A DOWKER PRODUCT

BY

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Abstract. ◊ implies that there is a (normal) countably paracompact space X such that $X^2$ is normal and not countably paracompact.

1. Introduction. A normal space which is not countably paracompact is called a Dowker space. It is well known that a normal space $X$ is countably paracompact iff $X \times (\omega + 1)$ is normal iff, for every decreasing sequence $\langle F_n : n \in \omega \rangle$ of closed subsets of $X$ such that $\cap \{ F_n : n \in \omega \} = 0$, there is a sequence $\langle O_n : n \in \omega \rangle$ of open sets such that $\cap \{ O_n : n \in \omega \} = 0$ and, for every $n \in \omega$, $F_n \subset O_n$. There is essentially only one known Dowker space in ZFC, constructed by Rudin [15], although numerous Dowker spaces have been constructed beyond ZFC (see [18] for more on this).

Rudin and Starbird have proved that if $X$ is countably paracompact (paracompact, collectionwise normal), $M$ metric, and $X \times M$ normal, then $X \times M$ is countably paracompact (paracompact, collectionwise normal), see [20]. At that time it was already known that under $\text{MA} + \neg \text{CH}$ there is a Lindelöf space $X$ such that $X^2$ is normal but not collectionwise normal [11]. (There also is, in ZFC, a Lindelöf space $X$ such that $X^2$ is collectionwise normal but not paracompact [13].) So they asked whether the product of two countably paracompact spaces can be Dowker. This was answered affirmatively by Wage [21] who gave, under the Continuum Hypothesis, an example of countably paracompact $X$ and $Y$ such that $X \times Y$ is Dowker. He used a space constructed in [5]. However, he never published details of his construction.

Here we assume ◊* and construct a countably paracompact space $X$ such that $X^2$ is Dowker (§2). In §3 we show how the construction from §2 can be modified to give, for any $n \in \omega$, a space $X$ such that $X^n$ is countably paracompact and $X^{n+1}$ is Dowker. All our results follow from ◊ only; see §5.

The concept of a Dowker space can be generalized a bit.

1.1 Definition. An open cover $\{ V_\alpha : \alpha \in \kappa \}$ of a space $X$ is a shrinking of the family $\{ U_\alpha : \alpha \in \kappa \}$ of subsets of $X$ iff, for every $\alpha \in \kappa$, $\overline{V_\alpha} \subset U_\alpha$.

A space $X$ is shrinking iff every open cover of $X$ has a shrinking. □

Observe that a Hausdorff space $X$ is normal iff every open cover of size 2 has a shrinking, and that a Hausdorff space $X$ is normal and countably paracompact iff every countable open cover of $X$ has a shrinking. Also, notice that a space $X$ is shrinking iff for every open cover $\{ U_\alpha : \alpha \in \kappa \}$ of $X$ there is a closed cover $\{ F_\alpha : \alpha \in \kappa \}$ such that $F_\alpha \subset U_\alpha$ for every $\alpha \in \kappa$.  

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In ZFC, there is only one known type of normal spaces that are not shrinking [17]. For more on shrinking spaces see [19].

It is known that the normal product of a shrinking space and a metric space is shrinking [1, 3.6]. Our examples are shrinking spaces so it is consistent that there is a shrinking space $X$ such that $X^2$ is normal but not shrinking.

1.2 Definition. A normal space $X$ is $\kappa$-Dowker iff $\kappa$ is the minimal cardinality of an increasing open cover of $X$ which does not have a shrinking. □

Thus $\omega$-Dowker is the same as Dowker. For each regular cardinal $\kappa$ we essentially know exactly one $\kappa$-Dowker space in ZFC [17]. Call a space $\kappa$-shrinking iff every open cover of cardinality $\kappa$ has a shrinking. Hence a normal space is $\omega$-Dowker iff it is not $\omega$-shrinking. But this does not generalize: for a regular infinite $\kappa$, $\Diamond^{++}$ on $\kappa^+$ implies that there is a normal, not $\kappa^+$-shrinking, not $\kappa^+$-Dowker space [2]. For more on $\kappa$-Dowker spaces, see [19].

In §4 we show, using the construction from §3, how to get for each regular uncountable $\kappa$ and $n \in \omega$, a space $X$ such that $X^n$ is shrinking but $X^{n+1}$ is $\kappa$-Dowker.

Recall that a space $X$ is called a $P$-space iff $X \times M$ is normal for every metric space $M$. Our examples are $P$-spaces; thus under $\Diamond^*$, there is a $P$-space $X$ such that $X^2$ is normal but not a $P$-space.

We use an Ostaszewski technique [10] to construct our spaces. The result in §4 implies the results in the preceding sections, but we decided to present the result in §2 in detail, as it is the most interesting and the least technical case. §§3 and 4 are rather sketchy; in them we mainly emphasize the differences from the second section.

Any undefined notion or fact that is used without mention can be found in one of [4 and 7]. Many more facts about normality in products can be found in [14 and 16].

2. A Dowker square. We first state a consequence of $\Diamond^*$ which is then used to construct a countably paracompact space $X$ such that $X^2$ is Dowker.

2.1 Definition. $\Diamond^*$ is the statement: There is a sequence $(\mathcal{A}_\alpha: \alpha \in \omega_1)$ such that
(i) for every $\alpha$, $\mathcal{A}_\alpha$ is a countable family of subsets of $\alpha \times \omega$, 
(ii) for every $X \subseteq \omega_1 \times \omega$, there is a closed and unbounded $C \subseteq \omega_1$ such that, for every $\alpha \in C$, $X \cap (\alpha \times \omega) \in \mathcal{A}_\alpha$.

The sequence $(\mathcal{A}_\alpha: \alpha \in \omega_1)$ is called a $\Diamond^*$ sequence. □

Our definition of $\Diamond^*$ is not the usual one but it is clearly equivalent to it. $\Diamond^*$ holds in the constructible universe $L$.

Following [3, 6, and 12] we have

2.2 Definition. A set $X \subseteq (\omega_1 \times \omega)^2$ is 2-uncountable iff, for every $\alpha \in \omega_1$, $X \cap [(\omega_1 \setminus \alpha) \times \omega]^2 \neq 0$. □

Let $E = \{ \alpha \in \omega_1: \text{cf}(\alpha) = \omega \}$. Also, if $B \subseteq X \times Y$, let $\text{dom}(B) = \{ x \in X: \exists y (\langle x, y \rangle \in B) \}$ and $\text{ran}(B) = \{ y \in Y: \exists x (\langle x, y \rangle \in B) \}$, and if a set is closed and unbounded, we say that it is a cub.

2.3 Definition. $\Delta^*$ is the statement: There are sequences $\langle A_\alpha^0: \alpha \in E \rangle$, $\langle A_\alpha^1: \alpha \in E \rangle$, and $\langle B_\alpha: \alpha \in E \rangle$ such that, for $\alpha \in E$,
(i) $A_\alpha^0$ and $A_\alpha^1$ are subsets of $\alpha \times \omega$, 
(ii) $B_\alpha$ is a subset of $(\alpha \times \omega)^2$. 

