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MULTIPARAMETER MAXIMAL FUNCTIONS ALONG DILATION-INVARIANT HYPERSURFACES

BY

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ABSTRACT. Consider the hypersurface $x_{n+1} = \prod_1^n x_i^{\alpha_i}$ in $\mathbf{R}^{n+1}$. The associated maximal function operator is defined as the supremum of means taken over those parts of the surface lying above the rectangles $\{0 \leq x_i \leq h_i,\ i = 1, \ldots, n\}$. We prove that this operator is bounded on L^p for $p > 1$. An analogous result is proved for a quadratic surface in $\mathbf{R}^3$.

1. Introduction. Let K be the right circular cone $z^2 = x^2 + y^2,\ z > 0$, in $\mathbf{R}^3$. We fix a generating ray, say $l_0 = \{(t, 0, t) \colon t > 0\}$, and consider portions of K bounded by l_0, a variable generating ray, and a sphere centered at the origin. Thus, we set

$$A_{r\theta} = \{(x, y, z) \in K \colon 0 < \arg(x + iy) < \theta,\ x^2 + y^2 + z^2 < r^2\}.$$

The area measure on K is $dS = \sqrt{2}\, dx\, dy$, and we study mean values

$$m_{r\theta} f(x, y, z) = \left(\int_{A_{r\theta}} dS \right)^{-1} \int_{A_{r\theta}} f(x - x', y - y', z - z')\, dS(x', y', z').$$

We ask whether $m_{r\theta} f \to f$ a.e. as $r \to 0$ and θ varies freely in $(0, 2\pi]$ when $f \in L^p$.

It is well known that to prove such a.e. convergence, one must show that the corresponding maximal function operator

$$M_K f(x, y, z) = \sup_{r,\theta} m_{r\theta} |f|(x, y, z)$$

is bounded on L^p or, at least, of weak type (p, p). When $\theta = 2\pi$, it is easy to control $m_{r\theta}|f|$ via the mean of the one-dimensional maximal functions of $|f|$ along all the generating rays. This mean can also be used when θ is large, say $\theta \geq \pi/4$. The corresponding part of M_K is therefore bounded on L^p, $p > 1$. To handle small θ, notice that $A_{r\theta}$ is the disjoint union of sets

$$A'_{r\gamma} = \{(x, y, z) \in K \colon x, y > 0,\ \gamma/2 < y/x < \gamma,\ r^2/4 < x^2 + y^2 + z^2 < r^2\}.$$

Therefore, we can use $A'_{r\gamma}$ instead of $A_{r\theta}$ when defining M_K.

If we change coordinates by letting

$$x_1 = \frac{z + x}{\sqrt{2}}, \qquad x_2 = \frac{z - x}{\sqrt{2}}, \qquad x_3 = y,$$

the equation of K becomes $x_3 = \pm\sqrt{2x_1x_2}$, $x_1, x_2 \geq 0$, and we take only the positive sign here. Now $A'_{r\gamma}$ corresponds to

$$A''_{r\beta} = \{(x_1, x_2, \sqrt{2x_1x_2}) \colon \beta' < x_2/x_1 < \beta,\ r/2 < x_1 + x_2 < r\},$$

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where $\beta = (\sqrt{\gamma^2+1}-1)/(\sqrt{\gamma^2+1}+1)$ is roughly $\gamma^2/4$, and $\beta' \approx \beta/4$. In each such set, $dS/dx_1\,dx_2 = (1+\frac{1}{2}(x_1/x_2+x_2/x_1))^{1/2}$ is approximately constant. Therefore, when forming means, we can use $dx_1\,dx_2$ instead of dS.

We next replace the sets $A''_{r\beta}$ by subsets of K whose projections on the x_1x_2 plane are rectangles; these are more convenient to work with. Since θ, γ, and β are small, x_1+x_2 is roughly x_1 in $A''_{r\beta}$, and x_1 and x_2 are approximately constant there. Thus, we can cover $A''_{r\beta}$ by a small number of disjoint sets of type

$$\{(x_1, x_2, \sqrt{2x_1x_2})\colon h_1/2 < x_1 < h_1,\ h_2/2 < x_2 < h_2\}, \qquad h_1, h_2 > 0,$$

whose total measure is not much larger than that of $A''_{r\beta}$. Therefore, $M_K f$ is dominated by the supremum of means over sets of this type or, equivalently, by Mf, where

$$Mf(y_1, y_2, y_3) = \sup_{h_1,h_2>0} \frac{1}{h_1h_2}\int_0^{h_1} dx_1 \int_0^{h_2} dx_2\, |f(y_1-x_1, y_2-x_2, y_3-\sqrt{2x_1x_2})|.$$

