TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 292, Number 2, December 1985

LOCAL AND GLOBAL ENVELOPES OF HOLOMORPHY OF DOMAINS IN $\mathbf{C}^2$

BY
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ABSTRACT. A criterion is given for a smoothly bounded domain $D \subset \mathbf{C}^2$ to be locally extendible to a neighborhood of a point $z_0 \in \partial D$. (This result may also be formulated in terms of extension of CR functions on ∂D.) This is related to the envelope of holomorphy of the semitubular domain

$$\Omega(\Phi) = \left\{(z,w) \in \mathbf{C}^2 : \operatorname{Re} w + r^k\Phi(\theta) < 0\right\},$$

where $r = |z|$, $\theta = \arg(z)$. Necessary and sufficient conditions are given for the envelope of holomorphy of $\Omega(\Phi)$ to be $\mathbf{C}^2$. These conditions are equivalent to the existence of a subharmonic minorant for $r^k\Phi(\theta)$.

1. Introduction. Let us consider a smoothly bounded domain $D \subset \mathbf{C}^2$ and ask whether D is locally extendible at $p \in \partial D$, i.e. for every open set U containing p do all holomorphic functions on $D \cap U$ extend holomorphically through p?

This question has been answered when ∂D is pseudoconcave and real analytic at p (see **[3]**) and when ∂D has so-called "type k" with k odd (see **[2, 5, 10]**). The question of local extension of holomorphic functions from D is essentially equivalent to the question of local extension of CR functions from ∂D (see **[1, 8]**). However, we do not discuss CR functions further since our contribution is to deal with the geometric structure of certain envelopes, and we would like our presentation to be as self-contained as possible.

We may make a holomorphic change of coordinates (z,w) in a neighborhood of p such that $p = (0,0)$, $w = u + iv$, and that ∂D is given near p by the equation $u + p_k(z) + R(z,v) < 0$, where

$$p_k(z) = \sum_{j=1}^{k-1} a_j z^j \bar{z}^{k-j}$$

is a real, homogeneous polynomial of degree k, and the remainder is given by

$$R(z,v) = O\left(v^2 + |vz| + |z|^{k+1}\right) \tag{1}$$

Received by the editors July 13, 1984 and, in revised form, February 5, 1985.
1980 *Mathematics Subject Classification*. Primary 32D10; Secondary 31A05, 30D99.
Key words and phrases. Envelope of holomorphy, local extendibility, subharmonic minorant, CR extendibility.
[1]Research supported in part by the NSF.

0002-9947/85 $1.00 + $.25 per page

(cf. [3]). With a holomorphic change of variables $w' = w + \alpha w^2 + \beta z w$, $z' = z$, ∂D can be given by

$$u + cv^2 + R'(z, v) + p_k(z) < 0,$$

where $c \in \mathbf{R}$ is arbitrary, and

$$R'(z, w) = O\left(|v|^3 + |vz^2| + |z|^{k+1}\right).$$

We will be interested in domains that satisfy the following stronger condition: For any $\varepsilon > 0$, there exists $c > 0$ such that

$$(2) \qquad |R'(z, v)| = O\left(cv^2 + \varepsilon |z|^k\right).$$

If (2) holds, then for every $\varepsilon > 0$, there exists $\eta > 0$ and a change of coordinates as above such that

$$(3) \qquad \{|(z, w)| < \eta\} \cap D \subset \left\{u + p_k(z) < \varepsilon |z|^k\right\}.$$

Writing $z = re^{i\theta}$ and $p_k(z) = r^k \Phi(\theta)$, we see that the local study of D at $(0, 0)$ is related to the domain

$$\Omega(\Phi) = \left\{(z, w) \in \mathbf{C}^2 \colon \operatorname{Re} w + r^k \Phi(\theta) < 0\right\}.$$

To make the connection between D and $\Omega(\Phi)$, we will need to discuss (global) envelopes of holomorphy. The envelope of holomorphy $E(D)$ of a domain $D \subset \mathbf{C}^n$ is a Riemann domain $\pi\colon E(D) \to \mathbf{C}^n$ with $i\colon D \to E(D)$ and $E(D)$ is the minimal domain of holomorphy such that every function $f \in \mathcal{O}(D)$ extends holomorphically to $E(D)$. A convenient method for staying within the class of domains in $\mathbf{C}^n$ while taking envelopes is to consider D which are starshaped with respect to the origin, i.e., $\delta_t(D) \subset D$, where $\delta_t(z) = (tz_1, \ldots, tz_n)$, $0 \leqslant t \leqslant 1$. *If $D \subset \mathbf{C}^n$ is starshaped, then the envelope is a starshaped domain in $\mathbf{C}^n$ with $D \subset E(D) \subset \mathbf{C}^n$.* To prove this assertion it suffices to show that the projection π is one-to-one. The mapping δ_t has a holomorphic continuation to a map $\tilde{\delta}_t$: $E(D) \to E(D)$. We note that $\pi\tilde{\delta}_t = \delta_t \pi$, $\tilde{\delta}_1$ is the identity map, and $\tilde{\delta}_0 = \lim_{t \to 0} \tilde{\delta}_t$ is the constant $i(0)$. Let z_1, $z_2 \in E(D)$ be points such that $\pi(z_1) = \pi(z_2)$, and let σ_j, $j = 1, 2$, be the path given by $\gamma_j(t) = \tilde{\delta}_t(z_j)$, $0 \leqslant t \leqslant 1$.

