

ISOMETRIC HOMOTOPY IN CODIMENSION TWO

BY

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ABSTRACT. This article investigates the structure of the space of isometric immersions from a simply connected n -dimensional Riemannian manifold with positive sectional curvatures into $(n + 2)$ -dimensional Euclidean space E^{n+2} . It is proven that if $n \geq 4$ and M^n is such a manifold which admits a C^∞ isometric immersion as a hypersurface in E^{n+1} , then any C^∞ isometric immersion from M^n into E^{n+2} is C^{2n-4} homotopic through isometric immersions to an immersion whose image lies in some hyperplane.

1. Introduction. According to well-known rigidity theorems, if M^n is an n -dimensional compact Riemannian manifold with positive sectional curvatures ($n \geq 2$), then any two isometric immersions from M^n into $(n + 1)$ -dimensional Euclidean space E^{n+1} differ by a rigid motion. In particular, the n -dimensional sphere S^n of constant curvature one is rigid in E^{n+1} . Now rigidity is a very "unstable" phenomenon: S^n is not rigid in E^{n+2} because there is an infinite-dimensional family of composition isometric immersions

$$S^n \subseteq E^{n+1} \rightarrow E^{n+2},$$

where E^{n+1} is isometrically immersed in E^{n+2} as a cylinder. Nevertheless, Whitt [W] was able to show that some vestige of rigidity remains in codimension two; he argued that any isometric immersion from S^n into E^{n+2} is homotopic through isometric immersions to an immersion whose image lies in some hyperplane of E^{n+2} , when $n \geq 4$. Although Whitt did not establish differentiability of the isometric homotopy, Whitt's work on this topic is very important because it suggests a new viewpoint for rigidity theory in codimension greater than one.

This article will prove an extension of Whitt's theorem including differentiability up to order $2n - 4$.

THEOREM. *Suppose that $n \geq 4$ and k is an integer, $1 \leq k \leq 2n - 4$. If M^n is a simply connected n -dimensional Riemannian manifold with positive sectional curvatures which possesses a C^{k+3} isometric immersion into E^{n+1} , then any C^{k+3} isometric immersion from M^n into E^{n+2} is C^k homotopic through isometric immersions to an immersion whose image lies in some hyperplane of E^{n+2} .*

In particular, any smooth isometric immersion from S^n into E^{n+2} is unknotted, when $n \geq 4$.

It is conjectured that a C^∞ version of the above theorem is true, but although Whitt was kind enough to credit us with a proof in the C^∞ case, we have not been able to construct such a proof. The difficulty occurs at the so-called "umbilic

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points”, points where the second fundamental form fails to generate the entire normal space. In this regard, we call attention to Henke’s important investigation [H2] regarding the possibility of extending an isometric immersion from S^n into E^{n+2} to an isometric immersion from an open subset of E^{n+1} containing S^n into E^{n+2} ; Henke found counterexamples to local extendability near umbilic points, and his work suggests that there may be difficulties in proving smoothness of isometric homotopies near these points.

Aside from differentiability, the geometric ideas needed to prove our theorem are contained in §2. Our differentiability results are established in §4 by means of the cone lemma of §3.

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2. The main ideas of the proof. By hypothesis, M^n possesses a C^{k+3} isometric immersion into E^{n+1} ; call this immersion \tilde{f} . We will show that if f is an arbitrary C^{k+3} isometric immersion from M^n into E^{n+2} , then f is C^k homotopic through isometric immersions to $\phi \circ i \circ \tilde{f}$, where $\phi: E^{n+2} \rightarrow E^{n+2}$ is a rigid motion and $i: E^{n+1} \rightarrow E^{n+2}$ is the inclusion map.

If $p \in M$, let

$$\alpha: T_p M \times T_p M \rightarrow N_p M \quad \text{and} \quad \tilde{\alpha}: T_p M \times T_p M \rightarrow \tilde{N}_p M$$

denote the second fundamental forms of f and \tilde{f} respectively, and let $W = N_p M \oplus \tilde{N}_p M$. Define a symmetric bilinear map $\beta: T_p M \times T_p M \rightarrow W$ by $\beta = \alpha \oplus \tilde{\alpha}$, and use the positive definite inner products in $N_p M$ and $\tilde{N}_p M$ to define a Lorentz inner product $\langle \cdot, \cdot \rangle$ on the three-dimensional vector space W ,

$$\langle u \oplus \tilde{u}, v \oplus \tilde{v} \rangle = \langle u, v \rangle - \langle \tilde{u}, \tilde{v} \rangle.$$

It follows immediately from the Gauss equations for f and \tilde{f} that

$$\langle \beta(x, z), \beta(y, w) \rangle - \langle \beta(x, w), \beta(y, z) \rangle = 0 \quad \text{for all } x, y, z, w \in T_p M,$$

and thus in the terminology of [M], β is flat with respect to $\langle \cdot, \cdot \rangle$.