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If $B^0_a = \text{dom}(B_a)$ and $B^1_a = \text{ran}(B_a)$ for each $a \in E$, then

(iii) $\text{ran}(B^0_a) \cup \text{ran}(B^1_a)$ is finite,

(iv) for every $\beta < \alpha$, $(A^0_a \cup A^1_a \cup B^0_a \cup B^1_a) \cap (\beta \times \omega)$ is finite,

(v) $A^0_a, A^1_a$, and $B^0_a \cup B^1_a$ are pairwise disjoint,

(vi) for every uncountable $X \subset (\omega_1 \times \omega)$ there is a cub $C \subset E$ such that, for every $\alpha \in E$ such that, for every $\beta < \alpha$, $X \cap [(\alpha \setminus \beta) \times \omega]^2 \cap B_a \neq 0$ and $Y \cap B_a \cap [(\alpha \setminus \beta) \times \omega]^2 \neq 0$.

2.4 Lemma. $\diamondsuit^*$ implies $\Delta^*$.

Proof. Let $\langle A_\alpha : \alpha \in \omega_1 \rangle$ be a $\diamondsuit^*$ sequence. Since $\diamondsuit^*$ implies $\diamondsuit$, there is a sequence $\langle (B_a, R_a) : a \in \omega_1 \rangle$ of pairs of subsets of $(\omega_1 \times \omega)^2$ such that, for every pair $\langle X, Y \rangle$ of subsets of $(\omega_1 \times \omega)^2$, there is a stationary $S \subset \omega_1$ such that, for every $a \in S$, $X \cap (\omega_1 \times \omega)^2 = B_a$ and $Y \cap (\omega_1 \times \omega)^2 = B_a$.

For $\alpha \in E$, let

$$\mathcal{A}_\alpha = \{ A \in \mathcal{A}_\alpha : \forall \beta < \alpha (A \cap [(\alpha \setminus \beta) \times \omega] \neq 0) \}$$

and

$$\mathcal{B}_a = \{ B_{a,i} : i \in 2 \land \forall \beta < \alpha (B_{a,i} \cap [(\alpha \setminus \beta) \times \omega]^2 \neq 0)$$

$$\land (|\text{ran}(\text{dom}(B_{a,i})) \cup \text{ran}(\text{ran}(B_{a,i}))| < \omega) \}.$$ 

List $\mathcal{A}_\alpha \cup \mathcal{B}_a$ as $\langle F_n : n \in \omega \rangle$ in such a way that each member of $\mathcal{B}_a$ is listed infinitely many times, and that for every $A \in \mathcal{A}_\alpha$ there are infinitely many even and infinitely many odd $n$'s for which $F_n = A$.

By induction on $n \in \omega$, pick $p_n \in F_n$ and $\alpha_n, \alpha_n \in \alpha$ such that

1. $\sup_n \alpha_n = \alpha$,

2. if $k < n$ then $p_k \in (\alpha_n \times \omega)$ or $p_k \in (\alpha_n \times \omega)^2$, and $p_n \in (\alpha_n \setminus \alpha_n) \times \omega$ or $p_n \in ((\alpha_n \setminus \alpha_n) \times \omega)^2$.

Let $A^0_a = \{ p_n : n \text{ is even and } F_n \in \mathcal{A}_\alpha \}$, $A^1_a = \{ p_n : n \text{ is odd and } F_n \in \mathcal{A}_\alpha \}$, and $B_a = \{ p_n : F_n \in \mathcal{B}_a \}$.

Now we show that these $A$'s and $B$'s satisfy Definition 2.3. Conditions (i)-(v) hold trivially. For (vi), if $X \subset (\omega_1 \times \omega)$ is uncountable, fix a cub $C_0 \subset E$ so that if $\alpha \in C_0$ then, for every $\beta < \alpha$, $X \cap [(\alpha \setminus \beta) \times \omega] \neq 0$. Let $C_1$ be a cub such that if $\alpha \in C_1$ then $X \cap (\alpha \times \omega) \subset A_\alpha$. Then $C = C_0 \cap C_1$ shows that (vi) holds.

For (vii), observe that if $X \subset (\omega_1 \times \omega)^2$ is 2-uncountable then there is a $n \in \omega$ such that $X \cap (\omega_1 \times n)^2$ is 2-uncountable. So we may assume that $\langle X, Y \rangle$ from (vii) is such that there is an $n \in \omega$ with $X \subset (\omega_1 \times n)^2$. Fix a cub $C \subset E$ such that if $\alpha \in C$ then, for every $\beta < \alpha$, $X \cap [(\alpha \setminus \beta) \times \omega]^2 \neq 0$ and $Y \cap [(\alpha \setminus \beta) \times \omega]^2 \neq 0$. Let $S \subset \omega_1$ be stationary such that, for every $\alpha \in S$, $X \cap (\alpha \times \omega)^2 = B_{a,0}$ and $Y \cap (\alpha \times \omega)^2 = B_{a,1}$. Then for any $\alpha \in S \cap C$, $X \cap B_a$ and $Y \cap B_a$ are as required.
2.5 Theorem ($\Delta^*$). There is a Hausdorff, first countable, locally compact, shrinking, $P$-space $X$ such that $X^2$ is collectionwise normal but not countably paracompact.

Proof. We construct two topologies $\tau^0$ and $\tau^1$ on $\omega_1 \times \omega$ in such a way that $T_0 = \langle \omega_1 \times \omega, \tau^0 \rangle$ and $T_1 = \langle \omega_1 \times \omega, \tau^1 \rangle$ are Hausdorff, first countable, locally compact, shrinking, $P$-spaces, and also make sure that $T_0^2$, $T_0 \times T_1$, and $T_1^2$ are collectionwise normal. At the same time we kill countable paracompactness of $T_0 \times T_1$ by showing that the diagonal $\{(x, y) \in T_0 \times T_1: x = y\}$ is Dowker. Then for $X$ we take the free sum of $T_0$ and $T_1$.

By induction on $\alpha \in \omega_1$, for $\langle \alpha, n \rangle \in \omega_1 \times \omega$ and $i \in 3$ we construct families $\{U_{\langle \alpha, n \rangle, k}^i: k \in \omega\}$ of subsets of $\langle \alpha + 1 \rangle \times \omega$ such that if $\tau^i_\alpha$ is the topology on $\alpha \times \omega$ having $\{U_{\langle \alpha, n \rangle, k}^i: k \in \omega \land p \in \alpha \times \omega\}$ as a (sub)basis then

(a) $\tau^i_\alpha$ is a Hausdorff topology on $\alpha \times \omega$,
(b) for all $\beta < \alpha, (\beta + 1) \times \omega$ is closed in $\langle \alpha \times \omega, \tau^i_\alpha \rangle$,
(c) for all $p \in \alpha \times \omega, \{U_{\langle \alpha, n \rangle, k}^i: k \in \omega\}$ is decreasing,
(d) for all $p \in \alpha \times \omega, \{U_{\langle \alpha, n \rangle, k}^i: k \in \omega\}$ is a clopen basis for $p$ consisting of compact sets in $\tau^i_\alpha$,
(e) for all $p \in \alpha \times \omega$ and $k \in \omega, U_{\langle \alpha, n \rangle, k}^{0, k} \cup U_{\langle \alpha, n \rangle, k}^{1, k} \subset U_{\langle \alpha, n \rangle, k}^{2, k}$,
(f) for all $\langle \beta, n \rangle \in \alpha \times \omega, U_{\langle \beta, n \rangle}^{0, 0} \cap U_{\langle \beta, n \rangle}^{1, 0} \subset (\beta + 1) \times (n + 1)$.

