

NONCOMMUTATIVE ALGEBRAS OF DIMENSION THREE OVER INTEGRAL SCHEMES

BY

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ABSTRACT. In this article we describe the algebraic data which is equivalent to giving an associative, noncommutative algebra \mathcal{O}_X over an integral k -scheme Y (where k is an algebraically closed field of characteristic $\neq 3$), which is locally free of rank 3. The description allows us to conclude that, essentially, all such are locally upper triangular 2×2 matrices, with degenerations of a restricted form allowed.

0. Introduction. Let k be an algebraically closed field of characteristic unequal to 3, and let Y be an integral k -scheme. In this article we will describe the data necessary and sufficient to construct a noncommutative \mathcal{O}_Y -algebra \mathcal{O}_X which is associative with identity, and which is locally free of rank 3 as an \mathcal{O}_Y -module.

The “obvious” construction for such an algebra is to take locally free \mathcal{O}_Y -module F of rank 2, and a nowhere zero section $s: \mathcal{O}_Y \rightarrow F$. The subbundle L of F generated by s will be of rank 1, and the algebra \mathcal{O}_X of endomorphisms of F preserving L will be locally free of rank 3 over \mathcal{O}_Y ; locally, \mathcal{O}_X is isomorphic to the algebra of upper triangular 2×2 matrices. Our main theorem is that all noncommutative algebras of rank 3 are obtained using a similar construction, where the section s is allowed to have zeros; a precise statement is given in Theorem 10.

The method used to analyze these algebras is similar to that employed in [1], where the commutative case was studied, and some algebro-geometric applications were made. It is somewhat surprising that the answer in the noncommutative case is much simpler than in the commutative case.

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1. The local analysis. Let \mathcal{O}_X be a noncommutative \mathcal{O}_Y -algebra of rank 3. Since $\text{char } k \neq 3$, the natural inclusion of \mathcal{O}_Y into \mathcal{O}_X is split by one-third of the trace map. Let $E \subset \mathcal{O}_X$ be the locally free rank 2 submodule of \mathcal{O}_X consisting (locally) of those elements whose trace is zero; in this case we have $\mathcal{O}_X \cong \mathcal{O}_Y \oplus E$ as \mathcal{O}_Y -modules. The multiplication in \mathcal{O}_X is an \mathcal{O}_Y -linear map $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{O}_X \rightarrow \mathcal{O}_X$, and is induced from the multiplication $\phi: E \otimes_{\mathcal{O}_Y} E \rightarrow \mathcal{O}_X \cong \mathcal{O}_Y \oplus E$ of elements of E ; the other factors of the multiplication in \mathcal{O}_X are the natural multiplication in \mathcal{O}_Y and the left and right \mathcal{O}_Y -module structure on E . We are thus naturally led to the following question: What properties does the map ϕ enjoy in this situation? Conversely, we can ask: Which maps ϕ induce a noncommutative associative multiplication on $\mathcal{O}_Y \oplus E$ for which E is the “trace zero” submodule? The answer, locally, is given by the following

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PROPOSITION 1. Let \mathcal{O}_Y be a local integral domain of finite type over k , E a free rank 2 \mathcal{O}_Y -module, and $\phi: E \otimes_{\mathcal{O}_Y} E \rightarrow \mathcal{O}_Y \oplus E$ an \mathcal{O}_Y -linear map. Let $\mathcal{O}_X = \mathcal{O}_Y \oplus E$ be the \mathcal{O}_Y -algebra whose multiplication is induced by ϕ . Then \mathcal{O}_X is noncommutative, associative, and has E as its trace zero elements if and only if ϕ has the form

$$\begin{aligned} \phi(z \otimes z) &= 2a^2 + az, & \phi(z \otimes w) &= 2ab + 2bz - aw, \\ \phi(w \otimes z) &= 2ab - bz + 2aw, & \phi(w \otimes w) &= 2b^2 + bw, \end{aligned}$$

where $\{z, w\}$ is any basis for E , and a and b are elements of \mathcal{O}_Y , not both zero.

REMARK. The above form is independent of the choice of basis for E . This is implied by the proposition, but can also be checked directly quite easily.