We claim that because of our curvature assumption on M , there exist orthonormal bases (e_{n+1}, e_{n+2}) and (\tilde{e}_{n+1}) for $N_p M$ and $\tilde{N}_p M$ respectively such that

$$(1) \quad \langle \alpha, e_{n+1} \rangle = \langle \tilde{\alpha}, \tilde{e}_{n+1} \rangle \quad \text{and} \quad \langle \alpha, e_{n+2} \rangle \text{ has rank } \leq 1.$$

Indeed, if $x \in T_p M$, we define $\beta(x): T_p(M) \rightarrow W$ by setting $\beta(x)(y) = \beta(x, y)$. Choose $x \in T_p M$ so that $\beta(x)(T_p M)$ has maximum possible dimension; then $\beta(y)(T_p M)$ will vary smoothly with y for y sufficiently close to x . Let

$$N(\beta(x)) = \{n \in T_p M: \beta(x)(n) = 0\},$$

and check that if $n \in N(\beta(x))$,

$$\beta(y, n) = \frac{1}{t} \beta(x + ty, n) \in \lim_{t \rightarrow 0} \beta(x + ty)(T_p M) = \beta(x)(T_p M),$$

whereas since β is flat,

$$\langle \beta(y, n), \beta(x)(T_p M) \rangle = \langle \beta(x, n), \beta(y)(T_p M) \rangle = 0,$$

Thus if the restriction of the Lorentz inner product $\langle \cdot, \cdot \rangle$ to $\beta(x)(T_p M)$ were nondegenerate, it would follow that

$$\begin{aligned} n \in N(\beta(x)) &\Rightarrow \beta(y, n) = 0 \quad \text{for all } y \in T_p M \\ &\Rightarrow (\text{via Gauss equation}) R(n, y, z, w) = 0 \quad \text{for all } y, z, w \in T_p M, \end{aligned}$$

where R is the Riemann–Christoffel curvature tensor of M^n . Since $n \geq 4$, and hence $N(\beta(x)) \neq 0$, this contradicts our hypothesis that all the sectional curvatures of M are positive.

We conclude that the restriction of $\langle \cdot, \cdot \rangle$ to $\beta(x)(T_p M)$ is degenerate, and hence there exist unit length vectors $e_{n+1} \in N_p M$ and $\tilde{e}_{n+1} \in \tilde{N}_p M$ such that $(e_{n+1}, \tilde{e}_{n+1})$ is a null vector in $\beta(x)(T_p M)$ satisfying

$$\langle (e_{n+1}, \tilde{e}_{n+1}), \beta(x)(T_p M) \rangle = 0.$$

If $n \in N(\beta(x))$,

$$\begin{aligned} \beta(n)(T_p M) \subseteq \beta(x)(T_p M) &\Rightarrow \langle (e_{n+1}, \tilde{e}_{n+1}), \beta(n)(T_p M) \rangle = 0 \\ &\Rightarrow \langle \alpha(n, y), e_{n+1} \rangle = \langle \tilde{\alpha}(n, y), \tilde{e}_{n+1} \rangle \quad \text{for all } y \in T_p M. \end{aligned}$$

Now choose $e_{n+2} \in N_p M$ perpendicular to e_{n+1} , and it is straightforward to verify (1) via the Gauss equation.

Orient M so that the corresponding unit normal \tilde{e}_{n+1} to the immersion $\tilde{f}: M \rightarrow E^{n+1}$ has the property that the symmetric bilinear form $\langle \tilde{\alpha}, \tilde{e}_{n+1} \rangle$ is positive definite. The orientation for M determines an orientation of the normal bundle NM of the immersion $f: M \rightarrow E^{n+2}$. Having chosen \tilde{e}_{n+1} , we can specify the orthonormal sections $e_{n+1}, e_{n+2}: M \rightarrow NM$ completely by requiring that $(e_{n+1}(p), e_{n+2}(p))$ be the unique positively oriented orthonormal basis for $N_p M$ such that $(e_{n+1}(p), e_{n+2}(p))$ and $(\tilde{e}_{n+1}(p))$ satisfy condition (1). It is not difficult to check that e_{n+1} and e_{n+2} are continuous sections (using an argument similar to that of [H1, Satz 2]), and indeed we will demonstrate in §4 that e_{n+1} and e_{n+2} are C^k , a fact that we will assume for the remainder of this section.

We will need to make use of the fundamental existence and uniqueness theorem for submanifolds [BC, §10.8]. Let $\text{Iso}(M^n, E^N)$ denote the equivalence classes of isometric immersions from M^n into E^N , two isometric immersion being regarded as equivalent if they differ by a rigid motion of E^N . According to the fundamental existence and uniqueness theorem, under the assumption that M^n is simply connected, elements of $\text{Iso}(M, E^N)$ are in one-to-one correspondence with triples $(E, \nabla^\perp, \alpha)$, where E is an $(N - n)$ -dimensional vector bundle over M with fiber metric $\langle \cdot, \cdot \rangle$ such that $TM \oplus E$ is trivial, ∇^\perp is a metric connection in E , and $\alpha: TM \times TM \rightarrow NM$ is a fiberwise symmetric bilinear map such that ∇^\perp and α satisfy the Cartan structure equations which will be described in the next paragraph. Under this correspondence, E becomes the normal bundle of the isometric immersion, ∇^\perp the normal connection, and α the second fundamental form.

