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A DOWKER PRODUCT

BY

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ABSTRACT. $\diamondsuit$ implies that there is a (normal) countably paracompact space X such that X^2 is normal and not countably paracompact.

1. Introduction. A normal space which is not countably paracompact is called a *Dowker* space. It is well known that a normal space X is countably paracompact iff $X \times (\omega + 1)$ is normal iff, for every decreasing sequence $\langle F_n: n \in \omega \rangle$ of closed subsets of X such that $\bigcap\{F_n: n \in \omega\} = 0$, there is a sequence $\langle O_n: n \in \omega \rangle$ of open sets such that $\bigcap\{O_n: n \in \omega\} = 0$ and, for every $n \in \omega$, $F_n \subset O_n$. There is essentially only one known Dowker space in ZFC, constructed by Rudin **[15]**, although numerous Dowker spaces have been constructed beyond ZFC (see **[18]** for more on this).

Rudin and Starbird have proved that if X is countably paracompact (paracompact, collectionwise normal), M metric, and $X \times M$ normal, then $X \times M$ is countably paracompact (paracompact, collectionwise normal), see **[20]**. At that time it was already known that under MA + $\neg$CH there is a Lindelöf space X such that X^2 is normal but not collectionwise normal **[11]**. (There also is, in ZFC, a Lindelöf space X such that X^2 is collectionwise normal but not paracompact **[13]**.) So they asked whether the product of two countably paracompact spaces can be Dowker. This was answered affirmatively by Wage **[21]** who gave, under the Continuum Hypothesis, an example of countably paracompact X and Y such that $X \times Y$ is Dowker. He used a space constructed in **[5]**. However, he never published details of his construction.

Here we assume $\diamondsuit^*$ and construct a countably paracompact space X such that X^2 is Dowker (§2). In §3 we show how the construction from §2 can be modified to give, for any $n \in \omega$, a space X such that X^n is countably paracompact and X^{n+1} is Dowker. All our results follow from $\diamondsuit$ only; see §5.

The concept of a Dowker space can be generalized a bit.

1.1 DEFINITION. An open cover $\{V_\alpha: \alpha \in \kappa\}$ of a space X is a *shrinking* of the family $\{U_\alpha: \alpha \in \kappa\}$ of subsets of X iff, for every $\alpha \in \kappa$, $\overline{V}_\alpha \subset U_\alpha$.

A space X is *shrinking* iff every open cover of X has a shrinking. □

Observe that a Hausdorff space X is normal iff every open cover of size 2 has a shrinking, and that a Hausdorff space X is normal and countably paracompact iff every countable open cover of X has a shrinking. Also, notice that a space X is shrinking iff for every open cover $\{U_\alpha: \alpha \in \kappa\}$ of X there is a closed cover $\{F_\alpha: \alpha \in \kappa\}$ such that $F_\alpha \subset U_\alpha$ for every $\alpha \in \kappa$.

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In ZFC, there is only one known type of normal spaces that are not shrinking [**17**]. For more on shrinking spaces see [**19**].

It is known that the normal product of a shrinking space and a metric space is shrinking [**1**, 3.6]. Our examples are shrinking spaces so it is consistent that there is a shrinking space X such that X^2 is normal but not shrinking.

1.2 DEFINITION. A normal space X is *κ-Dowker* iff κ is the minimal cardinality of an increasing open cover of X which does not have a shrinking. □

Thus ω-Dowker is the same as Dowker. For each regular cardinal κ we essentially know exactly one κ-Dowker space in ZFC [**17**]. Call a space κ-shrinking iff every open cover of cardinality κ has a shrinking. Hence a normal space is ω-Dowker iff it is not ω-shrinking. But this does not generalize: for a regular infinite κ, $\diamondsuit^{++}$ on κ^+ implies that there is a normal, not κ^+-shrinking, not κ^+-Dowker space [**2**]. For more on κ-Dowker spaces, see [**19**].

In §4 we show, using the construction from §3, how to get for each regular uncountable κ and $n \in \omega$, a space X such that X^n is shrinking but X^{n+1} is κ-Dowker.

Recall that a space X is called a *P-space* iff $X \times M$ is normal for every metric space M. Our examples are P-spaces; thus under $\diamondsuit^*$, there is a P-space X such that X^2 is normal but not a P-space.

We use an Ostaszewski technique [**10**] to construct our spaces. The result in §4 implies the results in the preceding sections, but we decided to present the result in §2 in detail, as it is the most interesting and the least technical case. §§3 and 4 are rather sketchy; in them we mainly emphasize the differences from the second section.

Any undefined notion or fact that is used without mention can be found in one of [**4** and **7**]. Many more facts about normality in products can be found in [**14** and **16**].

2. A Dowker square. We first state a consequence of $\diamondsuit^*$ which is then used to construct a countably paracompact space X such that X^2 is Dowker.

2.1 DEFINITION. $\diamondsuit^*$ is the statement: There is a sequence $\langle \mathscr{A}_\alpha : \alpha \in \omega_1 \rangle$ such that

(i) for every α, $\mathscr{A}_\alpha$ is a countable family of subsets of $\alpha \times \omega$,

(ii) for every $X \subset \omega_1 \times \omega$, there is a closed and unbounded $C \subset \omega_1$ such that, for every $\alpha \in C$, $X \cap (\alpha \times \omega) \in \mathscr{A}_\alpha$.

The sequence $\langle \mathscr{A}_\alpha : \alpha \in \omega_1 \rangle$ is called a $\diamondsuit^*$ sequence. □

Our definition of $\diamondsuit^*$ is not the usual one but it is clearly equivalent to it. $\diamondsuit^*$ holds in the constructible universe **L**.

Following [**3**, **6**, and **12**] we have

2.2 DEFINITION. A set $X \subset (\omega_1 \times \omega)^2$ is *2-uncountable* iff, for every $\alpha \in \omega_1$, $X \cap [(\omega_1 \setminus \alpha) \times \omega]^2 \neq 0$. □

Let $E = \{\alpha \in \omega_1 : \operatorname{cf}(\alpha) = \omega\}$. Also, if $B \subset X \times Y$, let $\operatorname{dom}(B) = \{x \in X : \exists y(\langle x, y \rangle \in B)\}$ and $\operatorname{ran}(B) = \{y \in Y : \exists x(\langle x, y \rangle \in B)\}$, and if a set is closed and unbounded, we say that it is a cub.

2.3 DEFINITION. Δ^* is the statement: There are sequences $\langle A_\alpha^0 : \alpha \in E \rangle$, $\langle A_\alpha^1 : \alpha \in E \rangle$, and $\langle B_\alpha : \alpha \in E \rangle$ such that, for $\alpha \in E$,

(i) A_α^0 and A_α^1 are subsets of $\alpha \times \omega$,

(ii) B_α is a subset of $(\alpha \times \omega)^2$.