Observe that (d) implies that, for all $\beta < \alpha, \beta \times \omega$ is open in $\langle \alpha \times \omega, \tau^i_\alpha \rangle$, and that (e) implies that $\tau^2_\alpha$ is coarser than both $\tau^0_\alpha$ and $\tau^1_\alpha$, i.e. $\tau^2_\alpha \subset \tau^0_\alpha \cap \tau^1_\alpha$.

Topologies $\tau^i$ for $i \in 3$ are defined by letting $\bigcup\{U_{\langle \alpha, n \rangle}^i: \alpha \in \omega_1\}$ be a basis for $\tau^i$. The topology $\tau^2$ helps us to carry the induction, and it plays a small role in the proof that $T_0 \times T_1$ is not countably paracompact.

It remains to define $U_{\langle \alpha, n \rangle, k}^{i, k}$ for $i \in 3, k \in \omega$, and $p \in \omega_1 \times \omega$. If $\alpha$ is a successor or 0, for $i \in 3$ and $k \in \omega$ let $U_{\langle \alpha, n \rangle, k}^{i, k} = \{(a, n)\}$. It is easy to check that all induction hypotheses are satisfied.

Let $\langle A^0_\alpha: \alpha \in E \rangle$, $\langle A^1_\alpha: \alpha \in E \rangle$, and $\langle B_\alpha: \alpha \in E \rangle$ be as in Definition 2.3. Recall that $B^0_\alpha = \text{dom}(B_\alpha)$ and $B^1_\alpha = \text{ran}(B_\alpha)$. For $\alpha$ a limit, i.e. $\alpha \in E$, pick $n_\alpha \in \omega$ such that $n_\alpha > \sup(\text{ran}(B^0_\alpha) \cup \text{ran}(B^1_\alpha))$. Notice that $n_\alpha > 0$. If $n \notin \{0, n_\alpha\}$, for $i \in 3$ and $k \in \omega$ let $U_{\langle \alpha, n \rangle, k}^{i, k} = \langle \alpha, n \rangle$. Let $\langle \alpha_k: k \in \omega\rangle$ be an increasing cofinal sequence in $\alpha$.

The set $D = A^0_\alpha \cup A^1_\alpha \cup B^0_\alpha \cup B^1_\alpha$ is closed and discrete in $\langle \alpha \times \omega, \tau^2_\alpha \rangle$ by 2.3(iv) and the fact that $\beta \times \omega$ is open for every $\beta < \alpha$. Since $\langle \alpha \times \omega, \tau^2_\alpha \rangle$ is metrizable and locally compact, for each $\alpha \in D$ there is a compact clopen neighborhood $U_\alpha$ of $\alpha$ in $\langle \alpha \times \omega, \tau^2_\alpha \rangle$ such that $\langle U_{\alpha_i}^i: d \in D \rangle$ is discrete in $\langle \alpha \times \omega, \tau^2_\alpha \rangle$ (hence discrete in $\langle \alpha \times \omega, \tau^i_\alpha \rangle$ for any $i \in 2$), and such that if $d = \langle \beta, n \rangle$ and $\alpha_k + 1 < \beta$ then $U_{\alpha_k} \cap [(\alpha_k + 1) \times \omega] = 0$ (by (b), this can be done). For $i \in 2$ and $d \in D$ let $U_{\alpha_i}^d = U_{\alpha_i}^{i, k}$, where $k \in \omega$ is the least number for which $U_{\alpha_i}^{i, k} \subset U_\alpha$.

For $k \in \omega$ define

$$U^2_{\langle \alpha, 0 \rangle, k} = \{\langle \alpha, 0 \rangle\} \cup \left(\bigcup\{U_d: d \in \left(\left(A^0_\alpha \cup A^1_\alpha\right) \setminus (\alpha_k \times \omega)\}\}\right), \text{ and}$$

$$U^2_{\langle \alpha, n_\alpha \rangle, k} = \{\langle \alpha, n_\alpha \rangle\} \cup \left(\bigcup\{U_d: d \in \left(\left(B^0_\alpha \cup B^1_\alpha\right) \setminus (\alpha_k \times \omega)\}\}\right).$$
For $i \leq 2$ and $k \in \omega$ define
\[
U^{i,k}_{\langle \alpha,0 \rangle} = \{ \langle \alpha,0 \rangle \} \cup \left( \bigcup \left\{ U_d^i : d \in \left[ A_\alpha^i \setminus (\alpha_k \times \omega) \right] \right\} \right), \quad \text{and}
\]
\[
U^{i,k}_{\langle \alpha,n_\alpha \rangle} = \{ \langle \alpha,n_\alpha \rangle \} \cup \left( \bigcup \left\{ U_d^i : d \in \left[ (B_\alpha^0 \cup B_\alpha^1) \setminus (\alpha_k \times \omega) \right] \right\} \right).
\]

We now check that the induction hypotheses are satisfied. The condition (a) is satisfied because of 2.3(v), and (b)–(e) are satisfied trivially. We check that (f) holds. For $n \not\in \{0, n_\alpha\}$, (f) trivially holds at $\langle \alpha, n \rangle$. Note that $U^{0,0}_{\langle \alpha,0 \rangle} \cap U^{1,0}_{\langle \alpha,0 \rangle} = \{ \langle \alpha,0 \rangle \}$ by 2.3(v). To see that $U^{0,0}_{\langle \alpha,n_\alpha \rangle} \cap U^{1,0}_{\langle \alpha,n_\alpha \rangle} \subseteq (\alpha + 1) \times (n_\alpha + 1)$, it is enough to show that $U_d^0 \cap U_e^1 \subseteq (\alpha + 1) \times (n_\alpha + 1)$ for $d, e \in B_\alpha^0 \cup B_\alpha^1$. Since $U_d^0 \cap U_e^1 \subseteq U_d \cap U_e$, if $d \neq e$, $U_d^0 \cap U_e^1 = 0$, so assume $d = e = \langle \beta, n \rangle \in B_\alpha^0 \cup B_\alpha^1$. But then $U_{\langle \beta,n \rangle}^0 \cap U_{\langle \beta,n \rangle}^1 \subseteq U_{\langle \beta,n \rangle}^0 \cup U_{\langle \beta,n \rangle}^1 \subseteq (\beta + 1) \times (n + 1)$ by (c) and (f) for $\langle \beta, n \rangle$, so since $n_\alpha > n$ we have that $U_{\langle \beta,n \rangle}^0 \cap U_{\langle \beta,n \rangle}^1 \subseteq (\alpha + 1) \times (n_\alpha + 1)$.

This finishes our construction of $T_0$ and $T_1$, and we now check that they have all of the desired properties. From now on $i$ and $j$ will stand for arbitrary members of $2$. Trivially, $T_i$ is Hausdorff, first countable, and locally compact.

2.6 Lemma. Let $\{ X_n : n \in \omega \}$ be a family of uncountable subsets of $T_i$. Then there is a cub $C \subseteq \omega_1$ such that, for every $\alpha \in C$ and $n \in \omega$, $\langle \alpha,0 \rangle$ is an accumulation point of $X_n$. In particular, $\langle C \times \{0\} \rangle \subseteq \bigcap \{ \overline{X}_n : n \in \omega \}$.

Proof. For each $X_n$ fix a cub $C_n \subseteq \omega_1$ satisfying 2.3(vi). Then $C = \bigcap_{n \in \omega} C_n$ is as required. □

(1) $T_i$ is normal.