Now σ_1 and σ_2 project under π to the same path in $\mathbf{C}^n$, and $\gamma_1(0) = \gamma_2(0) = i(0)$. Since π is locally invertible, the paths σ_1 and σ_2 coincide, and thus $z_1 = \gamma_1(1) = \gamma_2(1) = z_2$.

It follows (e.g. from a result of Docquier and Grauert [7]), that *if D is starshaped, then it is a Runge domain*, i.e. every holomorphic function on D may be uniformly approximated by polynomials on compact subsets.

The domain $\Omega(\Phi)$ is invariant under the transformations

$$(4) \qquad (z, w) \to (z, w + \zeta), \qquad \zeta \in \mathbf{C}, \operatorname{Re}\zeta < 0,$$

$$(5) \qquad (z, w) \to (tz, t^k w), \qquad 0 < t < \infty.$$

The envelope of holomorphy has the same invariance and is thus given by

$$E(\Omega(\Phi)) = \left\{(z, w) \in \mathbf{C}^2 \colon \operatorname{Re} w + r^k \tilde{\Phi}(\theta) < 0\right\} = \Omega(\tilde{\Phi}),$$

where $r^k \tilde{\Phi}(\theta)$ is the greatest subharmonic minorant of $r^k \Phi(\theta)$. (This is a special case of a result on semitubular domains, see [6].)

We may approximate $\Omega(\Phi)$ by the truncated domain

$$\Omega_\lambda(\Phi) = \Omega(\Phi) \cap \{|z| < \lambda, |v| < \lambda^k, |u| < c\lambda^k\}$$

for $0 < \lambda < \infty$. Since $\Omega_\lambda(\Phi)$ is starshaped with respect to $(0, -c\lambda^k/2)$ for c sufficiently large, the envelope is again starshaped. Further, $\Omega_1(\Phi)$ is mapped biholomorphically to $\Omega_t(\Phi)$ by the transformation (5), and so $E(\Omega_1(\Phi))$ is also mapped to $E(\Omega_t(\Phi))$. Thus

$$E(\Omega(\Phi)) = \bigcup_\lambda E(\Omega_\lambda(\Phi)),$$

and so $(0,0) \in E(\Omega(\Phi))$ if and only if $(0,0) \in E(\Omega_\lambda(\Phi))$ for all λ.

The question of local extendibility of D at $(0,0)$ is tied to the global question for $\Omega(\Phi)$: Does $(0,0)$ belong to the envelope of holomorphy $E(\Omega(\Phi))$ of $\Omega(\Phi)$? There are two possibilities:

(i) $(0,0) \in E(\Omega_\lambda(\Phi))$, and in this case $E(\Omega(\Phi)) = \mathbf{C}^2$.

(ii) $(0,0) \notin E(\Omega_\lambda(\Phi))$, and $E(\Omega(\Phi)) = \Omega(\tilde{\Phi})$ with $\tilde{\Phi}$ not identically $-\infty$.

PROPOSITION. *If there exists* $\varepsilon > 0$ *such that* $E(\Omega(\Phi + \varepsilon)) = \mathbf{C}^2$, *then for all open* U *continuing* $(0,0)$, *every analytic function on* $U \cap D$ *extends analytically to a neighborhood of* $(0,0)$.

Conversely, if D *satisfies* (2), *and if* $E(\Omega(\Phi - \varepsilon)) \neq \mathbf{C}^2$ *for some* $\varepsilon > 0$, *then there exists* $\eta > 0$ *and a function*

$$f \in \mathcal{O}\big(D \cap \{|(z,w)| < n\}\big)$$

which cannot be extended holomorphically past $(0,0)$.