PROOF. We begin by proving the ‘‘only if’’ part, and assume that ϕ induces a noncommutative associative algebra structure on $\mathcal{O}_X = \mathcal{O}_Y \oplus E$, with trace zero elements E .

The general map ϕ can be written in the form

$$\begin{aligned} \phi(z \otimes z) &= i + az + gw, & \phi(z \otimes w) &= j + cz + fw, \\ \phi(w \otimes z) &= k + ez + dw, & \phi(w \otimes w) &= l + hz + bw, \end{aligned}$$

where a, b, \dots, l are in \mathcal{O}_Y . Associativity in \mathcal{O}_X is implied by the associativity of triple products of basis elements $(e_1 e_2) e_3 = e_1 (e_2 e_3)$ for $e_i \in \{z, w\}$. Therefore we must have the following equations in \mathcal{O}_X :

$$\begin{aligned} (z^2)z &= z(z^2), & (w^2)w &= w(w^2), \\ (z^2)w &= z(zw), & (w^2)z &= w(wz), \\ (zw)z &= z(wz), & (wz)w &= w(zw), \\ (wz)z &= w(z^2), & (zw)w &= z(w^2). \end{aligned}$$

Using the above general form for the multiplication map ϕ , we can compute both sides of each of these 8 equations in terms of $1, z$, and w (which are a basis for \mathcal{O}_X), and equate the three coefficients: This produces the explicit conditions on a, b, \dots, l for the multiplication in \mathcal{O}_X to be associative.

For the two equations $(z^2)z = z(z^2)$ and $(w^2)w = w(w^2)$, this gives the following six conditions: $gj = gk, ge = gc, gd = gf, hj = hk, hf = hd$, and $he = hc$. If either g or h is nonzero, then $j = k, e = c$, and $d = f$ is forced, since \mathcal{O}_Y is an integral domain. However, this implies that $\phi(z \otimes w) = \phi(w \otimes z)$ and so the multiplication in \mathcal{O}_X would be commutative, contrary to assumption. Therefore, we must have

$$(1) \quad g = h = 0.$$

The other six associations produce 18 equations in the coefficients, of which 2 are identities and 2 are redundant; the remaining 14 are

$$\begin{aligned} aj &= ci + fj, & bk &= ek + dl, \\ j + fc &= 0, & k + ed &= 0, \\ i + af &= f^2, & l + be &= e^2, \\ ci + fk &= ei + dj, & ej + dl &= ck + fl, \\ j + ca + fe &= k + ea + dc, & k + ef + db &= j + cd + fb, \\ ei + dk &= ak, & cj + fl &= bj, \\ i + ad &= d^2, & l + bc &= c^2. \end{aligned}$$

One can now solve for the four coefficients $i, j, k,$ and $l,$ obtaining

$$(2) \quad i = f^2 - af, \quad j = -fc, \quad k = -ed, \quad l = e^2 - be.$$

Upon substituting these expressions into the 10 unused equations, 2 become identities and 4 are redundant; the remaining 4 are easily factored and can be expressed as

$$\begin{aligned} (c - e)(a - f - d) &= 0, & (c - e)(b - e - c) &= 0, \\ (d - f)(a - f - d) &= 0, & (d - f)(b - e - c) &= 0. \end{aligned}$$

Note that with these reductions, the multiplication in \mathcal{O}_X will be commutative if and only if $c = e$ and $d = f$; therefore, we may assume that either $c - e$ or $d - f$ is nonzero in \mathcal{O}_Y . In either case, the above equations imply that

$$(3) \quad f = a - d \quad \text{and} \quad e = b - c,$$

since \mathcal{O}_Y is an integral domain.

This completes the analysis of the conditions imposed by associativity. The final piece of data to be used is that E is the submodule of trace zero elements of \mathcal{O}_X ; this is equivalent to $\text{trace}(z) = \text{trace}(w) = 0,$ since trace is \mathcal{O}_Y -linear. A calculation shows immediately that, in our situation, $\text{trace}(z) = a + f$ and $\text{trace}(w) = b + e,$ so that

$$(4) \quad e = -b \quad \text{and} \quad f = -a.$$

Solving for the other coefficients in terms of a and $b,$ using (1)–(4), gives ϕ the form required by the proposition; moreover, a and b cannot both be zero (\mathcal{O}_X is commutative in this case).