Proof. Let $X$ and $Y$ be two closed disjoint subsets of $T_i$. By Lemma 2.6 we may assume that $X$ is countable, so there is an $\alpha \in \omega_1$ such that $X \subseteq (\alpha + 1) \times \omega$. Since the subspace topology on $(\alpha + 1) \times \omega$ is metrizable there is a clopen set $U$ in $(\alpha + 1) \times \omega$ containing $X$ such that $U \cap Y = 0$. This $U$ is clopen in $T_i$ since $(\alpha + 1) \times \omega$ is □

(2) Every closed discrete subset of $T_i$ is countable.

Proof. Let $X \subseteq T_i$ be uncountable. Let $X_0, X_1$ be two disjoint uncountable subsets of $X$. By Lemma 2.6, $\overline{X}_0 \cap \overline{X}_1 \neq 0$, hence $X$ is not closed discrete. □

(3) $T_i$ is countably paracompact.

Proof. Let $\{ F_n : n \in \omega \}$ be a decreasing family of closed subsets of $T_i$ such that $\bigcap_{n \in \omega} F_n = 0$. By Lemma 2.6 there is a $k \in \omega$ such that $F_k$ is countable. Let $\alpha \in \omega_1$ be such that $F_k \subseteq \alpha \times \omega$. Since $\alpha \times \omega$ is open and metrizable, there are, for $n \geq k$, open $O_n$ containing $F_n$ such that $\bigcap_{n \geq k} O_n = 0$. For $n < k$, let $O_n = T_i$. So $T_i$ is countably paracompact. □

(4) $T_i$ is shrinking.

Proof. Since $T_i$ is normal and countably paracompact every open cover of cardinality $\leq \omega$ has a shrinking. Let $\{ U_\alpha : \alpha \in \omega_1 \}$ be an open cover of $T_i$. It suffices to show that there is a closed cover $\{ F_\alpha : \alpha \in \omega_1 \}$ such that $F_\alpha \subseteq U_\alpha$ for every $\alpha \in \omega_1$.  

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Define $S = \{ t \in T_j \mid \| \langle \alpha \in \omega_1 \mid t \in U_\alpha \| \leq \omega \}$. List $T_j \setminus S$ as $\{ t_\alpha \mid \alpha \in \omega_1 \}$. For $\alpha \in \omega_1$ pick $f(\alpha) \in \omega_1$ such that $t_\alpha \in U_{f(\alpha)}$ and, for every $\beta < \alpha$, $f(\beta) < f(\alpha)$. Let $F_{f(\alpha)} = \{ t_\alpha \}$, and if $\beta \in \omega_1$ is not in the range of $f$, let $F_{f(\alpha)} = 0$. Then for every $\alpha \in \omega_1$, $F_{f(\alpha)} \subseteq U_\alpha$, and $\bigcup_{\alpha \in \omega_1} F_{f(\alpha)} = T_j \setminus S$.

Observe that $S$ is closed since $T_j$ is first countable. For $\alpha \in \omega_1$, let $S_\alpha = S \setminus U_\alpha$.

Claim. There is an $\alpha \in \omega_1$ such that $S_\alpha$ is countable.

Proof. Assume not. Using Lemma 2.6 fix, for each $\alpha \in \omega_1$, a cub $C_\alpha \subset \omega_1$ such that each point of $C_\alpha \times \{ 0 \}$ is an accumulation point of $S_\alpha$. Let $C$ be the diagonal intersection of $\{ C_\alpha \mid \alpha \in \omega_1 \}$; so $C = \{ \beta \in \omega_1 \mid \forall \alpha < \beta \ (\beta \in C_\alpha) \}$ is a cub subset of $\omega_1$. Since each $S_\alpha$ is closed, if $\beta \in C$ then $\langle \beta, 0 \rangle \in \bigcap_{\alpha < \beta} S_\alpha$, hence $\langle \beta, 0 \rangle \notin \bigcup_{\alpha < \beta} U_\alpha$. So for $\beta \in C$, pick $p_\beta = \langle a_\beta, m_\beta \rangle \in S_0$, and $c_\beta > \beta$ such that $p_\beta \in U_{c_\beta}$ and $a_\beta < \beta$. This can be done since $\langle \beta, 0 \rangle$ is an accumulation point of $S_0$. By the pressing down lemma followed by an easy counting argument there are $\omega_1$ many $\beta$’s with the same $p_\beta = p$. But then this $p$ is an element of $S$ which is in uncountably many $U_\alpha$’s. So there is an $\alpha$ such that $S_\alpha$ is countable. □

We may assume that $S \setminus U_0$ is countable. Fix a $\beta \in \omega_1$ such that $(S \setminus U_0) \subset \beta \times \omega_1$, and let $F_0 = S \cap [(\omega_1 \setminus \beta) \times \omega] \subset U_0$. Then $F_0$ is closed since $S$ is. For $\alpha > 0$, let $F_\alpha = 0$. Since metrizable spaces are shrinking there is a closed cover $\{ F_\alpha \mid \omega_1 \}$ of $(\beta + 1) \times \omega_1$ such that, for every $\alpha \in \omega_1$, $F_\alpha \subset U_\alpha$.

For $\alpha \in \omega_1$, let $F_{\alpha} = F_{\alpha}^0 \cup F_{\alpha}^1 \cup F_{\alpha}^2$. Then $\{ F_{\alpha} \mid \alpha \in \omega_1 \}$ is a closed cover of $T_i$ such that $F_{\alpha} \subseteq U_\alpha$ for every $\alpha \in \omega_1$. □

At this point we need

2.7 Lemma. Let $\langle X, Y \rangle$ be a pair of 2-uncountable subsets of $T_i \times T_j$. Then there is an $\alpha \in \omega_1$ such that $\langle \langle \alpha, n_\alpha \rangle, \langle \alpha, n_\alpha \rangle \rangle \in \bar{X} \cap \bar{Y}$.

Proof. Let $\alpha$ satisfy 2.3(vii). Then $\langle \langle \alpha, n_\alpha \rangle, \langle \alpha, n_\alpha \rangle \rangle \in \bar{X} \cap \bar{Y}$. □

(5) $T_i \times T_j$ is normal.

Proof. Let $X, Y$ be two closed disjoint subsets of $T_i \times T_j$. Using Lemma 2.7 we may assume that $X$ is not 2-uncountable. Fix an $\alpha \in \omega_1$ so that $X \subset \{(\alpha + 1) \times \omega \} \times T_j \cup [T_i \times ((\alpha + 1) \times \omega)]$. It follows from (8) that both $((\alpha + 1) \times \omega) \times T_j$ and $T_i \times ((\alpha + 1) \times \omega)$ are normal; however, for the convenience of the reader not interested in $P$-spaces we give a direct proof of this fact in Lemma 2.8 below. Assuming that $((\alpha + 1) \times \omega) \times T_j$ and $T_i \times ((\alpha + 1) \times \omega)$ are normal we show how to separate $X$ and $Y$ by two disjoint open sets.

Sets $[(\omega_1 \setminus (\alpha + 1)) \times \omega] \times ((\omega_1 \setminus (\alpha + 1)) \times \omega)$, $T_i \times ((\alpha + 1) \times \omega)$, and $((\alpha + 1) \times \omega) \times [(\omega_1 \setminus (\alpha + 1)) \times \omega]$ partition $T_i \times T_j$ into pairwise disjoint clopen sets, and in each of them we can separate $X$ and $Y$ in the last two since they are normal and in the first one since $X$ does not intersect it); hence $X$ and $Y$ can be separated in $T_i \times T_j$.