This leads us to a more general problem in $\mathbf{R}^{n+1} = \{x = (x', x_{n+1}) \in \mathbf{R}^n \times \mathbf{R}\}$. Let $x_{n+1} = F(x')$ be the equation of a hypersurface, defined at least for $x_1, \ldots, x_n > 0$. Define

$$(1.1)\quad Mf(y) = \sup_{h_1,\ldots,h_n>0} \frac{1}{h_1\cdots h_n}\int_0^{h_1} dx_1 \cdots \int_0^{h_n} dx_n\, |f(y'-x', y_{n+1}-F(x'))|.$$

In [**8**, Problem 8, p. 1289], Stein and Wainger raised the question of L^p boundedness for such maximal operators. We shall deal mainly with the case

$$(1.2)\qquad x_{n+1} = F(x') = \prod_{i=1}^n x_i^{\alpha_i}, \qquad \alpha_i \in \mathbf{R},\ i = 1, \ldots, n.$$

THEOREM 1. *If F is as in* (1.2), *M is well defined and bounded on $L^p(\mathbf{R}^{n+1})$ for all $p > 1$.*

This theorem applies to the maximal function associated with the surface $x_1x_2 = x_3x_4$ in $\mathbf{R}^4$, which appears in connection with admissible convergence in the symmetric space $\mathrm{SL}(4,\mathbf{R})/\mathrm{SO}(4,\mathbf{R})$. Sjögren [**6**] proves admissible convergence in a general symmetric space.

By standard techniques, Theorem 1 implies a convergence result. Assume $\alpha_i \geq 0$ for all i and $\alpha_i > 0$ for some i. If $f \in L^p(\mathbf{R}^{n+1})$, the mean

$$\frac{1}{h_1\cdots h_n}\int_0^{h_1} dx_1 \cdots \int_0^{h_n} dx_n\, f(y'-x', y_{n+1}-F(x'))$$

tends to $f(y)$ as $h_1, \ldots, h_n \to 0$, for almost all $y \in \mathbf{R}^{n+1}$.

The Hilbert transform associated with the hypersurface (1.2) and similar surfaces of lower dimension has been studied by Nagel and Wainger [**4**] and by Vance [**11**] and Strichartz [**10**].

Theorem 1 is proved in §2. The method is an adaptation of the known methods for maximal functions along curves, involving Fourier transforms and analytic interpolation as in [**8**]. Theorem 1 implies that the operator M_K defined above is bounded on L^p, $p > 1$. However, this boundedness can be proved in a simpler way by means of one-dimensional maximal functions in lacunar directions, as shown in

§3. When $n = 2$, this proof also applies to those surfaces of type (1.2) which are conical, i.e., for which $\alpha_1 + \alpha_2 = 1$. §3 also contains a proof of the following result for a second-degree surface in $\mathbf{R}^3$.

THEOREM 2. *Let $n = 2$ and $F(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$, where $a, b, c \in \mathbf{R}$. Then M is bounded on L^p for $p > 1$.*

The proof uses combinations of maximal functions along curves and Theorem 1.

2. Proof of Theorem 1. We first eliminate those α_i which are zero. If $\alpha_i = 0$, then F is independent of x_i. In the mean value in (1.1), we can then estimate the integral

$$\frac{1}{h_i}\int_0^{h_i} |f(y' - x', y_{n+1} - F(x'))|\, dx_i$$

by means of the one-dimensional maximal function taken in the x_i direction. The remaining integral defines, for each fixed y_i, a maximal function of the same type in a lower-dimensional space. Continuing in this way, we can reduce the problem to the case where no α_i vanishes. We also assume that we are not in the case $n = 1$, $\alpha_1 = 1$, which is trivial.