If $B_\alpha^0 = \text{dom}(B_\alpha)$ and $B_\alpha^1 = \text{ran}(B_\alpha)$ for each $\alpha \in E$, then

(iii) $\text{ran}(B_\alpha^0) \cup \text{ran}(B_\alpha^1)$ is finite,

(iv) for every $\beta < \alpha$, $(A_\alpha^0 \cup A_\alpha^1 \cup B_\alpha^0 \cup B_\alpha^1) \cap (\beta \times \omega)$ is finite,

(v) A_α^0, A_α^1, and $B_\alpha^0 \cup B_\alpha^1$ are pairwise disjoint,

(vi) for every uncountable $X \subset (\omega_1 \times \omega)$ there is a cub $C \subset E$ such that, for every $\alpha \in C$, $X \cap A_\alpha^0$ and $X \cap A_\alpha^1$ are infinite,

(vii) for every pair $\langle X, Y \rangle$ of 2-uncountable subsets of $(\omega_1 \times \omega)^2$, there is an $\alpha \in E$ such that, for every $\beta < \alpha$, $X \cap [(\alpha \setminus \beta) \times \omega]^2 \cap B_\alpha \neq 0$ and $Y \cap B_\alpha \cap [(\alpha \setminus \beta) \times \omega]^2 \neq 0$. □

2.4 Lemma. *$\diamondsuit^*$ implies Δ^*.*

Proof. Let $\langle \mathscr{A}_\alpha\colon \alpha \in \omega_1 \rangle$ be a $\diamondsuit^*$ sequence. Since $\diamondsuit^*$ implies $\diamondsuit$, there is a sequence $\langle \langle B_{\alpha,0}, B_{\alpha,1} \rangle\colon \alpha \in \omega_1 \rangle$ of pairs of subsets of $(\omega_1 \times \omega)^2$ such that, for every pair $\langle X, Y \rangle$ of subsets of $(\omega_1 \times \omega)^2$, there is a stationary $S \subset \omega_1$ such that, for every $\alpha \in S$, $X \cap (\alpha \times \omega)^2 = B_{\alpha,0}$ and $Y \cap (\alpha \times \omega)^2 = B_{\alpha,1}$.

For $\alpha \in E$, let

$$\bar{\mathscr{A}}_\alpha = \{A \in \mathscr{A}_\alpha\colon \forall \beta < \alpha\,(A \cap [(\alpha \setminus \beta) \times \omega] \neq 0)\}$$

and

$$\mathscr{B}_\alpha = \Big\{B_{\alpha,i}\colon i \in 2 \wedge \forall \beta < \alpha\big(B_{\alpha,i} \cap [(\alpha \setminus \beta) \times \omega]^2 \neq 0\big) \wedge \big(\big|\text{ran}(\text{dom}(B_{\alpha,i})) \cup \text{ran}(\text{ran}(B_{\alpha,i}))\big| < \omega\big)\Big\}.$$

List $\bar{\mathscr{A}}_\alpha \cup \mathscr{B}_\alpha$ as $\langle F_n\colon n \in \omega \rangle$ in such a way that each member of $\mathscr{B}_\alpha$ is listed infinitely many times, and that for every $A \in \bar{\mathscr{A}}_\alpha$ there are infinitely many even and infinitely many odd n's for which $F_n = A$.

By induction on $n \in \omega$, pick $p_n \in F_n$ and $\alpha_n \in \alpha$ such that

(1) $\sup_n \alpha_n = \alpha$,

(2) if $k < n$ then $p_k \in (\alpha_n \times \omega)$ or $p_k \in (\alpha_n \times \omega)^2$, and $p_n \in (\alpha \setminus \alpha_n) \times \omega$ or $p_n \in [(\alpha \setminus \alpha_n) \times \omega]^2$.

Let $A_\alpha^0 = \{p_n\colon n \text{ is even and } F_n \in \bar{\mathscr{A}}_\alpha\}$, $A_\alpha^1 = \{p_n\colon n \text{ is odd and } F_n \in \bar{\mathscr{A}}_\alpha\}$, and $B_\alpha = \{p_n\colon F_n \in \mathscr{B}_\alpha\}$.

Now we show that these A's and B's satisfy Definition 2.3. Conditions (i)–(v) hold trivially. For (vi), if $X \subset (\omega_1 \times \omega)$ is uncountable, fix a cub $C_0 \subset E$ so that if $\alpha \in C_0$ then, for every $\beta < \alpha$, $X \cap [(\alpha \setminus \beta) \times \omega] \neq 0$. Let C_1 be a cub such that if $\alpha \in C_1$ then $X \cap (\alpha \times \omega) \in \mathscr{A}_\alpha$. Then $C = C_0 \cap C_1$ shows that (vi) holds.

For (vii), observe that if $X \subset (\omega_1 \times \omega)^2$ is 2-uncountable then there is a $n \in \omega$ such that $X \cap (\omega_1 \times n)^2$ is 2-uncountable. So we may assume that $\langle X, Y \rangle$ from (vii) is such that there is an $n \in \omega$ with $X, Y \subset (\omega_1 \times n)^2$. Fix a cub $C \subset E$ such that if $\alpha \in C$ then, for every $\beta < \alpha$, $X \cap [(\alpha \setminus \beta) \times \omega]^2 \neq 0$ and $Y \cap [(\alpha \setminus \beta) \times \omega]^2 \neq 0$. Let $S \subset \omega_1$ be stationary such that, for every $\alpha \in S$, $X \cap (\alpha \times \omega)^2 = B_{\alpha,0}$ and $Y \cap (\alpha \times \omega)^2 = B_{\alpha,1}$. Then for any $\alpha \in S \cap C$, $X \cap B_\alpha$ and $Y \cap B_\alpha$ are as required. □

2.5 THEOREM (Δ^*). *There is a Hausdorff, first countable, locally compact, shrinking, P-space X such that X^2 is collectionwise normal but not countably paracompact.*

PROOF. We construct two topologies τ^0 and τ^1 on $\omega_1 \times \omega$ in such a way that $T_0 = \langle \omega_1 \times \omega, \tau^0 \rangle$ and $T_1 = \langle \omega_1 \times \omega, \tau^1 \rangle$ are Hausdorff, first countable, locally compact, shrinking, P-spaces, and also make sure that T_0^2, $T_0 \times T_1$, and T_1^2 are collectionwise normal. At the same time we kill countable paracompactness of $T_0 \times T_1$ by showing that the diagonal $\{\langle x, y \rangle \in T_0 \times T_1 : x = y\}$ is Dowker. Then for X we take the free sum of T_0 and T_1.

By induction on $\alpha \in \omega_1$, for $\langle \alpha, n \rangle \in \omega_1 \times \omega$ and $i \in 3$ we construct families $\{U^{i,k}_{\langle \alpha, n \rangle} : k \in \omega\}$ of subsets of $(\alpha + 1) \times \omega$ such that if τ^i_α is the topology on $\alpha \times \omega$ having $\{U^{i,k}_p : k \in \omega \wedge p \in \alpha \times \omega\}$ as a (sub)basis then

(a) τ^i_α is a Hausdorff topology on $\alpha \times \omega$,

(b) for all $\beta < \alpha$, $(\beta + 1) \times \omega$ is closed in $\langle \alpha \times \omega, \tau^i_\alpha \rangle$,

(c) for all $p \in \alpha \times \omega$, $\{U^{i,k}_p : k \in \omega\}$ is decreasing,

(d) for all $p \in \alpha \times \omega$, $\{U^{i,k}_p : k \in \omega\}$ is a clopen basis for p consisting of compact sets (in τ^i_α),

(e) for all $p \in \alpha \times \omega$ and $k \in \omega$, $U^{0,k}_p \cup U^{1,k}_p \subset U^{2,k}_p$,

(f) for all $\langle \beta, n \rangle \in \alpha \times \omega$, $U^{0,0}_{\langle \beta, n \rangle} \cap U^{1,0}_{\langle \beta, n \rangle} \subset (\beta + 1) \times (n + 1)$.