Conversely, it is an easy exercise to check that if ϕ is in that form, then \mathcal{O}_X is noncommutative and associative, with E as the trace-zero submodule. Q.E.D.

COROLLARY 2. *Let \mathcal{O}_Y be a local integral domain over $k.$ Then every noncommutative \mathcal{O}_Y -algebra $\mathcal{O}_X,$ which is locally free of rank 3 as an \mathcal{O}_Y -module, is isomorphic to $\mathcal{O}_Y\{z, w\}/I,$ where $\mathcal{O}_Y\{z, w\}$ is the polynomial ring over \mathcal{O}_Y in the noncommuting variables z and $w,$ and I is the 2-sided ideal generated by*

$$(z + a)(z - 2a), \quad (w + b)(z - 2a), \quad (z + a)(w - 2b), \quad (w + b)(z - 2a),$$

for some a, b in $\mathcal{O}_Y,$ not both zero.

PROOF. When expanded, the above four equations for I become exactly the 4 multiplication rules for computing $z^2, zw, wz,$ and w^2 in terms of $1, z,$ and w in $\mathcal{O}_X,$ as given by Proposition 1. Q.E.D.

2. The global analysis. By Proposition 1, a noncommutative \mathcal{O}_Y -algebra $\mathcal{O}_X,$ which is locally free of rank 3 over $\mathcal{O}_Y,$ is locally determined by two elements a and b in $\mathcal{O}_Y.$ Although, as was previously remarked, the form of the multiplication map ϕ does not depend on the local choice of basis for the trace zero submodule $E,$ these two elements a and b certainly do. In order to globalize this analysis, we must describe the maps ϕ without resorting to any choice of basis.

Write $\phi = \phi_1 \oplus \phi_2,$ where $\phi_1: E \otimes_{\mathcal{O}_Y} E \rightarrow \mathcal{O}_Y$ is the 1st coordinate of ϕ and $\phi_2: E \otimes_{\mathcal{O}_Y} E \rightarrow E$ is the 2nd coordinate. Let $\underline{H}(E)$ be the submodule of

$\underline{\text{Hom}}_{\mathcal{O}_Y}(E \otimes E, E)$ consisting of those maps ϕ_2 which are locally of the form

$$\begin{aligned} \phi_2(z \otimes z) &= az, & \phi_2(z \otimes w) &= 2bz - aw, \\ \phi_2(w \otimes z) &= -bz + 2aw, & \phi_2(w \otimes w) &= bw \end{aligned}$$

for some local basis $\{z, w\}$ of E .

PROPOSITION 3. $\underline{H}(E)$ is canonically isomorphic to E^* .

PROOF. Define the transformation $\gamma: E^* \rightarrow \underline{H}(E)$ by sending a functional $\alpha: E \rightarrow \mathcal{O}_Y$ to the map $\phi_2(\alpha): E \otimes E \rightarrow E$, defined by

$$\phi_2(\alpha)(e_1 \otimes e_2) = 2\alpha(e_2)e_1 - \alpha(e_1)e_2,$$

for $e_1, e_2 \in E$. The reader can check that $\phi_2(\alpha)$ is indeed in $\underline{H}(E)$, and that γ is an isomorphism, by choosing a basis $\{z, w\}$ for E , and using the dual basis $\{z^*, w^*\}$ of E^* . The map ϕ_2 in the local form above corresponds to the functional defined by $\alpha(z) = a, \alpha(w) = b$. Q.E.D.

This description of $\underline{H}(E)$ completes the analysis of the second coordinate ϕ_2 of the multiplication map ϕ for \mathcal{O}_X ; since the description given by Proposition 3 is independent of the choice of basis for E , and is natural, the local analysis sheafifies, and so in general, ϕ_2 is induced from a global section of E^* .

By Proposition 1, the first coordinate ϕ_1 of ϕ is locally determined by ϕ_2 . In fact, there is a global coordinate-free description of ϕ_1 also.

Let ϕ_2 be a global section of $\underline{H}(E)$, corresponding to a global section of E^* , or a map $\alpha: E \rightarrow \mathcal{O}_Y$.