PROOF. If $(0,0)$ is in the envelope of $E(\Omega(\Phi + \varepsilon))$, there is a compact $K \subset \Omega(\Phi + \varepsilon)$ such that $|f(0,0)| \leqslant |f|_K$ for all $f \in \mathcal{O}(\Omega(\Phi + \varepsilon))$. Since K is compact, we may shrink ε if necessary, so that $K \subset \omega_\varepsilon$, where

$$\omega_\varepsilon = \left\{u + p_k(z) + \varepsilon|z|^k + \varepsilon|v|\right\} < 0.$$

By (1), we may choose η sufficiently small such that $D \supset \{|(z,w)| < \eta\} \cap \omega_\varepsilon$. Now ω_ε is invariant under the transformation (5), so we may apply (5) to K with t small to have $K \subset \{|(z,w)| < \eta\} \cap \omega_\varepsilon$.

Finally, since $D \cap \{|(z,w)| < \eta\}$ is starshaped for η small, it is Runge. Thus, $f \in \mathcal{O}(D \cap \{|(z,w)| < \eta\})$ may be approximated by polynomials uniformly on K. Since $(0,0)$ is in the hull of K, we may extend f past $(0,0)$.

Now we prove the converse statement. If D satisfies (2), then we have (3), and so for $\Psi = \Phi - \varepsilon$

$$D \cap \{|(z,w)| < \eta\} \subset \Omega(\tilde{\Psi}).$$

Since $\Omega(\tilde{\Psi})$ is a domain of holomorphy there exists $f \in \mathcal{O}(\Omega(\tilde{\Psi}))$ which cannot be continued past $(0,0)$.

REMARKS. The first part of the Proposition can be used to give sufficient conditions for local extension of functions from domains $D \subset \mathbf{C}^n$. For this, let P be a complex 2-plane intersecting ∂D transversally at $z_0 \in \partial D$. If $D \cap P$ satisfies the first hypotheses of the Proposition in a neighborhood of z_0 in P, then there is a compact $K \subset D \cap P$ such that z_0 is in its polynomial hull. For $\varepsilon > 0$ sufficiently small, a closed ε-neighborhood K^ε of K is contained in D. Since K^ε contains all ε-translates of K, the polynomial hull of K^ε contains all ε-translates of z_0, i.e. an ε-neighborhood of z_0. Thus if we have local extension in a 2-dimensional slice of D, we have local extension from D.

By writing the Laplacian in polar coordinates,

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2},$$

we see that $\Delta(r^k\tilde{\Phi}(\theta)) \geqslant 0$ if and only if $\mathscr{L}\tilde{\Phi} \geqslant 0$, where $\mathscr{L} = d^2/d\theta^2 + k^2$. Of course, $\mathscr{L}\psi = 0$ if and only if $\psi(\theta) = c\sin(k\theta) + d\cos(k\theta)$ and in this case $r^k\psi(\theta) = \operatorname{Re}((d - ic)z^k)$. The intervals in θ where $\tilde{\Phi}$ is positive or negative are of some importance. If $\mathscr{L}\tilde{\Phi} > 0$, then the intervals where $\{\Phi < 0\}$ have length $< \pi/k$ and the intervals where $\{\Phi > 0\}$ have length $> \pi/k$. This follows from (10) below.

It is also useful to adjoin nearby intervals.

DEFINITION. Given an open set $\mathscr{O} \subset \mathbf{R}$, the *amalgamated component* $\tilde{I}$ of an interval $I \subset \mathscr{O}$ is the smallest connected, open interval $\tilde{I} \supset I$ with the property: If $J \subset \mathscr{O}$ is an open interval with $\operatorname{dist}(J, I) < \pi/k$, then $J \subset \tilde{I}$.

DEFINITION. An upper semicontinuous periodic function Φ on $\mathbf{R}$ with period 2π has a *wide* (*amalgamated*) *sector* if either

(i) $0 < k \leqslant 1/2$, and $\Phi(\theta) < 0$ for some θ, or

(ii) $k > 1/2$, and there exist c_1, $c_2 \in \mathbf{R}$ and $\varepsilon > 0$ such that an (amalgamated) component of

$$\mathscr{O}(\varepsilon, c_1, c_2) = \{\theta \in \mathbf{R}: \Phi(\theta) + \varepsilon + c_1\sin(k\theta) + c_2\cos(k\theta) < 0\}$$

has length $\geqslant \pi/k$.

Note that the length will be $> \pi/k$ if we take $\varepsilon > 0$ smaller. By this same remark we see also that if Φ is continuous and has no wide sectors, then for $0 < c < \infty$, there exists $\varepsilon_0 > 0$ such that every connected component of $\mathscr{O}(\varepsilon, c_1, c_2)$ has length $\leqslant \pi/k - \varepsilon_0$ if $|c_1| + |c_2| < c$ and $0 < \varepsilon \leqslant \varepsilon_0$.