2.8 Lemma. Assume that $X$ is normal and countably paracompact and $Y$ is a countable metric space. Then $X \times Y$ is normal.

Proof. For $y \in Y$ let $\{ B_n(y) \mid n \in \omega \}$ be a clopen decreasing basis for $y$. Let $H$ and $K$ be two closed disjoint subsets of $X \times Y$. Let $\pi: X \times Y \to X$ be the projection map. For $y \in Y$ let $F_n(y) = \pi(H \cap (X \times B_n(y))) \cap \pi(K \cap (X \times B_n(y)))$. Since
\{B_n(y): n \in \omega\} is decreasing, \{F_n(y): n \in \omega\} decreases, and since \{B_n(y): n \in \omega\} is a basis for \(y\), \(\bigcap_{n \in \omega} F_n(y) = 0\). Let \(\{O_n(y): n \in \omega\}\) be an open family in \(X\) such that \(F_n(y) \subset O_n(y)\) for every \(n \in \omega\) and \(\bigcap_{n \in \omega} O_n = 0\).

For \(n \in \omega\) and \(y \in Y\) let \(H_n(y) = \pi(H \cap (X \times B_n(y))) \setminus O_n(y)\) and \(K_n(y) = \pi(K \cap (X \times B_n(y))) \setminus O_n(y)\). Then \(H_n(y)\) and \(K_n(y)\) are closed disjoint subsets of \(X\), so there is an open \(U_n(y)\) such that \(H_n(y) \subset U_n(y) \subset U_n(y) \subset X \setminus \pi(K \cap (X \times B_n(y)))\). Similarly, there is an open \(V_n(y)\) such that \(K_n(y) \subset V_n(y) \subset X \setminus \pi(H \cap (X \times B_n(y)))\).

Then \(X = \{U_n(y) \times B_n(y): n \in \omega \land y \in Y\}\) and \(X = \{V_n(y) \times B_n(y): n \in \omega \land y \in Y\}\) are countable open families in \(X \times Y\) covering \(H\) and \(K\), respectively, such that, for every \(U \in X\) and \(V \in X\), \((U \cap K = 0)\) and \((H \cap V = 0)\). Hence \(X \times Y\) is normal.

\((6)\) \(T_i \times T_j\) is collectionwise normal.

**Proof.** We show that every closed discrete subset of \(T_i \times T_j\) is countable. This shows that \(T_i \times T_j\) is collectionwise normal since it is normal. Let \(X \subset T_i \times T_j\) be a 2-uncountable subset of \(T_i \times T_j\). By an easy induction one picks two disjoint 2-uncountable \(X_0, X_1 \subset X\). By Lemma 2.7, \(X_0 \cap X_1 \neq 0\) so \(X\) is not closed discrete. Let \(X\) be an uncountable subset of \(T_i \times T_j\) which is not 2-uncountable. Then there is a \(p \in \omega_1 \times \omega\) such that either \(X \cap (T_i \times \{p\})\) or \(X \cap (\{p\} \times T_j)\) is uncountable.

Then \(U = \{U_n(y) \times B_n(y): n \in \omega \land y \in Y\}\) and \(V = \{V_n(y) \times B_n(y): n \in \omega \land y \in Y\}\) are countable open families in \(X \times Y\) covering \(H\) and \(K\), respectively, such that, for every \(U \in U\) and \(V \in V\), \((U \cap K = 0)\) and \((H \cap V = 0)\). Hence \(X \times Y\) is normal. □

\((7)\) \(T_0 \times T_1\) is not countably paracompact.

**Proof.** We show that \(R = \{\langle x, y \rangle \in T_0 \times T_1: x = y\}\) is not countably paracompact. As \(R\) is a closed subset of \(\langle\langle \omega_1 \times \omega, \tau^2\rangle\rangle^2\) it is closed in \(T_0 \times T_1\), hence \(T_0 \times T_1\) is not countably paracompact.

Since \(R\) is homeomorphic to \(\omega_1 \times \omega\) having as a basis the family \(\{U \cap V: U \in \tau_0 \land V \in \tau_1\}\), we identify \(R\) with this space.

For \(n \in \omega\), let \(F_n = \omega_1 \times (\omega \setminus n)\). Trivially, \(F_{n+1} \subset F_n\) for \(n \in \omega\), and \(\bigcap_{n \in \omega} F_n = 0\).

The condition (f) implies that each \(F_i\) is closed. Next we show that there is no open family \(\{O_n: n \in \omega\}\) such that \(\bigcap_{n \in \omega} O_n = 0\) and, for all \(n \in \omega\), \(F_n \subset O_n\). We need

2.9 Lemma. Let \(X\) be an uncountable subset of \(R\) and \(n \in \omega\). Then there are an \(\alpha \in \omega_1\) and an \(m > n\) such that \(\langle \alpha, m \rangle \in \bar{X}\).

**Proof.** Let \(X_0 = \{\langle x, x \rangle \in T_0 \times T_1: x \in X\}\) and \(X_1 = \{\langle \langle \beta, n \rangle, \langle \beta, n \rangle \rangle: \beta \in \omega_1\}\). Both \(X_0\) and \(X_1\) are 2-uncountable subsets of \(T_0 \times T_1\). By Lemma 2.7 there is an \(\alpha \in \omega_1\) such that \(\langle \langle \alpha, n_\alpha \rangle, \langle \alpha, n_\alpha \rangle \rangle \in \text{cl}_{T_0 \times T_1}(X_0) \cap \text{cl}_{T_0 \times T_1}(X_1)\). But then \(\langle \alpha, n_\alpha \rangle \in \text{cl}_{T_0 \times T_1}(X_0) \cap \text{cl}_{T_0 \times T_1}(X_1)\).

Assume that \(\{O_n: n \in \omega\}\) is an open family in \(R\) for which \(\bigcap_{n \in \omega} O_n = 0\) and \(F_n \subset O_n\) for every \(n \in \omega\). There is an \(n \in \omega\) such that \(R \setminus O_n\) is uncountable. Observe that \(R \setminus O_n\) is closed and \(R \setminus O_n \subset \omega_1 \times n\). By Lemma 2.9 there are an \(\alpha \in \omega_1\) and an \(m > n\) such that \(\langle \alpha, m \rangle \in R \setminus O_n\). This contradiction finishes the proof. □

\((8)\) \(T_i\) is a P-space.

**Proof.** This follows from Lemma 2.6 and Lemma 2.11 below. In order to be able to use Lemma 2.11 in §4 we state it in a more general form than needed at this point.
2.10 Definition. Let $\kappa$ be a cardinal. A space $X$ is a $\pi_\kappa$-space iff, for every paracompact space $Y$ with character $\leq \kappa$, $X \times Y$ is normal. □

The following is Lemma 5 from [2].