PROPOSITION 4. With the above notation, the map $\phi_1: E \otimes E \rightarrow \mathcal{O}_Y$ is the composition of $2(\alpha \otimes \alpha): E \otimes E \rightarrow \mathcal{O}_Y \otimes \mathcal{O}_Y$ with the multiplication in \mathcal{O}_Y .

PROOF. This can be checked locally. Let z, w be a local basis for E . If ϕ has the form of Proposition 1, then, as remarked during the proof of Proposition 3, its second coordinate ϕ_2 corresponds to the element $az^* + bw^*$, i.e., α is the map $\alpha(z) = a, \alpha(w) = b$. Therefore, $(2\alpha \otimes \alpha)(z \otimes z) = 2a^2, (2\alpha \otimes \alpha)(z \otimes w) = 2ab, (2\alpha \otimes \alpha)(w \otimes z) = 2ba,$ and $(2\alpha \otimes \alpha)(w \otimes w) = 2b^2$, which is exactly the map ϕ_1 . Q.E.D.

Putting these propositions together, we have the

THEOREM 5. Let Y be an integral k -scheme. Then:

(1) Isomorphism classes of noncommutative \mathcal{O}_Y -algebras \mathcal{O}_X which are locally free of rank 3 as \mathcal{O}_Y -modules are in one-to-one correspondence with isomorphism classes of pairs (E, α) , where E is a locally free rank 2 \mathcal{O}_Y -module and $\alpha: E \rightarrow \mathcal{O}_Y$ is a nontrivial \mathcal{O}_Y -linear map (or, equivalently, a global section of E^*).

Write $\mathcal{O}_X(E, \alpha)$ for the algebra corresponding to the pair (e, α) .

(2) $\mathcal{O}_X(E, \alpha) \cong \mathcal{O}_Y \oplus E$ as \mathcal{O}_Y -modules, and E corresponds to the submodule of $\mathcal{O}_X(E, \alpha)$ consisting of elements of trace zero. The multiplication map $\phi = \phi_1 \oplus \phi_2: E \otimes E \rightarrow \mathcal{O}_Y \oplus E$ is locally in the form of Proposition 1, for some local basis $\{z, w\}$ of E . Globally, the map $\phi_2 \in \Gamma(\underline{H}(E))$ corresponds to the map α under the isomorphism of Proposition 3, and the map ϕ_1 is $2\alpha \otimes \alpha$.

An algebra over a k -scheme Y restricts to an algebra over each of its closed points, i.e. an algebra over the residue fields. In this way, an \mathcal{O}_Y -algebra can be viewed as a family of k -algebras parametrized by Y .

PROPOSITION 6. Let y be a closed point of the integral k -scheme Y , and let $\mathcal{O}_X(E, \alpha)$ be a noncommutative \mathcal{O}_Y -algebra of rank 3. Let $\mathcal{O}_X(y) = \mathcal{O}_X(E, \alpha) \otimes k(y)$ be the restriction of $\mathcal{O}_X(E, \alpha)$ to the residue field $k(y)$ at y .

(1) If $\alpha \neq 0$ at y , then $\mathcal{O}_X(y)$ is isomorphic to the algebra of 2×2 upper triangular matrices over $k(y)$.

(2) If $\alpha = 0$ at y , then $\mathcal{O}_X(y) \cong k[z, w]/(z^2, zw, w^2)$.

PROOF. We may of course work locally for these statements, and choose a basis $\{z, w\}$ for E over the local ring of Y at y . Write $\alpha(z) = a$, $\alpha(w) = b$; then the form of Proposition 1 applies, and we see that if a and b are both zero at y , then $z^2 = zw = wz = w^2 = 0$, proving (2). To prove (1), we may assume $a \neq 0$ at y . Then the map

$$1 \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z \rightsquigarrow \begin{pmatrix} -a & 0 \\ 0 & 2a \end{pmatrix}, \quad w \rightsquigarrow \begin{pmatrix} -b & 1 \\ 0 & 2b \end{pmatrix}$$

is an isomorphism of $\mathcal{O}_X(y)$ with the algebra of upper triangular 2×2 matrices over $k(y)$. Q.E.D.

That the degeneration of $\left\{ \begin{pmatrix} k & k \\ 0 & k \end{pmatrix} \right\}$ to $k[z, w]/(z^2, zw, w^2)$ is an essentially codimension 2 phenomenon is shown by the above proposition.