THEOREM. *Let Φ be periodic and u.s.c. on* $[0, 2\pi]$. *Then the envelope of holomorphy* $E(\Omega(\Phi)) = \mathbf{C}^2$ *if and only if $\Phi + \varepsilon$ has a wide amalgamated sector for some $\varepsilon > 0$.*

REMARK. The "only if" part of the Theorem is easily seen. If $E(\Omega(\Phi)) \neq \mathbf{C}^2$, then there is a subharmonic $r^k\tilde{\Phi}(\theta) \leqslant r^k\Phi(\theta)$. Thus each interval of $\{\Phi + \varepsilon < 0\}$ lies in an interval of $\{\tilde{\Phi} + \varepsilon < 0\}$, which has length $< \pi/k$, since $\mathscr{L}(\tilde{\Phi} + \varepsilon) > 0$. Further, since the sectors of $\{\tilde{\Phi} + \varepsilon < 0\}$ are separated by a distance $> \pi/k$, the amalgamated components of $\{\Phi + \varepsilon < 0\}$ lie in the components of $\{\tilde{\Phi} + \varepsilon < 0\}$.

REMARK. The works [**2** and **9, 10**] use the weaker "*sector property*", which is just that Φ has a wide sector. We note that if Φ does not have the sector property, and if I_1 and I_2 are intervals of $\mathscr{O}(\varepsilon, c_1, c_2)$, and if $\operatorname{dist}(I_1, I_2) < \pi/k$, then $I_1 \cup I_2$ is contained in an interval of length $< \pi/k$.

(To see this, we may assume, to the contrary, that $0 \in I_1$ and $\pi/k \in I_2$. Then we make c_1 very large and negative so that $[0, \pi/k] \subset \mathscr{O}(\varepsilon, c_1, c_2)$.)

From this we conclude that if Φ has the sector property, and if $\mathcal{O}(\varepsilon, c_1, c_2)$ contains no more than two intervals (for all ε, c_1, c_2), then Φ has a wide amalgamated sector. The case $k = 4$, which was treated in [**2**], is a special case of this situation.

ACKNOWLEDGEMENT. We wish to thank J.-P. Rosay for several stimulating conversations on this material, and we are grateful to J. E. Fornaess for a timely remark. Fornaess and Rea have recently obtained related results [**11**] (independently of our work) using methods of [**4**]. This paper was written while the author was visiting the University of North Carolina, and he is grateful for their hospitality.

2. Construction of the envelope. Since $r^k\tilde{\Phi}(\theta)$ is subharmonic and constant on the sets $\{\theta = \text{const}\}$, it follows that $\tilde{\Phi}$ is bounded. Further, since $\mathcal{L}\Phi \geqslant 0$, we have $\tilde{\Phi}'' \geqslant -\text{const}$, and so $\tilde{\Phi} \in C^1$. Thus if the envelope $E(\Omega(\tilde{\Phi})) \neq \mathbf{C}^2$, and if $k > 1$, the boundary $\partial E(\Omega(\tilde{\Phi}))$ is C^1 smooth. In general, however, $\tilde{\Phi} \notin C^2$.

We may approximate $\tilde{\Phi} + \delta$ from below by $\tilde{\Phi}_\varepsilon + \delta_\varepsilon$, where $\tilde{\Phi}_\varepsilon = \tilde{\Phi}^*\chi_\varepsilon$ is a usual smoothing in θ, and $0 < \delta_\varepsilon < \delta$, $\lim_{\varepsilon \to 0}\delta_\varepsilon = \delta$. Thus

$$(6) \qquad (\widetilde{\Phi + \delta})(\theta) = \sup\{h(\theta): h \text{ is of class } C^2,\ h \leqslant \Phi + \delta,\ \mathcal{L}h \geqslant 0\}.$$

REMARK. In terms of the envelope (6) our question is whether the competing family of subsolutions is nonempty. Thus an alternative statement of our Theorem is: *$r^k\Phi(\theta)$ has a subharmonic minorant if and only if $\Phi(\theta) + \varepsilon$ does not have a wide amalgamated sector for any $\varepsilon > 0$.*

The envelope formulation (6) also suggests the structure of $\tilde{\Phi}$:

$$(7) \qquad \mathcal{L}\tilde{\Phi} = 0 \quad \text{on } \mathcal{O} = \{\tilde{\Phi} < \Phi\},$$

$$(8) \qquad \Phi = \tilde{\Phi} \quad \text{and} \quad \nabla\Phi = \nabla\tilde{\Phi} \quad \text{on } \partial\mathcal{O}.$$

We will construct $\tilde{\Phi}$ in the manner suggested by Figure 1. If $E = \{\mathcal{L}\Phi < 0\}$ is the set where the Levi form is negative, we must have $E \subset \{\tilde{\Phi} < \Phi\}$, and $\tilde{\Phi}$ is obtained by patching solutions ψ_j of $\mathcal{L}\psi_j = 0$ onto Φ so that they satisfy (7) and (8) above.

