LOCAL AND GLOBAL ENVELOPES
OF HOLOMORPHY OF DOMAINS IN C^2

BY
ERIC BEDFORD

ABSTRACT. A criterion is given for a smoothly bounded domain D ⊂ C^2 to be locally extendible to a neighborhood of a point z_0 ∈ ∂D. (This result may also be formulated in terms of extension of CR functions on ∂D.) This is related to the envelope of holomorphy of the semitubular domain

Ω(Φ) = \{(z, w) ∈ C^2: \Re w + r^k ϕ(θ) < 0\},

where r = |z|, θ = arg(z). Necessary and sufficient conditions are given for the envelope of holomorphy of Ω(Φ) to be C^2. These conditions are equivalent to the existence of a subharmonic minorant for r^k ϕ(θ).

1. Introduction. Let us consider a smoothly bounded domain D ⊂ C^2 and ask whether D is locally extendible at p ∈ ∂D, i.e. for every open set U containing p do all holomorphic functions on D ∩ U extend holomorphically through p?

This question has been answered when ∂D is pseudoconcave and real analytic at p (see [3]) and when ∂D has so-called "type k" with k odd (see [2, 5, 10]). The question of local extension of holomorphic functions from D is essentially equivalent to the question of local extension of CR functions from ∂D (see [1, 8]). However, we do not discuss CR functions further since our contribution is to deal with the geometric structure of certain envelopes, and we would like our presentation to be as self-contained as possible.

We may make a holomorphic change of coordinates (z, w) in a neighborhood of p such that p = (0, 0), w = u + iv, and that ∂D is given near p by the equation u + p_k(z) + R(z, v) < 0, where

p_k(z) = \sum_{j=1}^{k-1} a_j z^j \bar{z}^{k-j}

is a real, homogeneous polynomial of degree k, and the remainder is given by

(1) \quad R(z, v) = O(v^2 + |vz| + |z|^{k+1})

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With a holomorphic change of variables \( w' = w + \alpha w^2 + \beta zw, \quad z' = z, \quad \partial D \) can be given by
\[
u + c v^2 + R'(z, v) + p_k(z) < 0,
\]
where \( c \in \mathbb{R} \) is arbitrary, and
\[
R'(z, w) = O \left( |v|^3 + |vz|^2 + |z|^{k+1} \right).
\]
We will be interested in domains that satisfy the following stronger condition: For any \( \varepsilon > 0 \), there exists \( c > 0 \) such that
\[
(2) \quad |R'(z, v)| = O \left( cv^2 + \varepsilon |z|^k \right).
\]
If (2) holds, then for every \( \varepsilon > 0 \), there exists \( \eta > 0 \) and a change of coordinates as above such that
\[
(3) \quad \{(z, w) \in \mathbb{C}^2 : |(z, w)| < \eta \} \cap D \subset \left\{ u + p_k(z) < \varepsilon |z|^k \right\}.
\]
Writing \( z = re^{i\theta} \) and \( p_k(z) = r^k \Phi(\theta) \), we see that the local study of \( D \) at \( (0, 0) \) is related to the domain
\[
\Omega(\Phi) = \{(z, w) \in \mathbb{C}^2 : \text{Re} w + r^k \Phi(\theta) < 0\}.
\]
To make the connection between \( D \) and \( \Omega(\Phi) \), we will need to discuss (global) envelopes of holomorphy. The envelope of holomorphy \( E(D) \) of a domain \( D \subset \mathbb{C}^n \) is a Riemann domain \( \pi : E(D) \to \mathbb{C}^n \) with \( i : D \to E(D) \) and \( E(D) \) is the minimal domain of holomorphy such that every function \( f \in \mathcal{O}(D) \) extends holomorphically to \( E(D) \). A convenient method for staying within the class of domains in \( \mathbb{C}^n \) while taking envelopes is to consider \( D \) which are starshaped with respect to the origin, i.e., \( \delta_t(D) \subset D \), where \( \delta_t(z) = (tz_1, \ldots, tz_n), \quad 0 \leq t \leq 1 \). If \( D \subset \mathbb{C}^n \) is starshaped, then the envelope is a starshaped domain in \( \mathbb{C}^n \) with \( D \subset E(D) \subset \mathbb{C}^n \). To prove this assertion it suffices to show that the projection \( \pi \) is one-to-one. The mapping \( \delta_t \) has a holomorphic continuation to a map \( \tilde{\delta}_t : E(D) \to E(D) \). We note that \( \pi \tilde{\delta}_t = \delta_t \pi \), \( \tilde{\delta}_1 \) is the identity map, and \( \tilde{\delta}_0 = \lim_{t \to 0} \tilde{\delta}_t \) is the constant \( i(0) \). Let \( z_1, z_2 \in E(D) \) be points such that \( \pi(z_1) = \pi(z_2) \), and let \( \sigma_j, \quad j = 1, 2, \) be the path given by \( \gamma_j(t) = \delta_t(z_j), \quad 0 \leq t \leq 1 \).