3. $\mathcal{O}_X(E, \alpha)$ as an algebra of endomorphisms. The aim of this final section is to globalize Proposition 6, i.e. to show that $\mathcal{O}_X = \mathcal{O}_X(E, \alpha)$ is an algebra of endomorphisms of a locally free rank 2 \mathcal{O}_Y -module, in fact of E . Let $\pi_2: \mathcal{O}_X \rightarrow E$ be the canonical projection, and let $r: \mathcal{O}_X \rightarrow \underline{\text{Hom}}(\mathcal{O}_X, \mathcal{O}_X)$ be the regular representation of \mathcal{O}_X . The projection π_2 induces a map $\pi: \underline{\text{Hom}}(\mathcal{O}_X, \mathcal{O}_X) \rightarrow \underline{\text{Hom}}(\mathcal{O}_X, E)$ and the inclusion of E into \mathcal{O}_X induces $i: \underline{\text{Hom}}(\mathcal{O}_X, E) \rightarrow \underline{\text{Hom}}(E, E)$. Let $\beta_1 = i \circ \pi \circ r: \mathcal{O}_X \rightarrow \underline{\text{Hom}}(E, E)$; it is an \mathcal{O}_Y -linear map, and locally, $\beta_1(x)$ sends $e \in E$ to $\pi_2(xe)$.

Let $\gamma: E \rightarrow \underline{\text{Hom}}(\mathcal{O}_Y, E)$ be the natural isomorphism, and let $\alpha': \underline{\text{Hom}}(\mathcal{O}_Y, E) \rightarrow \underline{\text{Hom}}(E, E)$ be the map given by composition with $\alpha: E \rightarrow \mathcal{O}_Y$. Then $\beta_2 = \alpha' \circ \gamma \circ \pi_2: \mathcal{O}_X \rightarrow \underline{\text{Hom}}(E, E)$ is \mathcal{O}_Y -linear and, locally, $\beta_2(x)$ sends $e \in E$ to $\alpha(e) \cdot \pi_2(x)$.

PROPOSITION 7. The \mathcal{O}_Y -linear map $\beta = \beta_1 + \beta_2: \mathcal{O}_X \rightarrow \underline{\text{Hom}}(E, E)$ is an \mathcal{O}_Y -algebra monomorphism.

PROOF. This can be checked locally. Choose a basis $\{z, w\}$ for E , and write $\alpha(z) = a$, $\alpha(w) = b$. By identifying $\underline{\text{Hom}}(E, E)$ with 2×2 matrices over \mathcal{O}_Y , we have

$$\beta(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \beta(z) = \begin{pmatrix} 2a & 3b \\ 0 & -a \end{pmatrix}, \quad \beta(w) = \begin{pmatrix} -b & 0 \\ 3a & 2b \end{pmatrix},$$

using the definition of $\beta(x)(e) = \pi_2(xe) + \alpha(e)\pi_2(x)$. It is an easy exercise to verify that $\beta(z^2) = \beta(z)^2$, $\beta(zw) = \beta(z)\beta(w)$, $\beta(wz) = \beta(w)\beta(z)$, and $\beta(w^2) = \beta(w)^2$, which we will leave to the reader. This suffices to prove the proposition. Q.E.D.

Note that the map α is recovered from this representation by the composition

$$E \rightarrow \mathcal{O}_X \xrightarrow{\beta} \underline{\text{Hom}}(E, E) \xrightarrow{\text{trace}} \mathcal{O}_Y.$$

The subalgebra of $\underline{\text{Hom}}(E, E)$ isomorphic to \mathcal{O}_X consists entirely of endomorphisms which locally “factor through α ”:

LEMMA 8. *Locally, every endomorphism g in $\beta(\mathcal{O}_X)$ is such that $\alpha \circ g$ factors through α , i.e. there exists an element $t \in \mathcal{O}_Y$, such that the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\alpha} & \mathcal{O}_Y \\ g \downarrow & & \downarrow \text{multiplication by } t \\ E & \xrightarrow{\alpha} & \mathcal{O}_Y \end{array}$$

commutes.