2.11 Lemma [2]. Let $X$ be a Hausdorff space and $\mathcal{F}$ a family of subsets of $X$ such that $\bigcap \{ \overline{F}_\alpha : \alpha \in \kappa \} \neq \emptyset$ for any $\{ F_\alpha : \alpha \in \kappa \} \subseteq \mathcal{F}$ of cardinality $\leq \kappa$. Also suppose that $M \subseteq X$ and $M \notin \mathcal{F}$ implies there is a $\pi_\kappa$-space $U$ which is clopen in $X$ and contains $M$. Then $X$ is a $\pi_\kappa$-space. □

We show that $T_\omega$ is a $\pi_\omega$-space and thus a $P$-space. Let $\mathcal{F} = \{ X \subseteq T_\omega : X$ is uncountable $\}$. By Lemma 2.6 any countable subfamily $\{ F_n : n \in \omega \} \subseteq \mathcal{F}$ is such that $\bigcap \{ \overline{F}_n : n \in \omega \} \neq \emptyset$. If $X$ is a countable subset of $T_\omega$ then, for some $\alpha \in \omega_1$, $X \subseteq (\alpha + 1) \times \omega$ and, by Lemma 2.8, $(\alpha + 1) \times \omega$ is a $\pi_\omega$-space. □

2.12 Remarks. Lemma 2.8 follows from [9]; see also [14, 4.13]. We included a short proof of it for the convenience of the reader.

One does not need Lemma 2.8 to show that $T_\omega$ is a $P$-space. Lemma 2.11 holds if $\kappa$ is replaced by $\omega$ and $\pi_\kappa$ by $P$. But then $(\alpha + 1) \times \omega$ trivially is a $P$-space.

We should also note that Lemma 5 from [2] (our Lemma 2.11) is stated there only for the case of $\pi_\omega$-spaces, but the same proof as the one given there works for arbitrary $\kappa$.

The space $X^2$ from Theorem 2.5 is strongly zero dimensional. □

3. Products with more than two factors. Here we show how to generalize the construction from the preceding section to get, for each $n \in \omega$, a space $X$ such that $X^n$ is countably paracompact and $X^{n+1}$ is Dowker.

We use a slightly more general consequence of $\Diamond^*$ which we state for arbitrary regular $\kappa \geq \omega$ in order to be able to use it in §4.

3.1. Definition. Let $\kappa$ be a cardinal and $n \in \omega$. A set $X \subseteq (\kappa^+ \times \kappa)^n$ has the $n$-size-$\kappa^+$ iff, for every $\alpha \in \kappa^+$, $X \cap ([\kappa^+ \setminus \alpha] \times \kappa)^n \neq \emptyset$. □

Let $\kappa \geq \omega$ be a regular cardinal. Define $E = \{ \alpha \in \kappa^+ : cf(\alpha) = \kappa \}$. For $k \in \omega$ and $l \in k$, let $\pi_l : (\kappa^+ \times \kappa)^k \to (\kappa^+ \times \kappa)$ be the projection map onto the $l$th coordinate.

3.2 Definition. Let $\kappa \geq \omega$ be a regular cardinal and $n \in \omega$. $A^*(\kappa^+, n)$ is the statement: There are sequences $\langle \mathcal{A}_\alpha : \alpha \in \omega \rangle$ for $i \in n + 1$ and a sequence $\langle \mathcal{B}_\alpha : \alpha \in \omega \rangle$ such that, for $\alpha \in \omega$,

(i) $\mathcal{A}_\alpha^i$ is a subset of $(\alpha \times \kappa)^n$ for each $i \in n + 1$,

(ii) $\mathcal{B}_\alpha$ is a subset of $(\alpha \times \kappa)^{n+1}$.

If for $\alpha \in \omega$ and $i \in n + 1$, $A^i_\alpha = \bigcup \{ \pi_l(\mathcal{A}_\alpha^i) : l \in n \}$ and $B^i_\alpha = \bigcup \{ \pi_l(\mathcal{B}_\alpha) : l \in n + 1 \}$, then, for $\alpha \in \omega$,

(iii) $|\text{ran}(B^i_\alpha)| < \kappa$,

(iv) $|\bigcup_{i \in n + 1} A^i_\alpha \cup B^i_\alpha| \cap (\beta \times \kappa) < \kappa$ for every $\beta < \alpha$,

(v) the family $\{ A^i_\alpha : i \in n + 1 \} \cup \{ B^i_\alpha \}$ consists of pairwise disjoint sets,

(vi) there is a cub $C \subseteq \kappa^+$ for every $X \subseteq (\kappa^+ \times \kappa)^n$ which has $n$-size-$\kappa^+$ such that, for every $\beta \in C \cap E$ and every $i \in n + 1$, if $\gamma < \beta$, $X \cap \mathcal{A}_\beta^i \cap ([\beta \setminus \gamma] \times \kappa)^n \neq \emptyset$,

(vii) for every family $\mathcal{F}$ of cardinality $< \kappa$ whose members are subsets of $(\kappa^+ \times \kappa)^{n+1}$ of $(n + 1)$-size-$\kappa^+$, there is a $\beta \in E$ such that, for every $X \in \mathcal{F}$ and $\gamma < \beta$, $X \cap \mathcal{B}_\beta \cap ([\beta \setminus \gamma] \times \kappa)^n \neq \emptyset$. □
3.3 Theorem ($\Delta^k(\omega_1, n)$). There is a Hausdorff, first countable, locally compact space $X$ such that $X^n$ is a shrinking $P$-space, and $X^{n+1}$ is collectionwise normal but not countably paracompact.

Proof. The construction is almost the same as the one in Theorem 2.5. We construct $n + 2$ topologies $\tau^i$ on $\omega_1 \times \omega$. Then we let $T_i = \langle \omega_1 \times \omega, \tau^i \rangle$ for $i \leq n + 1$, and define $X$ to be the free sum of the $T_i$’s.

For each $p \in \omega_1 \times \omega$ and $i \leq n + 2$ we construct a neighborhood basis $\{ U^i_{p,k} : k \in \omega \}$ for $p$ in $\tau^i$ as in 2.5. Conditions (a)–(d) are the same as (a)–(d) from 2.5. We also have

(e) for all $p \in \alpha \times \omega$ and $k \in \omega$, $\bigcup_{i \leq n+1} U^i_{p,k} \subset U^i_{p,n+1,k}$,

(f) for all $\langle \beta, m \rangle \in \alpha \times \omega$, $\bigcap_{i \leq n+1} U^i_{\beta,m} \subset (\beta + 1) \times (m + 1)$.

Let $D = B_n \cup (\bigcup_{i \leq n+1} A_n^i)$ and pick $n_a > \sup(\text{ran}(B_n))$. The construction of the $U$’s is the same as in 2.5 except that for $\langle \alpha, 0 \rangle$ if $i \leq n + 1$ and $k \in \omega$ we define

$U^i_{\langle \alpha,0 \rangle} = \{ \langle \alpha, 0 \rangle \} \cup \left( \bigcup \left\{ U^i_{d,j} : d \in \left( \bigcup \left\{ A^i_{d,j} : j \leq n + 1 \land j \neq i \right\} \right) \right\} \right)$.

Trivially each $T_i$ is Hausdorff, first countable and locally compact. Let $T^k$ stand for an arbitrary product of $k$ (not necessarily different) factors from $\{ T_i : i \in n + 1 \}$.

3.4 Lemma. (i) Let $k \leq n$ and let $\{ X_j : j \in \omega \}$ be a family of $k$-size-$\omega_1$ subsets of $T^k$. Then there is a cub $C \subset \omega_1$ such that, for every $\alpha \in C \cap E$ and $j \in \omega$, the point $p \in T^k$ having all its coordinates equal to $\langle \alpha, 0 \rangle$ is an accumulation point of $X_j$.