Observe that (d) implies that, for all $\beta < \alpha$, $\beta \times \omega$ is open in $\langle \alpha \times \omega, \tau^i_\alpha \rangle$, and that (e) implies that τ^2_α is coarser than both τ^0_α and τ^1_α, i.e. $\tau^2_\alpha \subset \tau^0_\alpha \cap \tau^1_\alpha$.

Topologies τ^i for $i \in 3$ are defined by letting $\bigcup\{\tau^i_\alpha : \alpha \in \omega_1\}$ be a basis for τ^i. The topology τ^2 helps us to carry the induction, and it plays a small role in the proof that $T_0 \times T_1$ is not countably paracompact.

It remains to define $U^{i,k}_p$ for $i \in 3$, $k \in \omega$, and $p \in \omega_1 \times \omega$. If α is a successor or 0, for $i \in 3$ and $n, k \in \omega$ let $U^{i,k}_{\langle \alpha, n \rangle} = \{\langle \alpha, n \rangle\}$. It is easy to check that all induction hypotheses are satisfied.

Let $\langle A^0_\alpha : \alpha \in E \rangle$, $\langle A^1_\alpha : \alpha \in E \rangle$, and $\langle B_\alpha : \alpha \in E \rangle$ be as in Definition 2.3. Recall that $B^0_\alpha = \operatorname{dom}(B_\alpha)$ and $B^1_\alpha = \operatorname{ran}(B_\alpha)$. For α a limit, i.e. $\alpha \in E$, pick $n_\alpha \in \omega$ such that $n_\alpha > \sup(\operatorname{ran}(B^0_\alpha) \cup \operatorname{ran}(B^1_\alpha))$. Notice that $n_\alpha > 0$. If $n \notin \{0, n_\alpha\}$, for $i \in 3$ and $k \in \omega$ let $U^{i,k}_{\langle \alpha, n \rangle} = \{\langle \alpha, n \rangle\}$. Let $\langle \alpha_k : k \in \omega \rangle$ be an increasing cofinal sequence in α.

The set $D = A^0_\alpha \cup A^1_\alpha \cup B^0_\alpha \cup B^1_\alpha$ is closed and discrete in $\langle \alpha \times \omega, \tau^2_\alpha \rangle$ by 2.3(iv) and the fact that $\beta \times \omega$ is open for every $\beta < \alpha$. Since $\langle \alpha \times \omega, \tau^2_\alpha \rangle$ is metrizable and locally compact, for each $\alpha \in D$ there is a compact clopen neighborhood U_d of d in $\langle \alpha \times \omega, \tau^2_\alpha \rangle$ such that $\{U_d : d \in D\}$ is discrete in $\langle \alpha \times \omega, \tau^2_\alpha \rangle$ (hence discrete in $\langle \alpha \times \omega, \tau^i_\alpha \rangle$ for any $i \in 2$), and such that if $d = \langle \beta, n \rangle$ and $\alpha_k + 1 < \beta$ then $U_d \cap [(\alpha_k + 1) \times \omega] = 0$ (by (b), this can be done). For $i \in 2$ and $d \in D$ let $U^i_d = U^{i,k}_d$, where $k \in \omega$ is the least number for which $U^{i,k}_d \subset U_d$.

For $k \in \omega$ define

$$U^{2,k}_{\langle \alpha, 0 \rangle} = \{\langle \alpha, 0 \rangle\} \cup \Big(\bigcup\big\{U_d : d \in \big[\big(A^0_\alpha \cup A^1_\alpha\big) \backslash (\alpha_k \times \omega)\big]\big\}\Big), \quad \text{and}$$

$$U^{2,k}_{\langle \alpha, n_\alpha \rangle} = \{\langle \alpha, n_\alpha \rangle\} \cup \Big(\bigcup\big\{U_d : d \in \big[\big(B^0_\alpha \cup B^1_\alpha\big) \backslash (\alpha_k \times \omega)\big]\big\}\Big).$$

For $i \in 2$ and $k \in \omega$ define

$$U^{i,k}_{\langle\alpha,0\rangle} = \{\langle\alpha,0\rangle\} \cup \left(\bigcup\left\{U^i_d \colon d \in \left[A^i_\alpha \setminus (\alpha_k \times \omega)\right]\right\}\right), \quad \text{and}$$

$$U^{i,k}_{\langle\alpha,n_\alpha\rangle} = \{\langle\alpha,n_\alpha\rangle\} \cup \left(\bigcup\left\{U^i_d \colon d \in \left[\left(B^0_\alpha \cup B^1_\alpha\right) \setminus (\alpha_k \times \omega)\right]\right\}\right).$$

We now check that the induction hypotheses are satisfied. The condition (a) is satisfied because of 2.3(v), and (b)–(e) are satisfied trivially. We check that (f) holds. For $n \notin \{0, n_\alpha\}$, (f) trivially holds at $\langle\alpha, n\rangle$. Note that $U^{0,0}_{\langle\alpha,0\rangle} \cap U^{1,0}_{\langle\alpha,0\rangle} = \{\langle\alpha,0\rangle\}$ by 2.3(v). To see that $U^{0,0}_{\langle\alpha,n_\alpha\rangle} \cap U^{1,0}_{\langle\alpha,n_\alpha\rangle} \subset (\alpha+1) \times (n_\alpha + 1)$, it is enough to show that $U^0_d \cap U^1_e \subset (\alpha+1) \times (n_\alpha+1)$ for $d, e \in B^0_\alpha \cup B^1_\alpha$. Since $U^0_d \cap U^1_e \subset U_d \cap U_e$, if $d \neq e$, $U^0_d \cap U^1_e = 0$, so assume $d = e = \langle\beta, n\rangle \in B^1_\alpha \cup B^1_\alpha$. But then $U^0_{\langle\beta,n\rangle} \cap U^1_{\langle\beta,n\rangle} \subset U^{0,0}_{\langle\beta,n\rangle} \cap U^{1,0}_{\langle\beta,n\rangle} \subset (\beta+1) \times (n+1)$ by (c) and (f) for $\langle\beta, n\rangle$, so since $n_\alpha > n$ we have that $U^0_{\langle\beta,n\rangle} \cap U^1_{\langle\beta,n\rangle} \subset (\alpha+1) \times (n_\alpha+1)$.

This finishes our construction of T_0 and T_1, and we now check that they have all of the desired properties. From now on i and j will stand for arbitrary members of 2. Trivially, T_i is Hausdorff, first countable, and locally compact.