Now \( \sigma_1 \) and \( \sigma_2 \) project under \( \pi \) to the same path in \( \mathbb{C}^n \), and \( \gamma_1(0) = \gamma_2(0) = i(0) \). Since \( \pi \) is locally invertible, the paths \( \sigma_1 \) and \( \sigma_2 \) coincide, and thus \( z_1 = \gamma_1(1) = \gamma_2(1) = z_2 \).

It follows (e.g. from a result of Docquier and Grauert [7]), that if \( D \) is starshaped, then it is a Runge domain, i.e. every holomorphic function on \( D \) may be uniformly approximated by polynomials on compact subsets.

The domain \( \Omega(\Phi) \) is invariant under the transformations
\[
(4) \quad (z, w) \to (z, w + \xi), \quad \xi \in \mathbb{C}, \text{Re} \xi < 0,
\]
\[
(5) \quad (z, w) \to (t z, r^k w), \quad 0 < t < \infty.
\]
The envelope of holomorphy has the same invariance and is thus given by
\[
E(\Omega(\Phi)) = \{(z, w) \in \mathbb{C}^2 : \text{Re} w + r^k \Phi(\theta) < 0\} = \Omega(\Phi),
\]
where \( r^k \Phi(\theta) \) is the greatest subharmonic minorant of \( r^k \Phi(\theta) \). (This is a special case of a result on semitubular domains, see [6].)
We may approximate $\Omega(\Phi)$ by the truncated domain

$$\Omega_\lambda(\Phi) = \Omega(\Phi) \cap \{|z| < \lambda, |u| < \lambda^k, |v| < c\lambda^k\}$$

for $0 < \lambda < \infty$. Since $\Omega_\lambda(\Phi)$ is starshaped with respect to $(0, -c\lambda^k/2)$ for $c$ sufficiently large, the envelope is again starshaped. Further, $\Omega_\lambda(\Phi)$ is mapped biholomorphically to $\Omega_\lambda(\Phi)$ by the transformation (5), and so $E(\Omega_\lambda(\Phi))$ is also mapped to $E(\Omega_\lambda(\Phi))$. Thus

$$E(\Omega(\Phi)) = \bigcup_{\lambda} E(\Omega_\lambda(\Phi)),$$

and so $(0,0) \in E(\Omega(\Phi))$ if and only if $(0,0) \in E(\Omega_\lambda(\Phi))$ for all $\lambda$.

The question of local extendibility of $D$ at $(0,0)$ is tied to the global question for $\Omega(\Phi)$: Does $(0,0)$ belong to the envelope of holomorphy $E(\Omega(\Phi))$ of $\Omega(\Phi)$? There are two possibilities:

(i) $(0,0) \in E(\Omega(\Phi))$, and in this case $E(\Omega(\Phi)) = C^2$.

(ii) $(0,0) \notin E(\Omega(\Phi))$, and $E(\Omega(\Phi)) = \Omega(\tilde{\Phi})$ with $\tilde{\Phi}$ not identically $-\infty$.

**Proposition.** If there exists $\varepsilon > 0$ such that $E(\Omega(\Phi + \varepsilon)) = C^2$, then for all open $U$ containing $(0,0)$, every analytic function on $U \cap D$ extends analytically to a neighborhood of $(0,0)$.