It is easiest to describe the Cartan structure equations by means of a locally defined adapted orthonormal moving frame $(e_1, \dots, e_n, e_{n+1}, \dots, e_N)$; here (e_1, \dots, e_n) are orthonormal sections of TM , whereas (e_{n+1}, \dots, e_N) are orthonormal

sections of E . Our index conventions will be

$$1 \leq I, J, K \leq N; \quad 1 \leq i, j, k \leq n; \quad n + 1 \leq \lambda, \mu, \nu \leq N.$$

We define one-forms $\omega_{ij} = -\omega_{ji}$ and $\omega_{\lambda\mu} = -\omega_{\mu\lambda}$ by

$$(2) \quad \omega_{ij} = \langle e_i, \nabla e_j \rangle, \quad \omega_{\lambda\mu} = \langle e_\lambda, \nabla^\perp e_\mu \rangle,$$

where ∇ is the Levi-Civita connection in TM , ∇^\perp the given connection in E , and we define $\omega_{i\lambda}$ and $\omega_{\lambda i}$ by

$$(3) \quad \omega_{\lambda i} = -\omega_{i\lambda} = \langle e_\lambda, \alpha(e_i, \cdot) \rangle.$$

Then the Cartan structure equations can be formulated as

$$(4) \quad d\omega_{IJ} = -\sum_K \omega_{IK} \wedge \omega_{KJ}.$$

Thus the problem of constructing a one-parameter family of isometric immersions $f_t: M^n \rightarrow E^{n+2}$ such that $f_1 = f$ and $f_0 = \phi \circ i \circ \tilde{f}$ is reduced to the equivalent problem of constructing a one-parameter family $(E, \nabla_t^\perp, \alpha_t)$ satisfying the Cartan structure equations described above, such that ∇_1^\perp and α_1 are the normal connection and second fundamental form of f and ∇_0^\perp , and α_0 are the normal connection and second fundamental form of $\phi \circ i \circ \tilde{f}$. Under the assumption that f and \tilde{f} are C^{k+3} , the ∇_t^\perp and α_t we construct will be C^k and will depend in a C^k fashion upon t , and the resulting homotopy $f_t: M^n \rightarrow E^{n+2}$ will also be C^k .

We now work locally, using an adapted orthonormal frame $(e_1, \dots, e_n, e_{n+1}, e_{n+2})$ such that (e_{n+1}, e_{n+2}) are the special orthonormal sections of the normal bundle NM for f described before. Using this frame we define ω_{IJ} by (2) and (3). Similarly, we define $\tilde{\omega}_{IJ}$ by using an adapted orthonormal frame $(e_1, \dots, e_n, \tilde{e}_{n+1}, \tilde{e}_{n+2})$ for the isometric immersion $i \circ \tilde{f}: M^n \rightarrow E^{n+2}$ such that \tilde{e}_{n+2} is perpendicular to E^{n+1} and \tilde{e}_{n+1} is the unit normal to M^n in E^{n+1} distinguished by the orientation. Then $\tilde{\omega}_{ij} = \omega_{ij}$ and $\tilde{\omega}_{n+2,j} = 0$. Moreover, it follows from (1) that

$$(5) \quad \omega_{n+1,i} = \tilde{\omega}_{n+1,i}, \quad \omega_{n+2,i} \wedge \omega_{n+2,j} = 0.$$

From (5) and the structure equations (4) we see that

$$\begin{aligned} -\sum \omega_{n+1,i} \wedge \omega_{ij} &= -\sum \tilde{\omega}_{n+1,i} \wedge \omega_{ij} = d\tilde{\omega}_{n+1,j} = d\omega_{n+1,j} \\ &= -\sum \omega_{n+1,i} \wedge \omega_{ij} - \omega_{n+1,n+2} \wedge \omega_{n+2,j}, \end{aligned}$$

from which we conclude that

$$(6) \quad \omega_{n+2,n+1} \wedge \omega_{n+2,j} = 0.$$

Now we define $\omega_{IJ}(t)$ by multiplying the last row and column of the matrix (ω_{IJ}) by t :

$$(\omega_{IJ}(t)) = \left(\begin{array}{ccc|cc} & & & \omega_{1,n+1} & t\omega_{1,n+2} \\ & \omega_{ij} & & \vdots & \vdots \\ & & & \omega_{n,n+1} & t\omega_{n,n+2} \\ \hline \omega_{n+1,1} & \cdots & \omega_{n+1,n} & 0 & t\omega_{n+1,n+2} \\ t\omega_{n+2,1} & \cdots & t\omega_{n+2,n} & t\omega_{n+2,n+1} & 0 \end{array} \right)$$

