminus V) = 0.$$

Since the supports of the distributions $B(\cdot - \alpha|X \setminus V)$ are contained in the $(s-1)$-dimensional hyperplanes $\alpha + \langle X \setminus V \rangle$, $\alpha \in Z^s$, we conclude that

$$\sum_{\mu \in Z^s \cap \langle X \setminus V \rangle} a_\mu B(x - \alpha - \mu|X \setminus V) = 0 \quad \text{for all } \alpha \in Z^s.$$

We can now combine these equations (as in Example 4.1) and obtain an infinite number of linearly independent relations for the translates

$$B(\cdot - \alpha|X \setminus V), \qquad \alpha \in Z^s.$$

Using the convolution identity

$$B(\cdot|X \setminus V) * B(\cdot|V) = B(\cdot|X),$$

we obtain identical relations for the translates of $B(\cdot|X)$.

THEOREM 4.2. *Suppose that for every $V \in \mathscr{Y}_s(X)$ the distributions*

$$B(\cdot - \alpha | X \setminus V), \qquad \alpha \in Z^s, \tag{4.5}$$

are linearly independent. Then

$$(Tf)(x) = \sum_{\alpha \in Z^s} f(\alpha) B(x - \alpha | X) = 0, \qquad x \in \mathbf{R}^s, \tag{4.6}$$

if and only if $f \in E(X)$.

PROOF. In view of Theorem 4.1 it suffices to show that (4.5), (4.6) imply $f \in E(X)$. Differentiating Tf and summing by parts yields, for $V \in \mathscr{Y}_s(X)$ and any test function $\Phi \in C_0(\mathbf{R}^s)$,

$$\sum_{\alpha \in Z^s} (\Delta_V f)(\alpha) \int_{\mathbf{R}^s} \Phi(x) B(x - \alpha | X \setminus V)\, dx = 0.$$

Hence, by hypothesis, $\Delta_V f = 0$ for every $V \in \mathscr{Y}_s(X)$ confirming that $f \in \Delta(X)$. Because of Proposition 2.2, Corollary 3.1 and Theorem 4.1, (4.6) assures that $f \in E(X)$ which completes the proof.

In order to determine the circumstances under which (4.5) is satisfied we recall the following result from [5]: When $X \subset Z^s$ the translates $B(\cdot - \alpha | X)$, $\alpha \in Z^s$, are locally linearly independent if and only if $|\det Y| = 1$, for all $Y \in \mathscr{B}(X)$. We reinterpret this result by defining, for a given set V, the lattice $\mathscr{L}(V) = \{V\beta\colon \beta \in Z^{|V|}\}$. Then the above condition is equivalent to saying $\mathscr{L}(Y) = Z^s$, for all $Y \in \mathscr{B}(X)$. In these terms, we see that (4.5) is equivalent to requiring that, for all $Y \subset X$, $|Y| = s - 1$, $\dim \langle Y \rangle = s - 1$,

$$\mathscr{L}(W) = Z^s \cap \langle Y \rangle \tag{4.7}$$

holds for all $W \subset X \cap \langle Y \rangle$ such that $|W| = s - 1$, $\dim \langle W \rangle = s - 1$. A simple sufficient condition for (4.7) to be true is that every $(s-1) \times (s-1)$ submatrix V of X with $\dim \langle V \rangle = s - 1$ satisfies $|\det V| = 1$. However, this condition is not necessary as is seen by the example $X = \{(0,1), (3,1)\}$, $s = 2$, $W = Y = \{(3,1)\}$ which certainly satisfies (4.7).

Nevertheless, this sufficient condition is easily verified in the following

EXAMPLE 4.2. Let $e^1, e^2 \in \mathbf{R}^2$ be as above and $M = (m_1, m_2, m_3, m_4) \in Z^4_+$. Suppose X_M is comprised of the vectors e^1, e^2, $e^1 + e^2$, $e^2 - e^1$ repeated m_i times, $i = 1, \ldots, 4$, respectively. Thus $c(X_M)$ consists of all multi-integer translates of the four lines $x = 0$, $y = 0$, $x = y$, $x = -y$ which induce a regular triangulation of the plane. The box spline $B(x | X_M)$ for the special case $M = (1,1,1,1)$ has been studied before by several authors interested in constructing finite element spaces (see [2, 4]).