2.6 Lemma. *Let $\{X_n \colon n \in \omega\}$ be a family of uncountable subsets of T_i. Then there is a cub $C \subset \omega_1$ such that, for every $\alpha \in C$ and $n \in \omega$, $\langle\alpha, 0\rangle$ is an accumulation point of X_n. In particular, $(C \times \{0\}) \subset \bigcap\{\overline{X}_n \colon n \in \omega\}$.*

Proof. For each X_n fix a cub $C_n \subset \omega_1$ satisfying 2.3(vi). Then $C = \bigcap_{n\in\omega} C_n$ is as required. □

(1) *T_i is normal.*

Proof. Let X and Y be two closed disjoint subsets of T_i. By Lemma 2.6 we may assume that X is countable, so there is an $\alpha \in \omega_1$ such that $X \subset (\alpha+1) \times \omega$. Since the subspace topology on $(\alpha+1) \times \omega$ is metrizable there is a clopen set U in $(\alpha+1) \times \omega$ containing X such that $U \cap Y = 0$. This U is clopen in T_i since $(\alpha+1) \times \omega$ is. □

(2) *Every closed discrete subset of T_i is countable.*

Proof. Let $X \subset T_i$ be uncountable. Let X_0, X_1 be two disjoint uncountable subsets of X. By Lemma 2.6, $\overline{X}_0 \cap \overline{X}_1 \neq 0$, hence X is not closed discrete. □

(3) *T_i is countably paracompact.*

Proof. Let $\{F_n \colon n \in \omega\}$ be a decreasing family of closed subsets of T_i such that $\bigcap_{n\in\omega} F_n = 0$. By Lemma 2.6 there is a $k \in \omega$ such that F_k is countable. Let $\alpha \in \omega_1$ be such that $F_k \subset \alpha \times \omega$. Since $\alpha \times \omega$ is open and metrizable, there are, for $n \geqslant k$, open O_n containing F_n such that $\bigcap_{n\geqslant k} O_n = 0$. For $n < k$, let $O_n = T_i$. So T_i is countably paracompact. □

(4) *T_i is shrinking.*

Proof. Since T_i is normal and countably paracompact every open cover of cardinality $\leqslant \omega$ has a shrinking. Let $\{U_\alpha \colon \alpha \in \omega_1\}$ be an open cover of T_i. It suffices to show that there is a closed cover $\{F_\alpha \colon \alpha \in \omega_1\}$ such that $F_\alpha \subset U_\alpha$ for every $\alpha \in \omega_1$.

Define $S = \{t \in T_i: |\{\alpha \in \omega_1: t \in U_\alpha\}| \leqslant \omega\}$. List $T_i \setminus S$ as $\langle t_\alpha: \alpha \in \omega_1\rangle$. For $\alpha \in \omega_1$ pick $f(\alpha) \in \omega_1$ such that $t_\alpha \in U_{f(\alpha)}$ and, for every $\beta < \alpha$, $f(\beta) < f(\alpha)$. Let $F^0_{f(\alpha)} = \{t_\alpha\}$, and if $\beta \in \omega_1$ is not in the range of f, let $F^0_\beta = 0$. Then for every $\alpha \in \omega_1$, $F^0_\alpha \subset U_\alpha$, and $\bigcup_{\alpha \in \omega_1} F^0_\alpha = T_i \setminus S$.

Observe that S is closed since T_i is first countable. For $\alpha \in \omega_1$, let $S_\alpha = S \setminus U_\alpha$.

Claim. There is an $\alpha \in \omega_1$ such that S_α is countable.

PROOF. Assume not. Using Lemma 2.6 fix, for each $\alpha \in \omega_1$, a cub $C_\alpha \subset \omega_1$ such that each point of $C_\alpha \times \{0\}$ is an accumulation point of S_α. Let C be the diagonal intersection of $\{C_\alpha: \alpha \in \omega_1\}$; so $C = \{\beta \in \omega_1: \forall \alpha < \beta\ (\beta \in C_\alpha)\}$ is a cub subset of ω_1. Since each S_α is closed, if $\beta \in C$ then $\langle\beta, 0\rangle \in \bigcap_{\alpha<\beta} S_\alpha$, hence $\langle\beta, 0\rangle \notin \bigcup_{\alpha<\beta} U_\alpha$. So for $\beta \in C$, pick $p_\beta = \langle\alpha_\beta, m_\beta\rangle \in S_0$ and $\gamma_\beta > \beta$ such that $p_\beta \in U_{\gamma_\beta}$ and $\alpha_\beta < \beta$. This can be done since $\langle\beta, 0\rangle$ is an accumulation point of S_0. By the pressing down lemma followed by an easy counting argument there are ω_1 many β's with the same $p_\beta = p$. But then this p is an element of S which is in uncountably many U_α's. So there is an α such that S_α is countable. □

We may assume that $S \setminus U_0$ is countable. Fix a $\beta \in \omega_1$ such that $(S \setminus U_0) \subset \beta \times \omega$, and let $F^1_0 = S \cap [(\omega_1 \setminus \beta) \times \omega] \subset U_0$. Then F^1_0 is closed since S is. For $\alpha > 0$, let $F^1_\alpha = 0$. Since metrizable spaces are shrinking there is a closed cover $\{F^2_\alpha: \omega \in \omega_1\}$ of $(\beta + 1) \times \omega$ such that, for every $\alpha \in \omega_1$, $F^2_\alpha \subset U_\alpha$.

For $\alpha \in \omega_1$, let $F_\alpha = F^0_\alpha \cup F^1_\alpha \cup F^2_\alpha$. Then $\{F_\alpha: \alpha \in \omega_1\}$ is a closed cover of T_i such that $F_\alpha \subset U_\alpha$ for every $\alpha \in \omega_1$. □

At this point we need

2.7 LEMMA. *Let $\langle X, Y\rangle$ be a pair of 2-uncountable subsets of $T_i \times T_j$. Then there is an $\alpha \in \omega_1$ such that $\langle\langle\alpha, n_\alpha\rangle, \langle\alpha, n_\alpha\rangle\rangle \in \overline{X} \cap \overline{Y}$.*

PROOF. Let α satisfy 2.3(vii). Then $\langle\langle\alpha, n_\alpha\rangle, \langle\alpha, n_\alpha\rangle\rangle \in \overline{X} \cap \overline{Y}$. □

(5) *$T_i \times T_j$ is normal.*

PROOF. Let X, Y be two closed disjoint subsets of $T_i \times T_j$. Using Lemma 2.7 we may assume that X is not 2-uncountable. Fix an $\alpha \in \omega_1$ so that $X \subset [((\alpha + 1) \times \omega) \times T_j] \cup [T_i \times ((\alpha + 1) \times \omega)]$. It follows from (8) that both $((\alpha + 1) \times \omega) \times T_j$ and $T_i \times ((\alpha + 1) \times \omega)$ are normal; however, for the convenience of the reader not interested in P-spaces we give a direct proof of this fact in Lemma 2.8 below. Assuming that $((\alpha + 1) \times \omega) \times T_j$ and $T_i \times ((\alpha + 1) \times \omega)$ are normal we show how to separate X and Y by two disjoint open sets.