Conversely, if $D$ satisfies (2), and if $E(\Omega(\Phi - \varepsilon)) \neq C^2$ for some $\varepsilon > 0$, then there exists $\eta > 0$ and a function

$$f \in \mathcal{O}(D \cap \{(z,w) \mid |n| < \eta\})$$

which cannot be extended holomorphically past $(0,0)$.

**Proof.** If $(0,0)$ is in the envelope of $E(\Omega(\Phi + \varepsilon))$, there is a compact $K \subset \Omega(\Phi + \varepsilon)$ such that $|f(0,0)| \leq |f|_K$ for all $f \in \mathcal{O}(\Omega(\Phi + \varepsilon))$. Since $K$ is compact, we may shrink $\varepsilon$ if necessary, so that $K \subset \omega_{\varepsilon}$, where

$$\omega_{\varepsilon} = \left\{u + p_k(z) + \varepsilon|z|^k + \varepsilon|v| \mid \right\} < 0.$$

By (1), we may choose $\eta$ sufficiently small such that $D \supset \{(z,w) \mid < \eta\} \subset \omega_{\varepsilon}$. Now $\omega_{\varepsilon}$ is invariant under the transformation (5), so we may apply (5) to $K$ with $t$ small to have $K \subset \{(z,w) \mid < \eta\} \cap \omega_{\varepsilon}$.

Finally, since $D \cap \{(z,w) \mid < \eta\}$ is starshaped for $\eta$ small, it is Runge. Thus, $f \in \mathcal{O}(D \cap \{(z,w) \mid < \eta\})$ may be approximated by polynomials uniformly on $K$. Since $(0,0)$ is in the hull of $K$, we may extend $f$ past $(0,0)$.

Now we prove the converse statement. If $D$ satisfies (2), then we have (3), and so for $\Psi = \Phi - \varepsilon$

$$D \cap \{(z,w) \mid < \eta\} \subset \Omega(\tilde{\Psi}).$$

Since $\Omega(\tilde{\Psi})$ is a domain of holomorphy there exists $f \in \mathcal{O}(\Omega(\tilde{\Psi}))$ which cannot be continued past $(0,0)$.

**Remarks.** The first part of the Proposition can be used to give sufficient conditions for local extension of functions from domains $D \subset C^n$. For this, let $P$ be a complex 2-plane intersecting $\partial D$ transversally at $z_0 \in \partial D$. If $D \cap P$ satisfies the
first hypotheses of the Proposition in a neighborhood of \( z_0 \) in \( P \), then there is a compact \( K \subset D \cap P \) such that \( z_0 \) is in its polynomial hull. For \( \varepsilon > 0 \) sufficiently small, a closed \( \varepsilon \)-neighborhood \( K^\varepsilon \) of \( K \) is contained in \( D \). Since \( K^\varepsilon \) contains all \( \varepsilon \)-translates of \( K \), the polynomial hull of \( K^\varepsilon \) contains all \( \varepsilon \)-translates of \( z_0 \), i.e. an \( \varepsilon \)-neighborhood of \( z_0 \). Thus if we have local extension in a 2-dimensional slice of \( D \), we have local extension from \( D \).

By writing the Laplacian in polar coordinates,
\[
\Delta = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2},
\]
we see that \( \Delta(r^k\Phi(\theta)) \geq 0 \) if and only if \( \mathcal{L}\Phi \geq 0 \), where \( \mathcal{L} = d^2/d\theta^2 + k^2 \). Of course, \( \mathcal{L}\psi = 0 \) if and only if \( \psi(\theta) = c\sin(k\theta) + d\cos(k\theta) \) and in this case \( r^k\psi(\theta) = \text{Re}((d - ic)z^k) \). The intervals in \( \theta \) where \( \Phi \) is positive or negative are of some importance. If \( \mathcal{L}\Phi > 0 \), then the intervals where \( \{\Phi < 0\} \) have length \( < \pi/k \) and the intervals where \( \{\Phi > 0\} \) have length \( > \pi/k \). This follows from (10) below.

It is also useful to adjoin nearby intervals.

**Definition.** Given an open set \( \mathcal{O} \subset \mathbb{R} \), the **amalgamated component** \( \tilde{I} \) of an interval \( I \subset \mathcal{O} \) is the smallest connected, open interval \( \tilde{I} \supseteq I \) with the property: If \( J \subset \mathcal{O} \) is an open interval with \( \text{dist}(J, I) < \pi/k \), then \( J \subset \tilde{I} \).