The remarkable fact is that $\omega_{IJ}(t)$ will satisfy the Cartan structure equations (4) for all t . Indeed, the equations

$$d\omega_{ij} = -\sum \omega_{ik} \wedge \omega_{kj} - \omega_{i,n+1} \wedge \omega_{n+1,j} - \omega_{i,n+2} \wedge \omega_{n+2,j}$$

and

$$d\omega_{n+1,i} = -\sum \omega_{n+1,j} \wedge \omega_{ji} - \omega_{n+1,n+2} \wedge \omega_{n+2,i}$$

remain unchanged by (5) and (6), while the equation

$$d\omega_{n+2,I} = -\sum \omega_{n+2,J} \wedge \omega_{JI}$$

is multiplied by t . We define ∇_t^\perp and α_t by

$$\nabla_t^\perp e_\mu = \sum_\lambda e_\lambda \omega_{\lambda\mu}(t), \quad \alpha_t(e_i, e_j) = \sum_\lambda e_\lambda \omega_{\lambda i}(t)(e_j).$$

According to the fundamental existence and uniqueness theorem, there is a C^k one-parameter family of isometric immersions $f_t: M^n \rightarrow E^{n+2}$ such that ∇_t^\perp and α_t are the normal connection and second fundamental form of f_t . We can arrange that $f_1 = f$, and by uniqueness there will be a Euclidean motion $\phi: E^{n+2} \rightarrow E^{n+2}$ such that $f_0 = \phi \circ i \circ \tilde{f}$. The theorem is thus completely proven except for smoothness which will be treated in §§3 and 4.

3. The Cone Lemma. In this section we will prove a technical lemma needed for the proof of differentiability in §4.

Let $\text{Sym}(n \times n)$ denote the vector space of real symmetric $n \times n$ matrices, and let

$$V^{n-1} = \{ A \in \text{Sym}(n \times n) : \exists B \in O(n) \text{ such that } A = BJB^{-1} \},$$

where $O(n)$ is the orthogonal group, and

$$J = \frac{1}{\sqrt{n^2 - n}} \begin{pmatrix} 1 - n & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Note that J is a trace-free matrix, and has length one in terms of the standard inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on $\text{Sym}(n \times n)$ defined by $\langle\langle A, B \rangle\rangle = \text{trace}(AB)$. We will call V^{n-1} the *Veronese manifold*. There is a map $\phi: O(n) \rightarrow V^{n-1}$ defined by $\phi(B) = BJB^{-1}$, which projects to an $O(n)$ -equivariant diffeomorphism

$$O(n)/(O(1) \times O(n-1)) \rightarrow V^{n-1};$$

thus V^{n-1} is diffeomorphic to $(n-1)$ -dimensional real projective space.

When $n \geq 3$, the Veronese manifold V^{n-1} has the important property that $V^{n-1} \cap (-V^{n-1}) = \emptyset$. Thus if we let CV be the cone on V^{n-1} , i.e.,

$$CV = \{ A \in \text{Sym}(n \times n) : A = tv \text{ for some } v \in V^{n-1}, t \in \mathbf{R} \},$$

we can define a continuous map $\tau: CV \rightarrow \mathbf{R}$ by requiring that $\tau(tv) = t$, when $v \in V^{n-1}$. Note that CV is an n -dimensional smooth submanifold of $\text{Sym}(n \times n)$ except for a singularity at the origin, and τ is smooth except at the origin.

CONE LEMMA. *Let r be an integer, $0 \leq r \leq 2n - 3$. If (a, b) is an open interval of real numbers, and $f: (a, b) \rightarrow CV \subseteq \text{Sym}(n \times n)$ is a C^r map, then $\tau \circ f$ is also C^r . Moreover, for each integer k , $1 \leq k \leq r$, there exists a constant C_k such that*

$$(E_k) \quad |(d^k/dt^k)(\tau \circ f)(t)| \leq C_k |(d^k/dt^k)(f)(t)|, \quad t \in (a, b).$$

If $U = \{t \in (a, b) | f(t) \neq 0\}$, then clearly $\tau \circ f$ is C^r on U , since τ is C^∞ except at the vertex of the cone. The proof of the Cone Lemma will be based upon the following

SUBLEMMA. *With the same assumptions as in the Cone Lemma, the estimates (E_k) hold for $t \in U$, $0 \leq k \leq r$.*

It is in the proof of the sublemma, given at the end of this section, that the hypothesis $r \leq 2n - 3$ is needed.

PROOF THAT THE SUBLEMMA IMPLIES THE CONE LEMMA: We will prove the Cone Lemma by induction on r . To start the induction, note that the Cone Lemma is true for $r = 0$ because τ is continuous and $|\tau \circ f| = |f|$.

Assume inductively that the Cone Lemma is true when r is replaced by $r - 1$.