Sets $[(\omega_1 \setminus (\alpha + 1)) \times \omega] \times [(\omega_1 \setminus (\alpha + 1)) \times \omega]$, $T_i \times ((\alpha + 1) \times \omega)$, and $((\alpha + 1) \times \omega) \times [(\omega_1 \setminus (\alpha + 1)) \times \omega]$ partition $T_i \times T_j$ into pairwise disjoint clopen sets, and in each of them we can separate X and Y (in the last two since they are normal and in the first one since X does not intersect it); hence X and Y can be separated in $T_i \times T_j$.

2.8 LEMMA. *Assume that X is normal and countably paracompact and Y is a countable metric space. Then $X \times Y$ is normal.*

PROOF. For $y \in Y$ let $\{B_n(y): n \in \omega\}$ be a clopen decreasing basis for y. Let H and K be two closed disjoint subsets of $X \times Y$. Let $\pi: X \times Y \to X$ be the projection map. For $y \in Y$ let $F_n(y) = \overline{\pi(H \cap (X \times B_n(y)))} \cap \overline{\pi(K \cap (X \times B_n(y)))}$. Since

$\{B_n(y): n \in \omega\}$ is decreasing, $\{F_n(y): n \in \omega\}$ decreases, and since $\{B_n(y): n \in \omega\}$ is a basis for y, $\bigcap_{n\in\omega} F_n(y) = 0$. Let $\{O_n(y): n \in \omega\}$ be an open family in X such that $F_n(y) \subset O_n(y)$ for every $n \in \omega$ and $\bigcap_{n\in\omega} O_n = 0$.

For $n \in \omega$ and $y \in Y$ let $H_n(y) = \overline{\pi(H \cap (X \times B_n(y)))} \setminus O_n(y)$ and $K_n(y) = \overline{\pi(K \cap (X \times B_n(y)))} \setminus O_n(y)$. Then $H_n(y)$ and $\overline{\pi(K \cap (X \times B_n(y)))}$ are closed disjoint subsets of X, so there is an open $U_n(y)$ such that $H_n(y) \subset U_n(y) \subset \overline{U_n(y)} \subset X \setminus \overline{\pi(K \cap (X \times B_n(y)))}$. Similarly, there is an open $V_n(y)$ such that $K_n(y) \subset V_n(y) \subset \overline{V_n(y)} \subset X \setminus \overline{\pi(H \cap (X \times B_n(y)))}$.

Then $\mathscr{U} = \{U_n(y) \times B_n(y): n \in \omega \wedge y \in Y\}$ and $\mathscr{V} = \{V_n(y) \times B_n(y): n \in \omega \wedge y \in Y\}$ are countable open families in $X \times Y$ covering H and K, respectively, such that, for every $U \in \mathscr{U}$ and $V \in \mathscr{V}$, $(\overline{U} \cap K = 0 = H \cap \overline{V})$. Hence $X \times Y$ is normal. □

(6) *$T_i \times T_j$ is collectionwise normal.*

Proof. We show that every closed discrete subset of $T_i \times T_j$ is countable. This shows that $T_i \times T_j$ is collectionwise normal since it is normal. Let $X \subset T_i \times T_j$ be a 2-uncountable subset of $T_i \times T_j$. By an easy induction one picks two disjoint 2-uncountable $X_0, X_1 \subset X$. By Lemma 2.7, $\overline{X}_0 \cap \overline{X}_1 \neq 0$ so X is not closed discrete. Let X be an uncountable subset of $T_i \times T_j$ which is not 2-uncountable. Then there is a $p \in \omega_1 \times \omega$ such that either $X \cap (T_i \times \{p\})$ or $X \cap (\{p\} \times T_j)$ is uncountable. Then, by (2), X is not closed discrete. □

(7) *$T_0 \times T_1$ is not countably paracompact.*

Proof. We show that $R = \{\langle x, y\rangle \in T_0 \times T_1: x = y\}$ is not countably paracompact. As R is a closed subset of $(\langle \omega_1 \times \omega, \tau^2\rangle)^2$ it is closed in $T_0 \times T_1$, hence $T_0 \times T_1$ is not countably paracompact.

Since R is homeomorphic to $\omega_1 \times \omega$ having as a basis the family $\{U \cap V: U \in \tau^0 \wedge V \in \tau^1\}$, we identify R with this space.

For $n \in \omega$, let $F_n = \omega_1 \times (\omega \setminus n)$. Trivially, $F_{n+1} \subset F_n$ for $n \in \omega$, and $\bigcap_{n\in\omega} F_n = 0$. The condition (f) implies that each F_n is closed. Next we show that there is no open family $\{O_n: n \in \omega\}$ such that $\bigcap_{n\in\omega} O_n = 0$ and, for all $n \in \omega$, $F_n \subset O_n$. We need

2.9 **Lemma.** *Let X be an uncountable subset of R and $n \in \omega$. Then there are an $\alpha \in \omega_1$ and an $m > n$ such that $\langle \alpha, m\rangle \in \overline{X}$.*

Proof. Let $X_0 = \{\langle x, x\rangle \in T_0 \times T_1: x \in X\}$ and $X_1 = \{\langle\langle \beta, n\rangle, \langle \beta, n\rangle\rangle: \beta \in \omega_1\}$. Both X_0 and X_1 are 2-uncountable subsets of $T_0 \times T_1$. By Lemma 2.7 there is an $\alpha \in \omega_1$ such that $\langle\langle \alpha, n_\alpha\rangle, \langle \alpha, n_\alpha\rangle\rangle \in \operatorname{cl}_{T_0\times T_1}(X_0) \cap \operatorname{cl}_{T_0\times T_1}(X_1)$. But then $\langle \alpha, n_\alpha\rangle \in \operatorname{cl}_R X$ and $n_\alpha > n$. □

Assume that $\{O_n: n \in \omega\}$ is an open family in R for which $\bigcap_{n\in\omega} O_n = 0$ and $F_n \subset O_n$ for every $n \in \omega$. There is an $n \in \omega$ such that $R \setminus O_n$ is uncountable. Observe that $R \setminus O_n$ is closed and $R \setminus O_n \subset \omega_1 \times n$. By Lemma 2.9 there are an $\alpha \in \omega_1$ and an $m > n$ such that $\langle \alpha, m\rangle \in \overline{R \setminus O_n} = R \setminus O_n \subset \omega_1 \times n$. This contradiction finishes the proof. □

(8) *T_i is a P-space.*

Proof. This follows from Lemma 2.6 and Lemma 2.11 below. In order to be able to use Lemma 2.11 in §4 we state it in a more general form than needed at this point.

2.10 Definition. Let κ be a cardinal. A space X is a *π_κ-space* iff, for every paracompact space Y with character $\leqslant \kappa$, $X \times Y$ is normal. □

The following is Lemma 5 from [2].