(ii) Let $\mathcal{F}$ be a finite collection of $(n + 1)$-size-$\omega_1$ subsets of $T^{n+1}$. Then there is an $\alpha \in E$ such that the point $p \in T^{n+1}$ with all its coordinates equal $\langle \alpha, n_a \rangle$ is in $\bigcap\{ X : X \in \mathcal{F} \}$. □

By induction on $k \leq n$ one shows (simultaneously) that $T^k$ is normal and countably paracompact. This is similar to 2.5(1), (3), and (6); by Lemma 3.4(i) every countable family of closed subsets of $T^k$ of $k$-size-$\omega_1$ has a nonempty intersection and Lemma 2.8 (together with the induction hypothesis) shows that any subset of $T^k$ which does not have $k$-size-$\omega_1$ is contained in a clopen, normal, and countably paracompact subspace of $T^k$. (For countable paracompactness, observe that by applying Lemma 2.8 to $X$ and $X \times (\omega + 1)$ one concludes that $X \times Y$ is normal and countably paracompact for normal countably paracompact $X$ and metric countable $Y$.) Then the normality of $T^{n+1}$ follows from Lemma 3.4(ii), and since every closed discrete subset of $T^{n+1}$ is countable, $T^{n+1}$ is collectionwise normal. The diagonal $R$ of $\Pi_{i \leq n+1} T_i$ is a closed Dowker subspace of $\Pi_{i \leq n+1} T_i$ since by (f) and Lemma 3.4(ii) the same argument as in 2.5(7) applies for $R$. By Lemma 2.11 and induction on $k \leq n$, $T^k$ is a $P$-space for $k \leq n$ (for this note that the product of a $P$-space and a metric space is a $P$-space).

We show that $T^k$ is shrinking by induction on $k \leq n$. The proof is similar to 2.5(4) so we use the same notation. Some $S_\alpha$ is not of $k$-size-$\omega_1$. Assume not. Then by Lemma 3.4(i) we may assume that each $S_\alpha$ is a subset of the diagonal of $T^k$.

3.5 Lemma. Let $X$ be shrinking and $Y$ a countable metric space. Then $X \times Y$ is shrinking.
PROOF. For \( y \in Y \) let \( \{ B_i(y); i \in \omega \} \) be a clopen decreasing basis for \( y \). Let \( \{ U_\alpha; \alpha \in \lambda \} \) be an open cover of \( X \times Y \). For \( \alpha \in \lambda, i \in \omega, \) and \( y \in Y \), let \( U_\alpha^i(y) \) be the maximal open set in \( X \) such that \( U_\alpha^i(y) \times B_i(y) \subseteq U_\alpha \), and define \( F_i(y) = X \setminus \bigcup \{ U_\alpha^i(y); \alpha \in \lambda \} \). By the countable paracompactness of \( X \) there are open \( U_i(y) \supseteq F_i(y) \) for \( i \in \omega, \) with \( \bigcap_{i \in \omega} U_i(y) = 0 \). Let \( \{ V_\alpha^i(y); \alpha \in \lambda \} \cup \{ V_i(y) \} \) be a shrinking of \( \{ U_\alpha^i(y); \alpha \in \lambda \} \cup \{ U_i(y) \} \). Then \( \{ V_\alpha^i(y) \times B_i(y); \alpha \in \lambda \land i \in \omega \land y \in Y \} \) is an open cover of \( X \times Y \) such that, for every \( y \in Y, i \in \omega, \) and \( \alpha \in \lambda, V_\alpha^i(y) \times B_i(y) \subseteq U_\alpha \). Since \( X \times Y \) is countably paracompact, the following trivial fact [1, 3.1] finishes the proof:

3.6 Lemma [1]. Let \( X \) be \( \kappa \)-paracompact, and \( \{ U_\alpha; \alpha \in \lambda \} \) and \( \{ V_\alpha, \beta; \alpha \in \lambda \land \beta \in \kappa \} \) two open covers of \( X \) such that, for every \( \alpha \in \lambda \) and \( \beta \in \kappa, \overline{V}_\alpha, \beta \subseteq U_\alpha \). Then \( \{ U_\alpha; \alpha \in \lambda \} \) has a shrinking. □

3.7 Remarks. It is known that if \( \prod_{i \in \omega} T_i \) is normal and \( \prod_{i \in k} T_i \) is countably paracompact (shrinking) for every \( k \in \omega \), then \( \prod_{i \in \omega} T_i \) is countably paracompact [14, 6.1] (shrinking [1, 3.4]). So Theorem 3.3 can not be pushed up to \( \omega \).

Lemma 3.5 is a corollary of [1, 3.6]. We gave a direct proof of it since we need it in the next section. □

4. \( \kappa \)-Dowker products. Assume that \( \kappa \) is a regular uncountable cardinal. Recall that a space is called a \( P_\kappa \)-space iff any intersection of \( < \kappa \) open sets is open. \( P_\omega \) spaces are usually called \( P \)-spaces, but here a \( P \)-space is a space which has the normal product with each metrizable space.

4.1 Theorem \( (\Delta^*(\kappa^+, n)) \). There is a Hausdorff \( P_\kappa \)-space \( X \) of character \( \kappa \) such that \( X^n \) is shrinking and \( \kappa \)-paracompact, and \( X^{n+1} \) is a collectionwise normal, \( < \kappa \)-paracompact, \( \kappa \)-Dowker, \( P \)-space.

PROOF. The proof is almost verbatim as Theorem 3.3. Here \( \kappa \) plays the role of \( \omega \) (hence “finite” becomes “\( < \kappa \)”). We need [2, Lemma 6]. To state it, recall that a (clopen) basis \( \mathcal{B} \) for a space \( X \) is called non-Archimedean iff, for all \( A \) and \( B \) in \( \mathcal{B} \), either \( A \subset B \) or \( B \subset A \) or \( A \cap B = 0 \).

4.2 Lemma [2]. Assume that \( Y = \{ y_\alpha; \alpha \in \kappa \} \) is a Hausdorff space with a non-Archimedean basis \( \mathcal{B} \) such that for every \( \alpha \in \kappa \) there is \( B_\alpha \in \mathcal{B} \) so that

(i) \( y_\alpha \in B_\alpha \),

(ii) \( B_\alpha \cap \{ y_\beta; \beta < \alpha \} = 0, \) and

(iii) \( \{ \beta < \alpha; y_\alpha \in B_\beta \} \) is finite.

Then for every paracompact space \( X, X \times Y \) is paracompact. □

Now we check that, for all \( \beta \in \kappa^+ \) and \( i \in n + 1, Y = \langle \beta \times \kappa, \tau_\gamma \rangle \) satisfies the conditions of Lemma 4.2. Being a regular \( P_\kappa \)-space of weight \( \leq \kappa \), \( Y \) has a non-Archimedean basis \( \mathcal{B} \). Let \( Y = \{ y_\alpha; \alpha \in \kappa \} \) and pick \( B_\alpha \in \mathcal{B} \) so that if \( y_\alpha = \langle \gamma, \delta \rangle \) then \( B_\alpha \subseteq (\gamma \times \delta) \cup \{ \langle \gamma, \delta \rangle \} \) and \( B_\alpha \cap \{ y_\beta; \beta < \alpha \} = 0 \). These \( B_\alpha \)'s satisfy 4.2(iii), since if (iii) fails for some \( \alpha \in \kappa \) there is an increasing sequence \( \{ \alpha_i; i \in \omega \} \) with \( y_\alpha \in B_{\alpha_i} \) for \( i \in \omega \); hence \( B_{\alpha_{i+1}} \subset B_{\alpha_i} \). So if \( y_\alpha = \langle \gamma_i, \delta_i \rangle \), then \( \gamma_i > \gamma_{i+1} \), a contradiction.
Also an obvious analogue of Lemma 3.4 holds (instead of $\omega$ write $\kappa$ and instead of "finite" write $< \kappa$). We call this new version of Lemma 3.4 Lemma 4.3.