We need some notation. As usual, C denotes various constants. Let

$$J = \left\{ j = (j_1, \dots, j_{n+1}) \in \mathbf{Z}^{n+1} \colon j_{n+1} = \sum_{i=1}^{n} j_i \operatorname{sgn} \alpha_i \right\}.$$

Define $a = (a_1, \dots, a_{n+1})$ by the conditions $a_1^{|\alpha_1|} = \dots = a_n^{|\alpha_n|} = a_{n+1} = 2$. The surface (1.2) is invariant under the dilations $\delta_j x = (a_1^{j_1} x_1, \dots, a_{n+1}^{j_{n+1}} x_{n+1})$, provided $j \in J$. The Jacobian of δ_j is $a^j = \prod_{i=1}^{n+1} a_i^{j_i}$. If $j' \in \mathbf{Z}^n$ and $x' = (x_1, \dots, x_n) \in \mathbf{R}^n$, we set $\delta_{j'} x' = (a_1^{j_1} x_1, \dots, a_n^{j_n} x_n)$. The dilations λ_j of a measure or a distribution λ in $\mathbf{R}^{n+1}$ are defined by

$$\int \varphi \, d\lambda_j = \int \varphi(\delta_j x)\, d\lambda$$

for test functions φ. The maximal function operator associated with λ is $M_\lambda \varphi = \sup_{j \in J} |\lambda_j * \varphi|$.

We shall dominate Mf by means of $M_\mu f$, where μ will be a compactly supported smooth measure on the surface. Let $\psi \in C_0^\infty(\mathbf{R}^n)$ be a nonnegative function with support in $\{x' \in \mathbf{R}^n \colon x_i > 0,\ i = 1, \dots, n\}$ which equals 1 in $R = \{x' \in \mathbf{R}^n \colon 1 \le x_i \le a_i\}$. Now μ is defined by

$$\int \varphi \, d\mu = \int \varphi(x', F(x'))\psi(x')\, dx', \qquad \varphi \in C(\mathbf{R}^{n+1}).$$

Let $f \ge 0$ be measurable in $\mathbf{R}^{n+1}$. Then $f(y' - x', y_{n+1} - F(x'))$ is measurable in x' for almost every y. In (1.1) it is enough to consider h_i which are of the form $a_i^{m_i}$, with integers $m_1, \dots, m_n$. Then the domain of integration $\{x' \in \mathbf{R}^n \colon 0 \le x_i \le h_i\}$ is the union of those sets $\delta_{-j'} R$ for which $j' = (j_1, \dots, j_n) \in \mathbf{Z}^n$ satisfies $j_i > -m_i$, $i = 1, \dots, n$. Thus, the mean in (1.1) is bounded by the supremum of the means over $\delta_{-j'} R$. But these last means are dominated by $\mu_j * f$ with $j \in J$ and, thus, by $M_\mu f$. Hence, $Mf \le C M_\mu f$, and the following result implies Theorem 1.

PROPOSITION 1. *If $f \in L^p(\mathbf{R}^{n+1})$, $p > 1$, the convolution $\mu_j * f$ is well defined almost everywhere for $j \in J$, and $\|M_\mu f\|_p \le C\|f\|_p$.*

The L^2-estimate here can be proved by means of a g-function argument. But for $1 < p < 2$, we need to consider an analytic family of operators. Let $\mu^z = G_z * \mu$, $z \in \mathbf{C}$, where the distribution G_z is defined by $\hat{G}_z(\xi) = (1 + |\xi|^2)^{-z/2}$. The following two lemmas will allow analytic interpolation.

LEMMA 1. *There exists a $\sigma > 0$ such that when $-\sigma < \operatorname{Re} z < 0$,*

$$\|M_{\mu^z} f\|_2 \le C(z)\|f\|_2, \qquad f \in \mathcal{S}.$$

LEMMA 2. *For $0 < \operatorname{Re} z < 1$ and each $p > 1$,*

$$\|M_{\mu^z} f\|_p \le C(z)\|f\|_p, \qquad f \in \mathcal{S}.$$

Here and in the sequel, $C(z)$ denotes constants which are bounded, for fixed $\operatorname{Re} z$, by a polynomial in $|z|$.