The last feature of the construction we shall require is

$$(9) \qquad \text{each interval in } \mathcal{O} = \{\Phi < \tilde{\Phi}\} \text{ has length } < \pi/k.$$

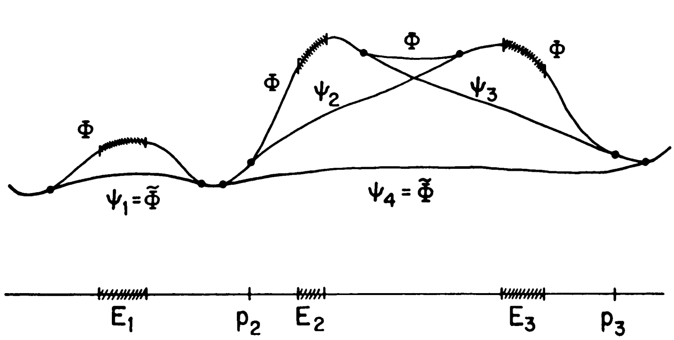

FIGURE 1

Without (9), the solution constructed according to Figure 1 is not unique. For instance, if $\Phi(\theta) = \sin(k\theta) + 1$, then $\mathscr{L}\Phi > 0$, and $\Phi = \tilde{\Phi}$. If we take $\psi_1(\theta) = 0$, $-\pi/2k < \theta < 3\pi/2k$, and equal to $\tilde{\Phi}$ for other values of θ, then the resulting solution $\tilde{\tilde{\Phi}}$ satisfies $\mathscr{L}\tilde{\tilde{\Phi}} \geqslant 0$, but $\{\tilde{\tilde{\Phi}} < \tilde{\Phi}\} = (-\pi/2k, 3\pi/2k)$.

We will use the following version of the Sturm Comparison Theorem (see [4]):

$$\text{(10)} \qquad \begin{aligned} &\text{if } \psi_1, \psi_2 \in C^2 \text{ and } \mathscr{L}\psi_1 \geqslant \mathscr{L}\psi_2, \text{ and if} \\ &\psi_1(\theta_0) = \psi_2(\theta_0),\ \psi_1'(\theta_0) \geqslant \psi_2'(\theta_0), \text{ then} \\ &\psi_1(\theta) \geqslant \psi_2(\theta) \text{ for } \theta_0 < \theta < \theta_0 + \pi/k. \end{aligned}$$

To prove (10), we consider $\psi = \psi_1 - \psi_2$, and we may add $\varepsilon((\theta - \theta_0) + (\theta - \theta_0)^2)$ so that $\psi'(\theta_0) > 0$ and $\mathscr{L}\psi > 0$ on $(\theta_0, \theta_0 + \pi/k)$. Now we will show that $\psi > 0$ on $(\theta_0, \theta_0 + \pi/k)$. Let $\theta_1 > \theta_0$ be the first point where $\psi(\theta_1) = 0$. We may assume $\psi'(\theta_1) < 0$. We set

$$h(\theta) = \arctan(\psi'(\theta)/k\psi(\theta)).$$

Since $h(\theta_0) = +\pi/2$ and $h(\theta_1) = -\pi/2$ we have

$$\int_{\theta_0}^{\theta_1} h'(\theta)\, d\theta = -\pi.$$

Further, since $\mathscr{L}\psi > 0$, we have $\psi\psi'' > -k^2\psi^2$, and with this we may compute that $h'(\theta) > -k$. Thus we have

$$-\pi = \int_{\theta_0}^{\theta_1} h'(\theta)\, d\theta > -(\theta_1 - \theta_0)k,$$

and so $\theta_1 - \theta_0 > \pi/k$ which yields (10).

We will use the notation ψ_p for the function

$$\psi_p(\theta) = c\sin(k\theta) + d\cos(k\theta)$$

such that $\psi_p(p) = \Phi(p)$ and $\psi_p'(p) = \Phi'(p)$.

Some properties of ψ_p are formulated in the following lemmas and are illustrated in Figure 2.