2.11 Lemma [2]. *Let X be a Hausdorff space and $\mathscr{F}$ a family of subsets of X such that $\bigcap\{\bar{F}_\alpha\colon \alpha \in \kappa\} \neq 0$ for any $\{F_\alpha\colon \alpha \in \kappa\} \subset \mathscr{F}$ of cardinality $\leqslant \kappa$. Also suppose that $M \subset X$ and $M \notin \mathscr{F}$ implies there is a π_κ-space U which is clopen in X and contains M. Then X is a π_κ-space.* □

We show that T_i is a π_ω-space and thus a P-space. Let $\mathscr{F} = \{X \subset T_i\colon X$ is uncountable$\}$. By Lemma 2.6 any countable subfamily $\{F_n\colon n \in \omega\}$ of $\mathscr{F}$ is such that $\bigcap\{\bar{F}_n\colon n \in \omega\} \neq 0$. If X is a countable subset of T_i then, for some $\alpha \in \omega_1$, $X \subset (\alpha + 1) \times \omega$ and, by Lemma 2.8, $(\alpha + 1) \times \omega$ is a π_ω-space. □

2.12 Remarks. Lemma 2.8 follows from [9]; see also [14, 4.13]. We included a short proof of it for the convenience of the reader.

One does not need Lemma 2.8 to show that T_i is a P-space. Lemma 2.11 holds if κ is replaced by ω and π_κ by P. But then $(\alpha + 1) \times \omega$ trivially is a P-space.

We should also note that Lemma 5 from [2] (our Lemma 2.11) is stated there only for the case of π_ω-spaces, but the same proof as the one given there works for arbitrary κ.

The space X^2 from Theorem 2.5 is strongly zero dimensional. □

3. Products with more than two factors. Here we show how to generalize the construction from the preceding section to get, for each $n \in \omega$, a space X such that X^n is countably paracompact and X^{n+1} is Dowker.

We use a slightly more general consequence of $\diamondsuit^*$ which we state for arbitrary regular $\kappa \geqslant \omega$ in order to be able to use it in §4.

3.1. Definition. Let κ be a cardinal and $n \in \omega$. A set $X \subset (\kappa^+ \times \kappa)^n$ *has the n-size-κ^+* iff, for every $\alpha \in \kappa^+$, $X \cap [(\kappa^+ \setminus \alpha) \times \kappa]^n \neq 0$. □

Let $\kappa \geqslant \omega$ be a regular cardinal. Define $E = \{\alpha \in \kappa^+\colon \mathrm{cf}(\alpha) = \kappa\}$. For $k \in \omega$ and $l \in k$, let $\pi_l\colon (\kappa^+ \times \kappa)^k \to (\kappa^+ \times \kappa)$ be the projection map onto the lth coordinate.

3.2 Definition. Let $\kappa \geqslant \omega$ be a regular cardinal and $n \in \omega$. $\Delta^*(\kappa^+, n)$ is the statement: There are sequences $\langle \mathscr{A}^i_\alpha\colon \alpha \in E\rangle$ for $i \in n + 1$ and a sequence $\langle \mathscr{B}_\alpha\colon \alpha \in E\rangle$ such that, for $\alpha \in E$,

(i) $\mathscr{A}^i_\alpha$ is a subset of $(\alpha \times \kappa)^n$ for each $i \in n + 1$,

(ii) $\mathscr{B}_\alpha$ is a subset of $(\alpha \times \kappa)^{n+1}$.

If for $\alpha \in E$ and $i \in n + 1$, $A^i_\alpha = \bigcup\{\pi_l(\mathscr{A}^i_\alpha)\colon l \in n\}$ and $B_\alpha = \bigcup\{\pi_l(\mathscr{B}_\alpha)\colon l \in n + 1\}$, then, for $\alpha \in E$,

(iii) $|\mathrm{ran}(B_\alpha)| < \kappa$,

(iv) $|[(\bigcup_{i\in n+1} A^i_\alpha) \cup B_\alpha] \cap (\beta \times \kappa)| < \kappa$ for every $\beta < \alpha$,

(v) the family $\{A^i_\alpha\colon i \in n + 1\} \cup \{B_\alpha\}$ consists of pairwise disjoint sets,

(vi) there is a cub $C \subset \kappa^+$ for every $X \subset (\kappa^+ \times \kappa)^n$ which has n-size-κ^+ such that, for every $\beta \in C \cap E$ and every $i \in n + 1$, if $\gamma < \beta$, $X \cap \mathscr{A}^i_\beta \cap [(\beta \setminus \gamma) \times \kappa]^n \neq 0$,

(vii) for every family $\mathscr{F}$ of cardinality $< \kappa$ whose members are subsets of $(\kappa^+ \times \kappa)^{n+1}$ of $(n + 1)$-size-κ^+, there is a $\beta \in E$ such that, for every $X \in \mathscr{F}$ and $\gamma < \beta$, $X \cap \mathscr{B}_\beta \cap [(\beta \setminus \gamma) \times \kappa]^{n+1} \neq 0$. □

3.3 THEOREM ($\Delta^*(\omega_1, n)$). *There is a Hausdorff, first countable, locally compact space X such that X^n is a shrinking P-space, and X^{n+1} is collectionwise normal but not countably paracompact.*

PROOF. The construction is almost the same as the one in Theorem 2.5. We construct $n + 2$ topologies τ^i on $\omega_1 \times \omega$. Then we let $T_i = \langle \omega_1 \times \omega, \tau^i \rangle$ for $i \in n + 1$, and define X to be the free sum of the T_i's.

For each $p \in \omega_1 \times \omega$ and $i \in n + 2$ we construct a neighborhood basis $\{U_p^{i,k}: k \in \omega\}$ for p in τ^i as in 2.5. Conditions (a)–(d) are the same as (a)–(d) from 2.5. We also have

(e) for all $p \in \alpha \times \omega$ and $k \in \omega$, $\bigcup_{i \in n+1} U_p^{i,k} \subset U_p^{n+1,k}$,

(f) for all $\langle \beta, m \rangle \in \alpha \times \omega$, $\bigcap_{i \in n+1} U_{\langle \beta, m \rangle}^{i,0} \subset (\beta + 1) \times (m + 1)$.

Let $D = B_\alpha \cup (\bigcup_{i \in n+1} A_\alpha^i)$ and pick $n_\alpha > \sup(\operatorname{ran}(B_\alpha))$. The construction of the U's is the same as in 2.5 except that for $\langle \alpha, 0 \rangle$ if $i \in n + 1$ and $k \in \omega$ we define

$$U_{\langle \alpha, 0 \rangle}^{i,k} = \{\langle \alpha, 0 \rangle\} \cup \Big(\bigcup \Big\{ U_d^i : d \in \Big[\Big(\bigcup \{ A_\alpha^j : j \in n + 1 \wedge j \neq i \} \Big) \setminus (\alpha_k \times \omega) \Big] \Big\} \Big).$$

Trivially each T_i is Hausdorff, first countable and locally compact. Let T^k stand for an arbitrary product of k (not necessarily different) factors from $\{T_i: i \in n + 1\}$.