PROOF OF LEMMA 1. We shall compare μ to a smooth measure ν such that the Fourier transform $\hat{\mu} - \hat{\nu}$ is small near the coordinate hyperplanes. This will allow us to estimate a g-function. Let $0 \le \phi \in C_0^\infty(-1, 1)$ with $\int \phi\, dx = 1$, and let δ denote the Dirac measure in $\mathbf{R}$. Then we set

$$\nu = \mu - \mu * \bigotimes_{i=1}^{n+1} (\delta - \phi).$$

The support of ν is compact, and we claim that ν is a C^∞ function. Clearly ν is a sum of terms obtained by convolving μ with ϕ in one or more variables. If we convolve μ with ϕ in the $(n+1)$st variable, we get the C^∞ function $\psi(x')\phi(x_{n+1} - F(x'))$. Convolution with ϕ in other variables preserves C^∞, so those terms where μ is convolved with ϕ in the $(n+1)$st variable are C^∞. Next, consider a term where μ is convolved with ϕ in the ith variable, $i \ne n+1$. Then we solve the equation $x_{n+1} = F(x')$ for x_i and replace x_{n+1} by x_i in the argument just given. Notice that this change of variables introduces a Jacobian in the expression for μ which is C^∞ in $\operatorname{supp}\mu$. Hence, $\nu \in C_0^\infty$.

We estimate $\hat{\mu} - \hat{\nu}$ next. Since $\hat{\mu}$ is bounded and $\hat{\phi}(0) = 1$, one has

$$|\hat{\mu}(\xi) - \hat{\nu}(\xi)| \le C|\xi_i|, \qquad i = 1, \ldots, n.$$

To estimate $\hat{\mu}$, and thus $\hat{\mu} - \hat{\nu}$, at infinity, we use a version of van der Corput's lemma (see Littman [**1**]). The determinant of the Hessian $(\partial^2 F/\partial x_i\, \partial x_j)_{i,j=1}^n$ is

$$(-1)^n \left(1 - \sum_{i=1}^n \alpha_i\right) F(x') \prod_{i=1}^n \frac{\alpha_i}{x_i^2}.$$

This expression vanishes only when $\sum_{i=1}^n \alpha_i = 1$, and then $\sum_{i=1}^{n-1} \alpha_i \ne 1$, so that the Hessian has rank at least $n - 1$. Therefore, [**1**] gives $|\hat{\mu}(\xi)| \le C|\xi|^{-\gamma}$, where $\gamma \ge \frac{1}{2}(n-1)$ and $\gamma \ge \frac{1}{2}$ since we have excluded the case $n = 1$, $\alpha_1 = 1$. Since $\hat{\nu} \in \mathcal{S}$, we may summarize:

$$|\hat{\mu}(\xi) - \hat{\nu}(\xi)| \le C \min((1 + |\xi|)^{-\gamma}, |\xi_1|, \ldots, |\xi_n|).$$

Here we can replace each $|\xi_i|$ by $|\xi_i|^\varepsilon$ for $0 < \varepsilon < 1$. Estimating the minimum by a geometric mean, one has

$$|\hat{\mu}(\xi) - \hat{\nu}(\xi)| \leq C(1+|\xi|)^{-\gamma/2} \prod_{i=1}^{n} |\xi_i|^{\varepsilon/2n}.$$

Letting $\nu^z = G_z * \nu$, we get

$$\begin{aligned} |\hat{\mu}^z(\xi) - \hat{\nu}^z(\xi)| &\leq C(1+|\xi|)^{-\gamma/2-\operatorname{Re} z} \prod_{i=1}^{n} |\xi_i|^{\varepsilon/2n} \\ &\leq C \prod_{i=1}^{n} (1+|\xi_i|)^{-(\gamma/2+\operatorname{Re} z)/n} |\xi_i|^{\varepsilon/2n} \end{aligned}$$

for $\operatorname{Re} z > -\frac{1}{2}\gamma$. If ε is small, the assumptions of the following lemma will be satisfied.

LEMMA 3. *If $\lambda \in S'(\mathbf{R}^{n+1})$ satisfies $|\hat{\lambda}(\xi)| \leq \prod_{i=1}^{n} \min(|\xi_i|^\eta, |\xi_i|^{-\eta})$ for some $\eta > 0$, then for $f \in S(\mathbf{R}^{n+1})$,*

$$\|M_\lambda f\|_2 \leq C\|f\|_2, \qquad C = C(\eta, n).$$

PROOF. We have $M_\lambda f \leq (\sum_{j\in J} |\lambda_j * f|^2)^{1/2}$, and thus, by Plancherel's theorem,

$$\|M_\lambda f\|_2 \leq \sup_\xi \left(\sum_{j\in J} |\hat{\lambda}_j(\xi)|^2 \right)^{1/2} \|f\|_2.$$

We must show that the supremum appearing here is finite. Since $\hat{\lambda}_j(\xi) = \hat{\lambda}(\delta_j \xi)$, we get

$$\sum_{j\in J} |\hat{\lambda}_j(\xi)|^2 \leq \prod_{i=1}^{n} \sum_{j_i \in \mathbf{Z}} \min(|a_i^{j_i} \xi_i|^{2\eta}, |a_i^{j_i} \xi_i|^{-2\eta}) \leq C,$$

which proves the lemma.