LEMMA 1. *If $\mathscr{L}\Phi(p) > 0$, then there exists $\varepsilon > 0$ such that $\psi_p(\theta) \leqslant \Phi(\theta)$ for $\theta \in (p - \varepsilon, p + \varepsilon)$. If $\mathscr{L}\Phi(p) > 0$ for $p_2 \leqslant p \leqslant p_1$, then $\psi_{p_2}(\theta) < \psi_{p_1}(\theta)$ for $p_1 < \theta < p_2 + \pi/k$.*

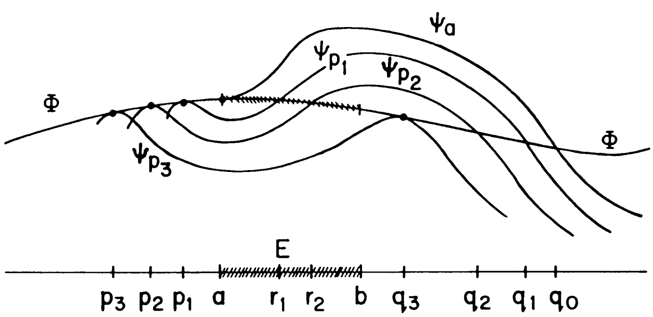

FIGURE 2

Proof. The first statement is just the comparison (10). The second statement also follows from (10). If we replace ψ_p by $\tilde{\psi}_q = \psi_q - \Phi$, then for $|q - p|$ small

$$\tilde{\psi}_q(\theta) = -k(q)(\theta - q)^2 + o((\theta - q)^2),$$

and $k(q) > 0$. If $p_2 < p_1 < p$, and $|p_2 - p|$ is small, then $\tilde{\psi}_{p_1}$ and $\tilde{\psi}_{p_2}$ will intersect at a point $q \in (p_2, p_1)$. Thus ψ_{p_1} and ψ_{p_2} will intersect as in Figure 2, and so by (10) we have $\psi_{p_1}(\theta) > \psi_{p_2}(\theta)$ for $\theta \in (q, q + \pi/k)$.

Lemma 2. *Let Φ have no wide sectors. If (a, b) is an open interval on which $\mathscr{L}\Phi < 0$, then $\psi_a(\theta) > \Phi(\theta)$ for $\theta \in (a, b]$.*

Proof. By (10), $\psi_a(\theta) > \Phi(\theta)$ holds for $a < \theta < \min(a + \pi/k,\ b)$. Thus the result holds unless $a + \pi/k < b$. But in this case we have $(a, a + \pi/k) \subset \{\Phi - \psi_a < 0\}$ which is a wide sector.

Lemma 3. *If $\Phi - \delta$ has no wide sectors for some $\delta > 0$ and if $E = \{\mathscr{L}\Phi < 0\}$ consists of a single interval $E = (a, b)$, then $\tilde{\Phi}$ exists.*

Proof. Note that if $E \neq \varnothing$, then by definition $k > 1/2$. By Lemma 2, $\psi_a(\theta) > \Phi(\theta)$ for $\theta \in E$. And by Lemma 1, $\psi_p(\theta) < \psi_a(\theta)$ holds for $p < a$ and $a < \theta < p + \pi/k$. Further, we claim that there is a wide sector unless $|q - p| < \pi/k$ holds for all p (q is the point where ψ_p crosses Φ from above). First, it is evident that $|a - q_0| < \pi/k$. Thus for p_1 near a, it follows that $|p_1 - r_1| < \pi/k$, where we write $\{\Phi < \psi_{p_1}\} \cap (p_1, q_0) = (r_1, q_1)$. Replacing Φ by

$$\Phi_1 = \Phi - \varepsilon \sin(k(\theta - p_1 + \varepsilon))$$

for $\varepsilon > 0$ small, we obtain a small interval $(p_1 - \delta, p_1 + \delta) \subset \{\Phi_1 < \psi_{p_1}\}$, in addition to $(\tilde{r}_1, \tilde{q}_1) \subset \{\Phi_1 < \psi_{p_1}\}$. Thus by the Remark at the end of the first section, we have

$$|(p_1 - \delta) - \tilde{q}_1| < \pi/k.$$

Letting ε tend to zero, we have $|q_1 - p_1| \leqslant \pi/k$. However, by the remark after the definition of wide sector, we see that $|q_1 - p_1| < \pi/k$.

We conclude from this that as we slide p_2 to the left, we must have $|p_2 - a| < |p_2 - q_2| < \pi/k$ unless the interval $(r_2, q_2) = \{\Phi < \psi_{p_2}\}$ disappears for some value, say $p = p_3$. It is clear, then, that the curve ψ_{p_3} satisfies (7)–(9).