3.4 LEMMA. (i) *Let $k \leqslant n$ and let $\{X_j: j \in \omega\}$ be a family of k-size-ω_1 subsets of T^k. Then there is a cub $C \subset \omega_1$ such that, for every $\alpha \in C \cap E$ and $j \in \omega$, the point $p \in T^k$ having all its coordinates equal to $\langle \alpha, 0 \rangle$ is an accumulation point of X_j.*

(ii) *Let $\mathscr{F}$ be a finite collection of $(n + 1)$-size-ω_1 subsets of T^{n+1}. Then there is an $\alpha \in E$ such that the point $p \in T^{n+1}$ with all its coordinates equal $\langle \alpha, n_\alpha \rangle$ is in $\bigcap \{ \overline{X}: X \in \mathscr{F} \}$.* □

By induction on $k \leqslant n$ one shows (simultaneously) that T^k is normal and countably paracompact. This is similar to 2.5(1), (3), and (6); by Lemma 3.4(i) every countable family of closed subsets of T^k of k-size-ω_1 has a nonempty intersection and Lemma 2.8 (together with the induction hypothesis) shows that any subset of T^k which does not have k-size-ω_1 is contained in a clopen, normal, and countably paracompact subspace of T^k. (For countable paracompactness, observe that by applying Lemma 2.8 to X and $Y \times (\omega + 1)$ one concludes that $X \times Y$ is normal and countably paracompact for normal countably paracompact X and metric countable Y.) Then the normality of T^{n+1} follows from Lemma 3.4(ii), and since every closed discrete subset of T^{n+1} is countable, T^{n+1} is collectionwise normal. The diagonal R of $\prod_{i \in n+1} T_i$ is a closed Dowker subspace of $\prod_{i \in n+1} T_i$ since by (f) and Lemma 3.4(ii) the same argument as in 2.5(7) applies for R. By Lemma 2.11 and induction on $k \leqslant n$, T^k is a P-space for $k \leqslant n$ (for this note that the product of a P-space and a metric space is a P-space).

We show that T^k is shrinking by induction on $k \leqslant n$. The proof is similar to 2.5(4) so we use the same notation. Some S_α is not of k-size-ω_1. Assume not. Then by Lemma 3.4(i) we may assume that each S_α is a subset of the diagonal of T^k. Proceed as in 2.5(4). To finish the proof that T^k is shrinking we use

3.5 LEMMA. *Let X be shrinking and Y a countable metric space. Then $X \times Y$ is shrinking.*

PROOF. For $y \in Y$ let $\{B_i(y): i \in \omega\}$ be a clopen decreasing basis for y. Let $\{U_\alpha: \alpha \in \lambda\}$ be an open cover of $X \times Y$. For $\alpha \in \lambda$, $i \in \omega$, and $y \in Y$, let $U_\alpha^i(y)$ be the maximal open set in X such that $U_\alpha^i(y) \times B_i(y) \subset U_\alpha$, and define $F_i(y) = X \setminus \bigcup\{U_\alpha^i(y): \alpha \in \lambda\}$. By the countable paracompactness of X there are open $U_i(y) \supset F_i(y)$ for $i \in \omega$, with $\bigcap_{i \in \omega} U_i(y) = 0$. Let $\{V_\alpha^i(y): \alpha \in \lambda\} \cup \{V_i(y)\}$ be a shrinking of $\{U_\alpha^i(y): \alpha \in \lambda\} \cup \{U_i(y)\}$. Then $\{V_\alpha^i(y) \times B_i(y): \alpha \in \lambda \wedge i \in \omega \wedge y \in Y\}$ is an open cover of $X \times Y$ such that, for every $y \in Y$, $i \in \omega$, and $\alpha \in \lambda$, $\overline{V_\alpha^i(y) \times B_i(y)} \subset U_\alpha$. Since $X \times Y$ is countably paracompact, the following trivial fact [**1**, 3.1] finishes the proof:

3.6 LEMMA [**1**]. *Let X be κ-paracompact, and $\{U_\alpha: \alpha \in \lambda\}$ and $\{V_{\alpha,\beta}: \alpha \in \lambda \wedge \beta \in \kappa\}$ two open covers of X such that, for every $\alpha \in \lambda$ and $\beta \in \kappa$, $\overline{V}_{\alpha,\beta} \subset U_\alpha$. Then $\{U_\alpha: \alpha \in \lambda\}$ has a shrinking.* □

3.7 REMARKS. It is known that if $\prod_{i \in \omega} T_i$ is normal and $\prod_{i \leqslant k} T_i$ is countably paracompact (shrinking) for every $k \in \omega$, then $\prod_{i \in \omega} T_i$ is countably paracompact [**14**, 6.1] (shrinking [**1**, 3.4]). So Theorem 3.3 can not be pushed up to ω.

Lemma 3.5 is a corollary of [**1**, 3.6]. We gave a direct proof of it since we need it in the next section. □

4. κ-Dowker products. Assume that κ is a regular uncountable cardinal. Recall that a space is called a P_κ-space iff any intersection of $< \kappa$ open sets is open. P_{ω_1} spaces are usually called P-spaces, but here a P-space is a space which has the normal product with each metrizable space.

4.1 THEOREM ($\Delta^*(\kappa^+, n)$). *There is a Hausdorff P_κ-space X of character κ such that X^n is shrinking and κ-paracompact, and X^{n+1} is a collectionwise normal, $< \kappa$-paracompact, κ-Dowker, P-space.*

PROOF. The proof is almost verbatim as Theorem 3.3. Here κ plays the role of ω (hence "finite" becomes "$< \kappa$"). We need [**2**, Lemma 6]. To state it, recall that a (clopen) basis $\mathscr{B}$ for a space X is called non-Archimedean iff, for all A and B in $\mathscr{B}$, either $A \subset B$ or $B \subset A$ or $A \cap B = 0$.

4.2 LEMMA [**2**]. *Assume that $Y = \{y_\alpha: \alpha \in \kappa\}$ is a Hausdorff space with a non-Archimedean basis $\mathscr{B}$ such that for every $\alpha \in \kappa$ there is a $B_\alpha \in \mathscr{B}$ so that*

(i) $y_\alpha \in B_\alpha$,

(ii) $B_\alpha \cap \{y_\beta: \beta < \alpha\} = 0$, *and*

(iii) $\{\beta < \alpha: y_\alpha \in B_\beta\}$ *is finite.*

Then for every paracompact space X, $X \times Y$ is paracompact. □

Now we check that, for all $\beta \in \kappa^+$ and $i \in n+1$, $Y = \langle \beta \times \kappa, \tau_\beta^i \rangle$ satisfies the conditions of Lemma 4.2. Being a regular P_κ-space of weight $\leqslant \kappa$, Y has a non-Archimedean basis $\mathscr{B}$. Let $Y = \{y_\alpha: \alpha \in \kappa\}$ and pick $B_\alpha \in \mathscr{B}$ so that if $y_\alpha = \langle \gamma, \delta \rangle$ then $B_\alpha \subset (\gamma \times \delta) \cup \{\langle \gamma, \delta \rangle\}$ and $B_\alpha \cap \{y_\beta: \beta < \alpha\} = 0$. These B_α's satisfy 4.2(iii), since if (iii) fails for some $\alpha \in \kappa$ there is an increasing sequence $\langle \alpha_i: i \in \omega \rangle$ with $y_\alpha \in B_{\alpha_i}$ for $i \in \omega$; hence $B_{\alpha_{i+1}} \subset B_{\alpha_i}$. So if $y_{\alpha_i} = \langle \gamma_i, \delta_i \rangle$, then $\gamma_i > \gamma_{i+1}$, a contradiction.