Theorem 2.1(i) gives $\dim D(X_M) = \sum_{0 \leqslant i < j \leqslant 4} m_i m_j$ which was also derived in [6] by different means. It is not hard to verify that

$$\operatorname{vol}_s(B(\cdot | X_M)) = \sum_{0 \leqslant i < j \leqslant 4} m_i m_j + m_3 m_4$$

which by Theorem 3.2 equals $\dim \Delta(X_M)$. Hence $\dim E(X_M) = m_3 m_4$ so that by Theorem 4.1 the translates $B(\cdot - i, \cdot - j | X_M)$, $(i, j) \in Z^2$, are *linearly dependent* iff $m_3 m_4 \neq 0$. Moreover, noting that $A(X_M) = \{(1,1), (-1,-1)\}$, we have

$$(X_M)_{(-1,-1)} = \Big\{\underbrace{e^1 + e^2, \ldots, e^1 + e^2}_{m_3}, \underbrace{e^2 - e^1, \ldots, e^2 - e^1}_{m_4}\Big\},$$

so that

$$D\big((X_M)_{(-1,-1)}\big) = \operatorname{span}\big\{(x+y)^r(y-x)^l \colon 0 \leqslant r < m_3, 0 \leqslant l < m_4\big\}.$$

Hence, by Corollary 3.1, $E(X_M)$ is spanned by the sequences

$$f_{r,l}(i,j) = (-1)^{i+j}(i+j)^r(j-i)^l, \qquad 0 \leqslant r < m_3, 0 \leqslant l < m_4.$$

Since X_M clearly satisfies (4.7), Theorem 4.2 says that

$$\sum_{(i,j)\in Z^2} f(i,j)B(x-i, y-j|X_M) = 0, \qquad (x,y) \in \mathbf{R}^2, \tag{4.8}$$

holds iff f has the form

$$f(i,j) = \sum_{r=0}^{m_3-1}\sum_{l=0}^{m_4-1}(-1)^{i+j}a_{rl}(i+j)^r(j-i)^l \tag{4.9}$$

for arbitrary real coefficients a_{rl}, $0 \leqslant r < m_3$, $0 \leqslant l < m_4$.

Our motivation to characterize all linear dependence relations among translates of box splines may be also explained in terms of the above example: Given any domain $\Omega \subset \mathbf{R}^2$, the results in [**5**] imply that

$$\Phi_\Omega = \big\{B(\cdot - i, \cdot - j|X_M) \colon i, j \in Z, \big(\operatorname{supp} B(\cdot - i, \cdot - j|X_M) \cap \Omega\big) \neq \varnothing\big\}$$

is *linearly dependent over* Ω if and only if $m_3 m_4 \neq 0$. Thus we wish to discard a suitable subset from Φ_Ω so that the remaining translates form a basis for $\operatorname{span}\{\Phi_\Omega\}$ over Ω. In order to accomplish this we have to characterize all sequence $f(i,j)$ for which (4.8) holds on *some domain* Ω. A consequence of the general result below is that the linear relations (4.8) are also the *only* ones which hold on Ω.

We will now address this question in generality. When (4.7) holds we shall prove that all possible *local* linear dependence relations for the translates $B(\cdot - \alpha|X)$, $\alpha \in Z^s$, with respect to any domain $\Omega \subset \mathbf{R}^s$ are obtained just as in Theorem 4.2. Thus the following local version of Theorem 4.2 may be viewed as an extension of the result in [**5**] that global linear independence of the translates of a box spline is equivalent to their local linear independence.

THEOREM 4.3. *Suppose X satisfies* (4.7) *and let Ω be any domain in $\mathbf{R}^s$ such that for*

$$\sum_{\alpha\in Z^s} d_\alpha B(x-\alpha|X) = 0, \qquad x \in \Omega. \tag{4.10}$$

Then there exists a unique sequence $f \in E(X)$ such that, for

$$b_\Omega = \big\{\alpha \colon \big(\operatorname{supp} B(\cdot - \alpha|X)\big) \cap \Omega \neq \varnothing\big\},$$

$$f(\alpha) = d_\alpha, \qquad \alpha \in b_\Omega,$$

and

$$(Tf)(x) = \sum_{\alpha\in Z^s} f(\alpha)B(x-\alpha|X) = 0, \qquad x \in \mathbf{R}^s.$$

PROOF. Our first step is to prove the assertion for the case that Ω is some *fundamental region* by which we mean any (open) domain Ω which is bounded by but not intersected by the cut regions in $c(X)$.