If $f(t_0) = 0$, the standard argument used to prove Taylor's theorem shows that there is a C^{r-1} function $f_1: (a, b) \rightarrow CV$ such that $f(t) = (t - t_0)f_1(t)$, and the fact that τ is homogeneous implies that

$$(\tau \circ f)(t) = (t - t_0)(\tau \circ f_1)(t).$$

By inductive assumption, we can differentiate $\tau \circ f$ and $\tau \circ f_1$ $r - 1$ times, and it follows from the Leibniz rule that

$$(7) \quad (\tau \circ f)^{(r-1)}(t) = (r - 1)(\tau \circ f_1)^{(r-2)}(t) + (t - t_0)(\tau \circ f_1)^{(r-1)}(t),$$

where the exponent in parentheses denotes differentiation. In particular,

$$(\tau \circ f)^{(r-1)}(t_0) = (r - 1)(\tau \circ f_1)^{(r-2)}(t_0).$$

It follows from (7) that

$$\begin{aligned} \frac{(\tau \circ f)^{(r-1)}(t) - (\tau \circ f)^{(r-1)}(t_0)}{t - t_0} &= \frac{(\tau \circ f)^{(r-1)}(t) - (r - 1)(\tau \circ f_1)^{(r-2)}(t_0)}{t - t_0} \\ &= \frac{(r - 1)(\tau \circ f_1)^{(r-2)}(t) - (r - 1)(\tau \circ f_1)^{(r-2)}(t_0)}{t - t_0} + (\tau \circ f_1)^{(r-1)}(t), \end{aligned}$$

which approaches $r(\tau \circ f_1)^{(r-1)}(t_0)$ as t approaches t_0 , because $\tau \circ f_1$ is C^{r-1} . Hence the r th derivative of $\tau \circ f$ at t_0 exists and is given by the formula

$$(\tau \circ f)^{(r)}(t_0) = r(\tau \circ f_1)^{(r-1)}(t_0).$$

Thus $(\tau \circ f)^{(r)}(t)$ exists for every $t \in (a, b)$.

It remains to show that if $t_0 \in (a, b)$, $(\tau \circ f)^{(r)}(t)$ is continuous for t in some neighborhood of t_0 , and the estimate (E_r) holds on this neighborhood. It will be convenient to divide the discussion into two cases, depending on whether $t_0 \in U_r$ or $t_0 \in F_r$, where

$$U_r = \{t \in (a, b) : f^{(k)}(t) \neq 0 \text{ for some } k, 0 \leq k \leq r\}, \quad F_r = (a, b) - U_r.$$

Case I: $t_0 \in U_r$. By the Sublemma, we can assume that $f(t_0) = 0$, so there is a smooth function $f_1: (a, b) \rightarrow CV$ such that $f(t) = (t - t_0)f_1(t)$. Since some derivative of f does not vanish at t_0 , there is an $\epsilon > 0$ such that

$$0 < |t - t_0| < \epsilon \Rightarrow f(t) \neq 0 \ \& \ f_1(t) \neq 0.$$

By the Sublemma again, f_1 is C^r on $\{t: 0 < |t - t_0| < \epsilon\}$ and satisfies the estimate (E_r) on this set, and hence

$$|(t - t_0)(\tau \circ f_1)^{(r)}(t)| \leq C_r |t - t_0| |f_1^{(r)}(t)| \rightarrow 0$$

as $t \rightarrow t_0$. Now we apply the Leibniz rule,

$$(\tau \circ f)^{(r)}(t) = r(\tau \circ f_1)^{(r-1)}(t) + (t - t_0)(\tau \circ f_1)^{(r)}(t)$$

for $0 < |t - t_0| < \epsilon$, to conclude that

$$(\tau \circ f)^{(r)}(t) \rightarrow r(\tau \circ f_1)^{(r-1)}(t_0) = (\tau \circ f)^{(r)}(t_0)$$

as $t \rightarrow t_0$, so continuity is established in this case. Moreover, by continuity, the estimate (E_r) holds on a neighborhood of t_0 .

Case II: $t_0 \in F_r$. In this case it follows from the estimate (E_{r-1}) that

$$\left| \frac{(\tau \circ f)^{(r-1)}(t) - (\tau \circ f)^{(r-1)}(t_0)}{t - t_0} \right| \leq C_{r-1} \left| \frac{f^{(r-1)}(t) - f^{(r-1)}(t_0)}{t - t_0} \right| \rightarrow 0$$

as $t \rightarrow t_0$, and hence the r th derivative of $\tau \circ f$ vanishes at t_0 . Thus $(\tau \circ f)^{(r)} \equiv 0$ on F_r and the estimate (E_r) holds on the entire interval (a, b) . If $t_0 \in F_r$,

$$|(\tau \circ f)^{(r)}(t)| \leq C_r |f^{(r)}(t)| \rightarrow 0$$

as $t \rightarrow t_0$, so $(\tau \circ f)^{(r)}$ is continuous at t_0 . This finishes the inductive step. Q.E.D.

PROOF OF THE SUBLEMMA. By continuity of derivatives on U , it will suffice to establish the estimates (E_r) on a dense open subset of U , and indeed we will establish these estimates on the dense open subset R of regular points, where R will be specified below.