PROOF. It suffices to check that, if $\{z, w\}$ is a local basis for E over \mathcal{O}_Y , then $\beta(1), \beta(z)$, and $\beta(w)$ satisfy the above diagram for some t . We leave it to the reader to check that $t = 1$ works for $\beta(1)$, $t = 2a$ works for $\beta(z)$, and $t = 2b$ works for $\beta(w)$. Q.E.D.

By Lemma 8, any endomorphism in $\beta(\mathcal{O}_X)$ must preserve the kernel of α , which is a rank 1 subsheaf of E ; at points where $\alpha \neq 0$, a basis for E may be extended from a generator for this kernel, and the elements of $\beta(\mathcal{O}_X)$ will be represented by upper triangular matrices in this basis. However, it is not true that $\beta(\mathcal{O}_X)$ consists of all such matrices, i.e. $\beta(\mathcal{O}_X)$ is not the algebra of all g in $\underline{\text{End}}(E)$ which locally satisfy Lemma 8. Our algebra $\beta(\mathcal{O}_X)$ is only the algebra of g 's which "obviously" satisfy the commutative diagram. Let us be more precise: Motivated by the construction of the Koszul complex, we make the following

DEFINITION 9. Let E be a locally free rank 2 \mathcal{O}_Y -module and $\alpha: E \rightarrow \mathcal{O}_Y$ a nontrivial \mathcal{O}_Y -linear map. The Koszul algebra of (E, α) , denoted by $K(E, \alpha)$ is the subalgebra of $\underline{\text{End}}(E)$ generated (locally) by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$$

with respect to some local basis $\{z, w\}$ of E , where $\alpha(z) = a, \alpha(w) = b$.

It can be easily checked that $K(e, \alpha)$ is locally free of rank 3 as an \mathcal{O}_Y -module, and that it is a subalgebra of $\underline{\text{End}}(E)$. Moreover, every element of $K(E, \alpha)$ locally satisfies Lemma 8, and the algebra is independent of the local choice of basis for E .

THEOREM 10. *The map β is an isomorphism of $\mathcal{O}_X(E, \alpha)$ onto the Koszul algebra $K(E, \alpha)$.*

PROOF. This can be checked locally; choose a basis $\{z, w\}$ for E . Then

$$\beta(z) = \begin{pmatrix} 2a & 3b \\ 0 & -a \end{pmatrix} = -a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 3 \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix},$$

and similarly for $\beta(w)$, showing that β maps $\mathcal{O}_X(E, \alpha)$ into $K(E, \alpha)$. On the other hand, $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \frac{1}{3}[\beta(z) + a\beta(1)]$ and similarly for $\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$ which proves surjectivity. Q.E.D.

It may seem that we have defined our way out of identifying the algebra $\mathcal{O}_X(E, \alpha)$ by the above, and that is a fair criticism. To be complete, we should answer the following question: How far is the Koszul algebra from the full subalgebra of $\underline{\text{End}}(E)$ consisting of elements which locally satisfy Lemma 8? Let us denote this algebra by $\hat{\mathcal{O}}_X(E, \alpha)$ and let \underline{C} be the cokernel of the inclusion of $\mathcal{O}_X(E, \alpha) \cong K(E, \alpha)$ into $\hat{\mathcal{O}}_X(E, \alpha)$. Our answer to the above question is to identify the algebra $\hat{\mathcal{O}}_X(E, \alpha)$.

PROPOSITION 11. (1) *The sheaf \underline{C} is supported on the zero locus Z of α (whose ideal sheaf is $\alpha(E) \subset \mathcal{O}_Y$).*

(2) *Let D be the divisor of zeroes of α , and write $E(D)$ for $E \otimes \mathcal{O}_Y(D)$. Note that α factors through $\mathcal{O}_Y(-D)$, thereby inducing a map $\alpha(D): E(D) \rightarrow \mathcal{O}_Y$. Then $\tilde{\mathcal{O}}_X(E, \alpha) \cong \tilde{\mathcal{O}}_X(E(D), \alpha(D))$.*

(3) *If Y is factorial and D is zero, then \underline{C} is zero and $\tilde{\mathcal{O}}_X(E, \alpha) \cong \mathcal{O}_X(E, \alpha)$.*

(4) *If Y is factorial, then $\tilde{\mathcal{O}}_X(E, \alpha) \cong \mathcal{O}_X(E(D), \alpha(D))$.*