LEMMA 4.2. *The assertion of Theorem* 4.3 *holds when* Ω *is any fundamental region.*

PROOF. If Ω is some fundamental region we have $b_\Omega = b(y|X)$ for any $y \in \Omega$. By Theorem 3.3 there exists a unique $f \in \Delta(X)$ such that $f(\alpha) = d_\alpha$, $\alpha \in b(y|X)$. By Corollary 3.1 we have $f = f_1 + f_2, f_1 \in D(X), f_2 \in E(X)$. Since Theorem 4.1 implies $(Tf_2)(x) = 0$, $x \in \mathbf{R}^s$, our hypothesis yields $(Tf_1)(x) = 0$, $x \in \Omega$. Therefore, by Proposition 2.2, we conclude that $f_1 = 0$. This completes the proof of Lemma 4.2.

The rest of the proof of Theorem 4.3 relies on the following observation:

LEMMA 4.3. *Let* Ω, Ω' *be any two adjacent fundamental regions, i.e.* $\mathrm{vol}_{s-1}(\overline{\Omega} \cap \overline{\Omega}') > 0$. *Suppose* $f \in E(X)$ *satisfies* $f(\alpha) = 0$, $\alpha \in b(y|X) \cap b(y'|X)$, *where* $y \in \Omega$, $y' \in \Omega'$. *Then* $f(\alpha) = 0, \alpha \in Z^s$.

PROOF. Let us begin with a general observation about the sets $b(y|X)$ and $b(y'|X)$.

If the common part of the boundaries of Ω and Ω' lies on the $(s-1)$-hyperplane $\alpha^0 + \langle Y \rangle \subset c(X)$, where $Y \subset X$, $\dim\langle Y \rangle = s-1$ and $|Y| = s-1$, then

$$(4.11) \qquad |b(y|X) \setminus b(y'|X)| = |b(y'|X) \setminus b(y|X)| = |b(x|X \cap \langle Y \rangle)|$$

holds for any x in $(\overline{\Omega} \cap \overline{\Omega}') \setminus c(X \cap \langle Y \rangle)$, because $(\mathrm{supp}\, B(\cdot - \alpha|X)) \cap (\alpha^0 + \langle Y \rangle)$ equals the support of some translate of $B(\cdot|X \cap \langle Y \rangle)$ for $\alpha \in b(y|X) \setminus b(y'|X)$. Furthermore, if (4.7) holds, we know from Theorem 3.1 in [5] that

$$(4.12) \qquad |\mathscr{B}_{s-1}(X \cap \langle Y \rangle)| = |b(x|X \cap \langle Y \rangle)|,$$

where x is as above and

$$\mathscr{B}_{s-1}(X) = \{Y \subset X: |Y| = s-1, \dim\langle Y \rangle = s-1\}.$$

Since $|\mathscr{B}_{s-1}(X \cap \langle Y \rangle)| \leqslant |\mathscr{B}(X)|$ and, by Theorem 2.1(i), $|\mathscr{B}(X)| = \dim D(X)$, we infer from Corollary 3.1 that

$$(4.13) \qquad |b(y|X) \cap b(y'|X)| \geqslant \dim E(X),$$

so that the claim in Lemma 4.3 is plausible.

We will now prove Lemma 4.3 by induction on $|X|$. For $|X| = s$, (4.5) implies, in view of (4.11) and (4.12), that

$$(4.14) \qquad |b(y|X) \setminus b(y'|X)| = |b(y'|X) \setminus b(y|X)| = 1.$$

As before we set $P(X) = \{Xt: t \in [0,1)^s\}$ and $\{\mu^1, \dots, \mu^l\} = Z^s \cap P(X)$ for $X = \{x^1, \dots, x^s\}$. Thus we know that $l = |b(y|X)| = |\det X|$. Recall that for each sublattice $Z_{\mu^i} = \{\alpha = \mu^i + X\beta: \beta \in Z^s\}$, $|b(y|X) \cap Z_{\mu^i}| = 1$, $i = 1, \dots, l$, i.e. $b(y|X)$ contains exactly one element of each sublattice Z_{μ^i}. Thus setting $b(y|X) = \{\alpha^i, \dots, \alpha^l\}$, $b(y'|X) = \{\beta^1, \dots, \beta^l\}$ where α^j, $\beta^j \in Z_{\mu^j}$, we infer from (4.14) that, for some m,

$$(4.15) \qquad \alpha^j = \beta^j, \qquad j = 1, \dots, l, j \neq m.$$

Suppose now $f \in E(X)$. Then by Theorem 4.1 we have

$$(4.16) \qquad (Tf)(x) = \sum_{\alpha \in Z^s} f(\alpha) B(x - \alpha|X) = 0, \qquad x \in \mathbf{R}^s.$$

Since for $|X| = s$, $B(x|X) = \chi_{P(X)}(x)/|\det X|$, evaluating (4.16) for $x \in \Omega$, $x \in \Omega'$, respectively, yields

$$\sum_{j=1}^{l} f(\alpha^j) = 0 = \sum_{j=1}^{l} f(\beta^j). \tag{4.17}$$

Hence assuming that $f(\alpha) = 0$, $\alpha \in b(y|X) \cap b(y'|X) = \{\alpha^j;\, j = 1,\ldots,l,\, j \neq m\}$, (4.15) and (4.17) provide

$$f(\alpha^m) = f(\beta^m) = 0. \tag{4.18}$$

Thus $f(\alpha) = 0$, $\alpha \in b(y|X)$, which in turn yields on account of Theorem 3.3 that $f = 0$ which proves the assertion for $|X| = s$.