It will be convenient to change notation slightly, by writing τ in place of $\tau \circ f$, and setting $f = \tau g$, where $g: U \rightarrow V^{n-1} \subseteq \text{Sym}(n \times n)$. If $x, y \in \mathbf{R}^n$, say $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, we let

$$x \circ y = \begin{pmatrix} x_1 y_1 & 1/2(x_1 y_2 + x_2 y_1) & \cdots \\ 1/2(x_2 y_1 + x_1 y_2) & x_2 y_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \in \text{Sym}(n \times n).$$

Given $t \in U$, we can define a unit length vector $e_1(t) \in \mathbf{R}^n$ in such a way that

$$(8) \quad g(t) = (\sqrt{n^2 - n})^{-1} [(1 - n)e_1(t) \circ e_1(t) + (e_2 \circ e_2 + \cdots + e_n \circ e_n)],$$

where e_2, \dots, e_n are any vectors completing $e_1(t)$ to an orthonormal basis for \mathbf{R}^n . Indeed, we can take $e_1(t)$ to be a unit-length vector in the one-dimensional eigenspace for $g(t)$ corresponding to eigenvalue $(1 - n)/\sqrt{n^2 - n}$, an eigenspace which depends smoothly on $g(t)$.

We will now define the set R of regular points, and if $t_0 \in R$, we will specify a moving orthonormal frame on a neighborhood of t_0 : Let $t_0 \in U$. We begin with $e_1(t)$, defined as in the preceding paragraph. If $e'_1(t) = 0$ on some neighborhood I of t_0 , we say that $t_0 \in R$, and we choose $e_2(t), \dots, e_n(t)$ for $t \in I$ so that $(e_1(t), \dots, e_n(t))$ is orthonormal and $e'_i(t) = 0$. If $e'_1(t_0) = 0$ but $e'_1(t) \neq 0$ for t arbitrarily near t_0 , we say that $t_0 \notin R$ and omit it from further consideration. Finally, if $e'_1(t_0) \neq 0$, we select a unit-length vector $e_2(t)$, for t in some neighborhood of t_0 , so that $(de_1/dt)(t) = \kappa_1(t)e_2(t)$ on this neighborhood, and we proceed to the next step in the construction of the moving orthonormal frame.

If $(e_1 \wedge e_2)'(t) = 0$ on a neighborhood of t_0 , we say that $t_0 \in R$ and we choose $e_3(t), \dots, e_n(t)$ on this neighborhood of t_0 so that $(e_1(t), \dots, e_n(t))$ is orthonormal and $e'_i(t) = 0$ for $3 \leq i \leq n$. If $(e_1 \wedge e_2)'(t_0) = 0$ but $(e_1 \wedge e_2)'(t) \neq 0$ for t arbitrarily near t_0 , we say that $t_0 \notin R$ and omit it from further consideration. Finally, if $(e_1 \wedge e_2)'(t_0) \neq 0$, we choose $e_3(t)$ for t in some neighborhood of t_0 , so that $e_3(t)$ is perpendicular to $e_1(t)$ and $e_2(t)$, and $de_2/dt = -\kappa_1 e_1 + \kappa_2 e_3$, and we proceed to the next step.

Continue in this fashion until either it is determined that $t_0 \notin R$, or a complete orthonormal frame $(e_1(t), \dots, e_n(t))$ is constructed on some neighborhood of t_0 . In the latter case, $t_0 \in R$ and the moving orthonormal frame satisfies the ‘‘Frenet-Serret formula’’,

$$\begin{pmatrix} e'_1 \\ e'_2 \\ e'_3 \\ \vdots \\ e'_n \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 & \cdots & 0 \\ -\kappa_1 & 0 & \kappa_2 & \cdots & 0 \\ 0 & -\kappa_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \\ e_n \end{pmatrix},$$

where some of the κ_i 's may be identically zero.

Using these formulae, it is possible to differentiate (8) and obtain quite explicit expressions for the derivatives $g^{(k)}(t_0)$, when $t_0 \in R$:

$$g'(t_0) = (\sqrt{n^2 - n})^{-1}[-2n\kappa_1 e_1 \circ e_2](t_0),$$

$$g''(t_0) = (\sqrt{n^2 - n})^{-1}[-2n\kappa'_1 e_1 \circ e_2 - 2n\kappa_1^2 e_2 \circ e_2 - 2n\kappa_1 e_1 \circ (-\kappa_1 e_1 + \kappa_2 e_3)](t_0),$$

etc. The crucial observation is that $g^{(k)}(t_0)$ is a linear combination of $e_i(t_0) \circ e_j(t_0)$, with the $e_n(t_0) \circ e_n(t_0)$ term making its first appearance only when k has reached at least $2n - 2$. Since

$$\langle\langle e_i(t_0) \circ e_j(t_0), e_n(t_0) \circ e_n(t_0) \rangle\rangle = 0 \quad \text{unless } i = j = n,$$

we conclude that

$$(9) \quad 1 \leq k < 2n - 3 \Rightarrow \langle\langle g^{(k)}(t_0), e_n(t_0) \circ e_n(t_0) \rangle\rangle = 0,$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ is the standard inner product on $\text{Sym}(n \times n)$ described earlier.