**Definition.** An upper semicontinuous periodic function \( \Phi \) on \( \mathbb{R} \) with period \( 2\pi \) has a **wide (amalgamated) sector** if either
1. \( 0 < k < 1/2 \), and \( \Phi(\theta) < 0 \) for some \( \theta \), or
2. \( k > 1/2 \), and there exist \( c_1, c_2 \in \mathbb{R} \) and \( \varepsilon > 0 \) such that an (amalgamated) component of
\[
\mathcal{O}(\varepsilon, c_1, c_2) = \{ \theta \in \mathbb{R} : \Phi(\theta) + \varepsilon + c_1\sin(k\theta) + c_2\cos(k\theta) < 0 \}
\]
has length \( \geq \pi/k \).

Note that the length will be \( > \pi/k \) if we take \( \varepsilon > 0 \) smaller. By this same remark we see also that if \( \Phi \) is continuous and has no wide sectors, then for \( 0 < c < \infty \), there exists \( \varepsilon_0 > 0 \) such that every connected component of \( \mathcal{O}(\varepsilon, c_1, c_2) \) has length \( \leq \pi/k - \varepsilon_0 \) if \( |c_1| + |c_2| < c \) and \( 0 < \varepsilon < \varepsilon_0 \).

**Theorem.** Let \( \Phi \) be periodic and u.s.c. on \( [0, 2\pi] \). Then the envelope of holomorphy \( E(\Omega(\Phi)) = C^2 \) if and only if \( \Phi + \varepsilon \) has a wide amalgamated sector for some \( \varepsilon > 0 \).

**Remark.** The “only if” part of the Theorem is easily seen. If \( E(\Omega(\Phi)) \neq C^2 \), then there is a subharmonic \( r^k\Phi(\theta) \leq r^k\Phi(\theta) \). Thus each interval of \( \{\Phi + \varepsilon < 0\} \) lies in an interval of \( \{\Phi + \varepsilon < 0\} \), which has length \( < \pi/k \), since \( \mathcal{L}(\Phi + \varepsilon) > 0 \). Further, since the sectors of \( \{\Phi + \varepsilon < 0\} \) are separated by a distance \( > \pi/k \), the amalgamated components of \( \{\Phi + \varepsilon < 0\} \) lie in the components of \( \{\Phi + \varepsilon < 0\} \).

**Remark.** The works \([2 \text{ and } 9, 10]\) use the weaker “sector property”, which is just that \( \Phi \) has a wide sector. We note that if \( \Phi \) does not have the sector property, and if \( I_1 \) and \( I_2 \) are intervals of \( \mathcal{O}(\varepsilon, c_1, c_2) \), and if \( \text{dist}(I_1, I_2) < \pi/k \), then \( I_1 \cup I_2 \) is contained in an interval of length \( < \pi/k \).

(To see this, we may assume, to the contrary, that \( 0 \in I_1 \) and \( \pi/k \in I_2 \). Then we make \( c_1 \) very large and negative so that \( [0, \pi/k] \subset \mathcal{O}(\varepsilon, c_1, c_2) \).)
From this we conclude that if $\Phi$ has the sector property, and if $\mathcal{O}(\varepsilon, c_1, c_2)$ contains no more than two intervals (for all $\varepsilon, c_1, c_2$), then $\Phi$ has a wide amalgamated sector. The case $k = 4$, which was treated in [2], is a special case of this situation.

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2. Construction of the envelope. Since $r^k \Phi(\theta)$ is subharmonic and constant on the sets $\{\theta = \text{const}\}$, it follows that $\Phi$ is bounded. Further, since $\mathcal{L}\Phi \geq 0$, we have $\Phi'' \geq -\text{const}$, and so $\Phi \in C^1$. Thus if the envelope $E(\Omega(\Phi)) \neq C^2$, and if $k > 1$, the boundary $\partial E(\Omega(\Phi))$ is $C^1$ smooth. In general, however, $\Phi \notin C^2$.