In order to advance the induction, suppose $|X| = n > s$ and assume the assertion holds for all $|X| < n$, X satisfying (4.7). Using again that for any $v \in X$

$$b(y|X\backslash\{v\}) \cup (b(y|X\backslash\{v\}) - v) \subseteq b(y|X)$$

(see the proof of Theorem 3.3) we conclude that $f(\alpha) = 0$, $\alpha \in b(y|X) \cap b(y'|X)$ implies

$$\nabla_v f(\alpha) = 0, \qquad \alpha \in b(y|X\backslash\{v\}) \cap b(y'|X\backslash\{v\}). \tag{4.19}$$

Note that $f \in E(X)$ gives

$$\nabla_v f \in E(X\backslash\{v\}). \tag{4.20}$$

In fact, $f \in E(X)$ yields, by Theorem 4.1, $Tf = 0$. Therefore,

$$0 = \sum_{\alpha \in Z^s} f(\alpha) D_v B(x - \alpha|X) = \sum_{\alpha \in Z^s} (\Delta_v f)(\alpha) B(x - \alpha|X\backslash\{v\}).$$

Since $X\backslash\{v\}$ satisfies (4.7), Theorem 4.2 confirms (4.20). Our induction hypothesis combined with (4.19) therefore yields $\nabla_v f = 0$, $v \in Y$, where Y is some element in $\mathscr{B}(X)$. On account of Lemma 3.3 we have

$$f(\alpha) = c_i, \qquad \alpha \in Z_{\mu^i}, \qquad i = 1,\ldots,l, \tag{4.21}$$

where we set again $\{\mu^1,\ldots,\mu^l\} = P(Y) \cap Z^s$, $l = |\det Y|$ and $Z_{\mu^i} = \{\mu^i + X\beta\colon \beta \in Z^s\}$. Clearly, when $|\det Y| = 1$, f is constant on Z^s and the assertion follows immediately. Since $b(y|Y) \cap b(y'|Y) \subseteq b(y|X) \cap b(y'|X)$ the same reasoning as in the case $|X| = s$ allows us to conclude that in the general case $|\det Y| = l > 1$

$$c_i = 0, \qquad i = 1,\ldots,l,\, i \neq m, \tag{4.22}$$

where m is some fixed element in $\{1,\ldots,l\}$ and the c_i's are given by (4.21). Since $f \in E(X)$, Theorem 4.1 assures that $Tf = 0$ which in view of (4.21) and (4.22) reads

$$c_m \sum_{\alpha \in Z_{\mu^m}} B(x - \alpha|X) = 0, \qquad x \in \mathbf{R}^s.$$

This clearly proves $c_m = 0$ which by (4.21) and (4.22) finally gives $f = 0$. This proves Lemma 4.3.

In order to proceed now with the proof of Theorm 4.3 we may assume without loss of generality that $\Omega = \bigcup\{\overline{\Omega}_i\colon i = 1,2,\ldots\}$ where the Ω_i's are fundamental regions. Then

$$b_\Omega = \bigcup\{b(y^i|X)\colon i = 1,2,\ldots\} \tag{4.23}$$

where y^i is some fixed point in Ω_i. According to Lemma 4.2 we may now associate with each Ω_i a unique sequence $f_i \in E(X)$ satisfying $f_i(\alpha) = d_\alpha$, $\alpha \in b(y^i|X)$. Hence, when Ω_i, Ω_j are any two adjacent fundamental regions, we have $f_i - f_j \in E(X)$ and

$$f_i(\alpha) - f_j(\alpha) = 0, \qquad \alpha \in b(y^i|X) \cap b(y^j|X).$$

Thus Lemma 4.3 yields

$$f_i = f_j, \quad f_i(\alpha) = d_\alpha, \qquad \alpha \in b(y^i|X) \cup b(y^j|X). \tag{4.24}$$

Since Ω is connected, we may find now for any fundamental region Ω_m in the above collection a subsequence $\{\Omega_{i_j}\}_{j=1}^N$ such that $i_1 = 1$, $i_N = m$ and any two $\Omega_{i_j}, \Omega_{i_{j+1}}$ are adjacent. Hence (4.24) readily provides that $f_1 = f_j$, $j = 2, 3, \ldots$ and in view of (4.23) that $f_1(\alpha) = d_\alpha$, $\alpha \in b_\Omega$, which establishes the theorem.