Now we apply the Leibniz formula,

$$f^{(k)} = \tau^{(k)}g + k\tau^{(k-1)}g' + \dots + k\tau'g^{(k-1)} + \tau g^{(k)},$$

take the inner product with $e_n(t_0) \circ e_n(t_0)$, and apply (9) to obtain

$$\begin{aligned} \langle\langle f^{(k)}(t_0), e_n(t_0) \circ e_n(t_0) \rangle\rangle &= \tau^{(k)}(t_0) \langle\langle g(t_0), e_n(t_0) \circ e_n(t_0) \rangle\rangle \\ &= (\sqrt{n^2 - n})^{-1} \tau^{(k)}(t_0), \end{aligned}$$

which implies that

$$|\tau^{(k)}(t_0)| \leq \sqrt{n^2 - n} |f^{(k)}(t_0)|, \quad \text{for } 0 \leq k \leq 2n - 3.$$

This gives the desired estimate (E_k) when we set $C_k = \sqrt{n^2 - n}$, and hence the Sublemma is proved.

4. Differentiability. To finish the proof of the theorem stated in the introduction, we need to show that the sections $e_{n+1}, e_{n+2}: M \rightarrow NM$ constructed in §2, where NM is the normal bundle of the isometric immersion $f: M^n \rightarrow E^{n+2}$, are differentiable of class C^k .

Let us consider the sections $\phi_1 = \langle \alpha, e_{n+1} \rangle$ and $\phi_2 = \langle \alpha, e_{n+2} \rangle$ of the bundle $L_s^2(TM)$ of symmetric covariant tensors of rank two on M . ϕ_1 is C^{k+1} because we also have $\phi_1 = \langle \tilde{\alpha}, \tilde{e}_{n+1} \rangle$, where $\tilde{\alpha}$ is C^{k+1} and \tilde{e}_{n+1} is C^{k+2} , and ϕ_1 is positive definite by our curvature assumption. Hence, ϕ_1 is a new Riemannian metric on M ; geometrically, it is the pullback of the standard unit curvature metric on S^n via the Gauss map of $\tilde{f}: M \rightarrow E^{n+1}$.

It will suffice to work locally. In a neighborhood U of a given point $p \in M$, we choose vector fields (e_1, \dots, e_n) which are orthonormal with respect to ϕ_1 . Set $H = \sum_{i=1}^n \alpha(e_i, e_i)$, a nonvanishing C^{k+1} section of the normal bundle over U , and let \hat{e}_{n+1} be the unit length normal field pointing in the direction of H . Finally, let \hat{e}_{n+2} be the normal field obtained from \hat{e}_{n+1} by counterclockwise rotation through $\pi/2$ in the direction determined by the orientation of the normal bundle. \hat{e}_{n+1} and \hat{e}_{n+2} are C^{k+1} sections of the normal bundle.

We define a function $\theta: U \rightarrow \mathbf{R}$ by requiring that $-\pi/2 < \theta < \pi/2$ and

$$\begin{pmatrix} e_{n+1} \\ e_{n+2} \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{e}_{n+1} \\ \hat{e}_{n+2} \end{pmatrix}.$$

To show that e_{n+1} and e_{n+2} are C^k , it will suffice to show that θ is C^k . Let $\psi_1 = \langle \alpha, \hat{e}_{n+1} \rangle$ and $\psi_2 = \langle \alpha, \hat{e}_{n+2} \rangle$, so that

$$(10) \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

We will need to use the C^{k+1} inner product $\langle\langle \cdot, \cdot \rangle\rangle$ defined on sections ϕ, ψ of $L_s^2(TM)|_U$ by

$$\langle\langle \phi, \psi \rangle\rangle = \sum_{i,j=1}^n \phi(e_i, e_j) \psi(e_i, e_j),$$

where (e_1, \dots, e_n) are orthonormal with respect to ϕ_1 . In terms of this inner product, $\langle\langle \phi_1, \phi_1 \rangle\rangle = n$ and $\langle\langle \psi_2, \phi_1 \rangle\rangle = 0$, and hence

$$\begin{aligned} 0 &= \langle\langle -\sin \theta \phi_1 + \cos \theta \phi_2, \phi_1 \rangle\rangle = -n \sin \theta + \cos \theta \langle\langle \phi_2, \phi_1 \rangle\rangle \\ &\Rightarrow \langle\langle \phi_1, \phi_2 \rangle\rangle = n \tan \theta, \\ \langle\langle \phi_1, \psi_1 \rangle\rangle &= \cos \theta \langle\langle \phi_1, \phi_1 \rangle\rangle + \sin \theta \langle\langle \phi_2, \phi_1 \rangle\rangle \\ &= n \cos \theta + n \sin \theta \tan \theta = n \sec \theta \\ &\Rightarrow \sec \theta = (1/n) \langle\langle \phi_1, \psi_1 \rangle\rangle. \end{aligned}$$

Since ϕ_1 and ψ_1 are C^{k+1} , so is $\sec \theta$. Thus $\cos \theta$ is C^{k+1} and θ itself is C^{k+1} except possibly at points where $\theta = 0$.