Proof of the Theorem. Let us start by choosing a sequence $\Phi_1 \geqslant \Phi_2 \geqslant \cdots$ of real analytic functions with $\Phi_j \to \Phi$. If there is an envelope $\tilde{\Phi}_j$ for each $j = 1, 2, \ldots$, then the sequence of envelopes $\tilde{\Phi}_1 \geqslant \tilde{\Phi}_2 \geqslant \cdots$ is decreasing and will converge to an upper semicontinuous function not identically $-\infty$, since $\int \tilde{\Phi}_j \, d\theta \geqslant 0$. Clearly $\tilde{\Phi} := \lim_{j \to \infty} \tilde{\Phi}_j$ will be our desired function. For the proof we will set $\Phi = \Phi_j$, and without loss of generality we assume $k > 1/2$.

Since we may replace Φ by a small C^2 perturbation, we assume that

$$\{\mathscr{L}\Phi < 0\} = E = E_1 \cup \cdots \cup E_m$$

is the union of a finite number of connected open intervals with $\overline{E}_i \cap \overline{E}_j = \varnothing$. Writing $E_j = (a_j, b_j)$, we suppose also that $\cdots < a_2 < b_2 < a_1 < b_1$. We will also define Φ to be a C^2 function on $\mathbf{R}$, which is periodic with period 2π.

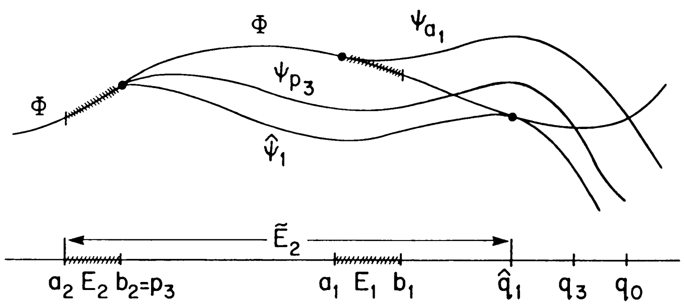

FIGURE 3

We start with ψ_a as in the proof of Lemma 3, and we slide p_3 to the left. If we obtain a tangency ψ_{p_3} for $b_2 \leqslant p_3 < a_1$ as in Figure 2, then the interval E has been elminated. The other possibility is that we arrive at $p = b$ without reaching a tangency. In this case, by the argument of Lemma 3, we have $|q_3 - p_3| < \pi/k$. Thus we may consider

$$\psi(\theta) = \psi_{p_3}(\theta) - \lambda \sin(k(\theta - p_3))$$

and increase λ until a tangency $\hat{q} \in (p_3, q_3)$ is obtained (see Figure 3).

In the first case above, we will say that E_1 is *covered* by ψ_{p_3}. We will replace Φ by ψ_{p_3} over the interval (p_3, q_3), and the resulting curve will be C^1, and piecewise C^2. Since $|p_3 - q_3| < \pi/k$ and $k > 1/2$, we may extend the replacement by ψ_{p_3} to be 2π-periodic on $\mathbf{R}$.

In the second case, we will replace Φ by the function $\hat{\psi}_1$ on the interval $\tilde{E}_2 = (a_2, \hat{q}_1)$, as in Figure 3. We will call $\tilde{E}_2$ a *temporary* interval. The new curve we obtain is piecewise C^2, with a downward-opening angle at a_2. The Sturm Comparison Theorem continues to hold in this nonsmooth case, so we may apply Lemma 3 to conclude that $\tilde{E}_2$ has length $< \pi/k < 2\pi$. Thus we can extend the temporary interval to have period 2π on $\mathbf{R}$.

Now we proceed by decreasing induction on the number of uncovered intervals. By Lemma 4 below, if there is only one interval left (temporary or not yet touched), it will be covered by the sliding procedure. In Lemma 4 we will show that if we start at a temporary interval and start sliding to the left, then we will produce another temporary interval containing both E_1 and E_2.

As long as we obtain only temporary intervals, without a covering, we may continue similarly to obtain a temporary interval $\tilde{E}_j$ containing $E_1 \cup \cdots \cup E_{j-1}$. By hypothesis, there is no wide amalgamated sector, so there exist $a \in \mathbf{R}$ and finitely many sectors $E_1 \cup \cdots \cup E_m$ such that