By this lemma, $M_{\mu^z - \nu^z}$ is bounded on $L^2(\mathbf{R}^{n+1})$, with norm depending only on $\operatorname{Re} z$. To finish the proof of Lemma 1, we must therefore estimate M_{ν^z}. By taking derivatives of $\hat{\nu}^z$ of order $n+2$, we get

$$|\nu^z(x)| \leq C(z)(1+|x|)^{-n-2}$$

because $\hat{\nu} \in S$. Therefore, M_{ν^z} is dominated by the strong maximal function, and Lemma 1 follows.

PROOF OF LEMMA 2. Let $0 < \operatorname{Re} z < 1$. The behavior of G_z is well known; see e.g. [**7**, Chapter V, §3]. One has

$$|G_z(x)| \leq C(z) \begin{cases} |x|^{\operatorname{Re} z - n - 1}, & |x| \leq 1, \\ |x|^{-N}, & |x| > 1, \end{cases}$$

for any N, which could also be verified directly. Choosing $N = n + 1 + \operatorname{Re} z$, we conclude

$$|G_z| \leq C(z) \sum_{m\in\mathbf{Z}} 2^{-|m|\operatorname{Re} z + m(n+1)} \chi_{|x|<2^{-m}},$$

where χ denotes characteristic function. This implies

$$|\mu_j^z * f(x)| = \left|\iint f(x-\delta_j(y+u))G_z(y)\,dy\,d\mu(u)\right|$$
$$\leq C(z)\sum_{m\in\mathbf{Z}} 2^{-|m|\mathrm{Re}\,z}\int d\mu(u)\int_{|y|<2^{-m}} 2^{m(n+1)}|f(x-\delta_j u-\delta_j y)|\,dy.$$

The inner integral here is at most

$$2^{m(n+1)}a^{-j}\int_{|y_i|<2^{-m}a_i^{ji}}|f(x-\delta_j u-y)|\,dy \leq M_1(u_1,m)\cdots M_{n+1}(u_{n+1},m)f(x).$$

By $M_i(u_i,m)$ we mean the one-dimensional maximal operator

$$M_i(u_i,m)g(t)=\sup_{k\in\mathbf{Z}} 2^m a_i^{-k}\int_{|s|<2^{-m}a_i^k}|g(t-a_i^k u_i-s)|\,ds, \qquad g\in L^1_{\mathrm{loc}}(\mathbf{R}),$$

acting in the ith variable. Thus,

$$\|M_{\mu^z}f\|_p \leq C(z)\sum_{m\in\mathbf{Z}}2^{-|m|\mathrm{Re}\,z}\int \|M_1(u_1,m)\cdots M_{n+1}(u_{n+1},m)f\|_p\,d\mu(u).$$

From Lemma 4 it follows that each $M_i(u_i,m)$ is bounded on L^p, $p>1$, with norm at most $C(\max(1,m))^{1/p}$, uniformly for $u\in\operatorname{supp}\mu$. This implies

$$\|M_{\mu^z}f\|_p \leq C(z)\sum_{m\in\mathbf{Z}}2^{-|m|\mathrm{Re}\,z}(\max(1,m))^{(n+1)/p}\|f\|_p \leq C(z)\|f\|_p,$$

which ends the proof of Lemma 2.

To state Lemma 4 we take a decreasing sequence $\omega=(\omega_k)_{-\infty}^{+\infty}$ of positive numbers and a sequence $\tau=(\tau_k)_{-\infty}^{+\infty}$ of real numbers. Consider the maximal function

$$M^{\omega,\tau}g(t)=\sup_{k\in\mathbf{Z}}\frac{1}{2\omega_k}\int_{-\omega_k}^{\omega_k}|g(t-\tau_k-s)|\,ds, \qquad g\in L^1_{\mathrm{loc}}(\mathbf{R}).$$

If the translations τ_k are no larger than the corresponding interval lengths ω_k, then $M^{\omega,\tau}$ is essentially the classical maximal function operator. But one can allow $|\tau_k|\gg\omega_k$ under a condition which compares τ_k to the quantities $\omega_{k'}$, $k'\leq k$.