We may approximate $\Phi + \delta$ from below by $\bar{\Phi} + \delta$, where $\bar{\Phi} = \Phi * \chi_\varepsilon$ is a usual smoothing in $\theta$, and $0 < \delta < \delta$, $\lim_{\varepsilon \to 0} \delta = \delta$. Thus

$$
(\Phi + \delta)(\theta) = \sup \{ h(\theta) : h \text{ is of class } C^2, h \leq \Phi + \delta, \mathcal{L}h \geq 0 \}. 
$$

Remark. In terms of the envelope (6) our question is whether the competing family of subsolutions is nonempty. Thus an alternative statement of our Theorem is: $r^k \Phi(\theta)$ has a subharmonic minorant if and only if $\Phi(\theta) + \varepsilon$ does not have a wide amalgamated sector for any $\varepsilon > 0$.

The envelope formulation (6) also suggests the structure of $\bar{\Phi}$:

$$
\mathcal{L}\bar{\Phi} = 0 \quad \text{on} \quad \mathcal{O} = \{ \bar{\Phi} < \Phi \}, 
$$
$$
\Phi = \bar{\Phi} \quad \text{and} \quad \nabla \Phi = \nabla \bar{\Phi} \quad \text{on} \quad \partial \mathcal{O}.
$$

We will construct $\Phi$ in the manner suggested by Figure 1. If $E = \{ \mathcal{L}\Phi < 0 \}$ is the set where the Levi form is negative, we must have $E \subset \{ \bar{\Phi} < \Phi \}$, and $\Phi$ is obtained by patching solutions $\psi_j$ of $\mathcal{L}\psi_j = 0$ onto $\Phi$ so that they satisfy (7) and (8) above.

The last feature of the construction we shall require is

$$
\text{each interval in } \mathcal{O} = \{ \Phi < \bar{\Phi} \} \text{ has length } < \pi/k.
$$

![Figure 1](image-url)
Without (9), the solution constructed according to Figure 1 is not unique. For instance, if \( \Phi(\theta) = \sin(k\theta) + 1 \), then \( \mathcal{L}\Phi > 0 \), and \( \Phi = \Phi_0 \). If we take \( \psi_1(\theta) = 0 \), \(-\pi/2k < \theta < 3\pi/2k\), and equal to \( \Phi \) for other values of \( \theta \), then the resulting solution \( \Phi_0 \) satisfies \( \mathcal{L}\Phi_0 > 0 \), but \( \{ \Phi < \Phi_0 \} = (-\pi/2k, 3\pi/2k) \).

We will use the following version of the Sturm Comparison Theorem (see [4]):

\[
\begin{align*}
\text{if } \psi_1, \psi_2 &\in C^2 \text{ and } \mathcal{L}\psi_1 \geq \mathcal{L}\psi_2, \text{ and if} \\
\psi_1(\theta_0) &= \psi_2(\theta_0), \psi_1'(\theta_0) \geq \psi_2'(\theta_0), \text{ then} \\
\psi_1(\theta) &\geq \psi_2(\theta) \text{ for } \theta_0 < \theta < \theta_0 + \pi/k.
\end{align*}
\]

To prove (10), we consider \( \psi = \psi_1 - \psi_2 \), and we may add \( \varepsilon((\theta - \theta_0) + (\theta - \theta_0)^2) \) so that \( \psi'(\theta_0) > 0 \) and \( \mathcal{L}\psi > 0 \) on \( (\theta_0, \theta_0 + \pi/k) \). Now we will show that \( \psi > 0 \) on \( (\theta_0, \theta_0 + \pi/k) \). Let \( \theta_1 > \theta_0 \) be the first point where \( \psi(\theta_1) = 0 \). We may assume \( \psi'(\theta_1) < 0 \). We set

\[
h(\theta) = \arctan(\psi'(\theta)/k\psi(\theta)).
\]

Since \( h(\theta_0) = +\pi/2 \) and \( h(\theta_1) = -\pi/2 \) we have

\[
\int_{\theta_0}^{\theta_1} h'(\theta) \, d\theta = -\pi.
\]

Further, since \( \mathcal{L}\psi > 0 \), we have \( \psi'' < -k^2\psi^2 \), and with this we may compute that

\[
h'(\theta) > -k.
\]

Thus we have

\[
-\pi = \int_{\theta_0}^{\theta_1} h'(\theta) \, d\theta > -(\theta_1 - \theta_0)k,
\]

and so \( \theta_1 - \theta_0 > \pi/k \) which yields (10).