$$\{\mathscr{L}\Phi < 0\} \cap (a, a + \pi/k) = E_1 \cup \cdots \cup E_m$$

and

$$\{\mathscr{L}\Phi \geqslant 0\} \supset [a - \pi/k, a].$$

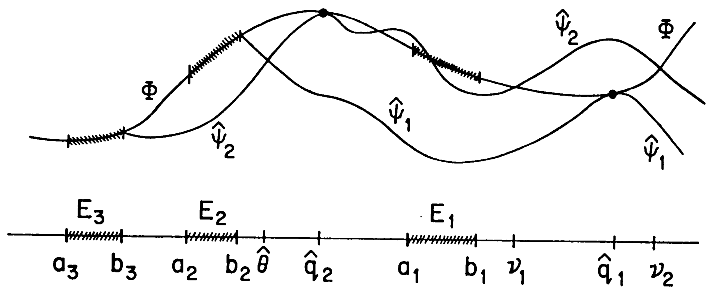

FIGURE 4

If the next interval $\tilde{E}_{m+1}$ is temporary, it must span $[a - \pi/k, a]$ and thus have length $> \pi/k$. On the other hand, by Lemma 4, a temporary interval $\tilde{E}_{m+1}$ would be forced to have length $< \pi/k$. Thus it follows from Lemma 4 that this sliding procedure must in fact produce intervals that cover E_j for $1 \leqslant j \leqslant m$.

Since, at each step, we reduce the total number of uncovered intervals, the proof is completed by Lemma 4.

LEMMA 4. *Let $\tilde{E}_2$ be a temporary interval given by $\hat{\psi}_1$. The procedure of sliding ψ_p, starting with $p = a_2$ and travelling to the left, will yield either a covering of $\tilde{E}_2$ or a new temporary interval $\tilde{E}_3$ containing $E_1 \cup E_2$. The interval $\tilde{E}_3$, if it exists, will have length $< \pi/k$. Thus if $\mathscr{L}\Phi \geqslant 0$ on $[a_2 - \pi/k, a_2]$ then this will yield a covering of $\tilde{E}_2$.*

PROOF. As in Lemma 2, we see that ψ_{a_2} lies above Φ over E_2 and above $\hat{\psi}_1$ over $(p_3, \hat{q}_1)$. Now we slide p to the left and obtain a function $\hat{\psi}_2$ which either covers $\tilde{E}_2$ or gives a temporary interval containing E_2. If the point $\hat{q}_2$, where $\hat{\psi}_2$ is tangent to Φ, lies to the right of E_1, then $\hat{\psi}_2$ gives a temporary interval $\tilde{E}_3$ containing both E_1 and E_2.

Otherwise, $\hat{q}_2$ lies between E_1 and E_2, and so $\hat{\psi}_1$ and $\hat{\psi}_2$ cross at a point $\hat{\theta}$ (see Figure 4). We show that in this case $|b_3 - \nu_2| \leqslant \pi/k$. By the construction of the temporary intervals, we have $|b_3 - \hat{q}_2| < \pi/k$, $|b_2 - \hat{q}_1| < \pi/k$.

Now for $\delta > 0$ we consider

$$\psi = \hat{\psi}_2 - \delta \sin(k(\theta - \hat{q}_2))$$

and note that for $\varepsilon > 0$ sufficiently small, $(\hat{q}_2 - \varepsilon, \hat{q}_2) \subset \{\Phi < \psi\}$. Thus the amalgamated interval of $(\hat{q}_2 - \varepsilon, \hat{q}_2)$ in $\{\Phi < \psi\}$ contains $(b_3, \nu_2 - \varepsilon')$. Letting δ tend to zero, we have $|b_3 - \nu_2| \leqslant \pi/k$.

Now we may replace $\hat{\psi}_2$ by $\psi^\lambda(\theta) = \hat{\psi}_2(\theta) - \lambda \sin(k(\theta - b_3))$ and lower $\hat{\psi}_2$ until we obtain a function $\hat{\psi}_3$ with a tangency $\hat{q}_3 \in (\hat{q}_2, \nu_2)$. If $\hat{q}_3$ lies to the right of E_1, then the new temporary interval $\tilde{E}_3$ contains $E_1 \cup E_2$, and the proof of the lemma is complete. Otherwise, if $\hat{q}_3 \in (\hat{q}_2, b_1)$ then it is evident from Figure 4 that $\hat{\psi}_3$ will intersect ψ_1 at a point $\hat{\theta}_3 \in (\hat{\theta}, b_1)$. By the comparison (10), we see that $\hat{\psi}_3(\theta) \geqslant \hat{\psi}_1(\theta)$ holds for $\hat{\theta}_3 < \theta < \hat{\theta}_3 + \pi/k$. In particular, $\hat{\psi}_3(q_1) > 0$, and so we may again increase λ to find another tangency.

Thus it follows that whenever we reach a tangency $\hat{p}_j < b_1$ we have $\hat{\psi}_j(\hat{q}_1) > 0$, and we may increase λ further to find another tangency $\hat{p}_{j+1} \in (\hat{p}_j, \nu_2)$. Clearly this process must end, i.e., we must have a tangancy $\hat{p}_j \geqslant b_1$, since for λ sufficiently large we have $\psi^\lambda(\hat{q}_1) < 0$. This completes the proof.

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