LEMMA 4. *Assume there exists an integer $m\geq 0$ such that for each k the number of l's with $l\geq k$ and $|\tau_l|>\omega_k$ is at most m. Then $M^{\omega,\tau}$ is of weak type $(1,1)$ with constant at most $C\cdot(m+1)$, and $M^{\omega,\tau}$ is bounded on L^p, $p>1$, with norm at most $C\cdot(m+1)^{1/p}$, $C=C(p)$.*

This lemma is a consequence of Theorem 4 in Nagel and Stein [**2**]. It can also be proved by means of Lemma 2 in Sjögren [**5**]. However, we give a direct proof here.

PROOF. Fix $g\in L^1(\mathbf{R})$ and $\alpha>0$. Let $\mathcal{F}_k$ be the family of all intervals I of length $2\omega_k$ such that $\int_I|g|\,ds>\alpha|I|$. Then

$$\{M^{\omega,\tau}g>\alpha\}\subset\bigcup_k\bigcup_{I\in\mathcal{F}_k}(I+\tau_k). \tag{2.1}$$

By a standard covering argument, one can find a disjoint subcollection $\{I_n\}_n$ of $\mathcal{F} = \bigcup_k \mathcal{F}_k$ such that each $I \in \mathcal{F}$ is contained in $5I_n$ for some n. We shall show that for each n,

$$\left| \bigcup_k \bigcup_{\substack{I \in \mathcal{F}_k \\ I \subset 5I_n}} (I + \tau_k) \right| \le C(m+1)|I_n|. \tag{2.2}$$

To this end, we fix n and let $k_0 \in \mathbf{Z} \cup \{\pm\infty\}$ be defined by

$$k_0 = \inf\{k \colon 2\omega_k \le 5|I_n|\}.$$

We shall divide the range of k in (2.2) into three subsets. Take $k \in \mathbf{Z}$, and let $I \in \mathcal{F}_k$, $I \subset 5I_n$. If $k < k_0$, then $I \in \mathcal{F}_k$ is too long to be contained in $5I_n$. If $k \ge k_0$ and $|\tau_k| \le 5|I_n|/2$, then $I + \tau_k \subset 10I_n$. In the remaining case, $k \ge k_0$, $|\tau_k| > 5|I_n|/2$, one has $|\tau_k| > \omega_{k_0}$. (If $k_0 = -\infty$, we define $\omega_{-\infty}$ as the obvious limit.) By hypothesis, this occurs for at most m values of k. For each such k, we get $I + \tau_k \subset 5I_n + \tau_k$.

By summation we obtain (2.2). Now (2.1), (2.2) imply

$$|\{M^{\omega,\tau} g > \alpha\}| \le C(m+1) \sum |I_n| \le C(m+1)\alpha^{-1} \|g\|_1,$$

and the weak type (1,1) inequality is proved.

The strong L^p estimate follows by interpolation, since $M^{\omega,\tau}$ is bounded on L^∞. This completes the proof of Lemma 4.

PROOF OF PROPOSITION 1. We shall apply the analytic interpolation theorem (see [**9**, Chapter V, §4]) to Lemmas 1 and 2. First we must linearize the operators. Let N be a natural number and $j(x)$ a measurable mapping of $\mathbf{R}^{n+1}$ into $\{j \in J \colon |j_i| \le N,\ i = 1, \ldots, n\}$. Then $\mu^z_{j(x)} * f(x)$ varies linearly with $f \in \mathcal{S}$. The interpolation theorem requires simple functions. Therefore, we let $\Phi \in C_0^\infty$ be nonnegative and satisfy $\int \Phi\, dx = 1$. Consider the mapping

$$T_z g(x) = \mu^z_{j(x)} * \Phi * g(x),$$

which is defined for simple g, since then $\Phi * g \in \mathcal{S}$. By Lemmas 1 and 2, the inequality $\|T_z g\|_p \le C(z)\|g\|_p$ holds for $p = 2$ if $-\sigma < \operatorname{Re} z < 0$ and for $p > 1$ if $0 < \operatorname{Re} z < 1$. Interpolation now yields $\|T_0 g\|_p \le C\|g\|_p$, $p > 1$, for simple g.