We will use the notation \( \psi_p \) for the function

\[
\psi_p(\theta) = c\sin(k\theta) + d\cos(k\theta)
\]

such that \( \psi_p(p) = \Phi(p) \) and \( \psi'_p(p) = \Phi'(p) \).

Some properties of \( \psi_p \) are formulated in the following lemmas and are illustrated in Figure 2.

**Lemma 1.** If \( \mathcal{L}\Phi(p) > 0 \), then there exists \( \varepsilon > 0 \) such that \( \psi_p(\theta) \leq \Phi(\theta) \) for \( \theta \in (p - \varepsilon, p + \varepsilon) \). If \( \mathcal{L}\Phi(p) > 0 \) for \( p_2 < p \leq p_1 \), then \( \psi_{p_2}(\theta) < \psi_{p_1}(\theta) \) for \( p_1 < \theta < p_2 + \pi/k \).
Proof. The first statement is just the comparison (10). The second statement also follows from (10). If we replace $\psi_p$ by $\tilde{\psi}_q = \psi_q - \Phi$, then for $|q - p|$ small

$$\tilde{\psi}_q(\theta) = -k(q)(\theta - q)^2 + o((\theta - q)^2),$$

and $k(q) > 0$. If $p_2 < p_1 < p$, and $|p_2 - p|$ is small, then $\tilde{\psi}_{p_1}$ and $\tilde{\psi}_{p_2}$ will intersect at a point $q \in (p_2, p_1)$. Thus $\psi_{p_1}$ and $\psi_{p_2}$ will intersect as in Figure 2, and so by (10) we have $\psi_{p_1}(\theta) > \psi_{p_2}(\theta)$ for $\theta \in (q, q + \pi/k)$.

**Lemma 2.** Let $\Phi$ have no wide sectors. If $(a, b)$ is an open interval on which $\mathcal{L}\Phi < 0$, then $\psi_a(\theta) > \Phi(\theta)$ for $\theta \in (a, b)$.

**Proof.** By (10), $\psi_a(\theta) > \Phi(\theta)$ holds for $a < \theta < \min(a + \pi/k, b)$. Thus the result holds unless $a + \pi/k < b$. But in this case we have $(a, a + \pi/k) \subset \{\Phi - \psi_a < 0\}$ which is a wide sector.

**Lemma 3.** If $\Phi - \delta$ has no wide sectors for some $\delta > 0$ and if $E = \{\mathcal{L}\Phi < 0\}$ consists of a single interval $E = (a, b)$, then $\Phi$ exists.

**Proof.** Note that if $E \neq \emptyset$, then by definition $k > 1/2$. By Lemma 2, $\psi_a(\theta) > \Phi(\theta)$ for $\theta \in E$. And by Lemma 1, $\psi_p(\theta) < \psi_a(\theta)$ holds for $p < a$ and $a < \theta < p + \pi/k$. Further, we claim that there is a wide sector unless $|q - p| < \pi/k$ holds for all $p$ ($q$ is the point where $\psi_p$ crosses $\Phi$ from above). First, it is evident that $|a - q_0| < \pi/k$. Thus for $p_1$ near $a$, it follows that $|p_1 - r_1| < \pi/k$, where we write $\{\Phi < \psi_{p_1}\} \cap (p_1, q_0) = (r_1, q_1)$. Replacing $\Phi$ by

$$\Phi_1 = \Phi - \varepsilon \sin(k(\theta - p_1 + \varepsilon))$$

for $\varepsilon > 0$ small, we obtain a small interval $(p_1 - \delta, p_1 + \delta) \subset \{\Phi_1 < \psi_{p_1}\}$, in addition to $(r_1, q_1) \subset \{\Phi_1 < \psi_{p_1}\}$. Thus by the Remark at the end of the first section, we have

$$|(p_1 - \delta) - q_1| < \pi/k.$$

Letting $\varepsilon$ tend to zero, we have $|q_1 - p_1| < \pi/k$. However, by the remark after the definition of wide sector, we see that $|q_1 - p_1| < \pi/k$.