PROOF. To prove (1), we may work locally, and choose a basis $\{z, w\}$ for E ; moreover, if we assume $\alpha \neq 0$ at $y \in Y$, then we may assume $a \neq 0$ at y , so that a is a unit in the local ring $\mathcal{O}_{Y,y}$ at y , and by replacing z by $a^{-1}z$ we may assume $a = 1$. In this case $K(E, \alpha)$ is generated by the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$, and $\begin{pmatrix} 0 & 0 \\ 1 & b \end{pmatrix}$. Assume finally that $\begin{pmatrix} p & q \\ u & v \end{pmatrix}$ is in $\tilde{\mathcal{O}}_X(E, \alpha)$. Hence there is a $t \in \mathcal{O}_{Y,y}$ such that $p + bu = t$ and $q + bv = tb$. Therefore,

$$\begin{pmatrix} p & q \\ u & v \end{pmatrix} = (v - ub) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (t - v) \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} + u \begin{pmatrix} 0 & 0 \\ 1 & b \end{pmatrix}$$

and is in $K(E, \alpha)$, proving (1).

To prove (2), consider the map $\bar{\alpha}: E \rightarrow \mathcal{O}_Y(-D)$ (which is $\alpha!$), and note that for an endomorphism g of E , $\alpha \circ g$ factors through α if and only if $\bar{\alpha} \circ g$ factors through $\bar{\alpha}$. In addition, the algebra of endomorphisms of E factoring through $\bar{\alpha}$ is isomorphic to the algebra of endomorphisms of $E(D)$ factoring through $\alpha(D)$; the isomorphism is obtained by twisting the maps by $\mathcal{O}_Y(D)$. Hence $\tilde{\mathcal{O}}_X(E, \alpha)$ is isomorphic to $\tilde{\mathcal{O}}_X(E(D), \alpha(D))$.

For the rest, assume Y is factorial. Working locally, if $D = 0$ then, when one factors a and b into irreducibles, there can be no common factors. Moreover, since α is not identically zero, we may assume that neither a nor b is zero. Let $\begin{pmatrix} p & q \\ u & v \end{pmatrix}$ be in $\tilde{\mathcal{O}}_X(E, \alpha)$. Then there exists t in \mathcal{O}_Y such that $ap + bu = at$ and $aq + bv = bt$. Therefore, $a|bu$ and $b|aq$, and since $D = 0$, $a|u$ and $b|q$. Write $u = ar$ and $q = bs$; then $p + br = t$ and $as + v = t$, so that $p - as = v - br$. Call this element x ; then

$$\begin{pmatrix} p & q \\ u & v \end{pmatrix} = x \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + s \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} + r \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix},$$

and is therefore in $K(E, \alpha) \cong \mathcal{O}_X(E, \alpha)$. Thus $\underline{C} = 0$, proving (3).

Statement (4) follows from applying (3) to the pair $(E(D), \alpha(D))$; by construction, the divisor of zeroes of $\alpha(D)$ is zero, so that

$$\tilde{\mathcal{O}}_X(E(D), \alpha(D)) \cong \mathcal{O}_X(E(D), \alpha(D)).$$

By (2), $\tilde{\mathcal{O}}_X(E, \alpha) \cong \tilde{\mathcal{O}}_X(E(D), \alpha(D))$ in any case; combining these isomorphisms proves (4). Q.E.D.

Finally, let us address the following situation. Suppose L is a locally free rank 1 subsheaf of a locally free rank 2 sheaf F . Let A be the sheaf of algebras of endomorphisms of F preserving L . The above proposition allows us to determine A in the case where Y is factorial, as follows. Let D be the divisor of zeroes of the inclusion $i: L \rightarrow F$. By twisting and dualizing, i corresponds to a map $\alpha: F^* \otimes L \rightarrow \mathcal{O}_Y$ whose divisor of zeroes is also D ; moreover, A is naturally isomorphic to $\tilde{\mathcal{O}}_X(F^* \otimes L, \alpha)$. Therefore by Proposition 11, (4), $A \cong \mathcal{O}_X(F^* \otimes L \otimes \mathcal{O}_Y(D), \alpha(D))$.

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