Given $f \in L^p$, $p > 1$, we can find functions of type $\Phi * g$ which converge to f in L^p and a.e. A simple limit argument then gives

$$\|\mu_{j(x)} * f(x)\|_p \le C\|f\|_p.$$

Letting $N \to \infty$ and choosing the optimal $j(x)$ for each N, we obtain Proposition 1 and, thus, Theorem 1.

3. Some proofs in $\mathbf{R}^3$. We first give a simple proof of Theorem 1 for $n = 2$ in the case $\alpha_1 + \alpha_2 = 1$. Write the integral in (1.1) as a sum of integrals over those subsets where $2^{-m} \le x_2/x_1 < 2^{-m+1}$. In these integrals, we change variables through $t = x_1$, $\lambda = 2^m x_2/x_1$. Hence,

$$\begin{aligned} Mf(y) &\le \sup_{h_1,h_2>0} \int_1^2 d\lambda \sum_{m\in\mathbf{Z}} \frac{2^{-m}}{h_1h_2} \\ &\qquad \cdot \int_0^{\min(h_1, 2^m h_2/\lambda)} t|f(y_1 - t, y_2 - 2^{-m}\lambda t, y_3 - 2^{-\alpha_2 m}\lambda^{\alpha_2} t)|\,dt \\ &\le \sup_{h_1,h_2>0} \int_1^2 d\lambda \sum_{m\in\mathbf{Z}} \frac{2^{-m}}{h_1h_2} \min\left(h_1, \frac{2^m h_2}{\lambda}\right)^2 \\ &\qquad \cdot \sup_{h>0} \frac{1}{h} \int_0^h |f(y_1 - t, y_2 - 2^{-m}\lambda t, y_3 - 2^{-\alpha_2 m}\lambda^{\alpha_2} t)|\,dt \\ &\le \sup_{h_1,h_2} \int_1^2 \sum_{m\in\mathbf{Z}} \min\left(2^{-m}\frac{h_1}{h_2}, 2^m \frac{h_2}{h_1}\lambda^{-2}\right) M_{\text{lac}}^\lambda f(y)\,d\lambda \\ &\le C \int_1^2 M_{\text{lac}}^\lambda f(y)\,d\lambda, \end{aligned}$$

where M_{lac}^λ denotes the maximal function along the lacunar directions $(1, 2^{-m}\lambda, 2^{-\alpha_2 m}\lambda^{\alpha_2})$, $m \in \mathbf{Z}$. It is shown in [**3**] that M_{lac}^λ is bounded on L^p, $p > 1$, and the bound is uniform in λ, as seen by linear mappings. An integration in λ now ends the proof.

PROOF OF THEOREM 2. Consider first the case $b = 0$. The two parabolic maximal function operators

$$M_a^1 f(x) = \sup_{h>0} \frac{1}{h} \int_0^h |f(x_1 - s, x_2, x_3 - as^2)|\,ds$$

and

$$M_c^2 f(x) = \sup_{h>0} \frac{1}{h} \int_0^h |f(x_1, x_2 - s, x_3 - cs^2)|\,ds$$

are bounded on L^p, $p > 1$; see [**8**]. Their composition dominates our operator; in fact, $Mf \le M_a^1 M_c^2 f$, and the conclusion follows in this case.

Assume now that $b \ne 0$. It is enough to consider h_1, h_2, with $h_1 < h_2$ in (1.1).

When $c \ne 0$, we write $F = a'(x_1')^2 + c(x_2')^2$, where $x_1' = x_1$ and $x_2' = x_2 + bx_1/2c$. The domain of integration in (1.1) is contained in $R' = \{(x_1', x_2') \colon 0 < x_1' < h_1,\ |x_2'| < Ch_2\}$, whose area is $2C$ times larger. The mean in (1.1) can thus be replaced by that over R', and the case $b = 0$ applies.

When $c = 0$, we write $F = x_1' x_2'$ instead where $x_1' = x_1$ and $x_2' = ax_1 + bx_2$. Now we can apply Theorem 1 to the mean over the same rectangle R' as before. This completes the proof of Theorem 2.

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