We conclude from this that as we slide $p_2$ to the left, we must have $|p_2 - a| < |p_2 - q_2| < \pi/k$ unless the interval $(r_2, q_2) = \{\Phi < \psi_{p_2}\}$ disappears for some value, say $p = p_2$. It is clear, then, that the curve $\psi_{p_2}$ satisfies (7)–(9).

**Proof of the Theorem.** Let us start by choosing a sequence $\Phi_1 \geq \Phi_2 \geq \cdots$ of real analytic functions with $\Phi_j \to \Phi$. If there is an envelope $\tilde{\Phi}_j$ for each $j = 1, 2, \ldots$, then the sequence of envelopes $\tilde{\Phi}_1 \geq \tilde{\Phi}_2 \geq \cdots$ is decreasing and will converge to an upper semicontinuous function not identically $-\infty$, since $\int \tilde{\Phi}_j d\theta \geq 0$. Clearly $\tilde{\Phi} := \lim_{j \to \infty} \tilde{\Phi}_j$ will be our desired function. For the proof we will set $\Phi = \tilde{\Phi}$, and without loss of generality we assume $k > 1/2$.

Since we may replace $\Phi$ by a small $C^2$ perturbation, we assume that

$$\{\mathcal{L}\Phi < 0\} = E = E_1 \cup \cdots \cup E_m$$

is the union of a finite number of connected open intervals with $\overline{E_i} \cap \overline{E_j} = \emptyset$. Writing $E_j = (a_j, b_j)$, we suppose also that $\cdots < a_2 < b_2 < a_1 < b_1$. We will also define $\Phi$ to be a $C^2$ function on $\mathbb{R}$, which is periodic with period $2\pi$. 
We start with $\psi_\alpha$ as in the proof of Lemma 3, and we slide $p_3$ to the left. If we obtain a tangency $\psi_{p_3}$ for $b_2 \leq p_3 < a_1$ as in Figure 2, then the interval $E$ has been eliminated. The other possibility is that we arrive at $p = b$ without reaching a tangency. In this case, by the argument of Lemma 3, we have $|q_3 - p_3| < \pi/k$. Thus we may consider

$$\psi(\theta) = \psi_{p_3}(\theta) - \lambda \sin(k(\theta - p_3))$$

and increase $\lambda$ until a tangency $\hat{q} \in (p_3, q_3)$ is obtained (see Figure 3).

In the first case above, we will say that $E_1$ is covered by $\psi_{p_3}$. We will replace $\Phi$ by $\psi_{p_3}$ over the interval $(p_3, q_3)$, and the resulting curve will be $C^1$, and piecewise $C^2$. Since $|p_3 - q_3| < \pi/k$ and $k > 1/2$, we may extend the replacement by $\psi_{p_3}$ to be $2\pi$-periodic on $\mathbb{R}$.

In the second case, we will replace $\Phi$ by the function $\hat{\psi}_1$ on the interval $\tilde{E}_2 = (a_2, \hat{q}_1)$, as in Figure 3. We will call $\tilde{E}_2$ a temporary interval. The new curve we obtain is piecewise $C^2$, with a downward-opening angle at $a_2$. The Sturm Comparison Theorem continues to hold in this nonsmooth case, so we may apply Lemma 3 to conclude that $\tilde{E}_2$ has length $< \pi/k < 2\pi$. Thus we can extend the temporary interval to have period $2\pi$ on $\mathbb{R}$.

Now we proceed by decreasing induction on the number of uncovered intervals. By Lemma 4 below, if there is only one interval left (temporary or not yet touched), it will be covered by the sliding procedure. In Lemma 4 we will show that if we start at a temporary interval and start sliding to the left, then we will produce another temporary interval containing both $E_1$ and $E_2$.

As long as we obtain only temporary intervals, without a covering, we may continue similarly to obtain a temporary interval $\tilde{E}_j$ containing $E_1 \cup \cdots \cup E_{j-1}$. By hypothesis, there is no wide amalgamated sector, so there exist $a \in \mathbb{R}$ and finitely many sectors $E_1 \cup \cdots \cup E_m$ such that

$$\{\mathcal{L} \Phi < 0\} \cap (a, a + \pi/k) = E_1 \cup \cdots \cup E_m$$

and

$$\{\mathcal{L} \Phi \geq 0\} \supset [a - \pi/k, a].$$
If the next interval $\tilde{E}_{m+1}$ is temporary, it must span $[a - \pi/k, a]$ and thus have length $> \pi/k$. On the other hand, by Lemma 4, a temporary interval $\tilde{E}_{m+1}$ would be forced to have length $< \pi/k$. Thus it follows from Lemma 4 that this sliding procedure must in fact produce intervals that cover $E_j$ for $1 \leq j \leq m$.