Note that Lemma 4.3 tells us where to look for a subset $U_y \subset b(y|X)$ with the property that

$$|U_y| = \dim E(X) \tag{4.25}$$

and for any basis $\{f_j : j = 1, \ldots, \dim E(X)\}$ of $E(X)$

$$f_j|_{U_y}, \qquad j = 1, \ldots, \dim E(X), \tag{4.26}$$

are linearly independent. In fact, whenever y, y' belong to any two adjacent fundamental regions, U_y can be chosen as a subset of $b(y|X) \cap b(y'|X)$.

Thus an immediate consequence of Theorem 4.3 is

COROLLARY 4.1. *Suppose X satisfies* (4.5) *and let Ω be any domain in $\mathbf{R}^s$. Suppose that for some $y \in \Omega \setminus c(X)$, $U_y \subset b(y|X)$ satisfies* (4.25) *and* (4.26). *Then $\{B(\cdot - \alpha|X) : \alpha \in b_\Omega \setminus U_y\}$ forms a basis for*

$$\mathscr{S}(X|\Omega) = \operatorname{span}\{B(\cdot - \alpha|X)|_\Omega : \alpha \in b_\Omega\}.$$

PROOF. In view of the relations

$$\sum_{\alpha \in b_\Omega \setminus U_y} f(\alpha) B(x - \alpha|X) = -\sum_{\alpha \in U_y} f(\alpha) B(x - \alpha|X), \qquad x \in \Omega,$$

and our choice of U_y it immediately follows that

$$B(\cdot - \alpha|X) \in \operatorname{span}\{B(\cdot - \beta|X) : \beta \in b_\Omega \setminus U_y\}, \qquad \alpha \in U_y.$$

Therefore it remains to show that the box splines $B(\cdot - \alpha|X)$, $\alpha \in b_\Omega \setminus U_y$, are linearly independent over Ω. To this end, assume that

$$\sum_{\alpha \in b_\Omega \setminus U_y} d_\alpha B(x - \alpha|X) = 0, \qquad x \in \Omega,$$

for some sequence $\{d_\alpha\}$. Then the sequence

$$\tilde{d}_\alpha = \begin{cases} d_\alpha, & \alpha \in b_\Omega \setminus U_y, \\ 0, & \alpha \in U_y, \end{cases}$$

has by Theorem 4.3 a *unique* extension. Again, by the choice of U_y this extension must be identically zero, confirming the linear independence of $B(\cdot - \alpha|X)$, $\alpha \in b_\Omega \setminus U_y$.

Consequently, we have shown that

$$\dim \mathscr{S}(X|\Omega) = |b_\Omega| - \dim E(X) = |b_\Omega| - \sum_{Y \in \mathscr{B}(X)} (|\det Y| - 1).$$

Returning to Example 4.2, Theorem 4.3 confirms that, whenever

$$\sum_{i,j \in Z} f(i, j) B(x - i, y - j | X_M) = 0, \qquad (x, y) \in \Omega,$$

holds for some domain $\Omega \subset \mathbf{R}^2$, f must have the form (4.9). For the special case $M = (1, 1, 1, 1)$ and Ω a rectangle this was shown by completely different means in [**2**]. In this case (4.9) readily says that a basis for $\mathscr{S}(X_M|\Omega)$ is obtained by discarding anyone of the translates $B(\cdot - i, \cdot - j | X_M)$, $(i, j) \in b_\Omega$. As for general $M \in Z_+^4$, we note that any subset of b_Ω of the form

$$L = \{(i, j) \colon k \leqslant i + j < k + m_3, m \leqslant j - i < m + m_4\}$$

is unisolvent for interpolation from $D((X_M)_{(-1,-1)}) = \text{span}\{(x + y)^r (y - x)^l \colon 0 \leqslant r < m_3, 0 \leqslant l < m_4\}$. Hence for any such L in b_Ω, in view of (4.9) one can solve for the translates $B(\cdot - i, \cdot - j | X_M)$, $(i, j) \in L$, in terms of the remaining translates $B(\cdot - i, \cdot - j | X_M)$, $(i, j) \in b_\Omega \setminus L$, providing a basis for $\mathscr{S}(X|\Omega)$.

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