Also an obvious analogue of Lemma 3.4 holds (instead of ω write κ and instead of "finite" write $< \kappa$). We call this new version of Lemma 3.4 Lemma 4.3.

By induction on $k \leqslant n$ we show that T^k is π_κ (Definition 2.10), κ-paracompact and shrinking.

Lemma 4.2 implies that if $\alpha \in \kappa^+$ then $\langle \alpha \times \kappa, \tau_\alpha^i \rangle \times X$ is paracompact for every paracompact space X. Hence if T^k is a π_κ-space and $\alpha \in \kappa^+$, then $T^k \times \langle \alpha \times \kappa, \tau_\alpha^i \rangle$ is a π_κ-space for any $i \in n + 1$.

In order to apply Lemma 2.11, let $\mathscr{F} = \{ X \subset T^k \colon X \text{ has } k\text{-size-}\kappa^+ \}$. The induction hypothesis, the preceding paragraph, and Lemma 4.3(i) show that $\mathscr{F}$ satisfies the hypothesis of 2.11, so T^k is a π_κ-space if $k \leqslant n$.

Observe that any π_κ-space is κ-paracompact. To see this one can use a theorem of Kunen which says that if $\kappa + 1$ has the order topology then $X \times (\kappa + 1)$ is normal iff X is κ-paracompact (for a proof see [**14**, 3.7 or **19**, Theorem 2]) or use a theorem of Morita: $X \times 2^\kappa$ is normal iff X is κ-paracompact [**8**, 2.4]; see also [**4**, 3.8]. Hence for $\kappa \leqslant n$, T^k is κ-paracompact.

To see that T^k is shrinking observe first that the proof of Lemma 3.5 shows that, for every $\alpha \in \kappa^+$, $i \in n + 1$, and $k \leqslant n$, $T^k \times \langle \alpha \times \kappa, \tau_\alpha^i \rangle$ is shrinking provided that T^k is. Then the same argument as in 3.3 shows T^k shrinking for $k \leqslant n$.

Using Lemma 4.3(ii), one shows that T^{n+1} is $< \kappa$-paracompact and that $\prod_{i \in n+1} T_i$ is κ-Dowker. To see T^{n+1} is a P-space use Lemma 2.11 for $\kappa = \omega$, and Lemma 4.3(ii). □

4.4 REMARK. The spaces X^{n+1} from this and the preceding section are strongly zero dimensional. □

5. Additional remarks. Ken Kunen observed that $\diamondsuit$ suffices in order to construct our examples. Instead of 2.3(vi) it is enough to have the following:

(*) There is a σ-complete, normal filter $\mathscr{F}$ on ω_1 consisting of stationary sets such that for every uncountable $X \subset (\omega_1 \times \omega)$ there is an $F \in \mathscr{F}$ so that, for every $\alpha \in F$, $X \cap A_\alpha^0$ and $X \cap A_\alpha^1$ are infinite.

To get (*) one uses the following unpublished result of Kunen.

5.1 THEOREM (KUNEN). *$\diamondsuit$ implies that there is a σ-complete, normal filter $\mathscr{F}$ on ω_1 containing the cub filter, and a sequence $\langle \mathscr{A}_\alpha \colon \alpha \in \omega_1 \rangle$ such that*

(i) *each $\mathscr{A}_\alpha$ is a countable family of subsets of α,*

(ii) *for every $X \subset \omega_1$, $\{\alpha \in \omega_1 \colon X \cap \alpha \in \mathscr{A}_\alpha\} \in \mathscr{F}$.*

PROOF. Let $\langle A_\alpha \colon \alpha \in \omega_1 \rangle$ be a $\diamondsuit$-sequence on $\omega_1 \times \omega_1$; so each A_α is a subset of $\alpha \times \alpha$ and, for every $X \subset \omega_1 \times \omega_1$, $\{\alpha \in \omega_1 \colon X \cap (\alpha \times \alpha) = A_\alpha\}$ is a stationary subset of ω_1.

For $\alpha \in \omega_1$ let $\mathscr{A}_\alpha = \{\text{dom}(A_\alpha \cap (\alpha \times \{\beta\})) \colon \beta \in \alpha\}$; and let $\mathscr{E} = \{S \subset \omega_1 \colon \exists X \subset \omega_1 \, (S = \{\alpha \in \omega_1 \colon X \cap \alpha \in \mathscr{A}_\alpha\})\}$.

We show that if $\{S_\alpha \colon \alpha \in \omega_1\} \subset \mathscr{E}$ then $\Delta_{\alpha \in \omega_1} S_\alpha = \{\beta \in \omega_1 \colon \forall \alpha < \beta \, (\beta \in S_\alpha)\}$ is stationary. This shows the existence of the required $\mathscr{F}$.

Fix $\{X_\alpha \colon \alpha \in \omega_1\}$ so that for every $\alpha \in \omega_1$, $S_\alpha = \{\beta \in \omega_1 \colon X_\alpha \cap \beta \in \mathscr{A}_\beta\}$. Define $X = \bigcup \{X_\alpha \times \{\alpha\} \colon \alpha \in \omega_1\} \subset \omega_1 \times \omega_1$, and let $S = \{\alpha \in \omega_1 \colon X \cap (\alpha \times \alpha) = A_\alpha\}$. Then S is stationary, and $S \subset \Delta_{\alpha \in \omega_1} S_\alpha$. □

Theorem 5.1 generalizes to show that if κ is regular and $E = \{\alpha \in \kappa^+ : \mathrm{cf}(\alpha) = \kappa\}$ then $\diamondsuit(E)$ implies $\Delta^*(\kappa^+, n)$ for all $n \in \omega$ (with the appropriate version of $(*)$ instead of 3.2(vi)).

It is easy to get Δ^* by forcing; add ω_1 Cohen reals. A similar forcing argument gives $\Delta^*(\kappa^+, n)$. This shows that Δ^* does not imply CH. Forcing with countable partial functions from ω_2 to 2 destroys Δ^*; hence CH does not imply Δ^*.

MA + $\neg$CH implies $\neg\Delta^*$. To see this recall that if $\mathscr{A}$ is a family of cardinality ω_1 consisting of countable subsets of ω_1 such that, for every $A \neq B \in \mathscr{A}$, $A \cap B$ is finite then MA_{ω_1} implies that there is an uncountable $X \subset \omega_1$ such that $X \cap A$ is finite for every $A \in \mathscr{A}$ [**22**, Theorem 1]. So under MA_{ω_1} the condition 2.3(vi) cannot hold (this can be shown directly, without using [**22**]).

If κ is regular one can force: $\forall\, n \in \omega\ \neg\Delta^*\ (\kappa^+, n)$.

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