By induction on $k < \eta$ we show that $T^k$ is $\pi_\kappa$ (Definition 2.10), $\kappa$-paracompact and shrinking.

Lemma 4.2 implies that if $\alpha \in \kappa^+$ then $\langle \alpha \times \kappa, \tau_\alpha \rangle \times X$ is paracompact for every paracompact space $X$. Hence if $T^k$ is a $\pi_\kappa$-space and $\alpha \in \kappa^+$, then $T^k \times \langle \alpha \times \kappa, \tau_\alpha \rangle$ is a $\pi_\kappa$-space for any $i \in n + 1$.

In order to apply Lemma 2.11, let $\mathcal{F} = \{ X \subset T^k : X \text{ has $k$-size-$\kappa^+$} \}$. The induction hypothesis, the preceding paragraph, and Lemma 4.3(i) show that $\mathcal{F}$ satisfies the hypothesis of 2.11, so $T^k$ is a $\pi_\kappa$-space if $k < \eta$.

Observe that any $\pi_\kappa$-space is $\kappa$-paracompact. To see this one can use a theorem of Kunen which says that if $\kappa + 1$ has the order topology then $X \times (\kappa + 1)$ is normal iff $X$ is $\kappa$-paracompact (for a proof see [14, 3.7 or 19, Theorem 2]) or use a theorem of Morita: $X \times 2^\kappa$ is normal iff $X$ is $\kappa$-paracompact [8, 2.4]; see also [4, 3.8]. Hence for $\kappa < \eta$, $T^k$ is $\kappa$-paracompact.

To see that $T^k$ is shrinking observe first that the proof of Lemma 3.5 shows that, for every $\alpha \in \kappa^+$, $i \in n + 1$, and $k < \eta$, $T^k \times \langle \alpha \times \kappa, \tau_\alpha \rangle$ is shrinking provided that $T^k$ is. Then the same argument as in 3.3 shows $T^k$ shrinking for $k < \eta$.

Using Lemma 4.3(ii), one shows that $T^{\eta + 1}$ is $< \kappa$-paracompact and that $\prod_{i \in n + 1} T_i$ is $\kappa$-Dowker. To see $T^{\eta + 1}$ is a $P$-space use Lemma 2.11 for $\kappa = \omega$, and Lemma 4.3(ii).

4.4 Remark. The spaces $X^{\eta + 1}$ from this and the preceding section are strongly zero dimensional. □

5. Additional remarks. Ken Kunen observed that $\diamondsuit$ suffices in order to construct our examples. Instead of 2.3(vi) it is enough to have the following:

$(*$) There is a $\sigma$-complete, normal filter $\mathcal{F}$ on $\omega_1$ consisting of stationary sets such that for every uncountable $X \subset (\omega_1 \times \omega)$ there is an $F \in \mathcal{F}$ so that, for every $\alpha \in F$, $X \cap A^0_\alpha$ and $X \cap A^1_\alpha$ are infinite.

To get $(*)$ one uses the following unpublished result of Kunen.

5.1 Theorem (Kunen). $\diamondsuit$ implies that there is a $\sigma$-complete, normal filter $\mathcal{F}$ on $\omega_1$ containing the cub filter, and a sequence $\langle A_\alpha : \alpha \in \omega_1 \rangle$ such that

(i) each $\mathcal{A}_\alpha$ is a countable family of subsets of $\alpha$,

(ii) for every $X \subset \omega_1$, $\{ \alpha \in \omega_1 : X \cap \alpha \in \mathcal{A}_\alpha \} \in \mathcal{F}$.

Proof. Let $\langle A_\alpha : \alpha \in \omega_1 \rangle$ be a $\diamondsuit$-sequence on $\omega_1 \times \omega_1$; so each $A_\alpha$ is a subset of $\alpha \times \alpha$ and, for every $X \subset \omega_1 \times \omega_1$, $\{ \alpha \in \omega_1 : X \cap (\alpha \times \alpha) = A_\alpha \}$ is a stationary subset of $\omega_1$.

For $\alpha \in \omega_1$ let $\mathcal{A}_\alpha = \{ \text{dom}(A_\alpha \cap (\alpha \times \{ \beta \})) : \beta \in \alpha \}$; and let $\mathcal{E} = \{ S \subset \omega_1 : \exists X \subset \omega_1 (S = \{ \alpha \in \omega_1 : X \cap \alpha \in \mathcal{A}_\alpha \}) \}$.

We show that if $\{ S_\alpha : \alpha \in \omega_1 \} \subset \mathcal{E}$ then $\Delta_{\alpha \in \omega_1} S_\alpha = \{ \beta \in \omega_1 : \forall \alpha < \beta (\beta \in S_\alpha) \}$ is stationary. This shows the existence of the required $\mathcal{F}$.

Fix $\{ X_\alpha : \alpha \in \omega_1 \}$ so that for every $\alpha \in \omega_1$, $S_\alpha = \{ \beta \in \omega_1 : X_\alpha \cap \beta \in \mathcal{A}_\beta \}$. Define $X = \bigcup \{ X_\alpha \times \{ \alpha \} : \alpha \in \omega_1 \}$, $X \cap (\alpha \times \alpha) = A_\alpha$. Then $S$ is stationary, and $S \subset \Delta_{\alpha \in \omega_1} S_\alpha$. □
Theorem 5.1 generalizes to show that if \( \kappa \) is regular and \( E = \{ \alpha \in \kappa^+: cf(\alpha) = \kappa \} \) then \( \Diamond(E) \) implies \( \Delta^*(\kappa^+, n) \) for all \( n \in \omega \) (with the appropriate version of (\( * \)) instead of 3.2(vi)).

It is easy to get \( \Delta^* \) by forcing; add \( \omega_1 \) Cohen reals. A similar forcing argument gives \( \Delta^*(\kappa^+, n) \). This shows that \( \Delta^* \) does not imply CH. Forcing with countable partial functions from \( \omega_2 \) to 2 destroys \( \Delta^* \); hence CH does not imply \( \Delta^* \).

\( \text{MA} + \neg\text{CH} \) implies \( \neg\Delta^* \). To see this recall that if \( \mathcal{A} \) is a family of cardinality \( \omega_1 \) consisting of countable subsets of \( \omega_1 \) such that, for every \( A \neq B \in \mathcal{A} \), \( A \cap B \) is finite then \( \text{MA}_{\omega_1} \) implies that there is an uncountable \( X \subseteq \omega_1 \) such that \( X \cap A \) is finite for every \( A \in \mathcal{A} \) [22, Theorem 1]. So under \( \text{MA}_{\omega_1} \) the condition 2.3(vi) cannot hold (this can be shown directly, without using [22]).

If \( \kappa \) is regular one can force: \( \forall n \in \omega \; \neg\Delta^*(\kappa^+, n) \).

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