Since, at each step, we reduce the total number of uncovered intervals, the proof is completed by Lemma 4.

**Lemma 4.** Let $\tilde{E}_2$ be a temporary interval given by $\hat{\psi}_1$. The procedure of sliding $\psi_p$, starting with $p = a_2$ and travelling to the left, will yield either a covering of $E_2$ or a new temporary interval $\tilde{E}_3$ containing $E_1 \cup E_2$. The interval $\tilde{E}_3$, if it exists, will have length $< \pi/k$. Thus if $\mathcal{P}\Phi > 0$ on $[a_2 - \pi/k, a_2]$ then this will yield a covering of $E_2$.

**Proof.** As in Lemma 2, we see that $\psi_{a_2}$ lies above $\Phi$ over $E_2$ and above $\psi_1$ over $(p_3, q_1)$. Now we slide $p$ to the left and obtain a function $\hat{\psi}_2$ which either covers $\tilde{E}_2$ or gives a temporary interval containing $E_2$. If the point $q_2$, where $\hat{\psi}_2$ is tangent to $\Phi$, lies to the right of $E_1$, then $\hat{\psi}_2$ gives a temporary interval $\tilde{E}_3$ containing both $E_1$ and $E_2$.

Otherwise, $q_2$ lies between $E_1$ and $E_2$, and so $\hat{\psi}_1$ and $\hat{\psi}_2$ cross at a point $\tilde{\theta}$ (see Figure 4). We show that in this case $|b_3 - v_2| < \pi/k$. By the construction of the temporary intervals, we have $|b_3 - q_2| < \pi/k, |b_2 - q_1| < \pi/k$.

Now we consider

$$\psi = \hat{\psi}_2 - \delta \sin(k(\theta - q_2))$$

and note that for $\varepsilon > 0$ sufficiently small, $(q_2 - \varepsilon, q_2) \subset (\Phi < \psi)$. Thus the amalgamated interval of $(q_2 - \varepsilon, q_2)$ in $(\Phi < \psi)$ contains $(b_3, v_2 - \varepsilon)$. Letting $\delta$ tend to zero, we have $|b_3 - v_2| \leq \pi/k$.

Now we may replace $\hat{\psi}_2$ by $\psi^\lambda(\theta) = \hat{\psi}_2(\theta) - \lambda \sin(k(\theta - b_3))$ and lower $\hat{\psi}_2$ until we obtain a function $\psi_3$ with a tangency $q_3 \in (q_2, v_2)$. If $q_3$ lies to the right of $E_1$, then the new temporary interval $\tilde{E}_3$ contains $E_1 \cup E_2$, and the proof of the lemma is complete. Otherwise, if $q_3 \in (q_2, b_1)$ then it is evident from Figure 4 that $\hat{\psi}_3$ will intersect $\psi_1$ at a point $\tilde{\theta}_3 \in (\hat{\theta}, b_1)$. By the comparison (10), we see that $\hat{\psi}_3(\theta) \geq \hat{\psi}_1(\theta)$ holds for $\hat{\theta}_3 < \theta < \hat{\theta}_3 + \pi/k$. In particular, $\hat{\psi}_3(q_1) > 0$, and so we may again increase $\lambda$ to find another tangency.
Thus it follows that whenever we reach a tangency $\hat{p}_j < b_1$ we have $\hat{\psi}_j(\hat{q}_1) > 0$, and we may increase $\lambda$ further to find another tangency $\hat{p}_{j+1} \in (\hat{p}_j, v_2)$. Clearly this process must end, i.e., we must have a tangency $\hat{p}_j \geq b_1$, since for $\lambda$ sufficiently large we have $\psi^\lambda(\hat{q}_1) < 0$. This completes the proof.

References

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DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47405