These points where $\theta = 0$ are the umbilic points mentioned in the introduction.

Again using our ϕ_1 -orthonormal frame (e_1, \dots, e_n) , we define a C^{k+1} map $F: U \rightarrow CV$, where CV is the cone on the Veronese manifold studied in §3, by

$$F(p) = \text{the symmetric matrix } (h_{ij}), \text{ where } h_{ij} = \psi_2(e_i(p), e_j(p)).$$

It follows from (10) that

$$F = -\sin \theta I + (\text{matrix of rank } \leq 1),$$

where I is the identity matrix, and it follows easily from the definition of τ in §3 that

$$(11) \quad \sin \theta = -(\sqrt{n^2 - n})^{-1} \tau \circ F.$$

Thus it remains only to show that $\tau \circ F$ is C^k on some neighborhood of the given point p . If $\eta: M \rightarrow [0, 1]$ is a smooth function which is identically one on a neighborhood of p and has compact support contained in the open set U considered above, it will suffice to show that $\eta(\tau \circ F)$ is C^k . We can assume that $\sigma = (x^1, \dots, x^n)$ is a local coordinate system defined on U and extend $(\eta(\tau \circ F)) \circ \sigma^{-1}$ to \mathbf{R}^n by setting it equal to zero outside $\sigma(U)$. By *abus de langage*, we will denote this extended function by $\eta(\tau \circ F)$; $\eta(\tau \circ F)$ will henceforth be regarded as a function with compact support defined on \mathbf{R}^n .

Now it follows immediately from the Cone Lemma of §3 that the classical partial derivatives $(\partial/\partial x^i)^j(\eta(\tau \circ F))$ exist at every point of U and are bounded, for $1 \leq i \leq n, 0 \leq j \leq k + 1$. (It is here that the assumption $k \leq 2n - 4$ is needed.) Note that the restriction of $(\partial/\partial x^i)^j \eta(\tau \circ F)$ to any coordinate line parallel to the x^i -axis is continuous. If ϕ is any C^∞ function with compact support in U , we can apply Fubini's theorem and integrate by parts to obtain

$$\begin{aligned} &\int_U (\partial/\partial x^i)^j (\eta(\tau \circ F)) \phi \, dx^1 \cdots dx^n \\ &= (-1)^j \int_U \eta(\tau \circ F) (\partial/\partial x^i)^j \phi \, dx^1 \cdots dx^n, \end{aligned}$$

and hence the classical partial derivatives are also distributional derivatives. In particular, we see that the distributional derivatives

$$(12) \quad (\partial/\partial x^i)^j (\eta(\tau \circ F)) \in L^p \text{ for all } p, 1 \leq i \leq n, 0 \leq j \leq k + 1.$$

If we were satisfied with a loss of about $n/2$ derivatives, we could apply Theorem 7.25 (the L^2 Sobolev imbedding theorem) of Rudin [R]; according to this theorem.

$$(\partial/\partial x^i)^j(\eta(\tau \circ F)) \in L^2 \text{ for } 1 \leq i \leq n, 0 \leq j \leq k+1 \Rightarrow \eta(\tau \circ F) \in C^r,$$

whenever r is a nonnegative integer satisfying the inequality $r + n/2 < k + 1$. Thus $\tau \circ F$ is C^r in a neighborhood of p .

However, by means of L^p regularity theory for elliptic operators, we can do somewhat better. Consider the elliptic operator

$$L: \{C^\infty \text{ real-valued functions on } \mathbf{R}^n\} \\ \rightarrow \{n\text{-tuples of } C^\infty \text{ real-valued functions on } \mathbf{R}^n\},$$

defined by

$$Lu = \left((\partial/\partial x^1)^{k+1}u, \dots, (\partial/\partial x^n)^{k+1}u \right).$$

It follows immediately from (12) that $L(\eta(\tau \circ F)) \in L^p$ for all p such that $1 < p < \infty$. Hence, the L^p regularity theorem for overdetermined elliptic systems (Theorem 7.1 of [A]) implies that $\eta(\tau \circ F) \in L_{k+1}^p$, where L_{k+1}^p is the Sobolev space of functions on \mathbf{R}^n whose distributional derivatives of order $\leq k+1$ belong to L^p . If p is sufficiently large, the Sobolev imbedding theorem implies that $L_{k+1}^p \subseteq C^k$; we thus conclude that $\eta(\tau \circ F) \in C^k$.

It now follows from (11) that $\sin \theta$ is C^k , and hence the Theorem stated in the introduction is completely proven.

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